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Stochastic Linear Quadratic Optimal Control Problems*

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Abstract. This paper is concerned with the stochastic linear quadratic optimal control problem (LQ problem, for short) for which the coefficients are allowed to be random and the cost functional is allowed to have a negative weight on the square of the control variable. Some intrinsic relations among the LQ problem, the stochastic maximum principle, and the (linear) forward–backward stochastic differential equations are established. Some results involving Riccati equation are discussed as well.

Key Words. Stochastic LQ problem, Stochastic maximum principle, Forward–backward stochastic differential equations, Riccati equation.

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1. Introduction—A General Formulation and Some Examples

Let $(\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_t\}_{t\geq 0})$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $w(\cdot)$ is defined such that $\{\mathcal{F}_t\}_{t\geq 0}$ is the

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natural filtration generated by $w(\cdot)$, augmented by all the $\mathbf{P}$-null sets in $\mathcal{F}$. We consider the following linear controlled stochastic differential equation:

$$\begin{cases} dx(t) = [A(t)x(t) + B(t)u(t)]\,dt \\ \qquad\qquad + [C(t)x(t) + D(t)u(t)]\,dw(t), \qquad t \in [\tau, T], \\ x(\tau) = \xi, \end{cases} \tag{1.1}$$

where $\tau \in \mathcal{T}[0,T]$, the set of all $\{\mathcal{F}_t\}_{t\geq 0}$-stopping times taking values in $[0,T]$, $\xi \in \mathcal{X}_\tau \overset{\Delta}{=} L^2_{\mathcal{F}_\tau}(\Omega;\mathbb{R}^n)$, the set of all $\mathbb{R}^n$-valued $\mathcal{F}_\tau$-measurable square-integrable random variables; A, B, C, D are matrix-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted bounded processes. In the above, $u(\cdot)$ is a *control process* and $x(\cdot)$ is the corresponding state process. Let $\mathcal{U}[\tau,T] = L^2_{\mathcal{F}}(\tau,T;\mathbb{R}^m)$, the set of all $\mathbb{R}^m$-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted square-integrable processes defined on the random interval $[\tau,T]$ (with $\tau \in \mathcal{T}[0,T]$). The control process $u(\cdot)$ is take from $\mathcal{U}[\tau,T]$.

Clearly, for any $(\xi, u(\cdot)) \in \mathcal{X}_\tau \times \mathcal{U}[\tau,T]$, there exists a unique (strong) solution $x(\cdot) \in L^2_{\mathcal{F}}(\tau,T;\mathbb{R}^n)$ to (1.1). Thus, we can define a *cost functional* as follows:

$$\begin{aligned} J(\tau,\xi;u(\cdot)) = E\Bigg\{\int_\tau^T [\langle Q(t)x(t),x(t)\rangle + \langle R(t)u(t),u(t)\rangle]\,dt \\ + \langle Gx(T),x(T)\rangle | \mathcal{F}_\tau\Bigg\}, \end{aligned} \tag{1.2}$$

where $Q(\cdot)$ and $R(\cdot)$ are symmetric matrix-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted bounded processes and G is a symmetric matrix-valued $\mathcal{F}_T$-measurable bounded random variable. It is seen that

$$J\colon \bigcup_{\tau\in\mathcal{T}[0,T]} (\{\tau\}\times\mathcal{X}_\tau\times\mathcal{U}[\tau,T]) \to L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}). \tag{1.3}$$

We now state the stochastic linear quadratic optimal control problem as follows:

Problem (LQ). For each $\tau \in \mathcal{T}[0,T]$ and $\xi \in \mathcal{X}_\tau$, find a $\overline{u}(\cdot) \in \mathcal{U}[\tau,T]$ such that

$$J(\tau,\xi;\overline{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[\tau,T]} J(\tau,\xi;u(\cdot)) \overset{\Delta}{=} V(\tau,\xi), \qquad \text{a.s.} \quad \omega\in\Omega. \tag{1.4}$$

We call V the *value function* of Problem (LQ). Note that

$$V(T,\xi) = \langle G\xi,\xi\rangle, \qquad \forall \xi \in \mathcal{X}_T. \tag{1.5}$$

For the state equation (1.1), one might introduce another cost functional:

$$\begin{aligned} \overline{J}(\tau,\xi;u(\cdot)) &= EJ(\tau,\xi;u(\cdot)) \\ &\equiv E\Bigg\{\int_\tau^T \Big[\langle Q(t)x(t),x(t)\rangle + \langle R(t)u(t),u(t)\rangle\Big]\,dt \\ &\qquad + \langle Gx(T),x(T)\rangle\Bigg\}, \end{aligned} \tag{1.6}$$

and pose a similar optimal control problem. Such a formulation has been used for the case of deterministic coefficients and within the *weak* solution framework for the state equation (see [11], [28], and the references cited therein). However, in this paper we study LQ problems with random coefficients. Thus, it is necessary to use the *strong* solution framework for which the probability space $(\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_t\}_{t\geq 0})$ has to be fixed. In such a framework, cost functional (1.2) is more appropriate than (1.6).

Stochastic LQ problems have been studied by many authors, among them we mention [25], [22], [9], and [13]. In [22] one can find some applications in engineering giving rise to problems with state/control dependent diffusion (see [16] also). On the other hand, in many recent works on mathematical finance, the portfolio regarded as the control appears in the diffusion (see [15], [11], [10], and [28] for extensive discussions). It is quite understandable that the coefficients of the system (as well as the cost functional) could depend on some other diffusion processes, and, therefore, they could be random. This is the case in problems like option pricing, utility optimization, etc. The point here is that the LQ problem with random coefficients has both mathematical interest and potential applications in other fields.

In what follows, we make the following convention: By $\tau \in [0, T]$ and $\xi \in \mathbb{R}^n$, we mean that τ and ξ are deterministic (compare with $\tau \in \mathcal{T}[0, T]$ and $\xi \in \mathcal{X}_\tau$). Next, for any $\sigma, \tau \in \mathcal{T}[0, T]$, with $\sigma \leq \tau$ almost surely, we let $\mathcal{T}[\sigma, \tau]$ be the set of all $\{\mathcal{F}_t\}_{t\geq 0}$-stopping times r such that $\sigma \leq r \leq \tau$, almost surely, and let $\Delta[\sigma, \tau] = \bigcup_{r\in\mathcal{T}[\sigma,\tau]}[\{r\} \times \mathcal{X}_r]$. The meanings of $\mathcal{T}[\sigma, \tau)$, $\Delta[\sigma, \tau)$, etc., are obvious. By (1.3), one has

$$V\colon \Delta[0, T] \to L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}).$$

We recall the corresponding deterministic LQ problem. Consider

$$\begin{cases} \dfrac{d}{dt}x(t) = A(t)x(t) + B(t)u(t), \qquad t \in [\tau, T], \\ x(\tau) = \xi, \end{cases} \tag{1.7}$$

with $A(\cdot)$ and $B(\cdot)$ being bounded (deterministic) matrix-valued functions, $(\tau, \xi) \in [0, T) \times \mathbb{R}^n$ and $u(\cdot) \in L^2(0, T; \mathbb{R}^m)$. The cost functional is

$$\begin{aligned} \widehat{J}(\tau, \xi; u(\cdot)) = \int_\tau^T \big[\langle Q(t)x(t), x(t)\rangle + \langle R(t)u(t), u(t)\rangle\big]\, dt \\ + \langle Gx(T), x(T)\rangle, \end{aligned} \tag{1.8}$$

for some bounded symmetric matrix-valued functions $Q(\cdot)$ and $R(\cdot)$, and a symmetric matrix G (all are deterministic). The deterministic LQ problem is to minimize (1.8) subject to (1.7). For such a problem, there is an extensive literature, see [14], [1], [6], and [7], to mention a few, and the references cited therein (see also [17] and [18] for infinite-dimensional cases). It is well known that (see [1], for example) for deterministic LQ problems, a necessary condition for the value function to be finite is

$$R(t) \geq 0, \quad \text{a.e.}; \tag{1.9}$$

and if (1.9) holds with $R(t)$ being degenerate on a set of positive Lebesgue measure, the LQ problem might have no optimal control in general (see an example below). Inspired by this, when people study stochastic LQ problems, the positive definiteness condition for (the deterministic function) $R(\cdot)$ was also assumed which led to some theories completely parallel to the deterministic one (see [25], [9], and [3]). Recently, it has been pointed out in [8] that condition (1.9) seems neither necessary for the infimum of the cost functional being finite, nor for the existence of optimal controls (see an example below). This reveals one of the significant differences between deterministic and stochastic LQ problems. To make the situation more appealing, we present several examples.

Example 1.1. Consider the following one-dimensional deterministic control system:

$$\begin{cases} \dot{x}(t) = u(t), \qquad t \in [\tau, T], \\ x(\tau) = \xi \in \mathbb{R}, \end{cases} \tag{1.10}$$

with $\tau \in [0, T)$ and the cost functional

$$\widehat{J}_1(\tau, \xi; u(\cdot)) = -\int_\tau^T u(t)^2\, dt + x(T)^2. \tag{1.11}$$

Since (1.11) has a negative weight on the term u^2, by a direct computation, we have

$$\widehat{V}_1(\tau, \xi) \equiv \inf_{u(\cdot)\in\mathcal{U}[\tau,T]} \widehat{J}_1(\tau, \xi; u(\cdot)) = -\infty, \qquad \forall (\tau, \xi) \in [0, T] \times \mathbb{R}, \tag{1.12}$$

and, of course, no optimal control exists.

Now, we consider the following stochastic control system (compare with (1.10)):

$$\begin{cases} dx(t) = u(t)\, dt + \delta u(t)\, dw(t), \qquad t \in [\tau, T], \\ x(\tau) = \xi \in \mathcal{X}_\tau, \end{cases} \tag{1.13}$$

for some $\delta \neq 0$, with the cost functionals (compare with (1.11))

$$J(\tau, \xi; u(\cdot)) = E\left\{ -\int_\tau^T u(t)^2\, dt + x(T)^2 \;\Big|\; \mathcal{F}_\tau \right\}. \tag{1.14}$$

It is seen from (1.13) that the control affects the size of the noise in the system. We can prove (see Section 5) that, for any $|\delta| > 1$ with

$$\delta^2\big(2 \ln|\delta| - 1\big) > T - 1, \tag{1.15}$$

Problem (LQ) admits optimal controls for any $\tau \in \mathcal{T}[0, T]$, and, thus, the value function is finite, in particular. An intuitive explanation is that even if the control is "rewarding" in the cost functional, due to the "noise" affected by the control in the system, it is *not* necessarily that "the bigger the control, the better."

