



Periodic Homogenization of the Principal Eigenvalue of Second-Order Elliptic Operators

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Abstract

In this paper we investigate homogenization results for the principal eigenvalue problem associated to 1-homogeneous, uniformly elliptic, second-order operators. Under rather general assumptions, we prove that the principal eigenpair associated to an oscillatory operator converges to the eigenpair associated to the effective one. This includes the case of fully nonlinear operators. Rates of convergence for the eigenvalues are provided for linear and nonlinear problems, under extra regularity/convexity assumptions. Finally, a linear rate of convergence (in terms of the oscillation parameter) of suitably normalized eigenfunctions is obtained for linear problems.

Keywords Second-order elliptic equations · Eigenvalue problem · Homogenization · Rate of convergence

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, and consider a non-divergence form, linear elliptic operator with the form

$$L = a_{ij}(x)\partial_{x_i x_j}^2 + b_j(x)\partial_{x_j} + c(x), \quad x \in \Omega, \quad (1.1)$$

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such that $a_{ij} \in C(\Omega)$, $b^j, c \in L^\infty(\Omega)$, satisfying the uniform ellipticity condition

$$\lambda |\xi|^2 \leq a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2, \quad x \in \Omega, \quad \xi \in \mathbb{R}^N, \tag{1.2}$$

for some given constants $0 < \lambda \leq \Lambda$. Here we have adopted the usual notation of sum over repeated indices.

In the pioneering work of Berestycki, Nirenberg and Varadhan [5], the authors provide a complete study of the principal eigenvalue problem associated to L and Ω , namely, the existence of a pair $(\phi_1, \lambda_1) \in C_+(\Omega) \times \mathbb{R}$ solving (in an adequate weak sense) the eigenvalue problem

$$\begin{cases} L\phi_1 = -\lambda_1\phi_1 & \text{in } \Omega \\ \phi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Here we have denoted by $C_+(\Omega)$ the space of continuous functions which are positive in Ω . For equations in divergence form, the principal eigenvalue coincides with the first eigenvalue in the sense of the classical Rayleigh-Ritz formula. In particular, it is *unique* (in the sense that it is the unique eigenvalue associated to an eigenfunction that is positive in Ω), and *simple* (in the sense that the associated eigenspace is one-dimensional).

One of the remarkable features of the methods presented in [5] is the possibility to extend the study of the principal eigenvalue problem for fully nonlinear, uniformly elliptic operators with the form

$$F = \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} L_{\alpha, \beta}, \tag{1.3}$$

where, given sets of indices \mathcal{A}, \mathcal{B} , $\{L_{\alpha, \beta}\}_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}}$ is a family of linear operators with the form (1.1). This has been studied, for instance, by Birindelli and Demengel [4], Armstrong [2] and Quaas-Sirakov [21]: under suitable assumptions on F , the authors prove the existence of a pair $(\phi_1, \lambda_1) \in C_+(\Omega) \times \mathbb{R}$, solving, in the viscosity sense, the Dirichlet problem

$$\begin{cases} F(x, \phi_1, D\phi_1, D^2\phi_1) = -\lambda_1\phi_1, & \text{in } \Omega \\ \phi_1 = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

In particular, the principal eigenvalue $\lambda_1 = \lambda_1^+(F, \Omega)$ is characterized as

$$\lambda_1^+(F, \Omega) = \sup\{\lambda : \exists \phi > 0 \text{ in } \Omega, F(x, \phi, D\phi, D^2\phi) \leq -\lambda\phi \text{ in } \Omega\}, \tag{1.5}$$

where the inequality is understood in the viscosity sense. This eigenvalue is unique and simple (the latter meaning that the eigenfunctions are unique up to a *positive* multiplicative constant). It is possible to consider the eigenvalue problem associated to *negative* eigenfunctions, but its analysis is somewhat analogous to that of its positive counterpart (at least in what respects our interest here) and therefore we concentrate on the positive eigenpair.

The purpose of this article is the study of stability results for the principal eigenvalue problem in the context of periodic homogenization. We introduce the main assumptions in order to present our results. Denote by \mathbb{T}^N the flat N -dimensional torus, and by S^N the space of $N \times N$ symmetric matrices. We consider $F \in C(\Omega \times \mathbb{T}^N \times \mathbb{R} \times \mathbb{R}^N \times S^N)$ and constants $0 < \lambda \leq \Lambda < +\infty$ and $C_1, C_2 > 0$ such that, for all $R > 0$ we have

$$\begin{aligned} & M_{\lambda, \Lambda}^-(N) - C_1 R(|z| + |w|) - C_2 R(|s| + |q|) \\ & \leq F(x + z, y + w, r + s, p + q, M + N) - F(x, y, r, p, M) \\ & \leq M_{\lambda, \Lambda}^+(N) + C_1 R(|z| + |w|) + C_2 R(|s| + |q|), \end{aligned} \tag{1.6}$$

for all $x \in \Omega, z \in \mathbb{R}^N$ with $x + z \in \Omega, y, w \in \mathbb{T}^N, r, s \in \mathbb{R}, p, q \in \mathbb{R}^N$ and $M, N \in S^N$ with $|r|, |s|, |p|, |q|, |M|, |N| \leq R$. Here, $M_{\lambda, \Lambda}^\pm$ denote the extremal Pucci operators (see, e.g., [8])

$$M_{\lambda, \Lambda}^+(X) = \sup_{\lambda I \leq A \leq \Lambda I} \text{Tr}(AX), \quad M_{\lambda, \Lambda}^-(X) = \inf_{\lambda I \leq A \leq \Lambda I} \text{Tr}(AX).$$

We also assume F is positively 1-homogeneous, that is

$$F(x, y, \alpha r, \alpha p, \alpha X) = \alpha F(x, y, r, p, X), \tag{1.7}$$

for all $\alpha \geq 0, x \in \Omega, y \in \mathbb{T}^N, p \in \mathbb{R}^N, X \in S^N$ and $r \in \mathbb{R}$.

