Applied Mathematics & Optimization (2023) 87:23
https://doi.org/10.1007/s00245-022-09930-8

On Approximate and Weak Correlated Equilibria in Constrained Discounted Stochastic Games

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Accepted: 10 October 2022 / Published online: 13 January 2023

Abstract

In this paper, we consider constrained discounted stochastic games with a countably generated state space and norm continuous transition probability having a density function. We prove existence of approximate stationary equilibria and stationary weak correlated equilibria. Our results imply the existence of stationary Nash equilibrium in $ARAT$ stochastic games.

Keywords Constrained discounted stochastic game · Approximate equilibrium · Nash equilibrium · Correlated equilibrium

Mathematics Subject Classification Primary: 91A15 · 91A10 · 60J10 · Secondary: 90C40 · 60J20

1 Introduction

Constrained Markov decision processes and stochastic games have numerous applications in operations research, economics, computer sciences, consult with [2, 3, 28, 37] and references cited therein. They arise in situations, in which a controller or player has many objectives. For example, when she or he wants to minimise one type of cost while keeping other costs lower than some given bounds. Constrained stochastic games with finite state and action spaces were first studied by Altman and Shwartz [3]. Their work was extended to some classes of games with countable state spaces in

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[4, 42] by finite state approximations. A more direct approach based on properties of measures induced by strategies and occupation measures was presented in [28].

In this paper, we study discounted constrained stochastic games with a general state space and the transition probability having a density function. Such two-person games with additive rewards and additive transition structure ($ARAT$ games) were recently studied by Dufour and Prieto-Rumeau [13]. They established the existence of stationary Nash equilibria generalising the result of Himmelberg et al. [25] proved for unconstrained games. Moreover, their theorem also holds for N-person $ARAT$ games satisfying the standard Slater condition. As shown in a highly non-trivial example by Levy and McLennan [29], the games under consideration in this paper may have no stationary Nash equilibrium in the unconstrained case. It can be seen that this example applies to the constrained case as well. Thus, results on approximate equilibria as in [34, 41] became more valuable. They are stated for the unconstrained case, and in this paper we extend the main result from [34] to a class of constrained games. In this way, we establish the existence of approximate stationary equilbria for discounted stochastic games with constraints and general state spaces. It should be noted that the existence of stationary equilibria in discounted unconstrained games was proved only in some special cases, for instance, for $ARAT$ games [25] or games with transitions having no conditional atoms [23]. For a survey of results on stationary and non-stationary Nash equilibria the reader is referred to [26].

The other group of papers comprise the ones on stationary equilibria with public signals, see [11, 22, 36]. Such solutions can be viewed as special communication or correlated equilibria widely discussed in dynamic frameworks (repeated, stochastic or extensive form games) in [20, 21, 31, 38, 39]. They were inspired by the seminal papers of Aumann [5, 6]. A weaker version of correlated equilibrium was proposed by Moulin and Vial [32]. According to their approach a correlated strategy in a finite (bimatrix) game is a probability distribution ν on the set of pure strategy pairs. Every player has to decide whether to accept ν or to use his or her individual strategy. If player i uses an individual strategy and player $j \neq i$ obeys ν, then a pure action for player j is selected by the marginal distribution of ν on his/her pure actions. Then ν is an equilibrium, if no unilateral deviations from it are profitable. This solution is called a weak correlated equilibrium or a correlated equilibrium with no exchange of information [32]. In contrast to Aumann's approach, the players who accepted ν cannot change actions after using the lottery ν. The solution proposed by Moulin and Vial [32] has an interesting property. Namely, the authors constructed a bimatrix game, in which the equilibrium payoffs in their equilibrium concept strictly dominate in the Pareto sense the payoffs in Aumann's equilibrium, see [30, 32].

In [35] the concept of Moulin and Vial is used to an unconstrained discounted stochastic game with a general state space. However, as shown by Solan and Vieille, [39], the notion of a weak correlated equilibrium can be also regarded as a special case of a general correlation scheme.

In this paper, we extend the result from [35] to a large class of discounted stochastic games with so-called integral constraints. We apply our recent result from [28] for games with discrete state spaces and use an approximation technique. A stationary weak correlated equilibrium is obtained as a limit (in the weak* sense) of approximate equilibria. Our result generalises the main theorem of Dufour and Prieto-Rumeau

[13] given for $ARAT$ games, if the action sets for players do not depend on the state. We wish to emphasise that the considerations of other classes of correlated equilibria in constrained stochastic games (like equilibria with public signals) seem to be very challenging for many reasons. Firstly, the integral constraints are difficult to apply. Secondly, the usual methods from dynamic programming (Bellman's principle) or backward and forward induction used in unconstrained cases are not applicable. Perhaps further possible results can be obtained for other correlated equilibria but under different type of constraints.

The paper is organised as follows. The model and main results on equilibria are contained in Sect. 2. Section 3 presents the approximation technique and the proofs of two main theorems. Section 4 is devoted to the proof on the existence of a weak correlated equilibrium and a discussion on our assumptions. In Sect. 5, we show that the example given in [29] can be used to show that discounted constrained stochastic games studied in this paper may not have stationary Nash equilibria. Section 6 discusses a useful transformation that shows how to easily extend our results formulated for bounded cost functions to unbounded ones. In Appendix (Sect. 7) we give a crucial lemma on a replacement of a general strategy by a piecewise constant strategy. It is used in the proofs of our main theorems on equilibria in constrained stochastic games.

2 The Game Model and Main Results

In this section, we describe constrained discounted stochastic games with general state space and our basic assumptions. We provide our main results in three cases. Firstly, we give a theorem on the existence of a stationary approximate equilibrium assuming that the players play the game independently. Secondly, we drop the constraints and give a theorem on the existence of a stationary ε-equilibrium for every initial state, extending the main result in [34]. Finally, we show that the constrained stochastic games under consideration possess stationary weak correlated equilibria introduced in the static (bimatrix) case by Moulin and Vial [32].

2.1 Approximate Nash Equilibria in Constrained Discounted Stochastic Games

The non-zero-sum *constrained stochastic game* (CSG) is described by the following objects:

- $\mathcal{N} = \{1, 2, ..., N\}$ is the *set of players*.
- X is a *state space* endowed with a countably generated σ-algebra $\mathcal{F}$.
- A_i is a compact metric *action space* for player $i \in \mathcal{N}$ endowed with the Borel σ-algebra. We put

$$A := \prod_{j\in\mathcal{N}} A_j \quad \text{and} \quad A_{-i} := \prod_{j\in\mathcal{N}\setminus\{i\}} A_j,$$
$$\mathbb{K}_i := \{(x, a_i) : x \in X,\ a_i \in A_i\}, \quad \mathbb{K} := \{(x, \boldsymbol{a}) : x \in X,\ \boldsymbol{a} = (a_1, ..., a_n) \in A\}.$$

- The real-valued functions $c_i^\ell : \mathbb{K} \to \mathbb{R}$, where $i \in \mathcal{N}$, $\ell \in \mathcal{L}_0 = \mathcal{L} \cup \{0\}$ with $\mathcal{L} = \{1, ..., L\}$, are product measurable. Here, c_i^0 is the *cost-per-stage function* for player $i \in \mathcal{N}$, and for each $\ell \in \mathcal{L}$, c_i^ℓ is a function used in the definition of the ℓ-th *constraint* for this player. It is assumed that there exists $b > 0$ such that

$$|c_i^\ell(x, \boldsymbol{a})| \le b, \quad \text{for all} \quad i \in \mathcal{N},\ \ell \in \mathcal{L}_0,\ (x, \boldsymbol{a}) \in \mathbb{K}.$$

- $p(dy|x, \boldsymbol{a})$ is the transition probability from x to $y \in X$, when the players choose a profile $\boldsymbol{a} = (a_1, a_2, ..., a_N)$ of actions in A.
- η is the *initial state distribution*.
- $\alpha \in (0, 1)$ is the *discount factor*.
- κ_i^ℓ are constraint constants, $i \in \mathcal{N}$, $\ell \in \mathcal{L}$.