Example 1.2. Consider control system (1.10) with the cost functional

$$\widehat{J}_2(\tau, \xi; u(\cdot)) = \int_\tau^T x(t)^2\, dt. \tag{1.16}$$

In this case, (1.9) holds with (actually) $R = 0$. A direct calculation shows that

$$\widehat{V}_2(\tau, \xi) \equiv \inf_{u(\cdot)\in\mathcal{U}[\tau,T]} \widehat{J}_2(\tau, \xi; u(\cdot)) = 0, \qquad \forall (\tau, \xi) \in [0, T) \times \mathbb{R}. \tag{1.17}$$

On the other hand, for any $\xi \neq 0$, one can show that there exists no optimal control. This control problem is known as a *singular* LQ problem.

Now, we consider the stochastic control system (1.13) with the cost functionals

$$J(\tau, \xi; u(\cdot)) = E\left\{\int_\tau^T x(t)^2\,dt \;\Big|\; \mathcal{F}_\tau\right\}, \tag{1.18}$$

Define

$$p(t) = \delta^2\left(1 - e^{(t-T)/\delta^2}\right), \qquad t \in [0, T]. \tag{1.19}$$

By Itô's formula, we have

$$\begin{aligned}
0 &= E\left\{p(T)x(T)^2 \;\big|\; \mathcal{F}_\tau\right\} \\
&= p(\tau)\xi^2 \\
&\quad + E\left\{\int_\tau^T \left[\left(\frac{1}{\delta^2}p(t) - 1\right)x(t)^2 + 2p(t)x(t)u(t) + \delta^2 p(t)u(t)^2\right]dt \;\Big|\; \mathcal{F}_\tau\right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
J(\tau, \xi; u) &= E\left\{\int_\tau^T x(t)^2\,dt \;\Big|\; \mathcal{F}_\tau\right\} \\
&= p(\tau)\xi^2 + E\left\{\int_\tau^T \delta^2 p(t)\left[u(t) + \frac{x(t)}{\delta^2}\right]^2 dt \;\Big|\; \mathcal{F}_\tau\right\},
\end{aligned}$$

which implies that Problem (LQ) admits an optimal control given by the following state feedback form:

$$u(t) = -\frac{x(t)}{\delta^2}, \qquad t \in [\tau, T], \tag{1.20}$$

and the value function is given by

$$V(\tau, \xi) = \delta^2\left(1 - e^{(\tau-T)/\delta^2}\right)\xi^2, \qquad \forall(\tau, \xi) \in \Delta[0, T]. \tag{1.21}$$

This example shows that a deterministic singular LQ problem may become *non-singular* if the noise exists in the control system.

Example 1.3. Consider control system (1.10) with the cost functional

$$\widehat{J}_3(\tau, \xi; u(\cdot)) = \int_\tau^T u(t)^2\,dt - x(T)^2. \tag{1.22}$$

This is an LQ problem with the weight on the square of the terminal state being negative. Let $T > 1$ and $\tau \in (T-1, T]$. For any $u(\cdot) \in L^2(\tau, T; \mathbb{R})$, let $x(\cdot)$ be the corresponding state trajectory. Applying the Newton–Leibniz formula to $x(t)^2/(t+1-T)$ over $[\tau, T]$, we have

$$x(T)^2 = \frac{\xi^2}{\tau + 1 - T} + \int_\tau^T \left\{\frac{2x(t)u(t)}{t+1-T} - \frac{x(t)^2}{(t+1-T)^2}\right\}dt.$$

Thus,

$$\widehat{J_3}(\tau,\xi;u(\cdot)) = \frac{-\xi^2}{\tau+1-T} + \int_\tau^T \left\{u(t) - \frac{x(t)}{t+1-T}\right\}^2 dt.$$

Consequently, the optimal control is given by

$$u(t) = \frac{x(t)}{t+1-T}, \qquad t \in [\tau, T], \tag{1.23}$$

and the corresponding value function is given by

$$\begin{aligned}\widehat{V_3}(\tau,\xi) &\equiv \inf_{u(\cdot)\in\mathcal{U}[\tau,T]} \widehat{J_3}(\tau,\xi;u(\cdot)) \\ &= \frac{-\xi^2}{\tau+1-T}, \qquad \forall(\tau,\xi)\in(T-1,T]\times\mathbb{R}.\end{aligned} \tag{1.24}$$

However, if we consider the stochastic control system (1.13) with the cost functionals

$$J(\tau,\xi;u(\cdot)) = E\left\{\int_\tau^T u(t)^2\,dt - x(T)^2 \;\Big|\; \mathcal{F}_\tau\right\}, \tag{1.25}$$

then for any $|\delta| > 1$, we can prove that $V(\tau,\xi)$ is not finite for any $\tau < T$, and, therefore, there will be no optimal controls (see Section 2).

This example shows that a well-posed deterministic LQ problem may become "ill-posed" if the noise gets into the control system.

From the above examples, we have seen that the stochastic LQ problem is quite different from its deterministic counterpart, mainly due to the appearance of the control in the diffusion. We will see more about this shortly.

The following simple example shows another interesting feature of Problem (LQ).

Example 1.4. Consider control system (1.10) with the cost functionals

$$J(\tau,\xi;u) = E\left\{Gx(T)^2 \;\big|\; \mathcal{F}_\tau\right\}, \tag{1.26}$$

where G is an $\mathcal{F}_\sigma$-measurable random variable such that

$$0 < \mathbf{P}(G < 0) < 1, \tag{1.27}$$

with $\sigma \in [0, T)$. Then, for any $\tau \ge \sigma$ and $\xi \in \mathcal{X}_\tau$, by taking

$$u_\varepsilon(\cdot) = -\frac{\xi}{T-\tau} + \frac{1}{\varepsilon(T-\tau)} I_{(G<0)} \in \mathcal{U}[\tau, T],$$

we have

$$\begin{aligned}J(\tau,\xi;u_\varepsilon) &= E\left\{G\left(\xi + \int_\tau^T u_\varepsilon(s)\,ds\right)^2 \;\Big|\; \mathcal{F}_\tau\right\} \\ &= E\left\{\frac{G}{\varepsilon^2} I_{(G<0)} \;\Big|\; \mathcal{F}_\tau\right\} = \frac{G}{\varepsilon^2} I_{(G<0)}.\end{aligned} \tag{1.28}$$

Letting $\varepsilon \to 0$, we see that

$$V(\tau, \xi) = \begin{cases} 0, & \text{on } (G \geq 0), \\ -\infty, & \text{on } (G < 0). \end{cases} \tag{1.29}$$

Hence, the value function $V(\tau, \xi)$ of Problem (LQ) is finite on a subset of Ω and (1.28)–(1.29) imply that

$$J(\tau, \xi; u_\varepsilon) = \inf_{u \in \mathcal{U}[\tau, T]} J(\tau, \xi; u), \qquad \text{a.s.} \quad \omega \in (G \geq 0). \tag{1.30}$$

Thus, u_ε is "partially" optimal.

From this example, we see that when the coefficients are allowed to be random, the situation could be very rich.

The rest of this paper is organized as follows. In Section 2, we introduce some basic notions and state the main results of this paper. In Section 3 we use functional analysis, backward, and/or forward–backward stochastic differential equations to study the LQ problem. A stochastic maximum principle is derived. In Section 4 we present a necessary condition for our LQ problem to be finite. In Section 5 we briefly discuss the case of constant coefficients via the Riccati equation. The results of this paper set up a solid base for further study of Problem (LQ) in our forthcoming publications.

We point out that all the results of this paper can be carried out for control systems with multidimensional Brownian motions. For the simplicity of presentation, we restrict ourselves to the case of one-dimensional Brownian motion.

2. Finiteness and Solvability

We let $\mathcal{S}^n$ be the set of all $(n \times n)$ symmetric matrices. Let $L^\infty_{\mathcal{F}}(0, T; X)$ (resp. $C_{\mathcal{F}}([0, T]; X)$) be the set of all X-valued $\{\mathcal{F}_t\}_{t \geq 0}$-adapted bounded (resp. bounded continuous) processes, and let $L^\infty_{\mathcal{F}_T}(\Omega; X)$ be the set of all X-valued $\mathcal{F}_T$-measurable bounded random variables, where X could be $\mathbb{R}^n$, $\mathbb{R}^{n \times n}$, $\mathcal{S}^n$, etc. Also, we recall, from Section 1 that $\mathcal{X}_\tau \stackrel{\Delta}{=} L^2_{\mathcal{F}_\tau}(\Omega; \mathbb{R}^n)$ for any $\tau \in \mathcal{T}[0, T]$, and $\Delta[\sigma, \tau] = \bigcup_{r \in \mathcal{T}[\sigma, \tau]} [\{r\} \times \mathcal{X}_r]$. We denote $\mathcal{X}[\tau, T] \stackrel{\Delta}{=} L^2_{\mathcal{F}}(\tau, T; \mathbb{R}^n)$, the set of all $\mathbb{R}^n$-valued $\{\mathcal{F}_t\}_{t \geq 0}$-adapted square integrable processes over $[\tau, T]$.

We introduce the following basic assumptions:

(S) Let

$$\begin{cases} A, C \in L^\infty_{\mathcal{F}}(0, T; \mathbb{R}^{n \times n}), \qquad B \in L^\infty_{\mathcal{F}}(0, T; \mathbb{R}^{n \times m}), \\ \qquad D \in C_{\mathcal{F}}([0, T]; \mathbb{R}^{n \times m}), \\ Q \in C_{\mathcal{F}}([0, T]; \mathcal{S}^n), \qquad R \in C_{\mathcal{F}}([0, T]; \mathcal{S}^m), \\ \qquad G \in L^\infty_{\mathcal{F}_T}(\Omega; \mathcal{S}^n). \end{cases} \tag{2.1}$$

We introduce the following definitions.

Definition 2.1. Problem (LQ) is said to be

(i) *partially finite* at $(\tau, \xi) \in \Delta[0, T]$ if

$$\mathbf{P}(V(\tau, \xi) > -\infty) > 0; \tag{2.2}$$

(ii) *(uniquely) partially solvable* at $(\tau, \xi) \in \Delta[0, T]$ if there exists a (unique) control $\overline{u}(\cdot) \in \mathcal{U}[\tau, T]$ such that

$$J(\tau, \xi; \overline{u}) = V(\tau, \xi), \qquad \text{a.s.} \quad \omega \in (V(\tau, \xi) > -\infty). \tag{2.3}$$

In case (ii), control $\overline{u}(\cdot)$ is called a *partially optimal control*, the corresponding $\overline{x}(\cdot)$ is called a *partially optimal state process*, and $(\overline{x}(\cdot), \overline{u}(\cdot))$ is called a *partially optimal pair*.

It is clear that (ii) implies (i), and the converse seems untrue. If one has

$$\mathbf{P}\big(V(\tau, \xi) > -\infty\big) = 1, \tag{2.4}$$

we omit the word "partial" in the above three notions.

