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Exact Controllability of the Superlinear Heat Equation*

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Abstract. The exact internal and boundary controllability of parabolic equations with superlinear nonlinearity is studied.

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1. Introduction

This work is concerned with the controllability of the equation

$$\begin{aligned}
&y_t(x,t) - \Delta y(x,t) + f(x,t,y(x,t)) = m(x)u(x,t),\\
&\qquad \forall (x,t) \in Q = \Omega \times (0,T),\\
&y(x,t) = 0, \qquad \forall (x,t) \in \Sigma = \partial\Omega \times (0,T),\\
&y(x,0) = y_0(x), \qquad \forall x \in \Omega,
\end{aligned} \tag{1.1}$$

where Ω is an open and bounded subset of R^n with a smooth boundary $\partial\Omega$ and m is the characteristic function of an open subset $\omega \subset \Omega$. Here Δ is the Laplace operator with respect to x.

The function $f\colon Q \times R \to R$ is continuous in y, measurable in (x,t), and satisfies the following conditions:

$$f(x,t,r)r \geq -\mu_0 r^2, \qquad \forall r \in R, \quad (x,t) \in Q, \tag{1.2}$$

$$|f(x,t,r)| \leq L|r|(\eta(|r|) + |f_0(x)|), \qquad \forall r \in R, \quad (x,t) \in Q. \tag{1.3}$$

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Here L is a positive constant, $\mu_0 \geq 0$, $f_0 \in L^n(\Omega)$, and η is a nonnegative, continuous, and increasing function.

We asume first that

$$\eta(r) \leq C(|r|^a + 1), \qquad \forall r \in R, \tag{1.4}$$

where

$$a \in (0, \infty) \qquad \text{if} \quad n = 1, 2; \qquad a = \frac{1}{n-2} \qquad \text{if} \quad n > 2.$$

We set

$$\rho(r) = \sup\left(\left(\int_\Omega (\eta(|w(x)|))^{2n}\, dx\right)^{1/2n}; \|w\|_{L^q(\Omega)} \leq r\right), \tag{1.5}$$

where

$$q = \frac{2n}{n-2} \quad \text{if } n > 2; \quad q \in [2, \infty) \quad \text{if } n = 2; \quad q = \infty \quad \text{if } n = 1. \tag{1.6}$$

The main results of this paper, Theorems 1 and 2 below, amount to saying that system (1.1) is null controllable for all $y_0 \in H_0^1(\Omega)$ satisfying the condition

$$\|y_0\|_{H_0^1(\Omega)} \leq C \sup(re^{-\mu\rho(r)}; r > 0) \tag{1.7}$$

and respectively for all $y_0 \in Y_n(\Omega) = W^2_{q(n)}(\Omega) \cap H_0^1(\Omega)$ such that

$$\|y_0\|_{Y_n(\Omega)} \leq C \sup(re^{-\mu\eta^{2/3}(r)}; r > 0) \tag{1.7$'$}$$

if $f_0 \in L^\infty(\Omega)$ and $1 \leq n < 6$. Here $\mu = \mu(L, f_0, T)$, C is independent of L and f_0, and

$$\frac{n+2}{2} < q(n) \leq \frac{2(n+2)}{n-2} \qquad \text{if} \quad n \geq 2; \qquad q(n) = 2 \qquad \text{if} \quad n = 1. \tag{1.8}$$

(In the latter case condition (1.4) is no longer necessary.) These theorems provide an estimate for the null controllability radius of system (1.1). The internal controllability implies the boundary controllability of (1.1) (Theorem 3) as well the controllability of the stationary solutions.

Null controllability of the linear heat equation was established by Lebeau and Robbiano [14] and was extended later to the sublinear heat equations, i.e., for $\eta = 0$, by Fursikov and Imanuvilov [7]). Null controllability for Lipschitzian nonlinearities f involving gradient terms was studied recently in [6] and [10]. Of course null controllability does not imply the controllability of an arbitrary smooth state y_1. However, if f is Lipschitzian it turns out (see [3] for (1.1) and [6] and [18] for nonlinearities with gradient terms) that system (1.1) is approximately controllable, i.e., the set of final states $y^u(T)$ of (1.1) is dense in $L^2(\Omega)$. A stronger version of this property (finite-approximate controllability) was studied by Zuazua [17].

Null controllability of superlinear control systems of the form (1.1) was recently proved by Fernandez-Cara [4] for nonlinearities of the form $f(y) = g(y)y$ where

$g(y)(\log(|y|+1))^{-1} \to 0$ as $|y| \to \infty$. The present approach is somewhat different and refers to more general nonlinearities f of accretive type and initial data y_0. In particular, if $1 \leq n < 6$ one derives from Theorem 2 the null controllability of (1.1) for $f = g(y)y$ where $g(y)(\log(|y|+1))^{-3/2} \to 0$ as $|y| \to \infty$ (Corollary 1.)

In general the best one can expect is local controllability [7] but we do not address this problem here.

The controllability problem for semilinear hyperbolic equations was previously studied in [16] (see also [13]). There is an extensive literature on the controllability of linear control systems of hyperbolic type and we refer the reader to the books by Lions [15] and Komornik [11] for general results and specific methods.

The paper is organized as follows. The main results, Theorems 1 and 2, are stated in Section 2 and proved in Section 3 (respectively Section 5) via the infinite-dimensional Kakutani fixed-point theorem (see, e.g., p. 310 of [2]) The proof relies on the Carleman inequality for the backward adjoint linearized system associated with (1.1), which is proved in Section 4. In fact, a large part of this paper is devoted to establishing a sharp estimate of the constant which appears on the right-hand side of the Carleman inequality proved earlier by Fursikov and Imanuvilov [7] (see also [9]).

In what follows we use the standard notations for the Sobolev spaces $H^k(\Omega)$, $H_0^1(\Omega)$ and the L^p spaces on Ω and Q, $1 \leq p \leq \infty$, with the norm denoted $\|\cdot\|_p$. Moreover, we set

$$W_p^2(\Omega) = \{y \in L^p(\Omega);\ D_{x_i}^s y \in L^p(\Omega);\ s = 1, 2;\ i = 1, \ldots, n\}, \qquad 2 \leq p \leq \infty,$$

$$W^{1,2}([0, T]; L^2(\Omega)) = \left\{y \in L^2(0, T; L^2(\Omega));\ \frac{dy}{dt} \in L^2(0, T; L^2(\Omega))\right\},$$

$$W_p^{2,1}(Q) = \{y \in L^q(Q);\ D_t^r D_{x_i}^s y \in L^p(Q);\ 2r + s \leq 2;\ i = 1, \ldots, n\},$$

where dy/dt and $D_t^r D_x^s y$ are taken in the sense of distributions.

