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An Observation About Weak Solutions of Linear Differential Equations in Hilbert Spaces

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Abstract

This note addresses the well-posedness of weak solutions for a general linear evolution problem on a separable Hilbert space. For this classical problem there is a well known challenge of obtaining a priori estimates, as a constructed weak solution may not be regular enough to be utilized as a test function. This issue presents an obstacle for obtaining uniqueness and continuous dependence of solutions. Utilizing a generic weak formulation (involving the adjoint of the system's evolution operator), the classical reference (Ball in Proceedings of the American Mathematical Society 63:370-373, 1977) provides a characterization which makes equivalent well-posedness of weak solutions and generation of a C_0-semigroup. On the other hand, the approach in (Ball in Proceedings of the American Mathematical Society 63:370-373, 1977) does not take into account any underlying energy estimate, and requires a characterization of the adjoint operator, the latter often posing a non-trivial task. We propose an alternative approach, when the problem is posed on a Hilbert space and admits an underlying "formal" energy estimate. For such a Cauchy problem, we provide a general notion of weak solution and through a straightforward observation, obtain that arbitrary weak solutions have additional time regularity and obey an a priori estimate. This yields weak well-posedness. Our result rests upon a central hypothesis asserting the existence of a "good" Galerkin basis for the construction of a weak solution. A posteriori, a C_0-semigroup may be obtained for weak solutions, and by uniqueness, weak and semigroup solutions are equivalent.

Keywords Linear differential equations · Weak solutions · Galerkin approximation

Mathematics Subject Classification 34G10 · 35L05 · 35K05 · 47D06 · 65L60

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1 Existence of Weak Solutions

Let H be a separable real Hilbert space with inner product $(\cdot, \cdot)_H$ and norm $\|\cdot\|_H$, respectively, and let $A : \mathcal{D}(A) \subset H \to H$ be a densely defined closed linear operator. The space H is thought to be embedded in a bigger Hilbert space, allowing us to continuously extend A outside of its domain, as is typically the case for differential operators. Assume that we are given the abstract evolution in H on the time-interval $[0, T]$

$$\begin{cases} \dot{x} = Ax + f, \\ x(0) = x_0 \in H, \end{cases} \tag{1.1}$$

in the unknown $x = x(t)$, where $f = f(t)$ is a forcing term. The aim of this note is to provide a convincing definition of weak solution which fully captures the structure of the Cauchy problem (1.1), and then present a corresponding existence and uniqueness result. We make a baseline assumption.

Assumption 1.1 Identifying H with its dual space H^*, there exist two Hilbert spaces V and W with dense and continuous embeddings

$$V \hookrightarrow W \hookrightarrow H = H^* \hookrightarrow W^* \hookrightarrow V^*$$

such that the following hold:

(i) A extends to a bounded operator from W onto V^*.
(ii) There exist $a \geq b > 0$ such that, for all $x \in \mathcal{D}(A)$,

$$-(Ax, x)_H \geq b\|x\|_W^2 - a\|x\|_H^2. \tag{1.2}$$

In the sequel, $\langle \cdot, \cdot \rangle$ will stand for the duality pairing between V^* and V, extending the inner product in H, whenever both defined. We introduce the bilinear form $a(\cdot, \cdot) : W \times V \to \mathbb{R}$

$$a(x, \tilde{x}) = -\langle Ax, \tilde{x} \rangle,$$

whose continuity follows from (i). Concerning the external force, we assume that

$$f \in L^2(0, T; W^*).$$

Definition 1.2 (W-Weak Solution) A function $x \in L^\infty(0, T; H) \cap L^2(0, T; W)$ is a W-weak solution of (1.1) if and only if $x(0) = x_0$ and, for a.e. $t \in [0, T]$ and every test $y \in V$,

$$\langle \dot{x}(t), y \rangle + a(x(t), y) = \langle f(t), y \rangle.$$

From our assumptions, such an equality dictates that $\dot{x} \in L^2(0, T; V^*)$, and the embedding $H^1(0, T; V^*) \hookrightarrow C([0, T]; V^*)$ ensures that $x(0)$ has meaning.