If, for $\tau \in \mathcal{T}[0, T]$, Problem (LQ) is finite (resp. (uniquely) solvable) at all (τ, ξ) with $\xi \in \mathcal{X}_\tau$, we say that Problem (LQ) is *finite* (resp. (*uniquely*) *solvable*) at τ. If Problem (LQ) is finite (resp. (uniquely) solvable) at all $\tau \in \mathcal{T}[0, T]$, we say that Problem (LQ) is *finite* (resp. (*uniquely*) *solvable*).

The first main result of this paper is the following:

Theorem 2.2. *Let* (S) *hold. Suppose Problem* (LQ) *is partially finite at some* $(\tau, \xi) \in \Delta[0, T)$. *Then*

$$R(T) + D(T)^T GD(T) \geq 0, \qquad a.s. \quad \omega \in (V(\tau, \xi) > -\infty). \tag{2.5}$$

In the case that R, D, and G are deterministic, (2.5) is equivalent to

$$R + D^T GD \geq 0. \tag{2.6}$$

From this, we see the role played by the appearance of the control in the diffusion (of the state equation). When $D = 0$, we recover the well-known condition (see (1.9)) for the deterministic LQ problem. Note that in Example 1.3, when $|\delta| > 1$,

$$R(T) + D(T)^T GD(T) = 1 - \delta^2 < 0.$$

Thus, (2.6) is violated and $V(\tau, \xi)$ must be equal to $-\infty$, as we claimed in Section 1. From (2.5), we also see that if G is positive and $D \neq 0$, R is allowed to be a little negative.

Next, we introduce the following stochastic differential equation:

$$\begin{cases} dp(t) = -[A^T p(t) + C^T q(t) + y(t)]dt + q(t)\, dw(t), & t \in [\tau, T], \\ p(T) = \eta \in \mathcal{X}_T. \end{cases} \tag{2.7}$$

This is a terminal value problem. Here, $y(\cdot) \in \mathcal{X}[\tau, T]$ and $\eta \in \mathcal{X}_T$. We are looking for a pair of $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes $(p(\cdot), q(\cdot))$, called an *adapted solution*, satisfying (2.7). Equation (2.7) is called a *backward stochastic differential equation* (BSDE, for short). The following is known (see [23]).

Proposition 2.3. *For any $y(\cdot) \in \mathcal{X}[\tau, T]$ and $\eta \in \mathcal{X}_T$, there exists a unique adapted solution $(p(\cdot), q(\cdot))$ to* (3.10) *satisfying*

$$E\left(\sup_{t\le s\le T} |p(s)|^2 + \int_t^T |q(s)|^2 ds\right) \le KE\left(|\eta|^2 + \int_t^T |y(s)|^2 ds\right),\quad \forall t \in [0, T]. \tag{2.8}$$

Equation (2.7) plays an interesting role in the next section.

To state our second main result, we need to introduce the following stochastic differential equation, which is closely related to (2.7):

$$\begin{cases} dx(t) = [Ax(t) + Bu(t)]\,dt + [Cx(t) + Du(t)]\,dw(t), \\ dp(t) = -[A^T p(t) + C^T q(t) + Qx(t)]\,dt + q(t)\,dw(t), \\ x(\tau) = \xi, \qquad p(T) = Gx(T). \end{cases} \tag{2.9}$$

Such a system is called a *forward–backward stochastic differential equation* (FBSDE, for short) since the equation for $x(\cdot)$ is *forward* (meaning that it is an initial value problem, which is to be solved forwardly) and the equation for $p(\cdot)$ (and $q(\cdot)$) is *backward* (meaning that it is a terminal value problem, which is to be solved backwardly). For given $(\xi, u(\cdot)) \in \mathcal{X}_\tau \times \mathcal{U}[\tau, T]$, an *adapted solution* of (2.9) is a triple of $\{\mathcal{F}_t\}_{t\ge 0}$-adapted processes $(x(\cdot), p(\cdot), q(\cdot))$ satisfying (2.9). It is clear that (2.9) is decoupled. Thus, for any $(\xi, u(\cdot)) \in \mathcal{X}_\tau \times \mathcal{U}[\tau, T]$, we can first solve the forward equation for $x(\cdot)$ and then solve the backward equation for $(p(\cdot), q(\cdot))$. Hence, by Proposition 2.3, for any $(\xi, u(\cdot)) \in \mathcal{X}_\tau \times \mathcal{U}[\tau, T]$, there exists a unique adapted solution $(x(\cdot), p(\cdot), q(\cdot))$ to (2.9). See [19]–[21], [26], and [27] for more information concerning the general theory of FBSDEs.

Our second main result of this paper is the following.

Theorem 2.4. *Let* (S) *hold. Then Problem* (*LQ*) *is* (*uniquely*) *partially solvable at $(\tau, \xi) \in \Delta[0, T]$ with a* (*the*) *partially optimal pair $(\overline{x}(\cdot), \overline{u}(\cdot))$ if and only if the following FBSDE,*

$$\begin{cases} d\overline{x}(t) = [A\overline{x}(t) + B\overline{u}(t)]\,dt + [C\overline{x}(t) + D\overline{u}(t)]\,dw(t), \\ d\overline{p}(t) = -[A^T \overline{p}(t) + C^T \overline{q}(t) + Q\overline{x}(t)]\,dt + \overline{q}(t)\,dw(t), \\ \overline{x}(\tau) = \xi, \qquad \overline{p}(T) = G\overline{x}(T), \end{cases} \tag{2.10}$$

admits a (*unique*) *adapted solution $(\overline{x}(\cdot), \overline{p}(\cdot), \overline{q}(\cdot))$ such that*

$$I_{\Omega(\tau,\xi)}\left[R\overline{u}(\cdot) + B^T\overline{p}(\cdot) + D^T\overline{q}(\cdot)\right] = 0, \qquad \text{in } L^2_{\mathcal{F}}(\tau, T; \mathbb{R}^m), \tag{2.11}$$

and, for any $u(\cdot) \in \mathcal{U}[\tau, T]$, the unique adapted solution $(x(\cdot), p(\cdot), q(\cdot))$ of (2.9) *with $\xi = 0$ satisfies*

$$E\left\{\int_\tau^T \langle Ru(t) + B^T p(t) + D^T q(t), u(t)\rangle\,dt \;\Big|\; \mathcal{F}_\tau\right\} \ge 0,\quad a.s. \quad \omega \in \Omega(\tau, \xi). \tag{2.12}$$

In addition, if $\Omega(\tau, \xi) = \Omega$ and $R(\cdot)^{-1} \in L^\infty_{\mathcal{F}}(0, T; \mathcal{S}^m)$, then an (*the*) *optimal control $\overline{u}(\cdot)$ admits a representation*:

$$\overline{u}(t) = -R(t)^{-1}[B(t)^T\overline{p}(t) + C(t)^T\overline{q}(t)], \qquad t \in [\tau, T]. \tag{2.13}$$

The above theorem is a version of the *stochastic maximum principle* (see [3]–[5], [12], [24], and [28]) for Problem (*LQ*), in which (2.10) is the Hamiltonian system and (2.11) is a direct consequence of the maximum condition. Condition (2.12) makes (2.10)–(2.11) sufficient for the pair $(\bar{x}(\cdot), \bar{x}(\cdot))$ to be optimal.

3. Approach by Functional Analysis and FBSDEs

In this section we use functional analysis and FBSDEs to approach Problem (*LQ*). At end of this section, Theorem 2.4 is proved.

Since the state equation is linear and the cost functional is quadratic in $(x(\cdot), u(\cdot))$, the cost functional $J(\tau, \xi; u(\cdot))$ is quadratic in $(\xi, u(\cdot))$ (τ is a parameter). Our first goal is to represent $J(\tau, \xi; u(\cdot))$ explicitly as a bilinear form in $(\xi, u(\cdot))$. To this end, we introduce the following:

$$\begin{cases} d\Phi(t) = A\Phi(t)\,dt + C\Phi(t)\,dw(t), & t \geq 0, \\ \Phi(0) = I. \end{cases} \tag{3.1}$$

Then it is known that (see [2], for example) $\Phi(t)^{-1}$ exists for all $t \geq 0$ satisfying the following:

$$\begin{cases} d\Phi(t)^{-1} = -\Phi(t)^{-1}[A - C^2]\,dt - \Phi(t)^{-1}C\,dw(t), & t \geq 0, \\ \Phi(0)^{-1} = I. \end{cases} \tag{3.2}$$

and the solution $x(\cdot)$ of (1.1) can be written as follows:

$$\begin{aligned} x(t) = \Phi(t)\Phi(\tau)^{-1}\xi + \Phi(t)\int_\tau^t \Phi(s)^{-1}(B - CD)u(s)\,ds \\ + \Phi(t)\int_\tau^t \Phi(s)^{-1}Du(s)\,dw(s), \qquad t \in [\tau, T]. \end{aligned} \tag{3.3}$$

By the Burkholder–Davis–Gundy inequality (see [15]), we have

$$E\left\{\sup_{\tau \leq s \leq t} |x(s)|^2 \,\Big|\, \mathcal{F}_\tau\right\} \leq KE\left\{|\xi|^2 + \int_\tau^t |u(s)|^2\,ds \,\Big|\, \mathcal{F}_\tau\right\}, \quad \forall t \in [\tau, T], \quad \text{a.s.} \tag{3.4}$$

Now, we define the following operators:

$$\begin{cases} (L_\tau u)(\cdot) = \Phi(\cdot)\int_\tau^\cdot \Phi(s)^{-1}(B - CD)u(s)\,ds \\ \qquad\qquad + \Phi(\cdot)\int_\tau^\cdot \Phi(s)^{-1}Du(s)\,dw(s), \\ \widehat{L}_\tau u = (L_\tau u)(T), \\ (S_\tau \xi)(\cdot) = \Phi(\cdot)\Phi(\tau)^{-1}\xi, \qquad \widehat{S}_\tau \xi = \Phi(T)\Phi(\tau)^{-1}\xi. \end{cases} \tag{3.5}$$