We set $H^{2,1}(Q) = W_2^{2,1}(Q) \cap L^2(0, T; H_0^1(\Omega))$.

2. The Main Results

Let $y_0 \in H_0^1(\Omega)$ be arbitrary but fixed. The control system (1.1) is said to be *null controllable* or *exactly null controllable* if there are $u \in L^2(Q)$ and $y \in L^2(0, T; H_0^1(\Omega) \cap H^2(\Omega)) \cap W^{1,2}([0, T]; L^2(\Omega))$ which satisfy (1.1) and

$$y(x, T) = 0, \qquad \text{a.e.} \quad x \in \Omega.$$

Now we are ready to formulate the main results of this paper.

Theorem 1. *Assume that conditions* (1.2)–(1.4) *hold. Then there is* $\mu = \mu(L, \|f_0\|_n, T)$ *bounded with respect to L and* $\|f_0\|_n$ *such that for all* $y_0 \in H_0^1(\Omega)$ *satisfying* (1.7) *system* (1.1) *is exactly null controllable. In particular, if*

$$\limsup_{r\to\infty}(re^{-\mu\rho(r)}) = +\infty, \qquad \forall\mu > 0, \tag{2.1}$$

then system (1.1) *is exactly null controllable for all* $y_0 \in H_0^1(\Omega)$.

Theorem 2. *Assume that conditions* (1.2) and (1.3) *hold and that* $f_0 \in L^\infty(\Omega)$. *Then for* $1 \le n < 6$ *system* (1.1) *is null controllable for all* $y_0 \in Y_n(\Omega) = W^2_{q(n)}(\Omega) \cap H^1_0(\Omega)$, *satisfying* (1.7)′. *If*

$$\limsup_{r\to\infty}(re^{-\mu\eta^{2/3}(r)}) = +\infty, \qquad \forall \mu > 0, \tag{2.1$'$}$$

then system (1.1) *is exactly null controllable for all* $y_0 \in Y_n(\Omega)$.

Now we derive some simple consequences of Theorem 1. First notice that since $\rho(r) \le C(r^a + 1)$, it follows by Theorem 1 that system (1.1) is exactly null controllable for all $y_0 \in H^1_0(\Omega)$ satisfying the condition

$$\|y_0\|_{H^1_0(\Omega)} \le \frac{1}{(\nu a)^{1/a}}, \tag{2.2}$$

where $\nu = \nu(L, \|f_0\|_n, T)$.

Corollary 1. *Assume that* $f_0 \in L^n(\Omega)$, f *satisfies* (1.2), *and*

$$|f(x,t,r)| \le L|r|(\varphi(|r|)\log(|r|+1) + |f_0(x)|), \qquad \forall (x,t,r) \in Q \times R, \tag{2.3}$$

where φ *is a continuous function such that* $\lim_{s\to\infty}\varphi(s) = 0$. *Then system* (1.1) *is exactly null controllable for all* $y_0 \in H^1_0(\Omega)$. *If* $y_0 \in Y_n(\Omega)$, $1 \le n < 6$, *and* $f_0 \in L^\infty(\Omega)$, *then condition* (2.3) *can be weakened to*

$$|f(x,t,r)| \le L|r|(\varphi(|r|)(\log(|r|+1))^{3/2} + |f_0(x)|), \qquad \forall (x,t,r) \in Q \times R. \tag{2.3$'$}$$

Proof of Corollary 1. By the mollifiers technique it follows that there is $C > 0$ such that $S_q(r) \subset \overline{S}(r)$ where $S(r) = \{w \in C(\overline{\Omega});\ \|w\|_\infty \le Cr\}$, $\overline{S}(r)$ is the closure of $S(r)$ in $L^q(\Omega)$ and $S_q(r) = \{w \in L^q(\Omega);\ \|w\|_q \le r\}$. We have, therefore,

$$\begin{aligned}\rho(r) &\le L \sup\left(\left(\int_\Omega (\varphi(|w|)(\log(|w|+1)))^{2n}\,dx\right)^{1/2n};\ w \in S(r)\right)\\ &\le C(\varphi(\theta_r)\log(|\theta_r|+1)), \qquad \forall r > 0,\end{aligned} \tag{2.4}$$

where $0 \le \theta_r \le Cr$. Hence condition (2.1) is satisfied and we conclude the proof by invoking Theorem 1. □

If $y_0 \in Y_n(\Omega)$ and $f_0 \in L^\infty(\Omega)$, then (2.3)′ implies (2.1)′.

Remark 1. It follows by (2.4) that if f satisfies condition (1.2) and

$$|f(x,t,r)| \le |r|(L\log(|r|+1) + |f_0(x)|), \qquad \forall (x,t,r) \in Q \times R,$$

then for L sufficiently small but independent of y_0 condition (2.1) is satisfied and so system (1.1) is exactly null controllable for all $y_0 \in H^1_0(\Omega)$.

Remark 2. The first part of Corollary 1, i.e., null controllability under condition (2.3) was previously proved by Fernandez-Cara [4] for L^∞ smooth initial data y_0. In general for nonlinearities f with polynomial growth of order $p > 1$, exact controllability may fail (see the examples given in [7], [8], and [5]). Moreover, analysis of the proofs of Theorems 1 and 2 seems to indicate that in general one cannot expect null controllability for functions f which grow to infinity faster than $r(\log(\mathrm{r}+1))^\alpha$ where $\alpha > \frac{3}{2}$.