In general, the space W complying with Assumption 1.1 is not unique. This will be clear in Example 2.1, where there are infinitely many choices of W. For any viable W, we produce a legitimate notion of a (weak) solution. Among them, we can locate the best possible one, corresponding to the minimum with respect to the relation of inclusion of such spaces W. This minimum provides the notion of solution that fully captures the features of the equation, as it exploits the maximal information which can be extracted weakly from the operator A. In principle, such a minimum might not exist (if it is only an infimum), but it seems very unlikely to find a concrete example where this sorry situation occurs. We agree to simply call *weak solution* the W-weak solution corresponding to the minimum W or, equivalently, the strongest among W-weak solutions.

Theorem 1.3 *Under Assumption 1.1, problem (1.1) admits at least one W-weak solution.*

Proof The proof is based on a standard Galerkin approximation scheme. We select a basis $\{h_n\}$ of the space H made by elements in V, and we denote by Π_n the self-adjoint projection from H onto F_n, where F_n is the n-dimensional subspace of H generated by the vectors $h_1, \dots, h_n$. We agree to call the pair of spaces and projections (F_n, Π_n) a *Galerkin family*. Usually, the basis $\{h_n\}$ is chosen to be orthonormal, although this is not strictly necessary. Appealing to the classical theory of ODEs, for every $n \in \mathbb{N}$ there is a unique function $x_n \in AC([0, T]; F_n)$ with prescribed initial condition $x_n(0) = \Pi_n x_0$ that solves the equation

$$\langle \dot{x}_n(t), y\rangle + a(x_n(t), y) = \langle f(t), y\rangle, \tag{1.3}$$

for a.e. $t \in [0, T]$ and every test $y \in F_n$. Observe that the duality pairings appearing above are actually just H inner products at the level of F_n. Such an x_n is sometimes called a *Galerkin approximate solution*. Choosing as test function $y = x_n$, from point (ii) of Assumption 1.1, together with Young's inequality, we obtain the differential inequality

$$\frac{d}{dt}\|x_n\|_H^2 = -2a(x_n, x_n) + 2\langle f, x_n\rangle \le -b\|x_n\|_W^2 + 2a\|x_n\|_H^2 + \frac{1}{b}\|f\|_{W^*}^2.$$

As $\|x_n(0)\|_H \le \|x_0\|_H$, an application of the standard Gronwall lemma yields an *energy estimate*

$$\|x_n\|_{L^\infty(0,T;H)} + \|x_n\|_{L^2(0,T;W)} \le C\big[\|x_0\|_H + \|f\|_{L^2(0,T;W^*)}\big], \tag{1.4}$$

for some $C > 0$ independent of n. Then, from (1.3) and the embedding $W^* \hookrightarrow V^*$, we also deduce the uniform bound

$$\|x_n\|_{H^1(0,T;V^*)} \le C\big[\|x_0\|_H + \|f\|_{L^2(0,T;W^*)}\big],$$

up to redefining C. Accordingly, there exists $x \in L^\infty(0, T; H) \cap H^1(0, T; V^*) \cap L^2(0, T; W)$ such that, up to a subsequence, we have the weak* and weak convergences

$$x_n \overset{*}{\rightharpoonup} x \in L^\infty(0,T;H), \qquad x_n \rightharpoonup x \in H^1(0,T;V^*), \qquad x_n \rightharpoonup x \in L^2(0,T;W).$$

At this point, we fix $m \in \mathbb{N}$ and we take $y \in F_m$. Passing to the limit in (1.3), we obtain

$$\langle \dot{x}(t), y\rangle + a(x(t), y) = \langle f(t), y\rangle.$$

Since this is true for every m, we conclude that the latter equality holds for all test $y \in V$. Finally, in light of the embedding $H^1(0,T;V^*) \hookrightarrow C([0,T];V^*)$, we establish the equality $x(0) = x_0$. Thus, x is a W-weak solution. □

Any solution obtained through this limiting procedure is called a *Galerkin solution*. On account of the weak lower semicontinuity of the norm, we infer from (1.4) an immediate corollary.

Corollary 1.4 *There exists $C > 0$ such that any Galerkin solution x satisfies the energy estimate*

$$\|x\|_{L^\infty(0,T;H)} + \|x\|_{L^2(0,T;W)} \leq C\big[\|x(0)\|_H + \|f\|_{L^2(0,T;W^*)}\big]. \tag{1.5}$$

Moreover, $x \in C([0,T];V^*)$.