Clearly, for any $\tau \in \mathcal{T}[0, T]$,

$$\begin{cases} L_\tau\colon \mathcal{U}[\tau, T] \to \mathcal{X}[\tau, T], & \widehat{L}_\tau\colon \mathcal{U}[\tau, T] \to \mathcal{X}_T, \\ S_\tau\colon \mathcal{X}_\tau \to \mathcal{X}[\tau, T], & \widehat{S}_\tau\colon \mathcal{X}_\tau \to \mathcal{X}_T, \end{cases} \tag{3.6}$$

and all are bounded linear operators (by (3.4)). We now want to find the bounded linear operators

$$\begin{cases} L_\tau^*\colon \mathcal{X}[\tau,T]\to\mathcal{U}[\tau,T], & \widehat{L}_\tau^*\colon \mathcal{X}_T\to\mathcal{U}[\tau,T],\\ S_\tau^*\colon \mathcal{X}[\tau,T]\to\mathcal{X}_\tau, & \widehat{S}_\tau^*\colon \mathcal{X}_T\to\mathcal{X}_\tau,\end{cases} \tag{3.7}$$

such that

$$\begin{cases} E\left\{\int_\tau^T \langle (L_\tau u)(t), y(t)\rangle\,dt \;\Big|\; \mathcal{F}_\tau\right\} = E\left\{\int_\tau^T \langle u(t), (L_\tau^* y)(t)\rangle\,dt \;\Big|\; \mathcal{F}_\tau\right\},\\ E\left\{\int_\tau^T \langle (S_\tau\xi)(t), y(t)\rangle\,dt \;\Big|\; \mathcal{F}_\tau\right\} = E\left\{\langle \xi, S_\tau^* y\rangle \;\big|\; \mathcal{F}_\tau\right\},\\ \qquad\qquad \forall u(\cdot)\in\mathcal{U}[\tau,T],\quad y(\cdot)\in\mathcal{X}[\tau,T],\quad \xi\in\mathcal{X}_\tau,\end{cases} \tag{3.8}$$

and

$$\begin{cases} E\left\{\langle \widehat{L}_\tau u, \eta\rangle \mid \mathcal{F}_\tau\right\} = E\left\{\int_\tau^T \langle u(t), (\widehat{L}_\tau^*\eta)(t)\rangle\,dt \mid \mathcal{F}_\tau\right\}\\ E\left\{\langle \widehat{S}_\tau\xi, \eta\rangle \mid \mathcal{F}_\tau\right\} = E\left\{\langle \xi, \widehat{S}_\tau^*\eta\rangle \mid \mathcal{F}_\tau\right\},\\ \qquad\qquad \forall u(\cdot)\in\mathcal{U}[\tau,T],\quad \xi\in\chi_\tau,\quad \eta\in\chi_T.\end{cases} \tag{3.9}$$

In the above, we have used $\langle\cdot\,,\cdot\rangle$ as inner products in different Euclidean spaces, which can be identified from the context.

Note that the operators in (3.7) satisfying (3.8)–(3.9) are *not* formal adjoint of those in (3.5). For example, the formal adjoint of $\widehat{S}_\tau$ is $[\Phi(T)\Phi(\tau)^{-1}]^T$ which maps from $\mathcal{X}_T$ to $\mathcal{X}_T$ (*not* to $\mathcal{X}_\tau$) in general. A similar situation happens for the other three operators in (3.5). To find the operators (3.7) satisfying (3.8)–(3.9), we need to use BSDE (2.7) and Proposition 2.3. We have the following result.

Proposition 3.1.

(i) *For any* $y(\cdot)\in\mathcal{X}[\tau,T]$, *let* $(p_0(\cdot),q_0(\cdot))\in\mathcal{X}[\tau,T]\times\mathcal{X}[\tau,T]$ *be the adapted solution of* (2.7) *with* $\eta=0$. *Define*

$$\begin{cases} (L_\tau^* y)(t) = B^T p_0(t) + D^T q_0(t), & t\in[\tau,T],\\ S_\tau^* y = p_0(\tau).\end{cases} \tag{3.10}$$

Then $L_\tau^*\colon \mathcal{X}[\tau,T]\to\mathcal{U}[\tau,T]$ *and* $S_\tau^*\colon \mathcal{X}[\tau,T]\to\mathcal{X}_\tau$ *are bounded, satisfying* (3.8).

(ii) *For any* $\eta\in\mathcal{X}_T$, *let* $(p_1(\cdot),q_1(\cdot))\in\mathcal{X}[\tau,T]\times\mathcal{X}[\tau,T]$ *be the solution of* (2.7) *with* $y(\cdot)=0$. *Define*

$$\begin{cases} (\widehat{L}_\tau^*\eta)(t) = B^T p_1(t) + D^T q_1(t), & t\in[\tau,T],\\ \widehat{S}_\tau^*\eta = p_1(\tau).\end{cases} \tag{3.11}$$

Then $\widehat{L}_\tau^*\colon \mathcal{X}_T\to\mathcal{U}[\tau,T]$ *and* $\widehat{S}_\tau^*\colon \mathcal{X}_T\to\mathcal{X}_\tau$ *are bounded, satisfying* (3.9).

Proof. For any $\xi \in \mathcal{X}_\tau$, $\eta \in \mathcal{X}_T$, $y(\cdot) \in \mathcal{X}[\tau, T]$, and $u(\cdot) \in \mathcal{U}[\tau, T]$, let $x(\cdot)$ and $(p(\cdot), q(\cdot))$ be the solutions of (1.1) and (2.7), respectively. By (2.8), all the operators defined in (3.10) and (3.11) are bounded. Next, applying Itô's formula to $\langle x(\cdot), p(\cdot)\rangle$, we obtain

$$\begin{aligned}&\langle x(T), \eta\rangle - \langle \xi, p(\tau)\rangle \\ &\quad = \int_\tau^T \big[\langle u(t), B^T p(t) + D^T q(t)\rangle - \langle x(t), y(t)\rangle\big]\,dt + \int_\tau^T [\cdots]\,dw(t).\end{aligned}$$

Using (3.3) and (3.5), and taking conditional expectation, one has:

$$\begin{aligned}&E\left\{\langle \widehat{S}_\tau\xi + \widehat{L}_\tau u, \eta\rangle - \langle \xi, p(\tau)\rangle \,\big|\, \mathcal{F}_\tau\right\} \\ &\quad = E\left\{\int_\tau^T \big[\langle u(t), B^T p(t) + D^T q(t)\rangle \right. \\ &\qquad\qquad \left. - \ \langle (S_\tau\xi)(t) + (L_\tau u)(t), y(t)\rangle\big]\ dt \,\Big|\, \mathcal{F}_\tau\right\}. \qquad (3.12)\end{aligned}$$

Then, in (3.12), by taking $\xi = 0$ and $\eta = 0$, we obtain the first relations in (3.8); by taking $u(\cdot) = 0$ and $\eta = 0$, we obtain the second relations in (3.8); by taking $\xi = 0$ and $y(\cdot) = 0$, we obtain the first relations in (3.9); and by taking $u(\cdot) = 0$ and $y(\cdot) = 0$, we obtain the second relations in (3.9). □

By the above result, we obtain the following representation for the cost functional (1.2):

$$\begin{aligned}J(\tau, \xi; u(\cdot)) &= E\Big\{\langle Q(S_\tau\xi + L_\tau u), S_\tau\xi + L_\tau u\rangle + \langle Ru, u\rangle \\ &\qquad + \langle G(\widehat{S}_\tau\xi + \widehat{L}_\tau u), \widehat{S}_\tau\xi + \widehat{L}_\tau u\rangle \,\big|\, \mathcal{F}_\tau\Big\} \\ &= E\Big\{\langle (R + L_\tau^* Q L_\tau + \widehat{L}_\tau^* G L_\tau)u, u\rangle + 2\langle (L_\tau^* Q S_\tau + \widehat{L}_\tau^* G \widehat{S}_\tau)\xi, u\rangle \\ &\qquad + \langle (S_\tau^* Q S_\tau + \widehat{S}_\tau^* G \widehat{S}_\tau)\xi, \xi\rangle \,\big|\, \mathcal{F}_\tau\Big\} \\ &\stackrel{\Delta}{=} E\Big\{\langle N_\tau u, u\rangle + 2\langle H_\tau \xi, u\rangle + \langle M_\tau \xi, \xi\rangle \,\big|\, \mathcal{F}_\tau\Big\}, \qquad (3.13)\end{aligned}$$

where

$$\begin{cases} N_\tau u = [R + L_\tau^* Q L_\tau + \widehat{L}_\tau^* G \widehat{L}_\tau]u, & \forall u \in \mathcal{U}[\tau, T], \\ H_\tau^* u = [S_\tau^* Q L_\tau + \widehat{S}_\tau^* G \widehat{L}_\tau]u, & \forall u \in \mathcal{U}[\tau, T], \\ H_\tau \xi = [L_\tau^* Q S_\tau + \widehat{L}_\tau^* G \widehat{S}_\tau]\xi, & \forall \xi \in \mathcal{X}_\tau, \\ M_\tau \xi = [S_\tau^* Q S_\tau + \widehat{S}_\tau^* G \widehat{S}_\tau]\xi, & \forall \xi \in \mathcal{X}_\tau, \end{cases} \qquad (3.14)$$

and $\langle \cdot\,, \cdot\rangle$ represents the inner products in different spaces. It is clear that the operators

$$\begin{cases} N_\tau\colon \mathcal{U}[\tau, T] \to \mathcal{U}[\tau, T], & H_\tau^*\colon \mathcal{U}[\tau, T] \to \mathcal{X}_\tau, \\ H_\tau\colon \mathcal{X}_\tau \to \mathcal{U}[\tau, T], & M_\tau\colon \mathcal{X}_\tau \to \mathcal{X}_\tau, \end{cases}$$

are all bounded.

We now look at some conditions for the (partial) finiteness and solvability of Problem (LQ).

Theorem 3.2. *Let* (S) *hold. Suppose Problem* (LQ) *is partially finite at* $(\tau,\xi)\in\Delta[0,T]$. *Then*

$$E\left\{\langle N_\tau u,u\rangle \,\Big|\, \mathcal{F}_\tau\right\}\ge 0,\qquad \forall u\in\mathcal{U}[\tau,T],\quad \text{a.s.}\quad \omega\in\Omega(\tau,\xi), \tag{3.15}$$

where $\Omega(\tau,\xi)\stackrel{\Delta}{=}(V(\tau,\xi)>-\infty)$. *Moreover, Problem* ($LQ$) *is* (*uniquely*) *partially solvable at* $(\tau,\xi)\in\Delta[0,T]$ *if and only if* (3.15) *holds and there exists a* (*unique*) $\overline{u}\in\mathcal{U}[\tau,T]$ *such that*

$$I_{\Omega(\tau,\xi)}\,(N_\tau\overline{u}+H_\tau\xi)=0,\qquad in\ L^2_{\mathcal{F}}(\tau,T;\mathbb{R}^m). \tag{3.16}$$

In this case, $\overline{u}$ *is a* (*the*) *partially optimal control.*

Proof. It is obvious that (3.15) is necessary for Problem (LQ) being partially finite at $(\tau,\xi)\in\Delta[\tau,T]$. Now, suppose Problem ($LQ$) is partially solvable at $(\tau,\xi)\in\Delta[\tau,T]$ with $\overline{u}\in\mathcal{U}[\tau,T]$ being a partially optimal control. Denote $\Omega_0=\Omega(\tau,\xi)$. Then, for any $u\in\mathcal{U}[\tau,T]$, we have

$$\begin{aligned} 0\ \le\ &\frac{1}{\varepsilon}I_{\Omega_0}\left[J(\tau,\xi;\overline{u}+\varepsilon u)-J(\tau,\xi;\overline{u})\right]\\ &\to 2I_{\Omega_0}E\left\{\langle N_\tau\overline{u}+H_\tau\xi,u\rangle \,\Big|\, \mathcal{F}_\tau\right\}. \end{aligned} \tag{3.17}$$

By taking $u=-(N_\tau\overline{u}+H_\tau\xi)\in\mathcal{U}[\tau,T]$ in the above, we obtain (note $\Omega_0\in\mathcal{F}_\tau$)

$$E\left\{I_{\Omega_0}|N_\tau\overline{u}+H_\tau\xi|^2 \,\Big|\, \mathcal{F}_\tau\right\}=0,\quad \text{a.s.} \tag{3.18}$$

Then (3.16) follows.