Theorem 3. *Under assumptions* (1.2)–(1.4), *there is* $\mu > 0$ *such that, for each* $y_0 \in H^1(\Omega)$ *satisfying the condition*

$$\|y_0\|_{H^1(\Omega)} \le \Theta(\mu),$$

there are $v \in L^2(\Sigma)$ *and* $y \in H^{2,1}(Q)$ *such that*

$$\begin{array}{ll} y_t - \Delta y + f(y) = 0 & \text{in } Q, \\ y = v & \text{in } \Sigma, \\ y(x,0) = y_0(x) & \text{in } \Omega, \\ y(x,T) = 0 & \text{in } \Omega, \end{array} \tag{2.5}$$

with the usual modification if $y_0 \in Y_n(\Omega)$, $1 \le n < 6$, *and* $f_0 \in L^\infty(\Omega)$. *Moreover, the conclusions of Corollary* 1 *remain valid in the present situation.*

Here Θ is the function defined by the right-hand side of (1.7)

Proof of Theorem 3. One applies Theorem 1 (respectively Theorem 2) on $\widetilde{\Omega} \supset \Omega$ where $\omega = \widetilde{\Omega}\backslash\overline{\Omega}$ and $\widetilde{y}_0$ (the initial data) is an extension of y_0 to $H_0^1(\widetilde{\Omega})$. If $\widetilde{y}$ is a corresponding solution to the controllability problem provided by Theorem 1 (respectively by Theorem 2), then by the trace theorem we see that

$$y = \widetilde{y}|_\Omega \quad \text{and} \quad v = \widetilde{y}|_{\partial\Omega}$$

satisfy (2.5). □

Remark 3. The above results also imply the exact controllability of the stationary solutions to (1.1). The function $y_1 \in H_0^1(\Omega)$ is said to be a *stationary solution* to system (1.1) if there is $w \in L^2(\Omega)$ such that

$$-\Delta y_1(x) + f(x, y_1(x)) = m(x)w(x), \qquad x \in \Omega.$$

We note also that these theorems extend to mutivalued functions f of the form $f(r) = f_0(r) + cr$ where f_0 is a maximal monotone graph such that its minimal section f_0^0 satisfies conditions (1.2)–(1.4). To this end one applies Theorem 1 to function $(f_0)_\varepsilon(r) + cr$ and lets ε tend to zero. Here $(f_0)_\varepsilon$ is the Yosida approximation of f_0.

3. Proof of Theorem 1

We fix $y_0 \in H_0^1(\Omega)$ and define the set

$$K = \{w \in L^\infty(0, T; L^q(\Omega)); \|w(t)\|_q \le M, \text{ a.e. } t \in (0, T)\}, \tag{3.1}$$

where M is a positive constant to be defined later and

$$q = \frac{2n}{n-2} \quad \text{if} \quad n > 2; \qquad q \in (2, \infty) \quad \text{if} \quad n = 2; \qquad q = \infty \quad \text{if} \quad n = 1.$$

We set

$$g(x, t, r) = \frac{f(x, t, r)}{r} \qquad \text{for} \quad |r| > 0; \qquad g(x, t, 0) = \lim_{r \to 0} g(x, t, r).$$

Without loss of generality we may assume that the above limit exists and so $g(x, t, r)$ is continuous in r. Otherwise we approximate f by a family of smooth functions in r and tend to limit in the corresponding controllability problem.

For $w \in K$ consider the linear system

$$\begin{aligned} &y_t - \Delta y + g(x, t, w)y = mu && \text{in } Q, \\ &y(x, 0) = y_0(x) && \text{in } \Omega, \\ &y = 0 && \text{in } \Sigma. \end{aligned} \tag{3.2}$$

Lemma 1 below is the main ingredient of the proof.

Lemma 1. *For each $y_0 \in H_0^1(\Omega)$ and $w \in K$ there are $y \in L^2(0, T; H_0^1(\Omega) \cap H^2(\Omega)) \cap W^{1,2}([0, T; L^2(\Omega))$ and $u \in L^2(Q)$ which satisfy* (3.2) *and*

$$y(x, T) = 0, \qquad \text{a.e.} \quad x \in \Omega, \tag{3.3}$$

$$\int_Q mu^2 \, dx \, dt \le \gamma e^{\mu\rho(M)} \|y_0\|_2^2, \tag{3.4}$$

where $\mu = \mu(L, \|f_0\|_n) > 0$ and γ are independent of y_0, w, M and bounded with respect to L and $\|f_0\|_n$.

Proof. We note first that for all $w \in K$, $u \in L^2(Q)$, and $y_0 \in H_0^1(\Omega)$, (3.2) has a unique solution

$$y = y^u \in L^2(0, T; H_0^1(\Omega) \cap H^2(\Omega)) \cap W^{1,2}([0, T]; L^2(\Omega)).$$

Here is the argument. Since $g \ge -\mu_0$ we get the a priori estimate

$$|y(t)|_2^2 + \int_0^t \int_\Omega |\nabla y(x, s)|^2 \, dx \, ds \le C \left(|y_0|_2^2 + \int_0^t \int_\Omega mu^2 \, dx \, ds \right), \quad \forall t \in [0, T], \tag{3.5}$$

while by (1.3) and (1.5) we have

$$\begin{aligned}
&\int_Q |g(x,t,w)y|^2\,dx\,dt \\
&\qquad \le C\int_0^T dt\int_\Omega (\eta^2(|w(x,t)|)+|f_0(x)|^2)|y|^2\,dx \\
&\qquad \le C\int_0^T dt\left(\int_\Omega |y|^{p^*}\,dx\right)^{2/p^*}\left(\left(\int_\Omega (\eta(|w|))^{2p^*/(p^*-2)}\,dx\right)^{(p^*-2)/p^*}+\|f_0\|_n^2\right),
\end{aligned}$$

where $p^* = 2n/(n-2)$ for $n > 2$ and p^* is arbitrary in $(1,\infty)$ if $1 \le n \le 2$.

By the Sobolev embedding theorem we have

$$\begin{aligned}
&\int_Q |g(x,t,w)y|^2\,dx\,dt \\
&\qquad \le C\int_0^T dt\left(\int_\Omega |\nabla y(x,t)|^2\,dx\right)\left(\left(\int_\Omega \eta(|w(x,t)|)^n\,dx\right)^{2/n}+1\right) \\
&\qquad \le C(\rho^2(M)+1)\left(\|y_0\|_2^2+\int_Q mu^2\,dx\,dt\right) \qquad (3.6)
\end{aligned}$$

and by a standard approximating argument the existence of a unique solution $y = y^u$ which satisfies the estimate

$$\begin{aligned}
&\|y(t)\|^2_{H_0^1(\Omega)}+\int_0^T\int_\Omega |\Delta y|^2\,dx\,dt+\int_Q y_t^2(x,t)\,dx\,dt \\
&\qquad \le C\left(\|y_0\|^2_{H_0^1(\Omega)}+\int_Q mu^2\,dx\,dt\right)(\rho^2(M)+1) \qquad (3.7)
\end{aligned}$$

follows. Here and throughout in what follows we denote by C several positive constants independent of u, y_0, w, M but bounded with respect to L and $\|f_0\|_n$. (The latter estimate follows in the usual way by multiplying (3.2) by y_t and using (3.6).)