We note that, at this point, the constructed Galerkin solution satisfies the above a priori estimate; however, an arbitrary weak solution as per Definition 1.2 need not.

The uniqueness problem is more subtle, since one would hope to test a solution x of the homogeneous problem (with a null initial datum) with x itself. This is possible only if x resides in the correct test space, V, which may not be the case. However, we will see that by introducing an additional straightforward hypothesis on $a(\cdot,\cdot)$, we automatically gain uniqueness and boosted regularity of the weak solution. Before stating this hypothesis, and the consequent result, we consider some motivational examples to establish the validity and applicability of our scheme.

2 Three Examples

Let X^0 be a separable real Hilbert space with inner product $\langle\cdot,\cdot\rangle_0$ and norm $\|\cdot\|_0$, and let B be a strictly positive self-adjoint linear operator on X^0 with domain $\mathcal{D}(B)$, compactly embedded into X^0 (hence the spectrum of B comprises only eigenvalues). For $r \in \mathbb{R}$, we introduce the compactly nested scale of Hilbert spaces

$$X^r = \mathcal{D}(B^{\frac{r}{2}}), \qquad (u,v)_r = (B^{\frac{r}{2}}u, B^{\frac{r}{2}}v)_0, \qquad \|u\|_r = \|B^{\frac{r}{2}}u\|_0.$$

If $r > 0$, it is understood that X^r is the completion of the domain, so that X^{-r} is the dual space of X^r. The duality pairing between X^{-r} and X^r will be still denoted by $\langle\cdot,\cdot\rangle$. The paradigmatic example we have in mind for such a B is the Dirichlet

Laplacian $-\Delta$ acting on the Hilbert space $L^2(\Omega)$ with $\mathcal{D}(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary.

2.1 The Abstract Heat Equation

Given $f \in L^2(0, T; X^{-1})$, we consider the Cauchy problem in $H = X^0$

$$\begin{cases} \dot{u} = -Bu + f, \\ u(0) = u_0 \in H, \end{cases}$$

which is in the form (1.1) with $A = -B$. For every $u \in X^2$ we have that

$$(Bu, u)_0 = \|u\|_1^2. \tag{2.1}$$

Accordingly, we take $W = V = X^1$ and $V^* = X^{-1}$, and we define the bilinear form on $X^1 \times X^1$

$$a(u, \tilde{u}) = (u, \tilde{u})_1.$$

A weak solution is then a function $u \in L^\infty(0, T; X^0) \cap L^2(0, T; X^1)$ such that $u(0) = u_0$ and

$$\langle \dot{u}(t), \phi \rangle + (u(t), \phi)_1 = \langle f(t), \phi \rangle,$$

for a.e. $t \in [0, T]$ and every test $\phi \in X^1$. From Theorem 1.3, such a solution exists. Moreover, considering a solution u of (1.1) with $u_0 = 0$ and $f = 0$, we are in the ideal case where we can utilize u itself as a test function, since $u(t) \in X^1$ for a.e. $t \in [0, T]$. This provides the energy estimate (1.4) with $C = 0$, establishing the uniqueness of the solution as well.

Remark 2.1 It is evident from (2.1) that Assumption 1.1 would be in place with any $W = X^r$ with $r \in [0, 1]$. Hence, although we obtained the weak solution (i.e., the strongest in this framework) by choosing $W = X^1$, we could have given the notion of X^r-solution for any $r \in [0, 1)$ as well. The weakest among those is the one corresponding to $W = X^0$. Namely, for $f \in L^2(0, T; X^0)$, a X^0-weak solution is a function $u \in L^\infty(0, T; X^0)$ such that $u(0) = u_0$ and

$$\langle \dot{u}(t), \phi \rangle - (u(t), B\phi)_0 = (f(t), \phi)_0,$$

for a.e. $t \in [0, T]$ and every test $\phi \in X^2$. (See also the second bullet in Sect. 4.)