Thus, for each $\epsilon \in (0, 1)$, we have the existence of an eigenpair $(u^\epsilon, \lambda^\epsilon) \in C_+(\Omega) \times \mathbb{R}$ solving

$$\begin{cases} F(x, x/\epsilon, u, Du, D^2u) = -\lambda^\epsilon u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

Our first goal is to understand the homogenization behavior of problem (1.8) as $\epsilon \rightarrow 0$, and explore the possibility of having rates of convergence whenever homogenization holds. In fact, it is well-known (see Evans [12]) that for each x, r, p, X , there exists a unique $c \in \mathbb{R}$ such that

$$F(x, y, r, p, D^2v(y) + X) = c \quad \text{in } \mathbb{T}^N, \tag{1.9}$$

has a viscosity solution. We can thus define $\bar{F} : \Omega \times \mathbb{R} \times \mathbb{R}^N \times S^N \rightarrow \mathbb{R}$ as $\bar{F}(x, r, p, X) = c$, where c is the unique constant for which (1.9) has a solution. We call \bar{F} the *effective Hamiltonian*. As it can be seen in [12], \bar{F} is continuous in all its arguments (in view of (1.6), \bar{F} is locally Lipschitz), uniformly elliptic and positively 1-homogeneous, and therefore, the effective eigenvalue problem

$$\begin{cases} \bar{F}(x, u, Du, D^2u) = -\bar{\lambda}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.10}$$

is solvable by the (positive) eigenpair $(u, \bar{\lambda}) \in C_+(\Omega) \times \mathbb{R}$. As before, $\bar{\lambda}$ is unique and simple.

The first main result of this paper is the following

Theorem 1.1 *Assuming F satisfies (1.6), (1.7). Let $(u^\epsilon, \lambda^\epsilon)$ be the solution of (1.8). Then, we have $\lambda^\epsilon \rightarrow \bar{\lambda}$ as $\epsilon \rightarrow 0$.*

Let u^ϵ with $\|u^\epsilon\|_\infty = 1$ and u solving (1.10) with $\|u\|_\infty = 1$, then $u^\epsilon \rightarrow u$ as $\epsilon \rightarrow 0$, uniformly on $\bar{\Omega}$.

For linear operators, results of this type dates back to Osborn [20], using the Spectral Theorem for compact operators in Banach spaces. In Kesavan [16], the author obtains the result for self-adjoint linear problems in divergence form. From here, an ample class of problems have been addressed for operators having an appropriate weak formulation (p -Laplace operator), and the analysis is extended to the entire spectrum of the operator. Here we follow the viscosity approach initiated by Lions, Papanicolaou and Varadhan [19], in junction with the tools of [5]. The proof of this theorem is an adaptation of the perturbed test function method introduced in [12]. To some extent, the problem can be regarded as a multi-scale homogenization problem as in [1], but the resonant nature of the problem prevents the use of comparison principles, which are replaced by the simplicity of the eigenvalue. We remark that operators like (1.3) are commonly referred as *Bellman-Isaacs* operators, and naturally arise in some applications like the study of zero-sum stochastic differential games, see [14]. In the case of a convex operator (namely, when \mathcal{A} is a singleton), F in (1.3) is known as a *Bellman* operator, and appears in the study of stochastic optimal control problems.

In the second part of the paper, we deal with rates of convergence for the principal eigenvalue problem. In the case of *proper problems*, namely, for equations with the form

$$F(x, \frac{x}{\epsilon}, u, Du, D^2u) = 0,$$

with F such that $r \mapsto F(x, y, r, p, X)$ is nonincreasing, rates of convergence are at disposal in various contexts. In [9], Camilli and Marchi obtain a polynomial rate for Hamilton-Jacobi-Bellmann problems posed in \mathbb{R}^N . The convexity of the nonlinearity allows the use of $C^{2,\alpha}$ estimates to construct approximated correctors for the problem, which in turn lead to the rate $|u^\epsilon - u| = O(\epsilon^{\alpha'})$ for some $0 < \alpha' < \alpha$. Also based on regularity, higher order expansions of the solution u^ϵ are obtained by Kim and Lee [18] for the Dirichlet problem on a bounded smooth domain. For the problem treated therein, the nonlinearity F is convex, smooth and contains no lower-order terms. In [7], Caffarelli and Souganidis introduce the concept of δ -viscosity solutions in order to tackle the non-convex case, and where comparison principle for this type of generalized solutions plays a key role.

Concerning the rate of convergence for the eigenvalues, we start focusing on the nonlinear eigenvalue problem

$$\begin{cases} F(x, \frac{x}{\epsilon}, u^\epsilon, Du^\epsilon, D^2u^\epsilon) = -\lambda^\epsilon u^\epsilon & \text{in } \Omega, \\ u^\epsilon = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.11)$$

and its associated effective problem

$$\begin{cases} \bar{F}(x, u, Du, D^2u) = -\bar{\lambda}u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.12}$$

Under convexity/regularity assumptions on F , we can prove the following rate of convergence for $\{\lambda^\epsilon\}_\epsilon$.

Theorem 1.2 *Assume $F \in C^{2,1}(\Omega \times \mathbb{T}^N \times \mathbb{R} \times \mathbb{R}^N \times S^N)$ is convex in its last three variables and satisfies (1.6), (1.7). Let $\lambda^\epsilon, \bar{\lambda}$ be, respectively, the principal eigenvalues associated to (1.11) and (1.12). Then, there exists a constant $C > 0$ just depending on F and Ω such that*

$$|\lambda^\epsilon - \bar{\lambda}| \leq C\epsilon,$$

for all $\epsilon \in (0, 1)$.

The proof of Theorem 1.2 is based on the Donsker-Varadhan variational characterization of the principal eigenvalue (see [2]), together with regularity estimates for the corrector that can be found in [18]. This is why we need to impose some additional hypotheses on the operator F . The same arguments allows us to prove the convergence of the principal eigenvalue for linear operators with lower order terms with the form

$$L^\epsilon u = a_{ij}(x, x/\epsilon)\partial_{x_i x_j}^2 u + b_j(x, x/\epsilon)\partial_{x_j} u + c(x, x/\epsilon)u, \tag{1.13}$$

in the case the coefficients a_{ij}, b_j, c are smooth enough.

In fact, considering L^ϵ as above and the corresponding principal eigenvalue problem

$$\begin{cases} L^\epsilon u^\epsilon = -\lambda^\epsilon u^\epsilon \text{ in } \Omega, \\ u^\epsilon = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.14}$$

we are able to provide a rate of convergence for the principal eigenfunctions associated to this problem. It is a well-known fact that the effective problem inherits the main structure of the original one, see [12]. In particular, the effective operator related to L^ϵ is linear and takes the form

$$\bar{L}u = \bar{a}_{ij}\partial_{x_i x_j}^2 u + \bar{b}_j\partial_{x_j} u + \bar{c}u,$$

for some continuous entries $\bar{a}_{ij}, \bar{b}_j, \bar{c}$. Moreover, the effective matrix \bar{a} is uniformly elliptic, with the same ellipticity constants of $a = (a_{ij})$. Thus, the effective problem in this case takes the form

$$\begin{cases} \bar{L}u = -\bar{\lambda}u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.15}$$

Of course, if we ask $u > 0$ in Ω in (1.15), then $\bar{\lambda}$ is the principal eigenvalue associated to \bar{L} and Ω . The same holds for the perturbed problem (1.14).