Now consider the optimal control problem ($\varepsilon > 0$)

$$\text{Minimize } \int_Q u^2\,dx\,dt+\frac{1}{\varepsilon}\int_\Omega y^2(x,T)\,dx \text{ subject to (3.2).} \qquad (3.8)$$

Let $(y_\varepsilon, u_\varepsilon)$ be an optimal pair (the existence follows in a standard way by estimate (3.7)) because in (3.2) the map $u \to y^u$ is closed in $(L^2(Q))_w \times L^2(Q)$. (Here $(L^2(Q))_w$ is the space $L^2(Q)$ endowed with the weak topology.)

By the maximum principle we have

$$u_\varepsilon(x,t) = m(x)p_\varepsilon(x,t), \qquad \text{a.e.} \quad (x,t)\in Q, \qquad (3.9)$$

where p_ε is the solution to the backward adjoint system

$$\begin{aligned}
&(p_\varepsilon)_t+\Delta p_\varepsilon - g(x,t,w)p_\varepsilon = 0 && \text{in } Q,\\
&p_\varepsilon = 0 && \text{in } \Sigma,\\
&p_\varepsilon(x,T) = -\frac{1}{\varepsilon}\,y_\varepsilon(x,T) && \text{in } \Omega.
\end{aligned} \qquad (3.10)$$

This yields

$$\int_Q mp_\varepsilon^2\,dx\,dt + \frac{1}{\varepsilon}\int_\Omega y_\varepsilon^2(x,T)\,dx = \int_\Omega y_0(x)p_\varepsilon(x,0)\,dx. \tag{3.11}$$

To continue the proof we need the following observability result for the solutions $p \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ to

$$p_t + \Delta p - g(x,t,w)p = 0 \qquad \text{in } Q. \tag{3.12}$$

Lemma 2. *There are $\mu = \mu(L, \|f_0\|_n)$ and γ independent of w, M, p and bounded with respect to L and $\|f_0\|_n$ such that*

$$\int_\Omega p^2(x,0)\,dx \le \gamma e^{\mu\rho(M)}\int_0^T\int_\omega p^2(x,t)\,dx\,dt. \tag{3.13}$$

This must be viewed as a uniform observability result for the linear adjoint system (3.12) with respect to $w \in K$. We postpone the proof of Lemma 2 until Section 4 where another version of this lemma will also be proved for the purposes of the proof of Theorem 2.

Now using (3.13) in (3.11) we get

$$\begin{aligned}\int_Q mp_\varepsilon^2\,dx\,dt + \frac{1}{\varepsilon}\int_\Omega y_\varepsilon^2(x,T)\,dx &\le \|y_0\|_2\left(\gamma e^{\mu\rho(M)}\right)^{1/2}\left(\int_Q mp_\varepsilon^2\,dx\,dt\right)^{1/2}\\ &\le 2^{-1}\int_Q mp_\varepsilon^2\,dx\,dt + 2^{-1}\gamma\|y_0\|_2^2 e^{\mu\rho(M)}.\end{aligned}$$

Hence

$$\int_Q u_\varepsilon^2\,dx\,dt + \frac{2}{\varepsilon}\int_\Omega y_\varepsilon^2(x,T)\,dx \le \gamma e^{\mu\rho(M)}\|y_0\|_2^2, \qquad \forall\varepsilon > 0. \tag{3.14}$$

By estimates (3.7) and (3.14) it follows that, selecting a subsequence, we have

$$\begin{aligned}&u_\varepsilon \to u \qquad \text{weakly in } L^2(Q),\\ &y_\varepsilon \to y \qquad \text{weakly in } L^2(0,T;H_0^1(\Omega)\cap H^2(\Omega)) \cap W^{1,2}([0,T];L^2(\Omega)),\end{aligned}$$

where (y,u) satisfy (3.2) and $y(T) \equiv 0$. Moreover, u satisfies estimate (3.4) as claimed. This completes the proof of Lemma 1. □

Proof of Theorem 1 (continued). For each $w \in K$ denote by $\Phi(w) \subset L^2(Q)$ the set of all solutions $y^u \in L^2(0,T;H_0^1(\Omega)\cap H^2(\Omega)) \cap W^{1,2}([0,T];L^2(\Omega))$ to (3.2) such that

$$y^u(T) \equiv 0; \qquad \|mu\|_{L^2(Q)}^2 \le \gamma e^{\mu\rho(M)}\|y_0\|_2^2, \tag{3.15}$$

where μ and γ are as in Lemma 1.

By Lemma 1 it follows that $\Phi(w) \ne \emptyset$ for each $w \in K$. Moreover, it is readily seen that $\Phi(w)$ is a closed and convex subset of $L^2(Q)$. (The fact that $\Phi(w)$ is closed follows by estimate (3.7) which implies that the map $u \to y^u$ is closed in $L^2(Q)$ for each $w \in K$.) By estimate (3.7) and by (3.15) we have

$$\|y(t)\|^2_{H^1_0(\Omega)} + \int_Q (y_t^2(x,t) + |\Delta y(x,t)|^2)\, dx\, dt \\ \leq C(\rho(M)+1)^2(\|y_0\|^2_{H^1_0(\Omega)} + e^{\mu\rho(M)}\|y_0\|_2^2) \tag{3.16}$$

and so by the Sobolev embedding theorem

$$\|y(t)\|^2_{2n/(n-2)} \leq Ce^{2\mu\rho(M)}\|y_0\|^2_{H^1_0(\Omega)}, \qquad \forall t \in [0,T], \tag{3.17}$$

where C is independent of M and w. If $n \leq 2$ we have

$$\|y(t)\|^2_s \leq Ce^{2\mu\rho(M)}\|y_0\|^2_{H^1_0(\Omega)}, \qquad \forall t \in [0,T], \tag{3.18}$$

where s is arbitrary in $(1,\infty)$. Now we choose

$$M = \arg\sup(re^{-\mu\rho(r)}; r > 0).$$

Thus it follows by (3.17) and (3.18) that if

$$\|y_0\|_{H^1_0(\Omega)} \leq C^{-1/2}\sup(re^{-\mu\rho(r)}; r > 0),$$

then $\Phi(K) \subset K$.