2.2 The Abstract Wave Equation

Given $g \in L^2(0, T; X^0)$, we consider the Cauchy problem

$$\begin{cases} \ddot{u} = -Bu + g, \\ u(0) = u_0, \quad \dot{u}(0) = u_1, \end{cases}$$

with $u_0 \in X^1$ and $u_1 \in X^0$. Setting $x = [u, v]^\top$ and $f = [0, g]^\top$, the above can given the form (1.1) in $H = X^1 \times X^0$, by setting

$$A = \begin{bmatrix} 0 & I \\ -B & 0 \end{bmatrix} \qquad \text{with domain} \qquad \mathcal{D}(A) = X^2 \times X^1.$$

For $x \in X^2 \times X^1$, we have

$$-(Ax, x)_{X^1 \times X^0} = 0.$$

In light of (1.2), this forces the equality $W = H = X^1 \times X^0$. Since $AW = X^0 \times X^{-1}$, and since the pivot space of the chain of embeddings is H, we obtain $V = X^2 \times X^1$ and $V^* = X^0 \times X^{-1}$, while, for $x = [u, v]^\top$ and $\tilde{x} = [\tilde{u}, \tilde{v}]^\top$, the bilinear form is given by

$$a(x, \tilde{x}) = -(v, B\tilde{u})_0 + (u, \tilde{v})_1.$$

Then $x = [u, v]^\top$ with $u \in L^\infty(0, T; X^1)$ and $v \in L^\infty(0, T; X^0)$ is a weak solution if $x(0) = [u_0, u_1]^\top$ and for every test $y = [\phi, \psi]^\top \in X^2 \times X^1$ the equality

$$(\dot{u}(t), B\phi)_0 + \langle \dot{v}(t), \psi \rangle - (v, B\phi)_0 + (u(t), \psi)_1 = (g(t), \psi)_0$$

holds for a.e. $t \in [0, T]$. Since the hypotheses on B ensure that the vectors of the form $B\phi$ with $\phi \in X^2$ span X^0, we can equivalently say that x is a weak solution if for every test $y = [\xi, \psi]^\top \in X^0 \times X^1$

$$(\dot{u}(t), \xi)_0 + \langle \dot{v}(t), \psi \rangle - (v, \xi)_0 + (u(t), \psi)_1 = (g(t), \psi)_0.$$

From this definition, we can easily recover the usual weak form of the wave equation. Indeed, choosing $\psi = 0$ above, we readily obtain the equality $v = \dot{u}$ in X^0. From there, we choose $\xi = 0$ to obtain what might be considered the weak formulation of the wave equation, that is,

$$\langle \ddot{u}(t), \psi \rangle + (u(t), \psi)_1 = (g(t), \psi)_0.$$

Although the existence follows from Theorem 1.3, the subtle point comes in obtaining the energy estimate (1.4) for any weak solution. In this case, one should choose $\psi = \dot{u}$ as test function in the latter equality, which is *forbidden*, since ψ is required to live in X^1, whereas $\dot{u}$ is in X^0 only. Thus, in order to obtain uniqueness of solutions, as well as a continuous dependence estimate, some additional (and nontrivial) work is needed. In the reference [4, pp.406–408] a temporal anti-differentiation is used to obtain uniqueness and by identifying the unique solution with the Galerkin solution, continuous dependence is obtained. In [8, II.4.1, pp.76–79] a well-cited hyperbolic regularization lemma is presented that yields an energy estimate for arbitrary weak solutions; uniqueness then follows.

Instead, by the method presented below, we will see that when one has existence of a weak solution via a “good" Galerkin construction, as it happens here, then the weak solution is automatically unique, satisfies the energy estimate, continuously dependent on the initial data, and has additional regularity.

2.3 The Abstract Damped Wave Equation

Let $\alpha \in [0, 1]$ be fixed. Given a forcing term $g \in L^2(0, T; X^{-\alpha})$, we consider the Cauchy problem

$$\begin{cases} \ddot{u} = -Bu - B^\alpha \dot{u} + g, \\ u(0) = u_0, \quad \dot{u}(0) = u_1, \end{cases}$$

with $u_0 \in X^1$ and $u_1 \in X^0$. Of particular interest are the two limit cases $\alpha = 0$, corresponding to the weakly damped wave equation or telegrapher’s equation, and $\alpha = 1$ corresponding to the strongly damped wave equation, also known as Kelvin-Voigt equation. The problem can be given the form (1.1) with $H = X^1 \times X^0$, by setting