Concerning the convergence of the eigenfunctions (and eigenvalues), our result is the following

Theorem 1.3 *Let L^ϵ in (1.13) with $a \in C^{2,1}(\bar{\Omega} \times \mathbb{T}^N; S^N)$, $b \in C^{2,1}(\bar{\Omega} \times \mathbb{T}^N; \mathbb{R}^N)$, and $c \in C^{2,1}(\bar{\Omega} \times \mathbb{T}^N)$. Let $(u, \bar{\lambda})$ be a solution to (1.15) with $u > 0$ in Ω . Then, there exists $C > 0$ depending on the coefficients a, b, c and Ω , such that, for all $\epsilon \in (0, 1)$, there exists u^ϵ solving (1.14) with $u^\epsilon > 0$ in Ω , satisfying*

$$|\lambda^\epsilon - \bar{\lambda}| + \|u^\epsilon - u\|_{L^\infty(\Omega)} \leq C\epsilon. \tag{1.16}$$

The rate of convergence for the principal eigenvalue is obtained in the same way as Theorem 1.2. For the rate of the eigenfunction, the result follows basically by Theorem 1 in [20], but we provide a pure PDE proof here, by introducing an auxiliary homogenization problem whose solution serves as a pivot between u^ϵ and u . This strategy was previously used in [16], and we combine it with the rates of convergence of [18] for the auxiliary problem. Again, the simplicity of the principal eigenvalue $\bar{\lambda}$ in (1.15) plays a crucial role.

The paper is organized as follows: in Sect. 2 we prove the general convergence result, Theorem 1.1. In Sect. 3, we analyze the convergence of the principal eigenvalue in some particular linear and nonlinear cases. Finally, in Sect. 4 we prove a rate of convergence of the eigenfunction in the linear case.

2 General Convergence Result

We start with the following result, see Lemma 3.1 in [12].

Proposition 2.1 *Under the assumptions of Theorem 1.1, for each $x, p \in \mathbb{R}^N, r \in \mathbb{R}, X \in \mathbb{S}^N$, there exists a unique constant $c = c(x, r, p, X)$ such that the problem (1.9) has a viscosity solution $v \in C^{1,\sigma}(\mathbb{T}^N)$. This solution is unique up to an additive constant.*

As we mentioned in the introduction, this result allows us to define the effective Hamiltonian $\bar{F} \in C(\Omega \times \mathbb{R} \times \mathbb{R}^N \times S^N)$, which satisfies the assumption in [21] to define the principal eigenvalue $\bar{\lambda} = \lambda_1^+(\bar{F}, \Omega)$ in the sense described in (1.5).

Next we prove uniform bounds for the eigenvalue λ^ϵ .

Lemma 2.2 *Assume F satisfies (1.6), (1.7). Let λ^ϵ be the principal eigenvalues associated to (1.11). Then, there exist universal constants $0 < c < C$ such that*

$$-c \leq \lambda^\epsilon \leq C$$

for all $\epsilon \in (0, 1)$.

Proof Write $y = x/\epsilon \in \mathbb{T}^N$ and note that the constant function $w \equiv 1$ satisfies

$$F(x, y, w, Dw, D^2w) \leq c := \|F(\cdot, \cdot, 1, 0, 0)\|_{L^\infty(\bar{\Omega} \times \mathbb{T}^N)}$$

Using the characterization of the eigenvalue (1.5), it is easy to see that $\lambda^\epsilon \geq -c$.

For the upper bound we use the technique presented in [5]. Up to a translation, we assume without loss of generality that the origin belongs to Ω . Let $r \in (0, 1)$ such that $B_r(0) \subset \Omega$ and define

$$\sigma(x) = (r^2 - |x|^2)_+^2,$$

A simple computation shows that for $|x| < r$, we have the eigenvalues of $D^2\sigma(x)$ are $8|x|^2 - 4(r^2 - |x|^2)$ (with multiplicity 1), and $-4(r^2 - |x|^2)$ (with multiplicity $N - 1$). Denote

$$r_0 = r \left(\frac{\Lambda N + C_1}{2\lambda + \Lambda N + C_1} \right)^{1/2}.$$

Let $|x| \leq r_0$. Then we have

$$\begin{aligned} \frac{F(x, x/\epsilon, \sigma(x), D\sigma(x), D^2\sigma(x))}{\sigma(x)} &\geq \frac{M_{\lambda, \Lambda}^-(D^2\sigma(x)) - C_1|D\sigma(x)|}{\sigma(x)} - C_2 \\ &\geq \frac{-4(N + C_1r)(r^2 - |x|^2)}{\sigma(x)} - C_2 \\ &= \frac{-4(N + C_1)}{r^2 - |x|^2} - C_2, \end{aligned}$$

from which the ratio is bounded below by $-(4(N + C_1)(r^2 - r_0^2)^{-1} + C_2)$.

Similarly, for $|x| > r_0$, we have that

$$\begin{aligned} \frac{F(x, x/\epsilon, \sigma(x), D\sigma(x), D^2\sigma(x))}{\sigma(x)} &\geq -\frac{8\lambda|x|^2 - 4(N\Lambda + C_1)(r^2 - |x|^2)}{\sigma(x)} - C_2 \\ &\geq -C_2. \end{aligned}$$

Then, there exists $C > 0$ just depending on the data such that

$$F(x, x/\epsilon, \sigma, D\sigma, D^2\sigma) \geq -C\sigma \quad \text{in } \Omega.$$

Since the support of σ is compact in Ω , up to a positive factor, we can assume σ touches u^ϵ from below at some point $x_1 \in B_r(0)$. Using the viscosity inequality for u^ϵ , we conclude that

$$F(x_1, x_1/\epsilon, \sigma(x_1), D\sigma(x_1), D^2\sigma(x_1)) \leq -\lambda^\epsilon \sigma(x_1),$$

and using the point wise inequality satisfied by σ , we conclude that

$$-C\sigma(x_1) \leq -\lambda^\epsilon \sigma(x_1).$$

Since $\sigma(x_1) > 0$, we conclude the upper bound for λ^ϵ . □

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1 We start by defining

$$\lambda^* = \limsup_{\epsilon \rightarrow 0} \lambda_\epsilon, \quad \lambda_* = \liminf_{\epsilon \rightarrow 0} \lambda_\epsilon.$$

Replacing F by $F - (C_1 + 1)u$, we can assume F is proper and $\lambda_* > 0$.

For each $\epsilon \in (0, 1)$, denote by u^ϵ the principal eigenfunction associated to λ^ϵ such that $\|u^\epsilon\|_\infty = 1$.