Moreover, by estimate (3.16) it follows via the Arzelà–Ascoli theorem that $\Phi(K)$ is a relatively compact subset of $L^2(Q)$.

Note also that Φ is upper semicontinuous in $L^2(Q) \times L^2(Q)$. Let $w_n \to w$ in $L^2(Q)$, $w_n \in K$, and $y_n \to y$ in $L^2(Q)$, $y_n \in \Phi(w_n)$, $y_n = y^{u_n}$. By Lemma 1 and by estimate (3.16) it follows (selecting a subsequence if necessary) that

$$\begin{aligned} u_n &\to u \qquad \text{weakly in } L^2(Q),\\ y_n &\to y \qquad \text{strongly in } C([0,T];L^2(\Omega))\\ &\qquad\quad \text{and weakly in } L^2(0,T;H^1_0(\Omega)\cap H^2(\Omega)) \cap W^{1,2}([0,T];L^2(\Omega)). \end{aligned}$$

We have

$$g(x,t,w_n(x,t))y_n(x,t) \to g(x,t,w(x,t))y(x,t), \quad \text{a.e. in } Q,$$

and by estimate (3.6) we have

$$g(x,t,w_n)y_n \to \eta \qquad \text{weakly in } L^2(Q).$$

Then by the Egorov theorem we infer that

$$\eta(x,t) = g(x,t,w(x,t))y(x,t), \qquad \text{a.e.} \quad (x,t) \in Q.$$

Thus letting n tend to $+\infty$ in the equations

$$\begin{array}{ll} (y_n)_t - \Delta y_n + g(x,t,w_n)y_n = mu_n & \text{in } Q, \\ y_n = 0 & \text{in } \Sigma, \\ y_n(x,0) = y_0(x), \quad y_n(x,T) = 0 & \text{in } \Omega, \end{array}$$

we conclude that (y,u) satisfy (3.2) and (3.15), i.e., $y \in \Phi(w)$ as claimed.

Then applying the Kakutani fixed-point theorem in the space $L^2(Q)$ to the mapping Φ we infer that there is at least one $w \in K$ such that $w \in \Phi(w)$. By definition of Φ this implies that there is at least one pair (y,u) satisfying the conditions of Theorem 1. This completes the proof. □

4. Proof of Lemma 2

It should be said that (3.13) follows by the Carleman inequality established in [7]. However, for the sake of completeness and for easy reference we give here a direct proof keeping the notations and the scheme developed in [7]. Namely, let $w_0 \subset\subset \omega$ and $\psi \in C^2(\overline{\Omega})$ be such that

$$\begin{array}{l} \psi(x) > 0, \qquad \forall x \in \Omega, \quad \psi = 0 \quad \text{in} \quad \partial\Omega, \\ |\nabla\psi(x)| > 0, \qquad \forall x \in \Omega_0 = \overline{\Omega}\backslash\omega_0. \end{array} \tag{4.1}$$

We set

$$\varphi(x,t) = \frac{e^{\lambda\psi(x)}}{t(T-t)}, \qquad \alpha(x,t) = \frac{e^{\lambda\psi(x)} - e^{2\lambda\|\psi\|_{C(\overline{\Omega})}}}{t(T-t)}.$$

Let $z = e^{s\alpha}p$ where s and λ are positive parameters which will be made precise later. Then z satisfies the equation

$$\begin{array}{l} z_t + \Delta z - 2s\lambda\varphi\nabla\psi\cdot\nabla z + (\lambda^2 s^2\varphi^2|\nabla\psi|^2 - \lambda^2 s\varphi|\nabla\psi|^2 - s\alpha_t - \lambda s\varphi\Delta\psi)z \\ \qquad = g(x,t,w)z \quad \text{in } Q, \\ z = 0 \quad \text{in } \Sigma; \qquad z(x,0) = z(x,T) = 0 \quad \text{in } \Omega. \end{array} \tag{4.2}$$

Arguing as in [1] we set

$$X(t)\zeta = -2(s\lambda^2\varphi|\nabla\psi|^2\zeta + s\lambda\varphi\nabla\zeta\cdot\nabla\psi)$$

and

$$F(t)z = -\Delta z - (\lambda^2 s^2\varphi^2|\nabla\psi|^2 + s\lambda^2\varphi|\nabla\psi|^2 - s\alpha_t - \lambda s\varphi\Delta\psi)z.$$

We have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega F(t)z(x,t)z(x,t)\,dx \\ &\quad = \int_\Omega F_t(t)z(x,t)z(x,t)\,dx \\ &\qquad + 2\int_\Omega F(t)z(x,t)(F(t)z - X(t)z + g(x,t,w)z(x,t))\,dx. \end{aligned}$$

Integrating on Q we obtain

$$-\frac{1}{2}\int_Q F_t(t)zz\,dx\,dt$$
$$= \int_Q (F(t)z(x,t))^2\,dx\,dt + \int_Q F(t)z(x,t)g(x,t,w)z(x,t)\,dx\,dt + Y, \tag{4.3}$$

where

$$Y = \int_Q (2s\lambda^2\varphi|\nabla\psi|^2 z + 2s\lambda\varphi\nabla\psi\cdot\nabla z)$$
$$\times\,(-\Delta z - s^2\lambda^2\varphi^2|\nabla\psi|^2 z - s\lambda^2\varphi|\nabla\psi|^2 z - s\lambda\varphi\Delta\psi z - s\alpha_t z)\,dx\,dt. \tag{4.4}$$

Since $|\varphi_t| \le C\varphi^2$, it follows by (4.3) that

$$Y \le C\int_Q (s^2\lambda^2\varphi^3 + |g(w)|^2)z^2\,dx\,dt. \tag{4.5}$$

We set

$$D(s,\lambda,z) = \int_Q ((s^3\lambda^3 + s^2\lambda^4)\varphi^3 z^2 + s\lambda\varphi|\nabla z|^2)\,dx\,dt.$$