$$A = \begin{bmatrix} 0 & I \\ -B & -B^\alpha \end{bmatrix} \qquad \text{with domain} \qquad \mathcal{D}(A) = \left\{ [u, v]^\top \,\middle|\, \begin{matrix} v \in X^1 \\ u + B^{\alpha-1} v \in X^2 \end{matrix} \right\},$$

which becomes $\mathcal{D}(A) = X^2 \times X^1$ whenever $\alpha \le 1/2$. For $x = [u, v]^\top \in \mathcal{D}(A)$, we now have

$$-(Ax, x)_{X^1 \times X^0} = \|v\|_\alpha^2.$$

Accordingly, $W = X^1 \times X^\alpha$ is the best possible space complying with (1.2). When $\alpha = 0$ the only possibility is $W = H$. Since $AW = X^\alpha \times X^{-1}$, and since the pivot space is H, we obtain

$$V = X^{2-\alpha} \times X^1, \qquad W = X^1 \times X^\alpha, \qquad W^* = X^1 \times X^{-\alpha}, \qquad V^* = X^\alpha \times X^{-1}.$$

For $x = [u, v]^\top$ and $\tilde{x} = [\tilde{u}, \tilde{v}]^\top$, the bilinear form is given by

$$a(x, \tilde{x}) = -(v, \tilde{u})_1 + (u, \tilde{v})_1 + (v, \tilde{v})_\alpha.$$

A function $x = [u, v]^\top$ with $u \in L^\infty(0, T; X^1)$ and $v \in L^\infty(0, T; X^0) \cap L^2(0, T; X^\alpha)$ is a weak solution if $x(0) = [u_0, u_1]^\top$ and for every test $y = [\phi, \psi]^\top \in X^{2-\alpha} \times X^1$ the equality

$$\begin{aligned}&(\dot{u}(t), B^{1-\alpha}\phi)_\alpha + \langle \dot{v}(t), \psi\rangle - (v(t), B^{1-\alpha}\phi)_1 + (u(t), \psi)_1 + (v(t), \psi)_\alpha \\ &\quad = \langle g(t), \psi\rangle\end{aligned}$$

holds for a.e. $t \in [0, T]$. Since $B^{1-\alpha}$ maps $X^{2-\alpha}$ onto X^α, this is the same as saying that x is a weak solution if for every test $y = [\xi, \psi]^\top \in X^\alpha \times X^1$

$$\begin{aligned}&(\dot{u}(t), \xi)_\alpha + \langle \dot{v}(t), \psi\rangle - (v(t), \xi)_1 + (u(t), \psi)_1 + (v(t), \psi)_\alpha \\ &\quad = \langle g(t), \psi\rangle.\end{aligned}$$

Similarly to the previous example, we obtain the equality $v = \dot{u}$ in X^α, along with the usual weak formulation of the damped wave equation, that is,

$$\langle \ddot{u}(t), \psi\rangle + (u(t), \psi)_1 + (\dot{u}(t), \psi)_\alpha = \langle g(t), \psi\rangle,$$

Again, existence follows from Theorem 1.3. When $\alpha = 1$ uniqueness is easily obtained as in Sect. 2.1, as the solution lives in the test space.

3 Uniqueness and Regularity

Whenever Assumption 1.1 is in force, Theorem 1.3 ensures the existence of a W-weak solution to the Cauchy problem in (1.1) via a standard Galerkin construction. The next assumption will automatically guarantee uniqueness of the solution, among several other improved properties.

Assumption 3.1 In the terminology of the proof of Theorem 1.3, let there exists a Galerkin family (F_n, Π_n) with the following property: for all vectors $x \in W$ and all $\tilde{x} \in V$,

$$a(x, \Pi_n\tilde{x}) = a(\Pi_n x, \tilde{x}), \quad \forall n \in \mathbb{N}.$$

Theorem 3.2 *Under Assumptions 1.1 and 3.1, the Cauchy problem* (1.1) *admits a unique W-weak solution x, for any admissible W space. Moreover, $x \in C([0, T]; H)$ and the map $x_0 \mapsto x(t)$ belongs to $C(H; H)$ for any fixed $t \in [0, T]$.*