We consider $\epsilon_k \rightarrow 0$ such that $\lambda^k := \lambda^{\epsilon_k} \rightarrow \lambda^*$, and write $u_k = u_{\epsilon_k}$. Given that the operator is elliptic we have by standard regularity theory that $\{u_k\}_k$ is precompact in $C^\alpha(\bar{\Omega})$ for some $\alpha > 0$. Then, up to subsequences, it converges uniformly to some $u \in C^{\alpha'}(\bar{\Omega})$, $0 \leq \alpha' < \alpha$, which is nonnegative in Ω . Using that the boundary of Ω is smooth, and the fact that u_k solves extremal inequalities involving the Pucci operators, independent of k , from available boundary regularity results (see Theorem 1.1 in [22]) it follows that the family u_k is uniformly Lipschitz on the boundary of the domain, hence every sequence of points $x_k \in \Omega$ such that $u_k(x_k) = 1$ remains uniformly away the boundary. Thus, taking \bar{x} as a limit point of the sequence x_k , we have shown that there exists $\bar{x} \in \Omega$ such that $u(\bar{x}) = 1$.

Now, we prove that the limit u solves

$$\bar{F}(x, u, Du, D^2u) \leq -\lambda^*u \quad \text{in } \Omega; \quad u = 0 \quad \text{on } \partial\Omega, \tag{2.1}$$

in the viscosity sense.

By contradiction, we assume this does not hold. Then, there exists $x_0 \in \Omega$ and a smooth function ϕ strictly touching u from below at x_0 such that

$$\bar{F}(x_0, \phi(x_0), D\phi(x_0), D^2\phi(x_0)) \geq -\lambda^*\phi(x_0) + 3\eta, \tag{2.2}$$

for some $\eta > 0$. For simplicity, we write $X_0 = (x_0, \phi(x_0), D\phi(x_0), D^2\phi(x_0))$.

Next, we follow the arguments in [1]. For $r \in (0, 1)$ to be chosen, we define the Hamiltonian $F_r \in C(\mathbb{T}^N \times S^N)$, given by

$$F_r(y, Y) = \min\{F(x, y, \phi(x) + s, D\phi(x) + q, D^2\phi(x) + Y) : |x - x_0|, |s|, |q| \leq r\},$$

Next, we state our

Claim 1: There exists $r > 0$ small enough for which there exists a function $\chi \in C(\mathbb{T}^N)$ solving, in the viscosity sense, the inequality

$$F_r(y, D^2\chi) \geq \bar{F}(X_0) - \eta \quad \text{in } \mathbb{T}^N. \tag{2.3}$$

In fact, notice that F_r is uniformly elliptic in view of (1.6), and that

$$F_r(y, Y) \rightarrow F(x_0, y, \phi(x_0), D\phi(x_0), D^2\phi(x_0) + Y) \quad \text{as } r \rightarrow 0,$$

locally uniform in (y, Y) . Then, for $\delta > 0$, we consider the problem

$$-\delta w + F_r(y, D^2w) = 0 \quad \text{in } \mathbb{T}^N,$$

which has a unique viscosity solution $w^{r,\delta}$, which has Lipschitz bounds independent of δ and r in view of the uniform ellipticity of F_r . By comparison, there exists $C > 0$ such that $\delta \|w^{r,\delta}\|_\infty \leq C$. Considering $\tilde{w}^{r,\delta}(y) = w^{r,\delta}(y) - w^{r,\delta}(0)$, we have $\tilde{w}^{r,\delta}$ is bounded and uniformly Lipschitz in \mathbb{T}^N . By standard stability results of viscosity solutions, $\tilde{w}^{r,\delta}$ converges (up to subsequences) to a solution w of the problem

$$F(x_0, y, \phi(x_0), D\phi(x_0), D^2\phi(x_0) + D^2w) = c \quad \text{in } \mathbb{T}^N,$$

as $r, \delta \rightarrow 0$ for some $c \in \mathbb{R}$. Since this constant is unique, it necessarily meets $\bar{F}(X_0)$.

Then, given $\eta > 0$, we can fix r, δ small enough to conclude our *Claim 1* with $\chi = \tilde{w}^{r,\delta}$. Notice that we can assume $\chi \in C^{1,\sigma}$ for some $\sigma > 0$ in view of classical regularity results in elliptic theory (namely, Theorem VII.2 in [15] together with Theorem 2.1 in [23]).

Consider the function

$$\phi_k(x) = \phi(x) + \epsilon_k^2 \chi\left(\frac{x}{\epsilon_k}\right), \quad x \in \Omega.$$

We now seek to establish the following

Claim 2: There exists $0 < r_1 < r$ for which there exists $K_0 \in \mathbb{N}$ such that, for all $k \geq K_0$, ϕ_k solves

$$F\left(x, \frac{x}{\epsilon_k}, \phi_k, D\phi_k, D^2\phi_k\right) \geq -\lambda^k \phi_k \quad \text{in } B_{r_1}(x_0),$$

in the viscosity sense.

This claim follows by the usual perturbed test function method, but we provide the details for completeness. Let $x_1 \in B_{r_1}(x_0)$ and ψ a smooth function touching ϕ_k from above. Then, we have $y_1 = x_1/\epsilon_k$ is a maximum point for the function

$$y \mapsto \chi(y) - \frac{1}{\epsilon_k^2}(\psi(\epsilon_k y) - \phi(\epsilon_k y)).$$

Using (2.3) and (2.2), we have

$$F_r\left(\frac{x_1}{\epsilon_k}, D^2\psi(x_1) - D^2\phi(x_1)\right) \geq -\lambda^* \phi(x_0) + 2\eta,$$

and from here, since $\chi, D\chi$ are uniformly bounded in terms of k , and using that

$$\epsilon_k D\chi(y_1) + D\phi(x_1) = D\psi(x_1),$$

for all k large in terms of r we conclude that

$$F(x_1, \frac{x_1}{\epsilon_k}, \phi_k(x_1), D\psi(x_1), D^2\psi(x_1)) \geq -\lambda^* \phi(x_0) + 2\eta. \tag{2.4}$$

Then, taking r_1 small enough just in terms of ϕ, η and the upper bounds for λ^k (which do not depend on k), and taking k large enough (just depending on η and ϕ), we conclude that

$$F(x_1, \frac{x_1}{\epsilon_k}, \phi_k(x_1), D\psi(x_1), D^2\psi(x_1)) \geq -\lambda^k \phi_k(x_1),$$

from which the *Claim 2* follows.