Then after some calculation involving the Gauss–Ostrogradski formula it follows by (4.4) that

$$Y \ge \int_Q (2s\lambda^2\varphi|\nabla\psi|^2|\nabla z|^2 + s^3\lambda^4\varphi^3|\nabla\psi|^4 z^2$$
$$- 2s\lambda\varphi(\nabla\psi\cdot\nabla z)\Delta z)\,dx\,dt - CD(s,\lambda,z). \tag{4.6}$$

On the other hand, we have

$$2s\lambda\int_Q \varphi\Delta z(\nabla\psi\cdot\nabla z)\,dx\,dt$$
$$= 2s\lambda\int_\Sigma \varphi(\nabla\psi\cdot\nabla z)(\nabla z\cdot\nu)\,dx\,dt - s\lambda\int_\Sigma \varphi|\nabla z|^2(\nabla\psi\cdot\nu)\,dx\,dt$$
$$+ s\lambda^2\int_Q \varphi|\nabla\psi|^2|\nabla z|^2\,dx\,dt - \int_Q 2s\lambda^2\varphi(\nabla z\cdot\nabla\psi)^2$$
$$+ s\lambda\varphi\left(|\nabla z|^2\Delta\psi + \sum_{i,j=1}^n z_{x_i}z_{x_j}\psi_{x_i,x_j}\right)\,dx\,dt,$$

where ν is the outward normal to Ω. Since $z = \psi = 0$ on $\partial\Omega$ and $\psi \ge 0$ in $\overline{\Omega}$, we have

$$(\nabla\psi\cdot\nabla z)(\nabla z\cdot\nu) = |\nabla z|^2(\nabla\psi\cdot\nu) = -|\nabla\psi|^2|\nabla z|^2.$$

Hence

$$2s\lambda\int_Q \varphi\Delta z(\nabla\psi\cdot\nabla z)\,dx\,dt \le s\lambda^2\int_Q \varphi|\nabla\psi|^2|\nabla z|^2\,dx\,dt + CD(s,\lambda,z).$$

Along with (4.5) and (4.6) the latter yields

$$\begin{aligned} &s^3\lambda^4\int_Q \varphi^3|\nabla\psi|^4 z^2\,dx\,dt + s\lambda^2\int_Q \varphi|\nabla\psi|^2|\nabla z|^2\,dx\,dt \\ &\quad\le C\left(D(s,\lambda,z)+\int_Q |g(x,t,w)|^2 z^2\,dx\,dt\right) \\ &\quad\le C\int_Q (s^3\lambda^3+s^2\lambda^4)\varphi^3 z^2 + s\lambda\varphi|\nabla z|^2 + (\eta^2(|w|)+|f_0|^2)z^2)\,dx\,dt. \end{aligned} \tag{4.7}$$

On the other hand, by the Holder inequality we have

$$\begin{aligned} \int_\Omega (\eta(|w|))^2 z^2\,dx &\le \left(\int_\Omega (\eta(|w|)^2 z)^{2n/(n+2)} dx\right)^{(n+2)/2n} \left(\int_\Omega |z|^{p^*} dx\right)^{1/p^*} \\ &\le \left(\int_\Omega |z|^{p^*}\,dx\right)^{1/p^*} \left(\int_\Omega |\eta(|w|)|^{2n}\,dx\right)^{1/n} \left(\int_\Omega z^2\,dx\right)^{1/2} \\ &\le \tfrac{1}{2}\left(\rho(M)\left(\int_\Omega |z|^{p^*}\,dx\right)^{2/p^*} + \rho^3(M)\int_\Omega z^2\,dx\right) \end{aligned}$$

and similarly,

$$\int_\Omega |f_0 z|^2\,dx \le \left(\int_\Omega |z^*|^{p^*}\,dx\right)^{2/p^*} \left(\int_\Omega |f_0|^n\,dx\right)^{2/n},$$

where $p^* = 2n/(n-2)$ if $n > 2$, $p^* \in (2,\infty)$ if $1 \le n \le 2$. Then by the Sobolev embedding theorem we get

$$\begin{aligned} &\int_Q ((\eta(|w|))^2 + |f_0|^2)|z|^2\,dx\,dt \\ &\quad\le \tfrac{1}{2}\rho(M)\int_Q |\nabla z|^2\,dx\,dt + (\tfrac{1}{2}\rho^3(M) + \|f_0\|_n^2)\int_Q z^2\,dx\,dt. \end{aligned}$$

Substituting the latter into (4.7) and recalling that $|\nabla\psi(x)| \ge \gamma_0 > 0$, $\forall x \in \Omega_0$, it follows that, for $\lambda \ge \lambda_0$ and $s \ge s_0 + C\rho(M)$ where λ_0, s_0 are sufficiently large but independent of M, z, we have

$$\begin{aligned} &\int_Q (s^3\lambda^4\varphi^3 z^2 + s\lambda^2\varphi|\nabla z|^2)\,dx\,dt \\ &\quad\le C\int_{Q_{\omega_0}} (((s^3\lambda^3+s^2\lambda^4)\varphi^3 + \rho^3(M))z^2 + (\rho(M)+s\lambda\varphi)|\nabla z|^2)\,dx\,dt, \end{aligned} \tag{4.8}$$

where $Q_{w_0} = \omega_0 \times (0,T)$. This yields

$$\begin{aligned} &\int_Q e^{2s\alpha}(s^3\lambda^4\varphi^3 p^2 + s\lambda^2\varphi|sp\nabla\alpha + \nabla p|^2)\,dx\,dt \\ &\quad\le C\int_{Q_{\omega_0}} (s^3\lambda^3+s^2\lambda^4)e^{2s\alpha}(\varphi^3+\rho^3(M))p^2\,dx\,dt \\ &\qquad + C\int_{Q_{\omega_0}} (\rho(M)+s\lambda\varphi)e^{2s\alpha}|sp\nabla\alpha + \nabla p|^2\,dx\,dt. \end{aligned}$$

Then recalling (4.1) and using Green's formula it follows as above that

$$\int_Q e^{2s\alpha}(\varphi^3 p^2 + \varphi|\nabla p|^2)\,dx\,dt \\ \le C\int_{Q_{\omega_0}} e^{2s\alpha}((\varphi^3 + \rho^3(M))p^2 + (\rho(M)+1)\varphi|\nabla p|^2)\,dx\,dt \tag{4.9}$$

for $s \ge s_0 + C\rho(M),\ \lambda \ge \lambda_0$. (Here C is independent of p, M, w.) Let $\chi \in C_0^\infty(\Omega)$ be such that $\chi = 1$ in $\overline{\omega}_0$ and $\chi = 0$ in $\Omega\backslash\omega$.