Let $\mathbb{N} = \{1, 2, ...\}$. Define $H^1 = X$ and $H^{t+1} = \mathbb{K} \times H^t$ for $t \in \mathbb{N}$. An element $h^t = (x^1, \boldsymbol{a}^1, \dots, x^t)$ of H^t represents a history of the game up to the t-th period, where $\boldsymbol{a}^k = (a_1^k, \dots, a_N^k)$ is the profile of actions chosen by the players in the state x^k on the k-th stage of the game, $h^1 = x^1$.

Strategies for the players are defined in the usual way. A *strategy* for player $i \in \mathcal{N}$ is a sequence $\pi_i = (\pi_i^t)_{t\in\mathbb{N}}$, where each π_i^t is a transition probability from H^t to A_i. By Π_i we denote the *set of all strategies* for player i. Let Φ_i be the set of transition probabilities from X to A_i. A *stationary strategy* for player i is a constant sequence $(\varphi_i^t)_{t\in\mathbb{N}}$, where $\varphi_i^t = \varphi_i$ for all $t \in \mathbb{N}$ and some $\varphi_i \in \Phi_i$. Furthermore, we shall identify a stationary strategy for player i with the constant element φ_i of the sequence. Thus, the *set of all stationary* strategies of player i is also denoted by Φ_i. We define

$$\Pi = \prod_{i=1}^N \Pi_i \quad \text{and} \quad \Phi = \prod_{i=1}^N \Phi_i.$$

Hence, Π (Φ) is the set of all (stationary) multi-strategies of the players.

Let $H^\infty = \mathbb{K} \times \mathbb{K} \times \cdots$ be the space of all infinite histories of the game endowed with the product σ-algebra. For any multi-strategy $\boldsymbol{\pi} \in \Pi$, a unique probability measure $\mathbb{P}_\eta^{\boldsymbol{\pi}}$ and a stochastic process $(x^t, \boldsymbol{a}^t)_{t\in\mathbb{N}}$ are defined on H^∞ in a canonical way, see the Ionescu-Tulcea theorem, e.g., Proposition V.1.1 in [33]. The measure $\mathbb{P}_\eta^{\boldsymbol{\pi}}$ is induced by $\boldsymbol{\pi}$, the transition probability p and the initial distribution η. The expectation operator with respect to $\mathbb{P}_\eta^{\boldsymbol{\pi}}$ is denoted by $\mathbb{E}_\eta^{\boldsymbol{\pi}}$.

Let $\boldsymbol{\pi} \in \Pi$ be any multi-strategy. For each $i \in \mathcal{N}$ and $\ell \in \mathcal{L}_0$, the *discounted cost functionals* are defined as

$$J_i^\ell(\boldsymbol{\pi}) = (1 - \alpha)\mathbb{E}_\eta^{\boldsymbol{\pi}} \left[\sum_{t=1}^\infty \alpha^{t-1} c_i^\ell(x^t, \boldsymbol{a}^t) \right].$$

We assume that $J_i^0(\boldsymbol{\pi})$ is the expected discounted cost of player $i \in \mathcal{N}$, who wishes to minimise it over $\pi_i \in \Pi_i$ in such a way that the following constraints are satisfied

$$J_i^\ell(\boldsymbol{\pi}) \le \kappa_i^\ell \quad \text{for all} \quad \ell \in \mathcal{L}.$$

A multi-strategy $\boldsymbol{\pi}$ is *feasible*, if the above inequality holds for each $i \in \mathcal{N}$, $\ell \in \mathcal{L}$. We denote by Δ the set of all feasible multi-strategies in the CSG.

As usual, for any $\boldsymbol{\pi} \in \Pi$, we denote by $\boldsymbol{\pi_{-i}}$ the multi-strategy of all players but player i, that is, $\boldsymbol{\pi_{-1}} = (\pi_2, ..., \pi_N)$, $\boldsymbol{\pi_{-N}} = (\pi_1, ..., \pi_{N-1})$, and for $i \in \mathcal{N} \setminus \{1, N\}$,

$$\boldsymbol{\pi_{-i}} = (\pi_1, \ldots, \pi_{i-1}, \pi_{i+1}, \ldots, \pi_N).$$

We identify $[\boldsymbol{\pi_{-i}}, \pi_i]$ with $\boldsymbol{\pi}$. For each $\boldsymbol{\pi} \in \Pi$, we define the set of feasible strategies for player i with $\boldsymbol{\pi_{-i}}$ as

$$\Delta_i(\boldsymbol{\pi_{-i}}) = \{\pi_i \in \Pi_i : \ J_i^\ell(\boldsymbol{\pi}) = J_i^\ell([\boldsymbol{\pi_{-i}}, \pi_i]) \le \kappa_i^\ell \quad \text{for all} \quad \ell \in \mathcal{L}\}.$$

Let $\boldsymbol{\pi} = (\pi_1, \pi_2, ..., \pi_N) \in \Pi$ and $\sigma_i \in \Pi_i$. By $[\boldsymbol{\pi_{-i}}, \sigma_i]$ we denote the multi-strategy, where player i uses σ_i and every player $j \neq i$ uses π_j.

Definition 2.1 A multi-strategy $\boldsymbol{\pi}^* \in \Pi$ is an *approximate equilibrium* in the CSG (for given $\varepsilon > 0$), if for every $i \in \mathcal{N}$ and $\ell \in \mathcal{L}$,

$$J_i^\ell(\boldsymbol{\pi}^*) \le \kappa_i^\ell + \varepsilon, \tag{2.1}$$

and for every $i \in \mathcal{N}$,

$$J_i^0(\boldsymbol{\pi}^*) - \varepsilon \le \inf_{\sigma_i \in \Delta_i(\boldsymbol{\pi^*_{-i}})} J_i^0([\boldsymbol{\pi^*_{-i}}, \sigma_i]). \tag{2.2}$$

A multi-strategy $\boldsymbol{\pi}^* \in \Pi$ is an ε-*equilibrium* in the CSG (for given $\varepsilon \ge 0$), if (2.2) holds and $J_i^\ell(\boldsymbol{\pi}^*) \le \kappa_i^\ell$ for every $i \in \mathcal{N}$ and $\ell \in \mathcal{L}$. A 0-equilibrium is called a Nash equilibrium in the CSG.

Note that, every ε-equilibrium is approximate, but not vice versa. For small $\varepsilon > 0$, condition (2.1) allows for a slight violation of the feasibility of $\boldsymbol{\pi}^*$. Further comments on this condition the reader will find in Remark 2.4.

We now formulate our basic assumptions.

Assumption A1 The functions $c_i^\ell(x, \cdot)$ are continuous on A for all $x \in X$, $i \in \mathcal{N}$ and $\ell \in \mathcal{L}_0$.

Assumption A2 The transition probability p is of the form

$$p(B|x, \boldsymbol{a}) = \int_B \delta(x, y, \boldsymbol{a})\mu(dy), \quad B \in \mathcal{F},$$

where μ is a probability measure on $\mathcal{F}$ and δ is a product measurable non-negative (density) function such that, if $\boldsymbol{a}^n \to \boldsymbol{a}$ as $n \to \infty$, then

$$\int_X |\delta(x, y, \boldsymbol{a}^n) - \delta(x, y, \boldsymbol{a})|\mu(dy) \to 0.$$

This assumption means the norm continuity of p with respect to action profiles.

Assumption A3 For each stationary multi-strategy $\boldsymbol{\varphi} \in \Phi$ and for each player $i \in \mathcal{N}$, there exists $\pi_i \in \Pi_i$ such that

$$J_i^\ell([\boldsymbol{\varphi_{-i}}, \pi_i]) \leq \kappa_i^\ell \quad \text{for all} \quad \ell \in \mathcal{L}.$$

Assumption A3 is standard in the theory of constrained decision processes and stochastic games [2, 3, 13, 28].

Remark 2.2 From Assumption A3, Lemma 2.3 in [13] and Lemma 24 in [37], it follows that the strategy $\pi_i \in \Pi_i$ can be replaced a stationary strategy $\sigma_i \in \Phi_i$ such that

$$J_i^\ell([\boldsymbol{\varphi_{-i}}, \pi_i]) = J_i^\ell([\boldsymbol{\varphi_{-i}}, \sigma_i]) \quad \text{for all} \quad \ell \in \mathcal{L}.$$

The proof of Lemma 24 in [37] on the equivalence of these strategies is formulated for models with Borel state spaces. However, it is also valid in our framework (see pp. 307–309 in [37]) with the exception that we need an appropriate disintegration result. In this matter, consult with Lemma 2.3 in [13] or Theorem 3.2 in [19].