Then, taking $r_1 > 0$ smaller if necessary, we can use maximum principles in small domains (see Theorem 3.5 in [21]) to conclude that

$$\sup_{B_{r_1}(x_0)} \{\phi_k - u_k\} \leq \sup_{\partial B_{r_1}(x_0)} \{\phi_k - u_k\}.$$

Passing to the limit as $k \rightarrow \infty$, and arranging terms, we conclude that

$$\inf_{B_{r_1}(x_0)} \{u - \phi\} \geq \inf_{\partial B_{r_1}(x_0)} \{u - \phi\},$$

which is a contradiction with the fact that x_0 is a strict test point. This concludes (2.1). Notice that in particular we have $\bar{F}(x, u, Du, D^2u) \leq 0$ in Ω , from which we see that $u > 0$ in Ω by Strong Maximum Principle in [6]. Then, by definition of $\lambda(\bar{F})$, we get

$$\lambda^* \leq \lambda(\bar{F}).$$

The same procedure above leads to the existence of a function $v \geq 0$ in Ω , not identically zero, viscosity solution to

$$\bar{F}(x, v, Dv, D^2v) \geq -\lambda_* v \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega.$$

Hence, if $\lambda_* < \lambda(\bar{F})$, the operator $\bar{F}(u) + \lambda_* u$ satisfies the maximum principle. Thus, $v \leq 0$ in Ω , which is a contradiction. Then,

$$\lambda(\bar{F}) \leq \lambda_* \leq \lambda^* \leq \lambda(\bar{F}),$$

and the convergence of the eigenvalues follows.

Now we tackle the convergence of the eigenfunctions. By the previous result, we have every converging subsequence of $\{u^\epsilon\}$ converges to a positive solution of the effective eigenvalue problem. By Strong Maximum Principle such solution is strictly positive in Ω , from which we conclude the result by the simplicity of the principal eigenfunction. □

Remark 2.3 It is possible to prove the convergence of the principal eigenfunctions with different normalizations. For instance, let $x_0 \in \Omega$ and consider the normalization $u^\epsilon(x_0) = 1$. Since u^ϵ solves

$$\begin{aligned} M_{\lambda, \Lambda}^-(D^2u) - C_1|Du| - \tilde{C}_2|u| &\leq 0 \quad \text{in } \Omega \\ M_{\lambda, \Lambda}^+(D^2u) + C_1|Du| + \tilde{C}_2|u| &\leq 0 \quad \text{in } \Omega, \end{aligned}$$

for some $\tilde{C}_2 > 0$, by Harnack inequality (c.f. Theorem 3.6 in [21]), for each $K \subset\subset \Omega$ containing x_0 , there exists a constant $C_K > 0$ depending on K and the data, but not on ϵ such that $\sup_K u^\epsilon \leq C_K \inf_K u^\epsilon \leq C_K$. Then, writing $F^\epsilon(x, r, p, X) = F(x, \frac{x}{\epsilon}, r, p, X)$, we consider K such that the operator $F^\epsilon + \lambda^\epsilon$ satisfies the maximum principle in $\Omega \setminus K$, (see Theorem 3.5 in [21]). This compact set K does not depend on ϵ . Then, using a suitable power of the distance to the boundary, we can construct a barrier function to conclude uniform bounds for the family $\{u^\epsilon\}_\epsilon$ in $\Omega \setminus K$. Hence, the family is uniformly bounded in $\bar{\Omega}$ and we can perform the same proof above to conclude the convergence of u^ϵ to the principal eigenfunction u with normalization $u(x_0) = 1$.

The following corollary is obtained from Theorem 1.1, resembling multiscale results in the spirit of Corollary 1 in [1]. It will be useful when we look for the rate of convergence for the first eigenfunction in Sect. 4.

Corollary 2.4 *Assume F satisfies the assumptions of Theorem 1.1, and denote by λ^ϵ its principal eigenvalue as in (1.8). Consider $\{f^\epsilon\}_\epsilon \subset C(\Omega)$ such that $f^\epsilon \rightarrow 0$ locally uniformly in Ω as $\epsilon \rightarrow 0$, and assume z^ϵ is a viscosity solution to the problem*

$$\begin{cases} F(x, \frac{x}{\epsilon}, u, Du, D^2u) = -\lambda^\epsilon u + f^\epsilon, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

with $\{z^\epsilon\}_\epsilon$ uniformly bounded in Ω . Then, up to subsequences, $z^\epsilon \rightarrow u$ uniformly in Ω , with u solving (1.10).

Proof We provide an sketch of the proof. It follows the lines of Theorem 1.1, but knowing from the start that $\lambda^\epsilon \rightarrow \bar{\lambda}$, the eigenvalue in (1.10). From this fact, and by the same compactness argument of Theorem 1.1, we may assume z^ϵ converges locally uniformly to some $z \in C(\bar{\Omega})$.

The contradiction argument starts by assuming the existence of $x_0 \in \Omega$ and a smooth function ϕ touching z from below, such that the following analogue of (2.2) holds:

$$\bar{F}(x_0, \phi(x_0), D\phi(x_0), D^2\phi(x_0)) \geq -\bar{\lambda}\phi(x_0) + 4\eta, \tag{2.5}$$

for some $\eta > 0$. The statement and proof of the analogue of Claim 1 in Theorem 1.1 are the same, since f^ϵ plays no role here.

The perturbed test function ϕ_k is constructed as before and, writing $f_k := f^{\epsilon_k}$, it remains to show the following, instead of Claim 2 in the previous proof: *there exists*

$0 < r_1 < r$ for which there exists $K_0 \in \mathbb{N}$ such that, for all $k \geq K_0$, ϕ_k solves

$$F(x, \frac{x}{\epsilon_k}, \phi_k, D\phi_k, D^2\phi_k) \geq -\lambda^k \phi_k + f_k \text{ in } B_{r_1}(x_0),$$

in the viscosity sense.

This is accomplished by taking k large enough so that $\|f_k\|_{B_{r_1}(x_0)} \leq \eta$, which implies that (2.4) holds exactly as before. The concluding argument is the same, so we omit further details. □

3 Rate of Convergence for the Principal Eigenvalue

In this section we assume that $F \in C^{2,1}(\Omega \times \mathbb{T}^N \times \mathbb{R} \times \mathbb{R}^N \times S^N)$. As in Proposition 2.1, for $x \in \Omega$, $r \in \mathbb{R}$, $p \in \mathbb{R}^N$ and $M \in S^N$, $\bar{F}(x, r, p, M)$ is defined as the unique constant c such that the problem

$$F(x, y, r, p, M + D_{yy}^2\chi(y)) = c, \quad y \in \mathbb{T}^N, \tag{3.1}$$

admits a viscosity solution $\chi \in C(\mathbb{T}^N)$ which is unique up to an additive constant. We denote it as $\chi(y; x, r, p, M)$.