Multiplying (3.12) by $\chi\varphi e^{2s\alpha}p$ and integrating on Q we get after some calculation that

$$\int_{Q_{\omega_0}} e^{2s\alpha}\varphi|\nabla p|^2\,dx\,dt \le C(\rho(M)+1)^3\int_{Q_\omega} e^{2s\alpha}\varphi^3 p^2\,dx\,dt$$

and substituting into (4.9) we get the Carleman inequality

$$\int_Q e^{2s\alpha}(\varphi^3 p^2 + \varphi|\nabla p|^2)\,dx\,dt \le C(\rho(M)+1)^3\int_{Q_\omega} e^{2s\alpha}\varphi^3 p^2\,dx\,dt \tag{4.10}$$

for $s \ge s_0 + C\rho(M)$ and $\lambda \ge \lambda_0$ where s_0, λ_0 are sufficiently large but likewise C are independent of M, p, w and bounded with respect to L and $\|f_0\|_n$.

Now by (3.12) we see that

$$\int_\Omega p^2(x,0)\,dx \le C\int_\Omega p^2(x,t)\,dx, \qquad \forall t \in [0,T]. \tag{4.11}$$

Integrating (4.11) on $(t_0,t_1) \subset (0,T)$, using (4.10), and

$$\inf_{t\in(t_0,t_1)}\{e^{2s\alpha(x,t)}\varphi^3(x,t)\} \ge Ce^{-C\rho(M)}, \qquad \forall x \in \Omega, \tag{4.12}$$

where $s = s_0 + C\rho(M)$ and $C = C(L, \|f_0\|)$ is independent of p, M, w and bounded with respect to $L, \|f\|_n$, we obtain inequality (3.13) as claimed. This completes the proof. □

By the previous proof we get the following sharpening of Lemma 2 in the case where

$$w \in K_\infty; \qquad f_0 \in L^\infty(\Omega). \tag{4.13}$$

Here $K_\infty = \{w \in L^\infty(Q);\ \|w\|_\infty \le M\}$.

Lemma 3. *Under assumptions* (4.13) *we have*

$$\int_Q e^{2s\alpha}(\varphi^3 p^2 + \varphi|\nabla p|^2)\,dx\,dt \le C(\eta(M)+1)^3\int_{Q_\omega} e^{2s\alpha}\varphi^3 p^2\,dx\,dt, \tag{4.14}$$

$$\int_\Omega p^2(x,0)\,dx \le \gamma e^{\beta\eta^{2/3}(M)}\int_{Q_\omega} e^{2s\alpha}\varphi^3 p^2(x,t)\,dx\,dt \tag{4.15}$$

for $s \ge s_0 + C\eta^{2/3}(M), \lambda \ge \lambda_0$. *Here* C, γ, β *are independent of* M *and* p.

Proof. Taking in account that $\eta(|w|) \le \eta(M)$ a.e. in Q for all $w \in K_\infty$ we get (see (4.7))

$$s^3\lambda^4 \int_Q \varphi^3 |\nabla\psi|^4 z^2\,dx\,dt + s\lambda^2 \int_Q \varphi|\nabla\psi|^2|\nabla z|^2\,dx\,dt$$
$$\le C\int_Q ((s^3\lambda^3 + s^2\lambda^4)\varphi^3 z^2 + s\lambda\varphi|\nabla z|^2 + (\eta^2(M) + |f_0|_\infty)z^2)\,dx\,dt.$$

Then arguing as above (see (4.8)) we obtain

$$\int_Q (s^3\lambda^4 z^2 + s\lambda^2\varphi|\nabla z|^2)\,dx\,dt \le C\left(\int_{Q_{\omega_0}} (\varphi^3 + \eta^2(M))z^2 + \varphi|\nabla z|^2\,dx\,dt\right)$$

for $s \ge s_0 + C\eta^{2/3}(M)$ and $\lambda \ge \lambda_0$. This implies (4.10) above which is just the desired inequality (4.14). Finally, by (4.14) and (4.11) where $s = s_0 + C\eta^{2/3}(M)$ one obtains (4.15) as claimed. □

5. Proof of Theorem 2

Let $q(n)$ be defined by (1.8) and let $y_0 \in W^2_{q(n)}(\Omega) \cap H^1_0(\Omega)$ be arbitrary but fixed. Consider the set

$$K_\infty = \{w \in L^\infty(Q);\ \|w\|_\infty \le M\} \tag{5.1}$$

and recall the notation $Y_n(\Omega) = W^2_{q(n)}(\Omega) \cap H^1_0(\Omega)$ where $q(n)$ is defined by (1.8).

Lemma 4. *For each $w \in K_\infty$ there are $y \in L^2(0,T;H^1_0(\Omega)) \cap W^{2,1}_{q(n)}(Q)$ and $u \in L^{q(n)}(Q)$ which satisfy system* (3.2) *and*

$$y(x,T) = 0, \qquad \text{a.e.} \quad x \in \Omega, \tag{5.2}$$
$$\|mu\|^2_{L^{q(n)}(Q)} \le \gamma e^{\mu\eta^{2/3}(M)}\|y_0\|_2^2, \tag{5.3}$$

where $\mu = \mu(L, \|f_0\|_\infty) > 0$ and γ are independent of y_0, w, M.

Proof. As noticed in the proof of Lemma 1, the solution y to (3.2) satisfies estimate (3.7).

Consider the optimal control problem

$$\text{Minimize} \int_Q e^{-2s\alpha}\varphi^{-3}u^2\,dx\,dt + \frac{1}{\varepsilon}\int_\Omega y^2(x,T)\,dx \text{ subject to (3.2).} \tag{5.4}$$

Here α, φ are defined by $(4.1)'$ and s, λ are chosen as in Lemma 3.