Proof Due to linearity, uniqueness follows once we prove that any solution to the homogeneous problem with null initial datum is trivial. Let then x be any such solution (namely, of (1.1) with $x_0 = 0$ and $f = 0$), and set $x_n(t) = \Pi_n x(t)$. In light of Assumption 3.1, taking a test $y \in F_n$, we have the equality

$$\langle \dot{x}_n(t), y\rangle + a(x_n(t), y) = 0.$$

This tells that x_n is a Galerkin approximate solution, which is known to converge (up to a subsequence) to some Galerkin solution $\hat{x}$. But since we already have that $x_n(t) \to x(t)$ in H, this forces the equality $\hat{x} = x$. Hence, x is a Galerkin solution of the homogeneous problem with initial datum $x_0 = 0$, and applying Corollary 1.4 allows us to conclude that x is the null solution. By the same token, once uniqueness is established (and so all solutions are Galerkin solutions) we establish the continuous dependence estimate. Indeed, let $x, \hat{x}$ be two solutions of (1.1) with initial data $x(0) = x_0$ and $\hat{x}(0) = \hat{x}_0$, respectively. Then, their difference $x - \hat{x}$ is the solution of the homogeneous problem with initial datum $x_0 - \hat{x}_0$, and Corollary 1.4 provides the estimate

$$\|x - \hat{x}\|_{L^\infty(0,T;H)} + \|x - \hat{x}\|_{L^2(0,T;W)} \le C\|x_0 - \hat{x}_0\|_H.$$

We are left to prove the continuity in time. To this end, let us assume first that $f \in L^2(0, T; H)$. For an arbitrarily given $x_0 \in H$, we construct the Galerkin approximate solutions x_n, based on the Galerkin family (F_n, Π_n). Now let $n \ge m$. Exploiting Assumption 3.1, for any test $y \in F_n$, we have the equality

$$\begin{aligned}\langle \dot{x}_m(t), y\rangle + a(x_m(t), y) &= \langle \dot{x}_m(t), \Pi_m y\rangle + a(x_m(t), \Pi_m y)\\ &= \langle f(t), \Pi_m y\rangle = \langle \Pi_m f(t), y\rangle.\end{aligned}$$

Hence, for any test $y \in F_n$, the difference $x_n - x_m$ fulfills

$$\langle \dot{x}_n(t) - \dot{x}_m(t), y\rangle + a(x_n(t) - x_m(t), y) = \langle f(t) - \Pi_m f(t), y\rangle.$$

Choosing $y = x_n - x_m$, and arguing as in the proof of Theorem 1.3, we obtain the estimate

$$\sup_{t\in[0,T]} \|x_n(t) - x_m(t)\|_H \le C\Big[\|\Pi_n x_0 - \Pi_m x_0\|_H + \|f - \Pi_m f\|_{L^2(0,T;H)}\Big].$$

Observe that the last term of the right-hand side above goes to zero as $m \to \infty$, by the Lebesgue dominated convergence theorem. Therefore, x_n is a Cauchy sequence in $C([0, T]; H)$, and so converges to an element x in that space. Such an x is exactly the previously established (unique) solution, taken with initial datum x_0. Now, to deal with the general case, let us consider a sequence $f_n \in L^2(0, T; H)$ converging to $f \in L^2(0, T; W^*)$ in the latter norm. This time, we call x_n the solution with initial datum x_0 corresponding to the forcing term f_n. A further application of Corollary 1.4 yields the estimate

$$\sup_{t\in[0,T]} \|x_n(t) - x_m(t)\|_H \le C\|f_n - f_m\|_{L^2(0,T;W^*)}.$$

Again, we conclude that x_n converges to $x \in C([0, T]; H)$. And it is standard matter to verify that x is the solution with initial datum x_0 corresponding to the forcing term f. □

Remark 3.3 It is clear from the above proof that, if one wants to prove only uniqueness and continuous dependence, then Assumption 3.1 is not needed in its full strength; it suffices to take only the weaker assumption

$$a(x, \Pi_n \tilde{x}) = a(\Pi_n x, \Pi_n \tilde{x}), \quad \forall n \in \mathbb{N}.$$