The solvability of (3.1) follows by well-known stability properties for the approximation problem

$$-\delta\chi^\delta + F(x, y, r, p, M + D_{yy}^2\chi^\delta) = 0 \quad y \in \mathbb{T}^N, \tag{3.2}$$

which, for each $\delta > 0$, has a unique solution that we denote by $\chi^\delta(y; x, r, p, M)$. By classical Evans-Krylov estimates for convex, uniformly elliptic operators (see [8, 13, 17]) we have $y \mapsto \chi^\delta(y) := \chi^\delta(y; x, r, p, M)$ is $C^{2,\alpha}(\mathbb{T}^N)$, and that by (1.6), for each $R > 0$ there exists $C_R > 0$ such that

$$\|\delta\chi^\delta\|_{C^{2,\alpha}(\mathbb{T}^N)} \leq C_R,$$

for all $x \in \bar{\Omega}$ and $|r|, |p|, |M| \leq R$. Moreover, denoting $\tilde{\chi}^\delta(y) = \chi^\delta(y) - \chi^\delta(0)$ we have that

$$\sup_{\delta \in (0,1)} \|\tilde{\chi}^\delta\|_{C^{2,\alpha}(\mathbb{T}^N)} \leq C_R.$$

This is a straightforward adaptation of Lemma 3.1.4 in [18]. We use this bounds define the function χ in (3.1), by means of stability as $\delta \rightarrow 0$, as the limit

$$\chi(y) = \lim_{\delta \rightarrow 0} \chi^\delta(y) - \chi^\delta(0),$$

where we have omitted the dependence on x, r, p, M for simplicity. Moreover, it is possible to prove that $c = \bar{F}$ in (3.1) is given by

$$\bar{F}(x, r, p, X) = \lim_{\delta \rightarrow 0} \delta \chi^\delta(y; x, r, p, X),$$

where the limit (up to subsequences) is uniform in y . By standard results in periodic homogenization, we have \bar{F} is convex, positively 1-homogeneous, uniformly elliptic (with the same ellipticity constants of F), see [12].

From now on, we concentrate on regularity estimates for the effective Hamiltonian \bar{F} and the other functions involved in its definition.

Given $x, x' \in \Omega, r, r' \in \mathbb{R}, p, p' \in \mathbb{R}^N, M, M' \in S^N$ (each of them bounded by $R > 0$), and denoting

$$v_1^\delta(y) = \chi^\delta(y; x, r, p, M); \quad v_2^\delta(y) = \chi^\delta(y; x', r', p', M')$$

we have $w^\delta = v_1^\delta - v_2^\delta$ solves

$$-\delta w^\delta + a_{ij} \partial_{y_i y_j}^2 w^\delta + f = 0 \quad \text{in } \mathbb{T}^N, \tag{3.3}$$

where we have adopted the notation

$$N_t(y) = (t(D_{yy}^2 v_1^\delta + M) + (1-t)(D_{yy}^2 v_2^\delta + M'), p, r, x, y), \quad t \in [0, 1],$$

$$n_t(y) = (y, tx + (1-t)x', tr + (1-t)r', tp + (1-t)p'), \quad t \in [0, 1],$$

$$a_{ij}(y) = \int_0^1 \frac{\partial F}{\partial m_{ij}}(N_t(y)) dt,$$

$$f(y) = a_{ij}(M - M')_{ij} + \int_0^1 \langle D_{(x,r,p)} F(n_t(y), X'), (x - x', r - r', p - p') \rangle dt,$$

where we have adopted the summation on repeated indices. Since $F \in C^{1,1}$ and v_1^δ, v_2^δ are $C^{2,\alpha}$, $a_{ij} \in C^\alpha(\mathbb{T}^N)$ (with C^α estimates depending on R) and is uniformly elliptic (with ellipticity constants not depending on R), see Lemma 3.1.6 in [18]. Then, by comparison principle on (3.3) we conclude that

$$\begin{aligned} & \| \delta \chi^\delta(\cdot; x, r, p, M) - \delta \chi^\delta(\cdot; x', r', p', M') \|_{C^{2,\alpha}(\mathbb{T}^N)} \\ & \leq C_R (|x - x'| + |r - r'| + |p - p'| + |M - M'|), \end{aligned}$$

for all $x, x' \in \bar{\Omega}, |r|, |r'|, |p|, |p'|, |M|, |M'| \leq R$. Taking limits in δ we conclude that \bar{F} is Lipschitz continuous, and therefore we conclude that the principal eigenfunction u to (1.10), normalized as $\|u\|_\infty = 1$, belongs to $C^{2,\alpha}(\Omega) \cap C^1(\bar{\Omega})$, with estimates depending only on Ω, N and F .

A linearization procedure involving partial derivatives of F [similar to the one leading to (3.3)] allows us to conclude that the first-order partial derivatives of \bar{F} and χ with respect to each of its variables are Lipschitz continuous (c.f. Lemma 3.2.5 in

[18]). Iterating the process for higher-order derivatives of \bar{F} and χ , we conclude the following

Proposition 3.1 *Under the assumptions of Theorem 1.2, we have*

$$\|\bar{F}\|_{C^{2,1}(\bar{\Omega} \times \bar{B}_R^3)} + \sup_{y \in \mathbb{T}^N} \|\chi(y; \cdot, \cdot, \cdot, \cdot)\|_{C^{2,1}(\bar{\Omega} \times \bar{B}_R^3)} \leq C_R;$$

here \bar{B}_R^3 denotes the product of the balls B_R in each \mathbb{R}, \mathbb{R}^N and S^N . In addition, the function u solving (1.10) is in $C^{4,\alpha}(\bar{\Omega})$.

For the latter fact, see Lemma 3.3.1 in [18].

Now, we write

$$w_2(x, y) := \chi(y; x, u(x), Du(x), D^2u(x)), \quad x \in \Omega, \quad y \in \mathbb{T}^N,$$

and note that w_2 is such that $D_x w_2, D_y w_2, D_{xx}^2 w_2, D_{xy}^2 w_2, D_{yy}^2 w_2$ are uniformly bounded in $\bar{\Omega} \times \mathbb{T}^N$ by Proposition 3.1.

Consider the function $w^\epsilon(x) = u(x) + \epsilon^2 w_2(x, x/\epsilon)$. Since $w^\epsilon \in C^2(\Omega)$, we can compute

$$\begin{aligned} Dw^\epsilon(x, x/\epsilon) &= Du(x) + \epsilon D_y w_2(x, x/\epsilon) + \epsilon^2 D_x w_2(x, x/\epsilon), \\ D^2 w^\epsilon(x, x/\epsilon) &= D^2 u(x) + D_{yy}^2 w_2(x, x/\epsilon) + 2\epsilon D_{xy}^2 w_2(x, x/\epsilon) + \epsilon^2 D_{xx}^2 w_2(x, x/\epsilon), \end{aligned}$$

for each $x \in \Omega$. Using (1.6), (1.9) and (1.10) (i.e., the equations solved by w_2 and u , respectively), that

$$\begin{aligned} F(x, x/\epsilon, w^\epsilon(x), D_x w^\epsilon(x), D_{xx}^2 w^\epsilon(x)) &= F(x, x/\epsilon, u(x), Du(x), D^2 u(x) + D_{yy}^2 w_2(x, x/\epsilon)) + O(\epsilon) \\ &= \bar{F}(x, u(x), Du(x), D^2 u(x)) + O(\epsilon) \\ &= -\bar{\lambda}u(x) + O(\epsilon), \end{aligned} \tag{3.4}$$

where the $O(\epsilon)$ -term depends only on the data.