We are ready to state our first main result.

Theorem 2.3 *Assume* **A1**, **A2** *and* **A3**. *Then, for each* $\varepsilon > 0$, *the* CSG *possesses a stationary approximate equilibrium.*

Remark 2.4 The proof of this result is given in Sect. 3. We prove that a stationary approximate equilibrium for given $\varepsilon > 0$ consists of strategies that are piecewise constant functions of the state variable. We observe that, under assumptions of Theorem 2.3, condition (2.1) with $\varepsilon = 0$ need not be satisfied by piecewise constant stationary multi-strategies. Therefore, the existence of an ε-equilibrium in the CSG is an open issue. We would like to emphasise that Theorem 2.3 is crucial in our proof of Theorem 2.13 on weak correlated equilibria, where we apply an asymptotic approach when $\varepsilon \to 0$.

Remark 2.5 The only result in the literature on the existence of stationary Nash equilibria in $CSGs$ with general state space was given by Dufour and Prieto-Rumeau [13]. It concerns so-called discounted additive rewards and additive transition ($ARAT$) stochastic games. In the two-person case the $ARAT$ assumption means that $c_i^\ell(x, a_1, a_2) = c_{1i}^\ell(x, a_1) + c_{2i}^\ell(x, a_2)$ and $p(\cdot|x, a_1, a_2) = p_1(\cdot|x, a_1) + p_2(\cdot|x, a_2)$, where p_1 and p_2 are transition subprobabilities. The results in [13] are given for two-person games satisfying the standard Slater condition (Assumption A3 with strict inequalities). However, they can be easily extended by the same methods to N-person $ARAT$ stochastic games. A simple adaptation of the counterexample by Levy and McLennan [29] given for unconstrained discounted stochastic games implies that stationary Nash equilibria may not exist in the constrained stochastic games studied in this paper. For more details see Sect. 5.

Remark 2.6 We wish to emphasise that the Slater condition is not needed for the establishing an approximate equilibrium in $CSGs$.

2.2 An Update on Stationary Equilibria in Unconstrained Discounted Stochastic Games

In this subsection, we drop the constraints. By the Ionescu–Tulcea theorem [33], any multi-strategy $\boldsymbol{\pi} \in \Pi$ and any initial state $x \in X$, induce a unique probability measure $\mathbb{P}_x^{\boldsymbol{\pi}}$ on H^∞. The expectation operator with respect to $\mathbb{P}_x^{\boldsymbol{\pi}}$ is denoted by $\mathbb{E}_x^{\boldsymbol{\pi}}$.

The *discounted cost* for player $i \in \mathcal{N}$ is defined as

$$J_i^0(\boldsymbol{\pi})(x) = (1-\alpha)\mathbb{E}_x^{\boldsymbol{\pi}}\left[\sum_{t=1}^{\infty} \alpha^{t-1} c_i^0(x^t, \boldsymbol{a}^t)\right].$$

Definition 2.7 Let $\varepsilon \geq 0$ be fixed. A multi-strategy $\boldsymbol{\pi}^* \in \Pi$ is an *ε-equilibrium* in the unconstrained discounted stochastic game, if

$$J_i^0(\boldsymbol{\pi}^*) - \varepsilon \leq \inf_{\sigma_i \in \Pi_i} J_i^0([\boldsymbol{\pi}^*, \sigma_i])$$

for every player $i \in \mathcal{N}$ and for all initial states $x \in X$. A 0-equilibrium is called a Nash equilibrium.

Theorem 2.8 *Under assumptions A1 and A2, for any* $\varepsilon > 0$, *the unconstrained discounted stochastic game has a stationary ε-equilibrium.*

The proof is given in Sect. 3.

Remark 2.9 Stationary Nash equilibria exist only in some special cases of stochastic games satisfying Assumptions A1 and A2, see [25] ($ARAT$ games), [23] (other classes of games) and [26] (a survey). As shown by Levy and McLennan [29] stationary Nash equilibria need not exist in general under assumptions of Theorem 2.8.

Remark 2.10 Theorem 2.8 is an extension of Theorem 3.1 in [34], where additionally it is assumed that

$$\int_X \sup_{\boldsymbol{a} \in A} \delta(x, y, \boldsymbol{a})\mu(dy) < \infty \quad \text{for each} \quad x \in X. \tag{2.3}$$

2.3 Weak Correlated Equilibria in Constrained Discounted Stochastic Games

Let Ψ be the set of all transition probabilities from X to A, that is, $\psi \in \Psi$ if $\psi(\cdot|x) \in \Pr(A)$ for every $x \in X$ and $\psi(D|\cdot)$ is $\mathcal{F}$-measurable for any Borel set $D \subset A$. A *stationary correlated strategy* for the players in the CSG is a constant sequence $(\psi, \psi, \ldots)$, where $\psi \in \Psi$. As in the case of stationary strategies, we shall identify a correlated strategy with the element ψ of this sequence.

By the Ionescu-Tulcea theorem [33], any correlated strategy $\psi \in \Psi$ and the initial distribution η, induce a unique probability measure $\mathbb{P}_\eta^\psi$ on H^∞. The expectation operator with respect to $\mathbb{P}_\eta^\psi$ is denoted by $\mathbb{E}_\eta^\psi$. Then the discounted cost functionals for player $i \in \mathcal{N}$ are defined as

$$J_i^\ell(\psi) = (1-\alpha)\mathbb{E}_\eta^\psi\left[\sum_{t=1}^{\infty} \alpha^{t-1} c_i^\ell(x^t, \boldsymbol{a}^t)\right]$$

for all $\ell \in \mathcal{L}_0$. Obviously, here at stage t the vector of actions $\boldsymbol{a}^t$ is chosen according to a probability measure $\psi(\cdot|x^t)$.

Furthermore, let ψ_i and i denote the projections for any x 2 X of (jx) on $A_{\psi i}$ and A_i, respectively. For any player $i \in \mathcal{N}$ and a strategy $\pi_i \in \Pi_i$ we denote by $[\psi_{-i}, \pi_i]$ a multi-strategy, where player i uses a strategy π_i and the other players act as one player applying ψ_{-i}. In this case, $J_i^0([\psi_{-i}, \pi_i])$ denotes the expected discounted cost for player i. Set

$$\Delta_i(\psi_{-i}) = \{\pi_i \in \Pi_i : J_i^\ell([\psi_{-i}, \pi_i]) \leq \kappa_i^\ell \text{ for all } \ell \in \mathcal{L}\}.$$

Definition 2.11 A strategy $\psi^* \in \Psi$ is called a *weak correlated equilibrium* in the CSG, if for every $i \in \mathcal{N}$ and $\ell \in \mathcal{L}$, $J_i^\ell(\psi^*) \leq \kappa_i^\ell$ and for every $i \in \mathcal{N}$,

$$J_i^0(\psi^*) \leq \inf_{\pi_i \in \Delta_i(\psi_{-i}^*)} J_i^0([\psi_{-i}^*, \pi_i]). \tag{2.4}$$

If all players but $i \in \mathcal{N}$ accept to use ψ^* to select an action profile in any state x and player $i \in \mathcal{N}$ decides to play independently of all of them by choosing a feasible strategy π_i, then the action profile for all players in $\mathcal{N} \setminus \{i\}$ is selected with respect to the marginal probability distribution $\psi_{-i}^*(\cdot|x)$ on A_{-i}. When ψ^* is a weak correlated equilibrium, then inequality (2.4) says that unilateral deviations from ψ^* are not profitable. This is an adaptation of the equilibrium concept, formulated by Moulin and Vial [32] for static games, to our dynamic game model.

In order to state our third main result, we define $\Phi_{-i} := \prod_{j \in \mathcal{N}\setminus\{i\}} \Phi_j$ and impose the following condition.

Assumption A4 For each player $i \in \mathcal{N}$,

$$\sup_{\boldsymbol{\varphi_{-i}} \in \Phi_{-i}} \min_{\sigma_i \in \Phi_i} \max_{\ell \in \mathcal{L}} (J_i^\ell([\boldsymbol{\varphi_{-i}}, \sigma_i]) - \kappa_i^\ell) < 0.$$

This assumption implies the standard Slater condition (see Assumption A5 below) widely used in the literature [2, 3, 13, 28].