Returning to the examples of Sect. 2, let $\lambda_1 \leq \cdots \leq \lambda_n \to \infty$ be the sequence of eigenvalues of B, counted with multiplicity, and let $\{b_n\}$ be the corresponding sequence of eigenvectors, which form (if normalized) a complete orthonormal family in X^0 belonging to X^r for all $r > 0$. For the Cauchy problem (2.1), the basis $\{h_n\}$ of $H = X^0$ needed to construct F_n is simply $\{b_n\}$. For the Cauchy problems (2.2) and (2.3), the orthonormal basis of $H = X^1 \times H^0$ is made by the vectors $h_{2n+1} = b_n/\sqrt{\lambda_n} \oplus 0$ and $h_{2n} = 0 \oplus b_n$. In all cases presented here, our Assumption 3.1 is then immediately verified.

4 Final Remarks

- Under Assumptions 1.1 and 3.1, we have a unique W-weak solution of the Cauchy problem (1.1). Since all the W-weak solutions arise from the same Galerkin scheme, they must coincide. Hence, the point of locating the best possible W is actually the one of giving a notion of solution apt to describe all the regularity properties satisfied by the solution itself.
- Although we stated our results under the hypothesis $f \in L^2(0, T; W^*)$, everything works if we take a more general forcing term $f \in L^1(0, T; H) + L^2(0, T; W^*)$. The only additional ingredient needed in the proofs is a modified form of the Gronwall lemma (see, e.g., [6]), in order to obtain the energy estimate (1.4).
- In a standard way, the existence of a unique W-weak solution satisfying (1.5) implies that the operator A is the infinitesimal generator of a C_0-semigroup $S(t)$ acting on H. In which case, it is well known that if $f \in L^1(0, T; H)$ then the Cauchy problem (1.1) admits a unique *mild solution*, given by the variation of parameters (Duhamel) formula (see, e.g., [3]),

$$x(t) = S(t)x + \int_0^t S(t-s)f(s)ds.$$

  Such a mild solution is nothing but the H-weak solution, hence, the weakest possible arising in the framework presented here.
- A different definition of weak solution for (1.1) has been proposed in [2]. Namely, a function $x \in C([0, T]; H)$ is a weak solution of (1.1) if and only if $x(0) = x_0$ and for every $y \in \mathcal{D}(A^*)$ the map $t \mapsto (x(t), y)_H$ is absolutely continuous on $[0, T]$ and

$$\frac{d}{dt}(x(t), y)_H = (x(t), A^* y)_H + (f(t), y)_H, \tag{4.1}$$

  for a.e. $t \in [0, T]$, where A^* is the adjoint of the operator A. Actually, in [2] it is shown that A is the generator of a C_0-semigroup if and only if for every initial datum $x_0 \in H$ there exists a unique weak solution in the sense of (4.1). The relation between a weak solution of type (4.1) and semigroup generation was also considered in [1]. This definition, although very elegant, is of less practical use since, in general, finding the adjoint of A is a formidable task. Among many others, an example in this direction is the infinitesimal generator A of the semigroup describing the evolution of the relative displacement in a linearly viscoelastic solid (see, e.g., [5, §3]), for which A^* seems to be particularly difficult to compute. In this case, generation of the semigroup is proved (not without some effort) via the classical Lumer-Phillips theorem. On the other hand, for this problem a Galerkin family is available (see [7]).

- As already mentioned, one way to establish the existence and uniqueness of weak (mild) solutions is to prove that the operator A is the infinitesimal generator of a C_0-semigroup $S(t)$. This requires us to apply the Feller–Miyadera–Phillips theorem. Unfortunately, as clearly stated in [3, Comment 3.9, pp.77–78], checking the hypotheses of this theorem is out of question for general operators. It can be reasonably accomplished only when the resulting $S(t)$ is an ω-contraction (in which case the Feller–Miyadera–Phillips theorem boils down to the Hille–Yosida theorem). This means that, in the operator norm, $S(t)$ must fulfill the estimate $\|S(t)\| \leq e^{\omega t}$, for some $\omega \in \mathbb{R}$. If so, such a semigroup generator A necessarily satisfies the estimate

$$(Ax, x)_H \leq \omega \|x\|_H^2, \quad \forall x \in \mathcal{D}(A).$$

This is (in fact, a particular case of) point (ii) of our Assumption 1.1.

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