With the above elements, we are able to provide the

Proof of Theorem 1.2: Using the Donsker-Varadhan characterization of the principal eigenvalue (see Theorem 1.1 in [3]), we have

$$-\lambda^\epsilon = \inf_{\substack{\phi \in C^2(\Omega), \\ \phi > 0}} \int_{\Omega} \frac{F(x, x/\epsilon, \phi(x), D\phi(x), D^2\phi(x))}{\phi(x)} d\mu^\epsilon(x),$$

for some probability measure $\mu^\epsilon \in \mathcal{P}(\bar{\Omega})$.

Let $\bar{C} > 0$ such that $\|w_2\|_\infty \leq \bar{C}$. For $\gamma \in (0, 1)$, consider the function

$$\phi_\epsilon(x) = w^\epsilon(x) + \bar{C}\epsilon + \epsilon d^\gamma(x), \quad x \in \Omega.$$

Here, d denotes the distance function to the boundary $\partial\Omega$. In what follows, for $\delta > 0$ we denote $\Omega_\delta = \{x \in \Omega : d(x) < \delta\}$, and $\delta_0 > 0$ is such that $d \in C^2(\Omega_{\delta_0})$. We extended d as a smooth function in Ω , positive in each compact subset of Ω .

Then, ϕ_ϵ is admissible in the infimum above. By (1.6) and (3.4), we see that

$$F(x, \frac{x}{\epsilon}, \phi_\epsilon(x), D\phi_\epsilon(x), D^2\phi_\epsilon(x)) \leq \bar{\lambda}u(x) + \epsilon M^+(D^2d^\gamma(x)) + C_2\epsilon|Dd^\gamma(x)| + C_2\epsilon,$$

where M^+ is the extremal Pucci operator. We have that there exists $c_1 > 0$ such that, for each $\delta \in (0, 1)$ small enough, we have

$$M^+(D^2d^\gamma(x)) \leq -c_1d^{\gamma-2}(x) \text{ for } x \text{ such that } d(x) \leq \delta,$$

and that

$$M^+(D^2d^\gamma(x)) \leq C_\delta \text{ for } x \text{ such that } d(x) \geq \delta,$$

for some $C_\delta > 0$. Also, since $|Dd^\gamma(x)| = \gamma d^{\gamma-1}|Dd(x)| = \gamma d^{\gamma-1}(x)$ for all x such that $d(x) \leq \delta$, we may take $c_1, \delta \in (0, 1)$ smaller and C_δ larger, if necessary, to have

$$M^+(D^2d^\gamma(x)) + C_2|Dd^\gamma(x)| \leq -c_1d^{\gamma-2}(x) \text{ for } x \text{ such that } d(x) \leq \delta, \tag{3.5}$$

and

$$M^+(D^2d^\gamma(x)) + C_2|Dd^\gamma(x)| \leq C_\delta \text{ for } x \text{ such that } d(x) \geq \delta. \tag{3.6}$$

Then, we have

$$\begin{aligned} -\lambda^\epsilon &\leq \int_\Omega \frac{-\bar{\lambda}u(x) + \epsilon M^+(D^2d^\gamma(x)) + C_2\epsilon|Dd^\gamma(x)| + C_2\epsilon}{\phi_\epsilon(x)} d\mu^\epsilon(x) \\ &\leq -\bar{\lambda} + \int_\Omega \frac{\epsilon M^+(D^2d^\gamma(x)) + \tilde{C}\epsilon}{\phi_\epsilon(x)} d\mu^\epsilon(x), \end{aligned}$$

for some $\tilde{C} > 0$ just depending on the data. Now, we can fix $\delta > 0$ small enough such that $M^+(D^2d^\gamma(x)) + C_2|Dd^\gamma(x)| + \tilde{C} \leq 0$ for all $d(x) \leq \delta$, from which we get

$$-\lambda_\epsilon \leq -\bar{\lambda} + \int_{\Omega \setminus \Omega_\delta} \frac{(C_\delta + \tilde{C})\epsilon}{\phi_\epsilon(x)} d\mu^\epsilon(x) \leq -\bar{\lambda} + C\epsilon,$$

where the last inequality follows by the fact that ϕ_ϵ is bounded by below by a positive constant not depending on ϵ over $\Omega \setminus \Omega_\delta$. Then, we conclude that

$$\lambda^\epsilon \geq \bar{\lambda} - C\epsilon,$$

for some $C > 0$.

For the upper bound, we consider the function

$$\phi_\epsilon(x) = w^\epsilon(x) - (\tilde{C} + 1)\epsilon - \epsilon d^\gamma(x), \quad x \in \Omega, \tag{3.7}$$

where d and γ are defined as before. Then, similarly to (3.5) and (3.6), there exists $c, C, \tilde{C} > 0$ just depending on the data such that, for each $\delta \in (0, \delta_0)$ and all $x \in \Omega$ we have

$$\begin{aligned} & F(x, \frac{x}{\epsilon}, \phi_\epsilon(x), D\phi_\epsilon(x), D^2\phi_\epsilon(x)) \\ & \geq -\bar{\lambda}u(x) - C_2\epsilon - C_2\epsilon|Dd^\gamma(x)| + \epsilon M^-(-D^2d^\gamma(x)) \\ & \geq -\bar{\lambda}\phi_\epsilon(x) - \tilde{C}\epsilon + c\epsilon\delta^{\gamma-2}\mathbf{1}_{\Omega_\delta}(x) - C\delta^{\gamma-2}\epsilon\mathbf{1}_{\Omega \setminus \Omega_\delta}(x) \\ & = -(\bar{\lambda} + \ell\epsilon)\phi_\epsilon(x) + \epsilon[\ell\phi_\epsilon(x) - \tilde{C} + c\delta^{\gamma-2}\mathbf{1}_{\Omega_\delta}(x) - C\delta^{\gamma-2}\mathbf{1}_{\Omega \setminus \Omega_\delta}(x)], \end{aligned} \tag{3.8}$$

for every $\ell > 0$, which is going to be fixed below. Here we have denoted as $\mathbf{1}_A$ the indicator function of the set A .