Assumption A5 For each player $i \in \mathcal{N}$ and any $\boldsymbol{\varphi_{-i}} \in \Phi_{-i}$, there exists $\sigma_i \in \Phi_i$ such that

$$J_i^\ell([\boldsymbol{\varphi_{-i}}, \sigma_i]) < \kappa_i^\ell \quad \text{for all} \quad \ell \in \mathcal{L}.$$

Assumptions A4 and A5 may seemingly be more general. Namely, we can formulate them for $\pi_i \in \Pi_i$ instead of $\sigma_i \in \Phi_i$ and replace the set Φ_i by Π_i. However, Remark 2.2 implies that these formulations are in fact equivalent.

Remark 2.12 From Assumption A4, it follows that there exists $\zeta > 0$ such that for every player $i \in \mathcal{N}$,

$$\sup_{\boldsymbol{\varphi_{-i}} \in \Phi_{-i}} \min_{\sigma_i \in \Phi_i} \max_{\ell \in \mathcal{L}} (J_i^{\ell}([\boldsymbol{\varphi_{-i}}, \sigma_i]) - \kappa_i^{\ell}) < -\zeta,$$

and consequently that for each player $i \in \mathcal{N}$ and any $\boldsymbol{\varphi_{-i}} \in \Phi_{-i}$, there exists $\sigma_i \in \Phi_i$ such that

$$J_i^{\ell}([\boldsymbol{\varphi_{-i}}, \sigma_i]) < \kappa_i^{\ell} - \zeta \quad \text{for all} \quad \ell \in \mathcal{L}.$$

Theorem 2.13 *Assume* **A1**, **A2** *and* **A4**. *Then, the CSG possesses a stationary weak correlated equilibrium.*

The proof is given in Sect. 4.

Remark 2.14 The existence of a weak correlated equilibrium in an unconstrained case was proved by Nowak [35] under additional integrability condition (2.3).

Remark 2.15 If ψ^* is a stationary weak correlated equilibrium in an $ARAT$ game, then $(\psi_1, \psi_2, ..., \psi_N)$ is a stationary Nash equilibrium in this game. Thus, Theorem 2.13 implies the main result of Dufour and Prieto-Rumeau [13], if the action sets are independent of the state. However, their proof is more direct in the sense that it is not based on an approximation by games with discrete state spaces. Instead, they directly apply a fixed point theorem. An extension to the case of action spaces depending on the state variable raises some additional technical issues.

3 Approximating Games with Countable State Spaces and Proofs of Theorems 2.3 and 2.8

In this section, we define a class of games that resemble stochastic games with a countable state space. Using them we can approximate the original game and apply the results on existence of stationary equilibria in discounted games with countably many states proved by Federgruen [15] (unconstrained case) and Jaśkiewicz and Nowak [28] (constrained case).

Let $\mathcal{C}(A)$ be the Banach space of all real-valued continuous functions on A endowed with the maximum norm $\|\cdot\|$. Let $\mathcal{C}_b = \{w_1, w_2, ...\}$ denote the countable dense subset in the ball $\{w \in \mathcal{C}(A) : \|w\| \le b\}$ in $\mathcal{C}(A)$, where $b \ge |c_i^{\ell}(x, \boldsymbol{a})|$ for all $i \in \mathcal{N}$, $\ell \in \mathcal{L}_0$, $(x, \boldsymbol{a}) \in \mathbb{K}$.

We write $\mathcal{L}^1$ to denote the Banach space $\mathcal{L}^1(X, \mathcal{F}, \mu)$ of all absolutely integrable real-valued measurable functions on X with the norm

$$\|v\|_1 = \int_X |v(y)|\mu(dy), \quad v \in \mathcal{L}^1.$$

Let $\mathcal{C}(A, \mathcal{L}^1)$ be the space of all $\mathcal{L}^1$-valued continuous functions on A with the norm

$$\|\lambda\|_c = \max_{\boldsymbol{a}\in A} \int_X |\lambda(y, \boldsymbol{a})|\mu(dy).$$

Here an element of $\mathcal{C}(A, \mathcal{L}^1)$ is written as a product measurable function $\lambda : X \times A \to \mathbb{R}$ such that $\lambda(\cdot, \boldsymbol{a}) \in \mathcal{L}^1$ for each $\boldsymbol{a} \in A$ and

$$\|\lambda(\cdot, \boldsymbol{a}^n) - \lambda(\cdot, \boldsymbol{a})\|_1 = \int_X |\lambda(y, \boldsymbol{a}^n) - \lambda(y, \boldsymbol{a})|\mu(dy) \to 0 \quad \text{as} \quad \boldsymbol{a}^n \to \boldsymbol{a},\ n \to \infty.$$

By Lemma 3.99 in [1], the space $\mathcal{C}(A, \mathcal{L}^1)$ is separable. Assumption A2 implies that $\mathcal{D} := \{\delta(x, \cdot, \cdot) : x \in X\} \subset \mathcal{C}(A, \mathcal{L}^1)$ is also a separable space when endowed with the relative topology. Therefore, there exists a subset $\{x_k : k \in \mathbb{N}\}$ of the state space X such that the set $\{\delta(x_k, \cdot, \cdot) : k \in \mathbb{N}\}$ is dense in $\mathcal{D}$.

For any player $i \in \mathcal{N}$, and positive integers $m_{i\ell}$, $\ell \in \mathcal{L}_0$, we put $\overline{m}_i = (m_{i0}, m_{i1}, ..., m_{iL})$. Then, given any $\gamma > 0$, we define $B^\gamma(i, \overline{m}_i)$ as the set of all states $x \in X$ such that

$$\sum_{\ell=0}^{L} \|c_i^\ell(x, \cdot) - w_{m_{i\ell}}\| < \gamma. \tag{3.1}$$

For any $k \in \mathbb{N}$, let

$$\begin{aligned} B_k^\gamma &:= \{x \in X : \|\delta(x, \cdot, \cdot) - \delta(x_k, \cdot, \cdot)\|_c \\ &= \max_{\boldsymbol{a}\in A} \int_X |\delta(x, y, \boldsymbol{a}) - \delta(x_k, y, \boldsymbol{a})|\mu(dy) < \gamma\}. \end{aligned} \tag{3.2}$$

It is obvious that the sets B_k^γ and $B^\gamma(i, \overline{m}_i)$ belong to $\mathcal{F}$ and the union of all sets

$$B_k^\gamma \cap B^\gamma(1, \overline{m}_1) \cap \ldots \cap B^\gamma(N, \overline{m}_N)$$

is the whole state space X. Indeed, if $x \in X$, then there exists $k \in \mathbb{N}$ such that $x \in B_k^\gamma$ and, for any player $i \in \mathcal{N}$, there exist functions $w_{m_{i\ell}} \in \mathcal{C}_b$, and thus $\overline{m}_i$ such that (3.1) holds.

Let ξ be a fixed one-to-one correspondence between the sets $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}^{N(L+1)}$. Assuming that $j \in \mathbb{N}$ and $\xi(j) = (k, \overline{m}_1, ..., \overline{m}_N)$, we put

$$Y_j^\gamma := B_k^\gamma \cap B^\gamma(1, \overline{m}_1) \cap \ldots \cap B^\gamma(N, \overline{m}_N).$$

We can assume without loss of generality that $Y_1^\gamma \neq \emptyset$. Next, we set $X_1^\gamma = Y_1^\gamma$ and

$$X_\tau^\gamma = Y_\tau^\gamma - \bigcup_{t<\tau} X_t^\gamma, \quad \text{for} \quad \tau \in \mathbb{N} \setminus \{1\}.$$