Conversely, suppose (3.15) holds and for some $\overline{u}\in\mathcal{U}[\tau,T]$, (3.16) holds. Then, for any $u\in\mathcal{U}[\tau,T]$, we have

$$\begin{aligned} &I_{\Omega_0}\{J(\tau,\xi;u)-J(\tau,\xi;\overline{u})\}\\ &\quad=I_{\Omega_0}E\left\{\langle N_\tau u,u\rangle+2\langle H_\tau\xi,u\rangle-\langle N_\tau\overline{u},\overline{u}\rangle-2\langle H_\tau\xi,\overline{u}\rangle \,\Big|\, \mathcal{F}_\tau\right\}\\ &\quad=E\left\{I_{\Omega_0}\langle N_\tau(u-\overline{u}),u-\overline{u}\rangle \,\Big|\, \mathcal{F}_\tau\right\}\ge 0. \end{aligned} \tag{3.19}$$

Hence, $\overline{u}$ is partially optimal and Problem (LQ) is partially solvable.

The uniqueness part is clear. □

Using Proposition 3.1, we can easily prove the following result, which gives a representation of operators N_τ, H^*_τ, H_τ, and M_τ defined in (3.14) in terms of adapted solutions of (2.9).

Proposition 3.3.

(i) *For any* $u(\cdot)\in\mathcal{U}[\tau,T]$, *let* $(x_0(\cdot),p_0(\cdot),q_0(\cdot))$ *be the unique adapted solution of* (2.9) *with* $\xi=0$. *Then*

$$\begin{cases}(N_\tau u)(t)=Ru(t)+B^Tp_0(t)+D^Tq_0(t), & t\in[\tau,T],\\ H^*_\tau u=p_0(\tau).\end{cases} \tag{3.20}$$

(ii) *For any* $\xi \in \mathcal{X}_\tau$, *let* $(x_1(\cdot), p_1(\cdot), q_1(\cdot))$ *be the unique adapted solution of* (2.9) *with* $u(\cdot) = 0$. *Then*

$$\begin{cases} (H_\tau\xi)(t) = B^T p_1(t) + D^T q_1(t), \qquad t \in [\tau, T], \\ M_\tau\xi = p_1(\tau). \end{cases} \tag{3.21}$$

(iii) *For any* $(\xi, u(\cdot)) \in \mathcal{X}_\tau \times \mathcal{U}[\tau, T]$, *let* $(x(\cdot), p(\cdot), q(\cdot))$ *be the unique adapted solution of* (2.9). *Then*

$$\begin{cases} (N_\tau u)(t) + (H_\tau\xi)(t) = Ru(t) + B^T p(t) + D^T q(t), \qquad t \in [\tau, T], \\ M_\tau\xi + H_\tau^* u = p(\tau). \end{cases} \tag{3.22}$$

Combining Theorem 3.2 and Proposition 3.3, we obtain a proof of Theorem 2.4. Now, we take a closer look at the case

$$R(\cdot)^{-1} \in L^\infty_{\mathcal{F}}(0, T; \mathcal{S}^m). \tag{3.23}$$

Note that we do not require the nonnegativity of $R(t)$ here. When (3.23) holds, (2.10) and (2.11) with $\Omega(\tau, \xi) = \Omega$ is equivalent to the following:

$$\begin{cases} d\overline{x}(t) = [A\overline{x}(t) - BR^{-1}B^T\overline{p}(t) - BR^{-1}D^T\overline{q}(t)]\,dt \\ \qquad\qquad + [C\overline{x}(t) - DR^{-1}B^T\overline{p}(t) - DR^{-1}D^T\overline{q}(t)]\,dw(t), \\ d\overline{p}(t) = [-Q\overline{x}(t) - A^T\overline{p}(t) - C^T\overline{q}(t)]\,dt + \overline{q}(t)\,dw(s), \\ \overline{x}(\tau) = \xi, \qquad \overline{p}(T) = G\overline{x}(T), \end{cases} \tag{3.24}$$

This is a *coupled* linear FBSDE. From Theorem 2.4, we have the following result.

Corollary 3.4. *Let* (S) *hold and* (2.12) *hold with* $\Omega(\tau, \xi) = \Omega$. *Then Problem* (*LQ*) *is* (*uniquely*) *solvable at* (τ, ξ) *if and only if the FBSDE* (3.24) *admits a* (*unique*) *adapted solution* $(\overline{x}(\cdot), \overline{p}(\cdot), \overline{q}(\cdot))$. *In this case, an* (*the*) *optimal control is given by* (2.13).

The above result gives an intrinsic relation between the solvability of Problem (*LQ*) and FBSDE (3.24). In [27], equations of form (3.24) were studied directly, where some necessary and sufficient conditions were obtained for the solvability of such linear FBSDEs.

Corollary 3.5. *Let* (S) *hold. Suppose further that for some* $\tau \in \mathcal{T}[0, T)$,

$$\begin{cases} G \geq 0, \qquad a.s. \quad \omega \in \Omega, \\ Q(t) \geq 0, \quad R(t) \geq 0, \qquad \forall t \in [\tau, T], \quad a.s. \quad \omega \in \Omega. \end{cases} \tag{3.25}$$

Then (3.15) *holds with* $\Omega(\tau, \xi) = \Omega$. *In addition, if* (3.23) *also holds, then the operator* $N_\tau \colon \mathcal{U}[\tau, T] \to \mathcal{U}[\tau, T]$ *is invertible, Problem* (*LQ*) *is uniquely solvable at* τ, *and for any* $\xi \in \mathcal{X}_\tau$, *the unique optimal control is given by*

$$\overline{u} = -N_\tau^{-1} H_\tau \xi. \tag{3.26}$$

In this case, the FBSDE (3.24) *also admits a unique adapted solution* $(\overline{x}(\cdot), \overline{p}(\cdot), \overline{q}(\cdot))$ *and the optimal control also admits representation* (2.13).

Proof. In the present case, it is easy to see that the operator $N_\tau\colon \mathcal{U}[\tau,T]\to\mathcal{U}[\tau,T]$ is invertible. Then the rest of the conclusions follows. $\square$

We refer to the case that (S), (3.23), and (3.25) hold as the *conventional case*.

By Corollary 3.5, if (3.25) holds, (3.15) (with $\Omega(\tau,\xi)=\Omega$) or, equivalently, (2.12) automatically hold. Thus, when (3.25) holds, (2.10)–(2.11) completely characterize the (partially) optimal pair $(\bar{x}(\cdot),\bar{u}(\cdot))$. This covers the conventional deterministic case completely in this context.

We now look at some further properties of the value functions.

Theorem 3.6. *Let Problem (LQ) be solvable at* $\tau\in\mathcal{T}[0,T]$. *Then there exists a* $P_\tau\in L^\infty_{\mathcal{F}_\tau}(\Omega;\mathcal{S}^n)$, *such that*

$$V(\tau,\xi)=\langle P_\tau\xi,\xi\rangle,\qquad\forall\xi\in\mathcal{X}_\tau. \tag{3.27}$$

From (3.13), we know that there exists a self-adjoint operator $P_\tau\colon\mathcal{X}_\tau\to\mathcal{X}_\tau$ satisfying (3.27). The above result asserts that the operator P_τ can actually be represented by a bounded $\mathcal{F}_\tau$-measurable $\mathcal{S}^n$-valued random variable.

To prove our result, we need the following seemingly obvious result.

Lemma 3.7. *Suppose* $\xi\mapsto V(\tau,\xi)$ *is continuous. Then, for any* $(\tau,\xi)\in\Delta[0,T]$,

$$V(\tau,x)(\omega)\big|_{x=\xi(\omega)}=V(\tau,\xi)(\omega),\qquad a.s.\quad\omega\in\Omega. \tag{3.28}$$

Proof. First, for any $(x(\cdot),u(\cdot))\in\mathcal{X}[0,T]\times\mathcal{U}[0,T]$, we have a unique adapted solution $(y(\cdot),z(\cdot))$ to the following BSDE:

$$\begin{cases}dy(t)=-[\langle Qx(t),x(t)\rangle+\langle Ru(t),u(t)\rangle]\,dt+z(t)\,dw(t), & t\in[0,T],\\ y(T)=\langle Gx(T),x(T)\rangle,\end{cases} \tag{3.29}$$

and, for any $\tau\in\mathcal{T}[0,T]$,

$$y(\tau)\equiv y(\tau;x(\cdot),u(\cdot))=J(\tau,x(\tau);u(\cdot)). \tag{3.30}$$

Now, let $(\tau,\xi)\in\Delta[0,T]$ be fixed. For any $\varepsilon>0$, we can find a partition $\{F_i^\varepsilon,i\ge1\}\subseteq\mathcal{F}_\tau$ of Ω and $x_i^\varepsilon\in\mathbb{R}^n$, such that

$$\xi^\varepsilon\equiv\sum_{i\ge1}I_{F_i^\varepsilon}x_i^\varepsilon\to\xi,\qquad\text{in } L^2_{\mathcal{F}_\tau}(\Omega;\mathbb{R}^n). \tag{3.31}$$

Let $x(\cdot\,;\tau,\xi,u)$ be the solution of (1.1) starting from (τ,ξ) under control $u(\cdot)$. Then, multiplying $I_{F_i^\varepsilon}$ on both sides of (1.1) (with $\xi=x_i$) and summing in $i\ge1$, using the uniqueness of the solutions, we have

$$\sum_{i\ge1}I_{F_i^\varepsilon}x(\cdot\,;\tau,x_i^\varepsilon,u)=x(\cdot\,;\tau,\xi^\varepsilon,u).$$