We start fixing $\delta > 0$ small such that $c\delta^{\gamma-2} \geq 2\tilde{C} + 1$. Once δ is fixed this way, we take ℓ such that $\ell \inf_{\Omega \setminus \Omega_\delta} \phi_\epsilon \geq \tilde{C} + C\delta^{\gamma-2}$. This is a constant not depending on ϵ . In particular, this makes the term in square brackets in (3.8) above non negative in $\Omega \setminus \Omega_\delta$. On the other hand, since $\phi_\epsilon \geq -C\epsilon$ in Ω for some $C > 0$, we have

$$\ell\phi_\epsilon(x) - \tilde{C} + c\delta^{\gamma-2} \geq -C\ell\epsilon + \tilde{C} + 1$$

for all $x \in \Omega_\delta$. Thus, for all ϵ small enough in terms of the data, we conclude that ϕ_ϵ satisfies

$$F(x, \frac{x}{\epsilon}, \phi_\epsilon, D\phi_\epsilon, D^2\phi_\epsilon) \geq -(\bar{\lambda} + \ell\epsilon)\phi_\epsilon \quad \text{in } \Omega.$$

Since $\phi_\epsilon(x) < 0$ for each x a neighborhood of the boundary, up to a multiplicative positive constant, we have ϕ_ϵ touches from below u^ϵ at a point $x_0 \in \Omega$ where $\phi_\epsilon(x_0) = u^\epsilon(x_0) > 0$, and therefore can be used as a test function for u . We thus have

$$-(\bar{\lambda} + \ell\epsilon)\phi_\epsilon(x_0) \leq F(x_0, x_0/\epsilon, \phi_\epsilon(x_0), D\phi_\epsilon(x_0), D^2\phi_\epsilon(x_0)) \leq -\lambda^\epsilon u^\epsilon(x_0),$$

and, using once more that $\phi_\epsilon(x_0) = u^\epsilon(x_0) > 0$, from here we get $\lambda^\epsilon \leq \bar{\lambda} + \ell\epsilon$. This concludes the proof. □

4 Proof of Theorem 1.3

This section is entirely devoted to the

Proof of Theorem 1.3: We have proved the rate of convergence for the eigenvalues in Theorem 1.2.

By replacing $\bar{\lambda}$ and λ^ϵ with $\bar{\lambda} + C_1 + 1$ and $\lambda^\epsilon + C_1 + 1$, respectively, we can assume that \bar{L} and L^ϵ are both proper operators. Let u be the solution to (1.15) with $\|u\|_\infty = 1$. Given $\epsilon \in (0, 1)$ consider the following homogenization problem

$$\begin{cases} L^\epsilon w^\epsilon = -\bar{\lambda}u \text{ in } \Omega \\ w^\epsilon = 0 \text{ in } \partial\Omega, \end{cases} \tag{4.1}$$

By Theorem 1.1.1 in [18], we have the existence of $C > 0$ just depending on the coefficients of L and Ω (but not on ϵ), such that

$$\|w^\epsilon - u\|_\infty \leq C\epsilon. \tag{4.2}$$

Assume u^ϵ is a normalized eigenfunction for (1.14) (take $\|u^\epsilon\|_\infty = 1$ for definiteness), and consider $z^\epsilon = u^\epsilon - w^\epsilon + t_\epsilon u^\epsilon$ for some $t_\epsilon \in \mathbb{R}$ to be fixed. It is easy to see that z^ϵ solves the problem

$$\begin{cases} L^\epsilon z^\epsilon + \lambda^\epsilon z^\epsilon = -\lambda^\epsilon w^\epsilon + \bar{\lambda}u \text{ in } \Omega, \\ z^\epsilon = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.3}$$

We consider t_ϵ such that

$$(z^\epsilon, u^\epsilon) = 0, \tag{4.4}$$

where (\cdot, \cdot) denotes the inner product in $L^2(\Omega)$. This is possible by taking

$$t_\epsilon = \frac{(w^\epsilon, u^\epsilon)}{\|u^\epsilon\|_{L^2}} - 1.$$

Note that, from the maximum principle applied to (4.3), $\|w_\epsilon\|_\infty$ is uniformly bounded; hence, so is t_ϵ , since $\|u^\epsilon\|_\infty = 1$. Thus the family $\{z^\epsilon\}_\epsilon$ is uniformly bounded, and by standard elliptic estimates we have it is equicontinuous in $\bar{\Omega}$.

With this choice, we claim the existence of $C_0 > 0$ such that

$$\|z^\epsilon\|_\infty \leq C_0\epsilon, \tag{4.5}$$

for all $\epsilon \in (0, 1)$. Let us argue by contradiction, assuming the existence of a sequence $\epsilon_n \in (0, 1)$ such that $z^n = z^{\epsilon_n}$ satisfies

$$\|z^n\|_\infty \geq n\epsilon_n,$$

as $n \rightarrow \infty$. Notice that, in particular, we have z^n is not identically zero in Ω for all n , and that $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Let $\hat{z}^n = z^n / \|z^n\|_\infty$ and note that it solves

$$\begin{cases} L^{\epsilon_n} \hat{z}^n + \lambda^{\epsilon_n} \hat{z}^n = \frac{-\lambda^{\epsilon_n} w^\epsilon + \bar{\lambda}u}{\|z^n\|} \text{ in } \Omega, \\ \hat{z}^n = 0 \text{ on } \partial\Omega, \end{cases}$$

Since $\|\lambda^{\epsilon_n} w^{\epsilon_n} - \bar{\lambda}u\|_{\infty} \leq C\epsilon_n$ for some $C > 0$, we have that

$$\frac{\|-\lambda^{\epsilon} w^{\epsilon} + \bar{\lambda}u\|}{\|z^n\|} \leq \frac{C}{n}.$$

Then, by Corollary 2.4, we have that (\hat{z}^n) converges uniformly, up to subsequences, to a nontrivial solution z to the eigenvalue problem

$$\begin{cases} \bar{L}z = -\bar{\lambda}z & \text{in } \Omega \\ z = 0 & \text{in } \partial\Omega, \end{cases}$$

hence z is an eigenfunction of L with $\|z\|_{\infty} = 1$. On the other hand, using (4.4) we have

$$(z, u) = 0,$$

but this implies that $z = 0$, a contradiction, and the claim is proven.

From (4.5), we get that

$$\|w^{\epsilon} - (1 + t_{\epsilon})u^{\epsilon}\|_{\infty} \leq C_0\epsilon,$$

and from here we get the result, by triangle inequality, (4.2) and replacing u^{ϵ} by $(1 + t_{\epsilon})u^{\epsilon}$ in the statement of the theorem. \square

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