Omitting empty sets X_τ^γ we obtain a subset $\mathbb{N}_0 \subset \mathbb{N}$ such that

$$\mathcal{P}^\gamma = \{X_j^\gamma : j \in \mathbb{N}_0\}$$

is a measurable partition of the state space X. Choose any $n \in \mathbb{N}_0$. Then, $\xi(n)$ is a unique sequence in $\mathbb{N} \times \mathbb{N}^{N(L+1)}$ that depends on n and, therefore, we can write $\xi(n) = (k^n, \overline{m}_1^n, ..., \overline{m}_N^n)$ where $\overline{m}_i^n = (m_{i0}^n, m_{i1}^n, ..., m_{iL}^n)$, $i \in \mathcal{N}$. Next, for each $x \in X_n^\gamma$, we define

$$\delta^\gamma(x, y, \boldsymbol{a}) := \delta(x_{k^n}, y, \boldsymbol{a}) \quad \text{for } y \in X \quad \text{and} \quad c_i^{\ell,\gamma}(x, \boldsymbol{a}) := w_{m_{i,\ell}^n}(\boldsymbol{a}) \\ \text{for all } \ell \in \mathcal{L}_0, \ i \in \mathcal{N}. \tag{3.3}$$

From (3.1), (3.2) and (3.3), it follows that for each $n \in \mathbb{N}_0$ and $x \in X_n^\gamma$, we have

$$\|c_i^\ell(x, \cdot) - c_i^{\ell,\gamma}(x, \cdot)\| < \gamma \quad \text{for all } \ell \in \mathcal{L}_0 \tag{3.4}$$

and

$$\|\delta(x, \cdot, \cdot) - \delta^\gamma(x, \cdot, \cdot)\|_c = \max_{\boldsymbol{a}\in A} \int_X |\delta(x, y, \boldsymbol{a}) - \delta^\gamma(x, y, \boldsymbol{a})|\mu(dy) < \gamma. \tag{3.5}$$

The original game defined in Sect. 2 is now denoted by $\mathcal{G}$. We use $\mathcal{G}^\gamma$ to denote the game, where the cost functions are $c_i^{\ell,\gamma}$, $\ell \in \mathcal{L}_0$ and $i \in \mathcal{N}$, and the transition probability is

$$p^\gamma(B|x, \boldsymbol{a}) = \int_X \delta^\gamma(x, y, \boldsymbol{a})\mu(dy), \quad B \in \mathcal{F}.$$

Note that $c_i^{\ell,\gamma}(x, \boldsymbol{a})$ and $p^\gamma(B|x, \boldsymbol{a})$ are constant functions of x on every set X_n^γ.

The discounted expected costs in the game $\mathcal{G}^\gamma$ under a multi-strategy $\boldsymbol{\pi} \in \Pi$ are denoted by

$$J_i^{\ell,\gamma}(\boldsymbol{\pi})(x) \quad \text{and} \quad J_i^{\ell,\gamma}(\boldsymbol{\pi}) = \int_X J_i^{\ell,\gamma}(\boldsymbol{\pi})(x)\eta(dx).$$

Let

$$\epsilon(\gamma) := \frac{\gamma(1 - \alpha + b\alpha)}{1 - \alpha}. \tag{3.6}$$

From (3.4), (3.5) and Lemma 4.4 in [34], we conclude the following auxiliary result.

Lemma 3.1 *For each $i \in \mathcal{N}$ and $\ell \in \mathcal{L}_0$, we have*

$$\sup_{x\in X} \sup_{\boldsymbol{\pi}\in\Pi} |J_i^\ell(\boldsymbol{\pi})(x) - J_i^{\ell,\gamma}(\boldsymbol{\pi})(x)| \le \epsilon(\gamma).$$

With $\mathcal{G}^\gamma$ we associate a stochastic game $\mathcal{G}_c^\gamma$ with the countable state space $\mathbb{N}_0 \subset \mathbb{N}$, the costs given by

$$\widehat{c}_i^{\ell,\gamma}(n, \boldsymbol{a}) := c_i^{\ell,\gamma}(x, \boldsymbol{a}), \quad x \in X_n^\gamma, \quad n \in \mathbb{N}_0, \quad \boldsymbol{a} \in A, \tag{3.7}$$

and transitions defined as

$$\widehat{p}^\gamma(\tau|n, \boldsymbol{a}) := \delta^\gamma(X_\tau^\gamma|x, \boldsymbol{a}), \quad x \in X_n^\gamma, \quad n, \tau \in \mathbb{N}_0, \quad \boldsymbol{a} \in A. \tag{3.8}$$

Note that the right-hand sides in (3.7) and (3.8) are independent of x in X_n^γ and thus the costs and transitions above are well-defined. A stationary strategy for player $i \in \mathcal{N}$ in the game $\mathcal{G}_c^\gamma$ is a transition probability f_i from $\mathbb{N}_0$ to A_i. The set of all stationary strategies for player $i \in \mathcal{N}$ in this game is denoted by F_i. We put $F := \prod_{i\in\mathcal{N}} F_i$.

The expected discounted costs in the game $\mathcal{G}_c^\gamma$ under stationary multi-strategy $\boldsymbol{\pi}$ are denoted by

$$\widehat{J}_i^{\ell,\gamma}(\boldsymbol{\pi})(n),\ n \in \mathbb{N}_0, \quad \text{and} \quad \widehat{J}_i^{\ell,\gamma}(\boldsymbol{\pi}) = \sum_{n\in\mathbb{N}_0} \widehat{J}_i^{\ell,\gamma}(\boldsymbol{\pi})(n)\eta(X_n^\gamma).$$

Let Φ_i^γ be the set of all piecewise constant stationary strategies of player $i \in \mathcal{N}$ in the game $\mathcal{G}^\gamma$. A strategy $\varphi_i \in \Phi_i^\gamma$, if, for each $n \in \mathbb{N}_0$, there exists a probability measure ν_n on A_i such that $\varphi_i(da_i|x) = \nu_n(da_i)$ for all $x \in X_n^\gamma$. We put $\Phi^\gamma = \prod_{i\in\mathcal{N}} \Phi_i^\gamma$.

Let $\boldsymbol{f} = (f_1, ..., f_N) \in F$ and $\boldsymbol{\varphi} = (\varphi_1, ..., \varphi_N) \in \Phi^\gamma$ be such that

$$\varphi_i(da_i|x) = f_i(da_i|n) \quad \text{for all} \quad i \in \mathcal{N},\ n \in \mathbb{N}_0,\ x \in X_n^\gamma. \tag{3.9}$$

Then, for each $i \in \mathcal{N}$, $\ell \in \mathcal{L}_0$, $n \in \mathbb{N}$ and $x \in X_n^\gamma$,

$$J_i^{\ell,\gamma}(\boldsymbol{\varphi})(x) = \widehat{J}_i^{\ell,\gamma}(\boldsymbol{f})(n) \tag{3.10}$$

and

$$J_i^{\ell,\gamma}(\boldsymbol{\varphi}) = \widehat{J}_i^{\ell,\gamma}(\boldsymbol{f}). \tag{3.11}$$

Equations (3.10) and (3.11) show that $\mathcal{G}^\gamma$ with the strategy sets Φ_i^γ can be recognised as a game with a countable state space. This observation plays an important role in the proof, because we can apply a result for games on countable state spaces.

Proof of Theorem 2.3 Let $\varepsilon > 0$ and $i \in \mathcal{N}$. Choose $\gamma > 0$ in (3.6) such that $\epsilon(\gamma) < \varepsilon/2$. By Assumption A3 and Remark 2.2 we imply that for any multi-strategy $\boldsymbol{\varphi} \in \Phi^\gamma$ there exists $\sigma_i \in \Phi_i$ such that

$$J_i^\ell([\boldsymbol{\varphi}_{-\boldsymbol{i}}, \sigma_i]) \le \kappa_i^\ell \quad \text{for all } \ell \in \mathcal{L}. \tag{3.12}$$

By Lemma 7.1 in Appendix, there exists a piecewise constant Markov strategy $\overline{\pi}_i$ such that

$$J_i^{\ell,\gamma}([\boldsymbol{\varphi_{-i}}, \sigma_i]) = J_i^{\ell,\gamma}([\boldsymbol{\varphi_{-i}}, \overline{\pi}_i])$$

for all $\ell \in \mathcal{L}_0$. By Lemma 3.1 and (3.12) we conclude that

$$J_i^{\ell,\gamma}([\boldsymbol{\varphi_{-i}}, \overline{\pi}_i]) < \kappa_i^\ell + \frac{\varepsilon}{2} \quad \text{for all } \ell \in \mathcal{L}.$$