Consequently, using (3.29) and (3.30) (together with the uniqueness of the adapted solutions to (3.29)), we have

$$\sum_{i\ge1}I_{F_i^\varepsilon}J(\tau,x_i^\varepsilon;\,;u(\cdot))=J(\tau,\xi^\varepsilon;u(\cdot)). \tag{3.32}$$

Therefore, by the continuity of $\xi \mapsto V(\tau, \xi)$, we obtain

$$\begin{aligned} V(\tau, x)\big|_{x=\xi} &= \lim_{\varepsilon\to 0} \sum_{i\geq 1} I_{F_i^\varepsilon} V(\tau, x_i^\varepsilon) = \lim_{\varepsilon\to 0} \sum_{i\geq 1} I_{F_i^\varepsilon} \inf_{u(\cdot)\in\mathcal{U}[\tau,T]} J(\tau, x_i^\varepsilon; u(\cdot)) \\ &= \lim_{\varepsilon\to 0} \inf_{u(\cdot)\in\mathcal{U}[\tau,T]} J(\tau, \xi^\varepsilon; u(\cdot)) = \lim_{\varepsilon\to 0} V(\tau, \xi^\varepsilon) = V(\tau, \xi). \end{aligned}$$

This proves (3.28). □

Proof of Theorem 3.6. First, it is clear that there exists a self-adjoint bounded operator $P_\tau\colon \mathcal{X}_\tau \to \mathcal{X}_\tau$, such that (3.27) holds. Consequently, $\xi \mapsto V(\tau, \xi)$ is continuous, and, for almost surely $\omega \in \Omega$, $x \mapsto V(\tau, x)(\omega)$ is a bilinear form on $\mathbb{R}^n$. Thus, there exists an $\mathcal{S}^n$-valued function $\widehat{P}(\omega)$ such that

$$V(\tau, x)(\omega) = \langle \widehat{P}(\omega)x, x\rangle, \qquad \forall x \in \mathbb{R}^n, \quad \text{a.s.} \quad \omega \in \Omega. \tag{3.33}$$

Then it is necessary that

$$\widehat{P}(\omega)x = (P_\tau x)(\omega), \qquad \forall x \in \mathbb{R}^n, \quad \text{a.s.} \quad \omega \in \Omega.$$

This implies that $\widehat{P} \in L^\infty_{\mathcal{F}_\tau}(\Omega; \mathcal{S}^n)$. Now, by (3.27) (with $P_\tau\colon \mathcal{X}_\tau \to \mathcal{X}_\tau$), (3.28), and (3.33), we have

$$\begin{aligned} \langle \widehat{P}(\omega)\xi(\omega), \xi(\omega)\rangle &= V(\tau, x)(\omega)\big|_{x=\xi(\omega)} \\ &= V(\tau, \xi)(\omega) \\ &= \langle (P_\tau \xi)(\omega), \xi(\omega)\rangle, \qquad \text{a.s.} \quad \omega \in \Omega, \quad \forall \xi \in \mathcal{X}_\tau. \end{aligned}$$

Thus,

$$\widehat{P}(\omega)\xi(\omega) = (P_\tau \xi)(\omega), \qquad \text{a.s.} \quad \omega \in \Omega, \quad \forall \xi \in \mathcal{X}_\tau.$$

This shows that we can take $P_\tau \in L^\infty_{\mathcal{F}_\tau}(\Omega; \mathcal{S}^n)$ in (3.27). □

4. A Necessary Condition for the Finiteness of LQ Problems

In this section we present a proof of Theorem 2.2.

We first state the following *dynamic programming principle* whose proof is pretty straightforward.

Lemma 4.1. *For any* $(\tau, \xi) \in \Delta[0, T)$ *and* $t \in \mathcal{T}[\tau, T]$, *it holds that*

$$V(\tau, \xi) = \inf_{u(\cdot)\in\mathcal{U}[\tau,t]} E\left\{ V(t, x(t)) + \int_\tau^t [\langle Qx(s), x(s)\rangle + \langle Ru(s), u(s)\rangle]\, ds \;\Big|\; \mathcal{F}_\tau \right\}, \quad \textit{a.s.} \tag{4.1}$$

The following corollary will be useful later.

Corollary 4.2. *For any $\tau \in \mathcal{T}[0,T)$, and $t \in \mathcal{T}[\tau,T]$,*

$$\begin{aligned}&(V(\tau,\xi) > -\infty) \\ &\quad \subseteq \bigcap_{u(\cdot)\in\mathcal{U}[\tau,T]} (V(t,x(t;\tau,\xi,u(\cdot))) > -\infty), \qquad \forall t \in \mathcal{T}[\tau,T].\end{aligned} \tag{4.2}$$

The above result implies that if Problem (*LQ*) is finite (resp. partially finite) at $(\tau,\xi) \in \Delta[0,T)$, then for any $u(\cdot) \in \mathcal{U}[\tau,T]$, along the trajectory $x(\cdot) \equiv x(\cdot\,;\tau,\xi,u(\cdot))$, $V(t,x(t))$ is finite (resp. finite on the set $(V(\tau,\xi) > -\infty)$) almost surely, for any $t \in \mathcal{T}[\tau,T]$. This fact will be useful below.

Proof of Theorem 2.2. We first prove the following:

$$\begin{aligned}&E\left\{\langle [R(T)+D(T)^T GD(T)]v,v\rangle \;\middle|\; \mathcal{F}_\tau\right\} \geq 0, \\ &\quad \text{a.s.} \quad \omega \in (V(\tau,\xi) > -\infty), \quad \forall v \in L^2_{\mathcal{F}_\tau}(\Omega;\mathbb{R}^m).\end{aligned} \tag{4.3}$$

Suppose (4.3) does not hold. Then we may find a $v \in L^2_{\mathcal{F}_\tau}(\Omega;\mathbb{R}^m)$, a constant $\delta > 0$, and some $\Omega_\tau \in \mathcal{F}_\tau$ with $\Omega_\tau \subseteq (V(\tau,\xi) > -\infty)$, $\mathcal{P}(\Omega_\tau) > 0$, such that

$$E\left\{\langle [R(T)+D(T)^T GD(T)]v,v\rangle \;\middle|\; \mathcal{F}_\tau\right\} \leq -\delta, \qquad \forall \omega \in \Omega_\tau. \tag{4.4}$$

From [23] we know that the there exists a pair of $\mathcal{S}^n$-valued square integrable $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes $(P(\cdot),\Lambda(\cdot))$ such that

$$\begin{cases} dP = -[PA + A^T P + C^T PC + \Lambda C + C^T \Lambda + Q]\,dt \\ \qquad\quad + \Lambda\, dw(t), \quad t \in [\tau,T], \\ P|_{t=T} = G. \end{cases} \tag{4.5}$$

By Itô's formula, Burkholder–Davis–Gundy's inequality [15], and Gronwall's inequality, we have

$$\begin{aligned}&E\left\{\sup_{t\leq s\leq T} |P(s)|^2 + \int_t^T |\Lambda(s)|^2\,ds \;\middle|\; \mathcal{F}_\tau\right\} \\ &\quad \leq K_0 E\left\{|G|^2 \;\middle|\; \mathcal{F}_\tau\right\}, \qquad \forall t \in [\tau,T],\end{aligned} \tag{4.6}$$

for some constant $K_0 > 0$, where $|\Theta|^2 \overset{\Delta}{=} \operatorname{tr}[\Theta\Theta^T]$, for any $\Theta \in \mathbb{R}^{n\times n}$.

Next, we take $\varepsilon \in (0,1]$ and let $\tau_\varepsilon = \tau \vee (T-\varepsilon) \in \mathcal{T}[\tau,T)$. Also, let $\lambda > 0$ be $\mathcal{F}_\tau$-measurable, time-independent, and undetermined. Let $x^\varepsilon_\lambda(\cdot)$ be the state process of the system (1.1) starting from (τ,ξ), under the control

$$u^\varepsilon_\lambda(t) = \lambda v \chi_{[\tau_\varepsilon,T]}(t), \qquad t \in [\tau,T]. \tag{4.7}$$

Then, by Itô's formula, Burkholder–Davis–Gundy's inequality, and Gronwall's inequality, we obtain (note that τ_ε is $\mathcal{F}_\tau$-measurable)

$$\begin{aligned}&E\left\{\sup_{\tau\leq r\leq t} |x^\varepsilon_\lambda(t)|^{2k} \;\middle|\; \mathcal{F}_\tau\right\} \leq K_{2k}\left\{|\xi|^{2k} + (\lambda|v|)^{2k}(t-\tau_\varepsilon)^+\right\}, \\ &\quad t \in [\tau,T], \quad \text{a.s.} \quad k \geq 1,\end{aligned} \tag{4.8}$$

for some constants $K_{2k} > 0$, independent of λ and ε. Now, by Itô's formula and (4.5), we have (suppressing s in some of the integrands)

$$\begin{aligned} & E\left\{\langle Gx_\lambda^\varepsilon(T), x_\lambda^\varepsilon(T)\rangle \,\middle|\, \mathcal{F}_\tau\right\} \\ &\quad = E\Bigg\{\langle P(\tau_\varepsilon)x_\lambda^\varepsilon(\tau_\varepsilon), x_\lambda^\varepsilon(\tau_\varepsilon)\rangle \\ &\qquad + \int_{\tau_\varepsilon}^T \big\{-\langle Qx_\lambda^\varepsilon(s), x_\lambda^\varepsilon(s)\rangle + 2\lambda\langle [B^T P + D^T PC + D^T\Lambda]x_\lambda^\varepsilon(s), v\rangle \\ &\qquad\qquad + \lambda^2\langle D^T PDv, v\rangle\big\}\, ds \Big| \mathcal{F}_\tau\Bigg\}. \end{aligned}$$