Let $(y_\varepsilon, u_\varepsilon)$ be an optimal pair. By the maximum principle we have

$$u_\varepsilon(x,t) = m(x)p_\varepsilon(x,t)e^{2s\alpha}\varphi^3, \qquad \text{a.e.} \quad (x,t) \in Q, \tag{5.5}$$

where p_ε is the solution to system (3.10). This yields

$$\int_{Q_\omega} e^{2s\alpha}\varphi^3 p_\varepsilon^2\,dx\,dt + \frac{1}{\varepsilon}\int_\Omega y_\varepsilon^2(x,T)\,dx = \int_\Omega y_0(x)p_\varepsilon(x,0)\,dx. \tag{5.6}$$

Then by Lemma 3 (inequality (4.15)) it follows that

$$\int_{Q_\omega} e^{2s\alpha}\varphi^3 p_\varepsilon^2\,dx\,dt + \frac{1}{\varepsilon}\int_\Omega y_\varepsilon^2(x,T)\,dx \le Ce^{\beta\eta^{2/3}(M)}\|y_0\|_2^2, \qquad \forall\varepsilon > 0. \tag{5.7}$$

Moreover, by (4.14) and (5.7) we see that (for simplicity we write $p_\varepsilon = p$)

$$\int_Q e^{2s\alpha}(\varphi^3 p^2 + \varphi|\nabla p|^2)\,dx\,dt \le Ce^{2\beta\eta^{2/3}(M)}\|y_0\|_2^2.$$

This yields

$$\int_Q e^{2s\alpha}\varphi^3(p_t + \Delta p)^2\,dx\,dt \le CA(M, y_0),$$

where $A(M, y_0) = e^{2\beta\eta^{2/3}(M)}\|y_0\|_2^2$ and C is independent of y_0 and M. Therefore

$$\int_Q e^{2s\alpha}\varphi^{-3}(p_t^2 + |\Delta p|^2)\,dx\,dt \le CA(M, y_0). \tag{5.8}$$

We set $v = e^{2s\alpha}\varphi^3 p$. Then by (5.7) and (5.8) we see that

$$\|v\|_{H^{2,1}(Q)}^2 \le CA(M, y_0).$$

Since $H^{2,1}(Q) \subset L^{q(n)}(Q)$ we infer that

$$\|mu_\varepsilon\|_{L^{q(n)}(Q)}^2 \le CA(M, y_0). \tag{5.9}$$

This estimate and the existence theory of parabolic boundary value problems in $L^{q(n)}(Q)$ (see [12]) imply that on a subsequence we have

$$\begin{aligned} u_\varepsilon &\to u \qquad \text{weakly in } L^{q(n)}(Q),\\ y_\varepsilon &\to y \qquad \text{weakly in } L^2(0,T; H_0^1(\Omega)) \cap W_{q(n)}^{2,1}(Q), \end{aligned}$$

where (y, u) satisfy (3.2) and $y(T) \equiv 0$. Moreover, we see that u satisfies the estimate (5.3) for $\mu = 2\beta$ and γ suitably chosen. This completes the proof of Lemma 4. □

Proof of Theorem 2 (continued). We proceed as in the proof of Theorem 1. Namely, for each $w \in K_\infty$ denote by $\Phi(w) \subset L^2(Q)$ the set of all solutions $y^u \in L^2(0,T; H_0^1(\Omega)) \cap W_{q(n)}^{2,1}(Q)$ to (3.2) such that

$$y^u(T) \equiv 0; \qquad \|mu\|_{L^{q(n)}(Q)}^2 \le \gamma e^{\mu\eta^{2/3}(M)}\|y_0\|_2^2, \tag{5.10}$$

where μ and γ are as in Lemma 4.

By Lemma 4 it follows that $\Phi(w) \ne \emptyset$ for each $w \in K_\infty$. Notice also that $\Phi(w)$ is a closed and convex subset of $L^2(Q)$. Moreover, we have

$$\|y^u\|_\infty^2 \le C\|y^u\|_{W_{q(n)}^{2,1}(Q)}^2 \le C(\eta(M) + 1)^2\|y_0\|_{Y_n(\Omega)}^2 e^{\mu\eta^{2/3}(M)} \tag{5.11}$$

where C is independent of M and w. Here is the argument. As in the proof of Lemma 1 it follows by (1.2) (condition (1.4) is unnecessary in this case) that

$$\|y^u\|_{H^{2,1}(Q)} \leq C(\eta(M)+1)(\|y_0\|_{H_0^1(\Omega)} + \|mu\|_2)$$

and therefore

$$\|g(w)y^u\|_{q(n)} \leq C(\eta(M)+1)(\|y_0\|_{H_0^1(\Omega)} + \|mu\|_2)$$

which in virtue of (5.10) and [12] implies that

$$\|y^u\|^2_{W^{2,1}_{q(n)}(Q)} \leq C(\eta(M)+1)^2 e^{\mu\eta^{2/3}(M)} \|y_0\|^2_{Y_n(\Omega)}.$$

Since $W^{2,1}_{q(n)}(Q) \subset C(\overline{Q})$ for $1 \leq n < 6$ the latter implies (5.11) as claimed.

We take

$$M = \arg\sup(re^{-\mu\eta^{2/3}(r)}; r > 0).$$

Thus it follows by (5.11) that if

$$\|y_0\|_{Y_n(\Omega)} \leq C \sup(re^{-\mu\eta^{2/3}(r)}; r > 0),$$

then $\Phi(K_\infty) \subset K_\infty$.

Moreover, by estimate (5.11) it follows that $\Phi(K_\infty)$ is a relatively compact subset of $L^2(Q)$ (as a matter of fact in $C(\overline{Q})$) and, as seen in the proof of Theorem 1, Φ is upper semicontinuous in $L^2(Q) \times L^2(Q)$.

Then applying the Kakutani fixed-point theorem in the space $L^2(Q)$ we infer that there is at least one $w \in K_\infty$ such that $w \in \Phi(w)$ and therefore there is at least one pair (y,u) satisfying the conditions of Theorem 2. This completes the proof. □

Remark 4. Recently Fernandez-Cara and Zuazua have shown that the null controllability result in Corollary 1 remains true without the dissipativity condition (1.2).

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