This means that the approximating game $\mathcal{G}^\gamma$ satisfies the Slater condition with the constants $\kappa_i^\ell + \frac{\varepsilon}{2}$, $\ell \in \mathcal{L}$. Note that the constraint constants in $\mathcal{G}^\gamma$ are also equal $\kappa_i^\ell + \frac{\varepsilon}{2}$, $\ell \in \mathcal{L}$. Therefore, the associated game $\mathcal{G}_c^\gamma$ also satisfies the Slater condition with the same constants $\kappa_i^\ell + \frac{\varepsilon}{2}$, $\ell \in \mathcal{L}$. Making use of Corollary 2 in [28], we infer that the game $\mathcal{G}_c^\gamma$ possesses a stationary Nash equilibrium $\boldsymbol{f}^* = (f_1^*, ..., f_N^*)$. Define $\boldsymbol{\varphi}^* = (\varphi_1^*, ..., \varphi_N^*) \in \Phi^\gamma$ as in (3.9) with $\boldsymbol{\varphi} = \boldsymbol{\varphi}^*$ and $f = f^*$. Then,

$$J_i^{0,\gamma}(\boldsymbol{\varphi}^*) \le J_i^{0,\gamma}([\boldsymbol{\varphi^*_{-i}}, \hat{\pi}_i])$$

for any piecewise constant strategy $\hat{\pi}_i$ such that

$$J_i^{\ell,\gamma}([\boldsymbol{\varphi^*_{-i}}, \hat{\pi}_i]) \le \kappa_i^\ell + \frac{\varepsilon}{2} \quad \text{for all } \ell \in \mathcal{L}.$$

We now show that $\boldsymbol{\varphi}^*$ is an approximate equilibrium in the original game. Note that for every player $i \in \mathcal{N}$

$$J_i^{\ell,\gamma}(\boldsymbol{\varphi^*}) = J_i^{\ell,\gamma}([\boldsymbol{\varphi^*_{-i}}, \varphi_i^*]) \le \kappa_i^\ell + \frac{\varepsilon}{2} \quad \text{for all } \ell \in \mathcal{L}.$$

Hence, for every player $i \in \mathcal{N}$

$$J_i^\ell(\boldsymbol{\varphi^*}) \le \kappa_i^\ell + \varepsilon \quad \text{for all } \ell \in \mathcal{L},$$

i.e., condition (2.1) holds. Consider any feasible strategy $\pi_i \in \Delta_i(\boldsymbol{\varphi^*_{-i}})$, i.e.,

$$J_i^\ell([\boldsymbol{\varphi^*_{-i}}, \pi_i]) \le \kappa_i^\ell, \quad \text{for all} \quad \ell \in \mathcal{L}. \tag{3.13}$$

Applying Remark 2.2, we deduce that there exists a strategy $\sigma_i \in \Phi_i$ such that

$$J_i^\ell([\boldsymbol{\varphi^*_{-i}}, \pi_i]) = J_i^\ell([\boldsymbol{\varphi^*_{-i}}, \sigma_i]) \quad \text{for all} \quad \ell \in \mathcal{L}_0. \tag{3.14}$$

Then, by Lemma 7.1 in Appendix, there exists a piecewise constant Markov strategy $\overline{\pi}_i$ such that

$$J_i^{\ell,\gamma}([\boldsymbol{\varphi^*_{-i}}, \sigma_i]) = J_i^{\ell,\gamma}([\boldsymbol{\varphi^*_{-i}}, \overline{\pi}_i]) \quad \text{for all } \ell \in \mathcal{L}_0. \tag{3.15}$$

Moreover, by (3.15), Lemma 3.1, (3.14) and (3.13), for every $\ell \in \mathcal{L}$, we have

$$J_i^{\ell,\gamma}([\boldsymbol{\varphi}^*_{-\boldsymbol{i}}, \overline{\pi}_i]) \le J_i^{\ell}([\boldsymbol{\varphi}^*_{-\boldsymbol{i}}, \sigma_i]) + \frac{\varepsilon}{2} = J_i^{\ell}([\boldsymbol{\varphi}^*_{-\boldsymbol{i}}, \pi_i]) + \frac{\varepsilon}{2} \le \kappa_i^{\ell} + \frac{\varepsilon}{2}.$$

In other words, $\overline{\pi}_i$ is a feasible strategy in $\mathcal{G}^{\gamma}$. Therefore, by Lemma 3.1, (3.15) and (3.14), we infer

$$\begin{aligned} J_i^{0}(\boldsymbol{\varphi}^*) &\le J_i^{0,\gamma}(\boldsymbol{\varphi}^*) + \frac{\varepsilon}{2} \le J_i^{0,\gamma}([\boldsymbol{\varphi}^*_{-\boldsymbol{i}}, \overline{\pi}_i]) + \frac{\varepsilon}{2} = J_i^{0,\gamma}([\boldsymbol{\varphi}^*_{-\boldsymbol{i}}, \sigma_i]) + \frac{\varepsilon}{2} \\ &< J_i^{0}([\boldsymbol{\varphi}^*_{-\boldsymbol{i}}, \sigma_i]) + \varepsilon = J_i^{0}([\boldsymbol{\varphi}^*_{-\boldsymbol{i}}, \pi_i]) + \varepsilon. \end{aligned}$$

This fact together with (3.13) implies that (2.2) holds. □

Proof of Theorem 2.8 Let $\varepsilon > 0$ be fixed. Choose $\gamma > 0$ in (3.6) such that $\epsilon(\gamma) < \varepsilon/2$. By Theorem 2.3 in [15], the game $\mathcal{G}_c^{\gamma}$ has a stationary equilibrium $\boldsymbol{f}^* = (f_1^*, ..., f_N^*)$. Define $\boldsymbol{\varphi}^* = (\varphi_1^*, ..., \varphi_N^*) \in \Phi^{\gamma}$ as in the proof of Theorem 2.3. Then we have

$$J_i^{0,\gamma}(\boldsymbol{\varphi}^*)(x) = \inf_{\phi_i \in \Phi_i^{\gamma}} J_i^{0,\gamma}([\boldsymbol{\varphi}^*_{-\boldsymbol{i}}, \phi_i])(x), \quad i \in \mathcal{N}, \ x \in X. \tag{3.16}$$

As in Lemma 4.1 in [34], we can prove that

$$\inf_{\phi_i \in \Phi_i^{\gamma}} J_i^{0,\gamma}([\boldsymbol{\varphi}^*_{-\boldsymbol{i}}, \phi_i])(x) = \inf_{\phi_i \in \Phi_i} J_i^{0,\gamma}([\boldsymbol{\varphi}^*_{-\boldsymbol{i}}, \phi_i])(x), \quad i \in \mathcal{N}, \ x \in X. \tag{3.17}$$

By (3.16) and (3.17), we get

$$J_i^{0,\gamma}(\boldsymbol{\varphi}^*)(x) = \inf_{\phi_i \in \Phi_i} J_i^{0,\gamma}([\boldsymbol{\varphi}^*_{-\boldsymbol{i}}, \phi_i])(x), \quad i \in \mathcal{N}, \ x \in X.$$

This equality and Lemma 3.1 imply that

$$J_i^{0}(\boldsymbol{\varphi}^*)(x) - \varepsilon \le \inf_{\phi_i \in \Phi_i} J_i^{0}([\boldsymbol{\varphi}^*_{-\boldsymbol{i}}, \phi_i])(x), \quad i \in \mathcal{N}, \ x \in X. \tag{3.18}$$

By standard methods in discounted dynamic programming [8, 34], we have

$$\inf_{\phi_i \in \Phi_i} J_i^{0}([\boldsymbol{\varphi}^*_{-\boldsymbol{i}}, \phi_i])(x) = \inf_{\sigma_i \in \Pi_i} J_i^{0}([\boldsymbol{\varphi}^*_{-\boldsymbol{i}}, \sigma_i])(x), \quad i \in \mathcal{N}, \ x \in X.$$

This fact and (3.18) imply that

$$J_i^{0}(\boldsymbol{\varphi}^*)(x) - \varepsilon \le \inf_{\sigma_i \in \Pi_i} J_i^{0}([\boldsymbol{\varphi}^*_{-\boldsymbol{i}}, \sigma_i])(x), \quad i \in \mathcal{N}, \ x \in X,$$

which completes the proof. □

Remark 3.2 The proof of Theorem 2.8 is similar to that of Theorem 3.1 in [34], but it has one important change implying that the restrictive condition (2.3) can be dropped.