Hence, by the definition of $V(\tau, \xi)$ and (S), as well as (4.6) and (4.8), we obtain (note $0 \le T - \tau_\varepsilon \le \varepsilon$ almost surely and τ_ε is $\mathcal{F}_\tau$-measurable)

$$\begin{aligned} V(\tau, \xi) &\le E\left\{\langle Gx_\lambda^\varepsilon(T), x_\lambda^\varepsilon(T)\rangle + \int_\tau^T \big[\langle Qx_\lambda^\varepsilon(s), x_\lambda^\varepsilon(s)\rangle + \langle Ru_\lambda^\varepsilon(s), u_\lambda^\varepsilon(s)\rangle\big]\, ds \Big| \mathcal{F}_\tau\right\} \\ &= E\Bigg\{\langle P(\tau_\varepsilon)x_\lambda^\varepsilon(\tau_\varepsilon), x_\lambda^\varepsilon(\tau_\varepsilon)\rangle + \int_\tau^{\tau_\varepsilon} \langle Qx_\lambda^\varepsilon(s), x_\lambda^\varepsilon(s)\rangle\, ds \\ &\qquad + \int_{\tau_\varepsilon}^T \big\{2\lambda\langle [B^T P + D^T PC + D^T\Lambda]x_\lambda^\varepsilon(s), v\rangle \\ &\qquad\qquad + \lambda^2\langle [R + D^T PD]v, v\rangle\big\}\, ds \Big| \mathcal{F}_\tau\Bigg\} \\ &\le E\Bigg\{\tfrac{1}{2}|P(\tau_\varepsilon)|^2 + \tfrac{1}{2}|x_\lambda^\varepsilon(\tau_\varepsilon)|^4 + \|Q\|_\infty \int_\tau^{\tau_\varepsilon} |x_\lambda^\varepsilon(s)|^2\, ds \\ &\qquad + \lambda \int_{\tau_\varepsilon}^T \big\{|x_\lambda^\varepsilon(s)|^2 + |(PB + C^T PD + \Lambda D)v|^2\big\}\, ds \\ &\qquad + \lambda^2 \int_{\tau_\varepsilon}^T \langle (R + D^T PD)v, v\rangle\, ds \,\Big|\, \mathcal{F}_\tau\Bigg\} \\ &\le \frac{K_0}{2} E\big\{|G|^2 \,\big|\, \mathcal{F}_\tau\big\} + \frac{K_4}{2}|\xi|^4 + \|Q\|_\infty K_2|\xi|^2 T \\ &\quad + \lambda(T - \tau_\varepsilon)K_2[|\xi|^2 + \lambda^2|v|^2(T - \tau_\varepsilon)] + \lambda\rho_1(\varepsilon) \\ &\quad + \lambda^2(T - \tau_\varepsilon)\big\{E\big[\langle [R(T) + D(T)^T GD(T)]v, v\rangle \,\big|\, \mathcal{F}_\tau\big] + \rho_2(\varepsilon)\big\} \\ &\le K(\omega) + \lambda(T - \tau_\varepsilon)K_2|\xi|^2 + K_2|v|^2\lambda^3(T - \tau_\varepsilon)^2 + \lambda\rho_1(\varepsilon) \\ &\quad + \lambda^2(T - \tau_\varepsilon)\Big\{E\big\{\langle [R(T) + D(T)^T GD(T)]v, v\rangle \,\big|\, \mathcal{F}_\tau\big\} + \rho_2(\varepsilon)\Big\}, \end{aligned} \tag{4.9}$$

where

$$\begin{cases} K(\omega) = \dfrac{K_0}{2} E\left\{|G|^2 \,\middle|\, \mathcal{F}_\tau\right\} + \dfrac{K_4}{2}|\xi|^4 + \|Q\|_\infty K_2|\xi|^2 T, \\ \rho_1(\varepsilon) = E\left\{\displaystyle\int_{\tau_\varepsilon}^T |(PB + C^T PD + \Lambda D)v|^2\, ds \,\middle|\, \mathcal{F}_\tau\right\}, \\ \rho_2(\varepsilon) = \dfrac{1}{T - \tau_\varepsilon} E\Bigg\{\displaystyle\int_{\tau_\varepsilon}^T \langle [R(s) + D(s)^T P(s)D(s) \\ \qquad\qquad - R(T) - D(T)^T GD(T)]v, v\rangle\, ds \,\Big|\, \mathcal{F}_\tau\Bigg\}. \end{cases} \tag{4.10}$$

By the continuity of R, D, and P, as well as (4.8), we see that

$$\rho_1(\varepsilon) + \rho_2(\varepsilon) \to 0, \qquad \text{a.s.} \quad \omega \in \Omega \quad (\varepsilon \to 0). \tag{4.11}$$

Now, for any $\omega \in \Omega_\tau$ (on which (4.4) holds), such that (4.11) holds, we take (note that $\rho_1(\varepsilon)$ is independent of $\lambda > 0$)

$$\begin{aligned} \lambda \equiv \lambda(\varepsilon) &\overset{\Delta}{=} (T-\tau_\varepsilon)^{-1}[T-\tau_\varepsilon+\rho_1(\varepsilon)]^{1/4} \\ &\geq (T-\tau_\varepsilon)^{-3/4} \to \infty, \qquad \text{as} \quad \varepsilon \to 0, \quad \text{a.s.} \quad \omega \in \Omega. \end{aligned} \tag{4.12}$$

Then λ is $\mathcal{F}_\tau$-measurable, and, as $\varepsilon \to 0$,

$$\begin{cases} \lambda(T-\tau_\varepsilon) = [T-\tau_\varepsilon+\rho_1(\varepsilon)]^{1/4} \to 0, \\ \lambda^2(T-\tau_\varepsilon) = (T-\tau_\varepsilon)^{-1}[T-\tau_\varepsilon+\rho_1(\varepsilon)]^{1/2} \\ \qquad\qquad \geq (T-\tau_\varepsilon)^{-1/2} \to \infty, \qquad \text{a.s.} \quad \omega \in \Omega. \\ \dfrac{\rho_1(\varepsilon)}{\lambda(T-\tau_\varepsilon)} = \rho_1(\varepsilon)[T-\tau_\varepsilon+\rho_1(\varepsilon)]^{-1/4} \leq \rho_1(\varepsilon)^{3/4} \to 0, \end{cases} \tag{4.13}$$

Hence, (4.9) together with (4.4) and (4.13) yields

$$\begin{aligned} V(\tau,\xi) &\leq K(\omega) + \lambda(T-\tau_\varepsilon)K_2|\xi|^2 \\ &\quad + \lambda^2(T-\tau_\varepsilon)\left\{\frac{K_2}{2}|v|^2\lambda(T-\tau_\varepsilon) + \frac{\rho_1(\varepsilon)}{\lambda(T-\tau_\varepsilon)} + \rho_2(\varepsilon) - \delta\right\} \\ &\to -\infty, \end{aligned} \tag{4.14}$$

as $\varepsilon \to 0$, almost surely on Ω_τ, which is a contradiction, proving (2.5). From Corollary 4.2, we know that for any $\bar\tau \in \mathcal{T}(\tau, T)$, there exists a $\bar\xi \in \mathcal{X}_{\bar\tau}$ such that $V(\bar\tau, \bar\xi)$ is finite. Thus, the above proof shows that we must have

$$\begin{aligned} &E\left\{\langle [R(T) + D(T)^T G D(T)]v, v\rangle \;\middle|\; \mathcal{F}_{\bar\tau}\right\} \geq 0, \\ &\qquad \text{a.s.} \quad \omega \in (V(\tau,\xi) > -\infty), \quad \forall v \in L^2_{\mathcal{F}_{\bar\tau}}(\Omega; \mathbb{R}^m), \end{aligned} \tag{4.15}$$

for all $\bar\tau \in \mathcal{T}[\tau, T)$. Now, taking $\bar\tau = \tau \vee (T-\varepsilon)$ and sending $\varepsilon \to 0$, by the continuity of $\{\mathcal{F}_t\}_{t\geq 0}$, we obtain (2.5). □

The following example shows that condition (2.5) is really only necessary.

Example 4.3. Consider the following one-dimensional control system:

$$\begin{cases} dx(t) = \frac{1}{2}x(t)\,dt + u(t)\,dw(t), \qquad t \in [\tau, T], \\ x(\tau) = \xi, \end{cases} \tag{4.16}$$

with the cost functional:

$$J(\tau,\xi;u(\cdot)) = E\left\{\int_\tau^T u(t)^2 dt - x(T)^2 \;\middle|\; \mathcal{F}_\tau\right\}. \tag{4.17}$$

In the current case, $A = \frac{1}{2}$, $B = C = Q = 0$, $D = R = 1$, and $G = -1$. Thus,

$$R + D^T G D = 0.$$

Applying Itô's formula to $-e^{T-t}x(t)^2$, we have that

$$J(\tau,\xi;u(\cdot)) = E\left\{\int_\tau^T u(t)^2(1-e^{T-t})\,dt - e^{T-\tau}\xi^2 \;\Big|\; \mathcal{F}_\tau\right\},\quad \forall u(\cdot)\in\mathcal{U}[0,T]. \tag{4.18}$$

Since $1-e^{T-t}<0$ for all $t<T$, Problem (LQ) is not (partially) finite at any $(\tau,\xi)\in\Delta[0,T)$.

Hence, in general, in order to have the solvability of Problem (LQ), we had better assume that

$$R(T)+D(T)^T GD(T)\geq\delta_0,\quad \text{a.s.}, \tag{4.19}$$

which is a little stronger than (2.5).

5. A Sufficient Condition for the Solvability of LQ Problems

In this section we briefly consider a sufficient condition for the solvability of LQ problems. We discuss only the case of constant coefficients, i.e., A, B, C, D, Q, R, G are all constant matrices. We introduce the following terminal value problem of a differential equation for the matrix-valued function $P(\cdot)$:

$$\begin{cases} \dot P(t) = -P(t)A - A^T P(t) - Q + [P(t)B + C^T P(t)D] \\ \qquad\qquad \times [R + D^T P(t)D]^{-1}[B^T P(t) + D^T P(t)C], \qquad t\in[\tau,T], \\ P(T) = G, \end{cases} \tag{5.1}$$

where $\tau\in[0,T)$. The above is called the *Riccati equation* for Problem (LQ) (with constant coefficients). In the present case, condition (4.19) can be replaced by

$$R + D^T GD > 0. \tag{5.2}$$

Then one can always find some $\tau\in[0,T)$, such that (5.1) admits a solution $P(\cdot)$ satisfying

$$R + D^T P(t)D > 0,\qquad t\in[\tau,T]. \tag{5.3}$$

The following result can be found in [8].

Proposition 5.1. *Suppose, for* $\tau\in[0,T)$, *there exists a solution* $P(\cdot)\colon[\tau,T]\to\mathcal{S}^n$ *to* (5.1) *satisfying* (5.3). *Then Problem* (LQ) *is solvable at* τ *with the optimal control* $\overline{u}(\cdot)$ *being of state feedback form*:

$$\overline{u}(t) = -[R + D^T P(t)D]^{-1}[B^T P(t) + D^T P(t)C]x(t),\qquad t\in[\tau,T], \tag{5.4}$$

and with the value function represented by

$$V(\tau,\xi) = \langle P(\tau)\xi,\xi\rangle,\qquad \forall\xi\in\chi_\tau. \tag{5.5}$$

From the above result, we see that the interval on which the Riccati equation (5.1) admits a solution $P(\cdot)$ satisfying condition (5.3) is closely related to the solvability of Problem (LQ). We now define

$$\begin{cases} I_R \overset{\Delta}{=} \{\tau \in [0,T) \bigm| (5.1) \text{ admits a solution } P(\cdot) \text{ satisfying } (5.3)\}, \\ I_S \overset{\Delta}{=} \{\tau \in [0,T) \bigm| \forall t \in [\tau,T], \xi \in \mathcal{X}_t, \\ \qquad \exists \overline{u} \in \mathcal{U}[t,T],\ V(t,\xi) = J(t,\xi;\overline{u}(\cdot))\}, \\ I_F \overset{\Delta}{=} \{\tau \in [0,T) \bigm| \forall t \in [\tau,T], \xi \in \mathcal{X}_t,\ V(\tau,\xi) > -\infty,\ \text{a.s.}\}. \end{cases} \tag{5.6}$$

By Proposition 5.1, we see that $I_R \subseteq I_S \subseteq I_F$. Now, we define

$$\sigma = \inf I_R. \tag{5.7}$$

Thus, $(\sigma, T]$ is the maximum interval on which (5.1) admits a solution $P(\cdot)$ such that (5.3) holds on $(\sigma, T]$. From the definition of σ, and Proposition 5.1, we see that for any $\tau \in (\sigma, T]$, Problem (LQ) is solvable at τ. We would like to know the solvability and/or finiteness of Problem (LQ) at σ.