4 Young Measures and the Proof of Theorem 2.13

Let $\vartheta := (\eta+\mu)/2$. A function $c : \mathbb{K} \to \mathbb{R}$ is Carathéodory, if it is product measurable on $\mathbb{K}$, $c(x, \cdot)$ is continuous on A for each $x \in X$ and

$$\int_X \max_{\boldsymbol{a}\in A} |c(x, \boldsymbol{a})|\vartheta(dx) < \infty.$$

Let Ψ^ϑ be the space of all ϑ-equivalence classes of functions in Ψ. The elements of Ψ^ϑ are called Young measures. Note that the expected discounted cost functionals are well-defined for all elements of Ψ^ϑ. More precisely, if $\psi^\vartheta \in \Psi^\vartheta$, then $J_i^\ell(\psi)$ is the same for all representatives ψ of ψ^ϑ in Ψ and we can understand $J_i^\ell(\psi^\vartheta)$ as $J_i^\ell(\psi)$. We shall identify in notation ψ^ϑ with its representative ψ and omit the superscript ϑ.

We assume that the space Ψ^ϑ is endowed with the weak* topology. Since $\mathcal{F}$ is countably generated, Ψ^ϑ is metrisable. Moreover, since the set A is compact, Ψ^ϑ is a *compact convex* subset of a locally convex linear topological space. For a detailed discussion of these issues consult with [7] or Chapter 3 in [19]. Here, we recall that $\psi^n \to^* \psi^0$ in Ψ^ϑ as $n \to \infty$ if and only if for every Carathéodory function $c : \mathbb{K} \to \mathbb{R}$, we have

$$\lim_{n\to\infty} \int_X \int_A c(x, \boldsymbol{a})\psi^n(d\boldsymbol{a}|x)\vartheta(dx) = \int_X \int_A c(x, \boldsymbol{a})\psi^0(d\boldsymbol{a}|x)\vartheta(dx).$$

We now choose $\varepsilon_n > 0$ such that $\varepsilon_n \searrow 0$ as $n \to \infty$ and define

$$\gamma_n := \frac{\varepsilon_n(1-\alpha)}{(1-\alpha+b\alpha)}. \tag{4.1}$$

In other words, $\epsilon(\gamma_n) = \varepsilon_n$ or $\gamma_n = \epsilon^{-1}(\varepsilon_n)$. From Theorem 2.3, it follows that there exists a profile of stationary piecewise constant strategies

$$\boldsymbol{\psi^n} = (\psi_1^n, \ldots, \psi_N^n) \in \Phi^{\gamma_n},$$

which comprises an approximate equilibrium in the CSG for ε_n and at the same time an equilibrium in the corresponding constrained game $\mathcal{G}^{\gamma_n}$ with γ_n as in (4.1) and the constraint constants $\kappa_i^\ell + \frac{\varepsilon_n}{2}$.

Define the product measure on A, for every $x \in X$ and $n \in \mathbb{N}$, as

$$\psi^n(\cdot|x) := \psi_1^n(\cdot|x) \otimes \ldots \otimes \psi_N^n(\cdot|x). \tag{4.2}$$

We use ψ^n to denote the class in Ψ^ϑ whose representative is this transition probability. Without loss of generality, we may assume that ψ^n converges in the weak* topology to some $\psi^* \in \Psi^\vartheta$ as $n \to \infty$.

We shall need the following results. The first one is a consequence of Lemma 3.1 and the fact that $J_i^{\ell,\gamma_n}(\boldsymbol{\psi^n}) = J_i^{\ell,\gamma_n}(\psi^n)$ and $J_i^\ell(\boldsymbol{\psi^n}) = J_i^\ell(\psi^n)$.

Lemma 4.1 *For each $i \in \mathcal{N}$ and $\ell \in \mathcal{L}_0$, we have*

$$\sup_{\psi\in\Psi} |J_i^\ell(\psi) - J_i^{\ell,\gamma_n}(\psi)| \le \varepsilon_n,$$
$$\sup_{\psi_{-i}\in\Psi_{-i}} \sup_{\pi_i\in\Pi_i} |J_i^\ell([\psi_{-i},\pi_i]) - J_i^{\ell,\gamma_n}([\psi_{-i},\pi_i])| \le \varepsilon_n,$$

where γ_n is as in (4.1).

Lemma 4.2 *If $n \to \infty$, then for any $\ell \in \mathcal{L}_0$*
(a) $J_i^{\ell,\gamma_n}(\psi^n) \to J_i^\ell(\psi^*)$,
(b) $J_i^{\ell,\gamma_n}([\psi_{-i}^n,\phi_i]) \to J_i^\ell([\psi_{-i}^*,\phi_i])$ for any $\phi_i \in \Phi_i$.

Proof For part (a) we first use the triangle inequality

$$|J_i^{\ell,\gamma_n}(\psi^n) - J_i^\ell(\psi^*)| \le |J_i^{\ell,\gamma_n}(\psi^n) - J_i^\ell(\psi^n)| + |J_i^\ell(\psi^n) - J_i^\ell(\psi^*)|.$$

The first term on the right-hand side converges to 0 by Lemma 4.1 and the definition of ψ^n, whereas the convergence to 0 of the the second term follows from Lemma 4.1 in [27] and the fact that $|J_i^\ell(\cdot)| \le b$ for every $i \in \mathcal{N}$ and $\ell \in \mathcal{L}_0$. Part (b) is proved as point (a) by using the Fubini theorem and noting that the elements in Ψ^ϑ induced by ψ_{-i}^n in (4.2) and ϕ_i converge in the weak* sense to the element of Ψ^ϑ induced by ψ_{-i}^* and ϕ_i. □ □

Let $i \in \mathcal{N}$. Consider a Markov decision process with player i as a decision maker and the transition probability

$$q^{\gamma_n}(dy|x,a_i) = \int_{A_{-i}} p^{\gamma_n}(dy|x,[\boldsymbol{a}_{-\boldsymbol{i}},a_i])\psi_{-i}^n(d\boldsymbol{a}_{-\boldsymbol{i}}|x), \quad (x,a_i)\in\mathbb{K}_i.$$

Let 1_D be the indicator of the set $D \subset X \times A$. The associated occupation measure, when player i uses a stationary strategy $\varphi_i \in \Phi_i$ is defined as follows

$$\theta_{\varphi_i}^{\gamma_n}(B\times C) = (1-\alpha)\sum_{t=1}^{\infty}\alpha^{t-1}\mathcal{E}_\eta^{\varphi_i}1_{B\times C}(x^t,a_i^t) \tag{4.3}$$

for any $B \in \mathcal{F}$ and a Borel set C in A_i. We use the symbol $\mathcal{E}_\eta^{\varphi_i}$ to denote the expectation operator corresponding to the unique probability measure induced by $\varphi_i \in \Phi_i$, the initial distribution η and the transition probability q^{γ_n}. For $\ell \in \mathcal{L}_0$, $x \in X$ and $a_i \in A_i$, set

$$c_i^{\ell,\gamma_n}(x,a_i) := \int_{A_{-i}} c_i^{\ell,\gamma_n}(x,[\boldsymbol{a}_{-\boldsymbol{i}},a_i])\psi_{-i}^n(d\boldsymbol{a}_{-\boldsymbol{i}}|x).$$

Proof of Theorem 2.13 Observe that Assumption A4 implies A3. We consider the weak* limit $\psi^* \in \Psi^\vartheta$ mentioned above and denote its representative in Ψ by the same letter.

We shall show that ψ^* is a weak correlated equilibrium. By Theorem 2.3, $J_i^\ell(\boldsymbol{\psi^n}) = J_i^\ell(\psi^n) \le \kappa_i^\ell + \varepsilon_n$ for all $i \in \mathcal{N}$ and $\ell \in \mathcal{L}$. Using Lemma 4.2(a), we conclude that

$$J_i^\ell(\psi^*) = \lim_{n\to\infty} J_i^\ell(\psi^n) \le \kappa_i^\ell, \quad i \in \mathcal{N},\ \ell \in \mathcal{L},$$

i.e., ψ^* is feasible.