Theorem 5.2. *Let all the coefficients be constants and let* (5.2) *hold. Then the following are equivalent*:

(i) $\sigma \in I_F$.
(ii) *There exists a sequence* $\tau_k \downarrow \sigma$ *and* $P_\sigma \in \mathcal{S}^n$ *such that*

$$\lim_{k\to\infty} P(\tau_k) = P_\sigma. \tag{5.8}$$

In this case, it is necessary that

$$V(\sigma,\xi) = \langle P_\sigma \xi, \xi\rangle, \qquad a.s. \quad \omega \in \Omega, \quad \forall \xi \in \mathcal{X}_\sigma. \tag{5.9}$$

(iii) *There exists a* $P_\sigma \in \mathcal{S}^n$ *such that*

$$\lim_{\tau\downarrow\sigma} P(\tau) = P_\sigma. \tag{5.10}$$

Proof. (iii) ⇒ (ii) The implication of (5.10) to (5.8) is clear. The proof of (5.9) is contained in the following.

(ii) ⇒ (i) For any $\xi \in \mathcal{X}_\sigma \subseteq \mathcal{X}_{\tau_k}$, by Proposition 5.1,

$$V(\tau_k,\xi) = \langle P(\tau_k)\xi,\xi\rangle, \qquad \forall k \geq 1.$$

Thus, for any $u(\cdot) \in \mathcal{U}[\sigma,T]$, we have (note $u\big|_{[\tau_k,T]} \in \mathcal{U}[\tau_k,T]$)

$$J(\tau_k,\xi;u\big|_{[\tau_k,T]}) \geq V(\tau_k,\xi) \to \langle P_\sigma\xi,\xi\rangle,$$

which yields

$$J(\sigma,\xi;u(\cdot)) \geq \langle P_\sigma\xi,\xi\rangle, \qquad \forall u(\cdot) \in \mathcal{U}[\sigma,T]. \tag{5.11}$$

Hence, $\sigma \in I_S$. We note that in this case (by Theorem 3.6)

$$V(\sigma,\xi) = \langle \widetilde{P}_\sigma \xi, \xi \rangle \geq \langle P_\sigma \xi, \xi \rangle, \qquad \forall \xi \in \mathcal{X}_\sigma, \tag{5.12}$$

for some $\widetilde{P}_\sigma \in L^\infty_{\mathcal{F}_\sigma}(\Omega; \mathcal{S}^n)$. On the other hand, by the dynamic programming principle (Lemma 4.1), and taking $u(\cdot) = 0$, we have

$$V(\sigma,\xi) \leq E\left\{ V(\tau_k, x(\tau_k)) + \int_\sigma^{\tau_k} \langle Qx(s), x(s)\rangle\, ds \,\Big|\, \mathcal{F}_\sigma \right\} \to \langle P_\sigma \xi, \xi \rangle. \tag{5.13}$$

Thus, combining (5.12)–(5.13), we see that

$$\widetilde{P}_\sigma \xi = P_\sigma \xi, \qquad \forall \xi \in \mathcal{X}_\sigma,$$

which leads to the first equality in (5.9). The second relation in (5.9) can be proved similarly.

(i) ⇒ (iii) Suppose Problem (LQ) is finite at σ. Take any $\xi \in \mathcal{X}_\sigma$. Let $x(\cdot)$ be the state process starting from (σ, ξ) under the control $u(\cdot) = 0$. Then

$$x(t) = \Phi(t)\Phi(\sigma)^{-1}\xi \stackrel{\Delta}{=} \Phi(t,\sigma)\xi, \qquad t \in [\sigma, T],$$

where $\Phi(\cdot)$ is the solution of

$$d\Phi(t) = A\Phi(t)\,dt + C\Phi(t)\,dw(t), \qquad \Phi(0) = I.$$

It is known that

$$E\Big\{ |\Phi(t,\sigma)^{-1}| \,\Big|\, \mathcal{F}_\sigma \Big\} \leq K_0, \qquad \forall t \in [\sigma, T], \quad \text{a.s.} \quad \omega \in \Omega, \tag{5.14}$$

for some absolute constant $K_0 > 0$. A direct computation shows that the solution $P(\cdot)$ of (5.1) satisfies the following:

$$\begin{aligned} P(t) &= e^{A^T(T-t)} G e^{A(T-t)} + \int_t^T e^{A^T(s-t)} Q e^{A(s-t)}\, ds \\ &\quad - \int_t^T e^{A^T(s-t)} [PB + C^T PD][R + D^T PD]^{-1} [B^T P + D^T PC] e^{A(s-t)}\, ds \\ &\leq e^{A^T(T-t)} G e^{A(T-t)} + \int_t^T e^{A^T(s-t)} Q e^{A(s-t)}\, ds \stackrel{\Delta}{=} \overline{P}(t). \end{aligned} \tag{5.15}$$

On the other hand, by Theorem 3.6, there exists a $\widetilde{P}_\sigma \in L^\infty_{\mathcal{F}_\sigma}(\Omega; \mathcal{S}^n)$ such that the first equality in (5.12) holds. Thus, by the dynamic programming principle again, we obtain that, for any $t \in (\sigma, T]$,

$$\begin{aligned} \langle \widetilde{P}_\sigma \xi, \xi \rangle = V(\sigma,\xi) &\leq E\left\{ V(t, x(t)) + \int_\sigma^t \langle Qx(s), x(s)\rangle\, ds \,\Big|\, \mathcal{F}_\sigma \right\} \\ &= E\left\{ \langle P(t)x(t), x(t)\rangle + \int_\sigma^t \langle Qx(s), x(s)\rangle\, ds \,\Big|\, \mathcal{F}_\sigma \right\} \\ &\leq E\left\{ \langle \overline{P}(t)x(t), x(t)\rangle + K_0|\xi|^2(t-\sigma) \,\Big|\, \mathcal{F}_\sigma \right\} \leq K_0|\xi|^2. \end{aligned}$$

Consequently,

$$E|\langle \Phi(t,\sigma)^T P(t)\Phi(t,\sigma)\xi,\xi\rangle| \le K_0 E|\xi|^2, \qquad \forall \xi \in \mathcal{X}_\sigma, \quad \forall t \in [\sigma,T], \tag{5.16}$$

which, together with (5.14), implies that

$$|P(t)| \le K_0, \qquad \forall t \in [\sigma,T]. \tag{5.17}$$

Then there exists a sequence $\tau_k \downarrow \sigma$, and some $P_\sigma \in \mathcal{S}^n$, such that (5.8) holds. Thus, (5.9) has to be true, and the whole sequence has to be convergent. □

The following example shows that $\sigma \notin I_F$ is possible.

Example 5.3. Let $n = m = 1$, $A = D = Q = 0$, $B = C = R = 1$, $G = -1$, and $T > 1$. Then (5.1) becomes

$$\begin{cases} \dot P(t) = P(t)^2, & t \in (\sigma,T], \\ P(T) = -1. \end{cases} \tag{5.18}$$

A direct computation gives us that

$$P(t) = \frac{1}{T-t-1}, \qquad t \in (\sigma,T], \quad \sigma = T-1. \tag{5.19}$$

Thus, $P(t) \to -\infty$ as $t \downarrow \sigma$, whereas (5.3), which is simply $R > 0$, remains true.

Now we return to Example 1.1. The state equation is (1.13) and the cost functional is (1.14). We assume (1.15). In this case, the Riccati equation reads (note $n = m = 1$, $A = C = Q = 0$, $B = G = 1$, $R = -1$, and $D = \delta$)

$$\begin{cases} \dot P(t) = \dfrac{P(t)^2}{\delta^2 P(t) - 1}, & t \le T, \\ P(T) = 1. \end{cases} \tag{5.20}$$

A direct computation shows that the solution $P(\cdot)$ satisfies

$$\begin{cases} \delta^2 \ln P(t) + \dfrac{1}{P(t)} = t + 1 - T, \\ \delta^2 P(t) > 1, \end{cases} \qquad t \in (\sigma,T].$$

We see easily that the σ defined by (5.6) satisfies $\delta^2 P(\sigma) = 1$ and thus

$$\delta^2\big(1 - 2\ln\delta\big) = \sigma + 1 - T$$

or

$$\sigma = T - 1 - \delta^2\big(2\ln|\delta| - 1\big).$$

Hence, (1.15) ensures that $\sigma < 0$, which guarantees the solvability of Problem (LQ) at all $\tau \in \mathcal{T}[0,T]$, by Proposition 5.1.

To conclude this paper, we make the following remark. The Riccati equation for Problem (LQ) with general random coefficients takes the following form:

$$\begin{cases} dP = -\{PA + A^T P + C^T PC + \Lambda C + C^T \Lambda + Q \\ \qquad\qquad - (PB + C^T PD + \Lambda D)(R + D^T PD)^{-1} \\ \qquad\qquad \times (B^T P + D^T PC + D^T \Lambda)\}\, dt + \Lambda\, dw(t), \qquad t \in [\tau, T], \\ P(T) = G, \\ \det[R(t) + D(t)^T P(T) D(t)] \neq 0, \qquad t \in [\tau, T], \quad \text{a.s.} \quad \omega \in \Omega. \end{cases} \tag{5.21}$$

This is again a BSDE, an adapted solution of which is a pair (P, Λ) of $\{\mathcal{F}_t\}_{t\geq 0}$-adapted $\mathcal{S}^n$-valued processes. Once an adapted solution (P, Λ) is obtained, one can construct an optimal state feedback control (see [8]). However, in (5.21) we note that the drift term depends Λ quadratically, and, moreover, it contains some possible singularity in P. Thus, the general solvability for such a BSDE remains open. Some results concerning the local/global solvability of (5.21) under certain conditions have been obtained by the authors and will be published elsewhere.

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