Take (if possible) any feasible strategy in the CSG for player $i \in \mathcal{N}$, i.e., $\pi_i \in \Pi_i$ such that

$$J_i^\ell([\psi_{-i}^*, \pi_i]) \le \kappa_i^\ell \quad \text{for all} \quad \ell \in \mathcal{L}.$$

By Remark 2.2 there exists a strategy $\phi_i \in \Phi_i$ such that

$$J_i^\ell([\psi_{-i}^*, \pi_i]) = J_i^\ell([\psi_{-i}^*, \phi_i]) \quad \text{for all } \ell \in \mathcal{L}_0.$$

1° Assume first that

$$J_i^\ell([\psi_{-i}^*, \pi_i]) = J_i^\ell([\psi_{-i}^*, \phi_i]) < \kappa_i^\ell \quad \text{for all} \quad \ell \in \mathcal{L}. \tag{4.4}$$

From this inequality and Lemma 4.2(b), we infer that there exists $N_1 \in \mathbb{N}$ such that

$$J_i^{\ell,\gamma_n}([\psi_{-i}^n, \phi_i]) < \kappa_i^\ell \quad \text{for all} \quad \ell \in \mathcal{L} \quad \text{and} \quad n \ge N_1.$$

For every $n \ge N_1$ and Lemma 7.1 in Appendix we conclude the existence of a piecewise constant Markov strategy $\overline{\pi}_i$ (that may depend on n) such that

$$J_i^{\ell,\gamma_n}([\psi_{-i}^n, \phi_i]) = J_i^{\ell,\gamma_n}([\psi_{-i}^n, \overline{\pi}_i]) \quad \text{for all} \quad \ell \in \mathcal{L}_0.$$

Hence, it must hold

$$J_i^{0,\gamma_n}(\psi^n) \le J_i^{0,\gamma_n}([\psi_{-i}^n, \overline{\pi}_i]) = J_i^{0,\gamma_n}([\psi_{-i}^n, \phi_i]).$$

In other words, for every $n \ge N_1$ we have

$$J_i^{0,\gamma_n}(\psi^n) \le J_i^{0,\gamma_n}([\psi_{-i}^n, \phi_i]).$$

Letting $n \to \infty$ and making use of Lemma 4.2, we infer

$$J_i^0(\psi^*) \le J_i^0([\psi_{-i}^*, \phi_i]) = J_i^0([\psi_{-i}^*, \pi_i])$$

for any feasible strategy $\pi_i \in \Pi_i$ such that (4.4) holds.

2° Assume now that there is player $i \in \mathcal{N}$ and an index $\ell_0 \in \mathcal{L}$ such that

$$J_i^{\ell_0}([\psi_{-i}^*, \pi_i]) = J_i^{\ell_0}([\psi_{-i}^*, \phi_i]) = \kappa_i^{\ell_0}. \tag{4.5}$$

From the proof of Lemma 4.2(b), it follows that there exists a sequence $e_n \to 0$ as $n \to \infty$, $e_n > 0$, such that

$$J_i^\ell([\psi_{-i}^n, \phi_i]) \le J_i^\ell([\psi_{-i}^*, \phi_i]) + e_n \le \kappa_i^\ell + e_n \qquad \text{for all} \quad \ell \in \mathcal{L}.$$

By Remark 2.12, we can find $\zeta > 0$ such that for every $n \in \mathbb{N}$ there exists a strategy $\sigma_i^n \in \Phi_i$ such that

$$J_i^\ell([\psi_{-i}^n, \sigma_i^n]) < \kappa_i^\ell - \zeta \quad \text{for all} \quad \ell \in \mathcal{L}.$$

Hence, by Lemma 4.1, we conclude

$$J_i^{\ell,\gamma_n}([\psi_{-i}^n, \phi_i]) - \varepsilon_n \le J_i^\ell([\psi_{-i}^n, \phi_i]) \le \kappa_i^\ell + e_n \quad \text{for all} \quad \ell \in \mathcal{L}$$

and

$$J_i^{\ell,\gamma_n}([\psi_{-i}^n, \sigma_i^n]) - \varepsilon_n \le J_i^\ell([\psi_{-i}^n, \sigma_i^n]) < \kappa_i^\ell - \zeta \quad \text{for all} \quad \ell \in \mathcal{L}.$$

Let $N_2 \in \mathbb{N}$ be such that $\varepsilon_{N_2} < \zeta$ for all $n \ge N_2$ set

$$\xi_n := \frac{\varepsilon_n + e_n}{\zeta + e_n}$$

and observe that $\xi_n \to 0$ as $n \to \infty$ and $\xi_n \in (0, 1)$ for all $n > N_2$. Let $\theta_{\phi_i}^{\gamma_n}$ and $\theta_{\sigma_i^n}^{\gamma_n}$ be two occupation measures defined as in (4.3). By Proposition 3.9 in [13], we define a sequence of occupation measures as follows

$$\theta^n := \xi_n \theta_{\sigma_i^n}^{\gamma_n} + (1 - \xi_n)\theta_{\phi_i}^{\gamma_n}.$$

Then, for all $\ell \in \mathcal{L}_0$ it holds

$$\int_{X \times A_i} c_i^{\ell,\gamma_n}(x, a_i)\theta^n(dx \times da_i) = \xi_n J_i^{\ell,\gamma_n}([\psi_{-i}^n, \sigma_i^n]) + (1 - \xi_n) J_i^{\ell,\gamma_n}([\psi_{-i}^n, \phi_i]). \tag{4.6}$$

Hence, for $n \ge N_2$ and all $\ell \in \mathcal{L}$, from (4.6), we have

$$\begin{aligned}\int_{X \times A_i} c_i^{\ell,\gamma_n}(x, a_i)\theta^n(dx \times da_i) &\le \xi_n(\kappa_i^\ell + \varepsilon_n - \zeta) + (1 - \xi_n)(\kappa_i^\ell + \varepsilon_n + e_n) \\ &= -\xi_n(e_n + \zeta) + \kappa_i^\ell + \varepsilon_n + e_n \le \kappa_i^\ell < \kappa_i^\ell + \frac{\varepsilon_n}{2}.\end{aligned} \tag{4.7}$$

By Lemma 2.3 in [13] or Theorem 3.2 in [19] for every $n \ge N_2$, there exists a stationary strategy $\chi_i^n \in \Phi_i$ such that θ^n can be written as in (4.3) with $\mathcal{E}_\eta^{\varphi_i}$ replaced by $\mathcal{E}_\eta^{\chi_i^n}$. In other words $\theta^n = \theta^{\gamma_n}_{\chi^n_i}$. Therefore, for all $\ell \in \mathcal{L}_0$, we obtain

$$\int_{X\times A_i} c_i^{\ell,\gamma_n}(x, a_i)\theta^n(dx \times da_i) = J_i^{\ell,\gamma_n}([\psi^n_{-i}, \chi^n_i]). \tag{4.8}$$

By Lemma 7.1 in Appendix for every $n \in \mathbb{N}$ there exists a piecewise constant Markov strategy $\overline{\pi}^n_i$ such that

$$J_i^{\ell,\gamma_n}([\psi^n_{-i}, \chi^n_i]) = J_i^{\ell,\gamma_n}([\psi^n_{-i}, \overline{\pi}^n_i]) \quad \text{for all} \quad \ell \in \mathcal{L}_0.$$

By (4.7) and (4.8)

$$J_i^{\ell,\gamma_n}([\psi^n_{-i}, \chi^n_i]) \leq \kappa_i^\ell < \kappa_i^\ell + \frac{\varepsilon_n}{2} \qquad \text{for all} \quad \ell \in \mathcal{L}.$$

Hence, it must hold

$$J_i^{0,\gamma_n}(\psi^n) \leq J_i^{0,\gamma_n}([\psi^n_{-i}, \overline{\pi}^n_i]) = J_i^{0,\gamma_n}([\psi^n_{-i}, \chi^n_i]). \tag{4.9}$$

We know that

$$J_i^{\ell,\gamma_n}([\psi^n_{-i}, \chi^n_i]) = \xi_n J_i^{\ell,\gamma_n}([\psi^n_{-i}, \sigma^n_i]) + (1-\xi_n) J_i^{\ell,\gamma_n}([\psi^n_{-i}, \phi_i]).$$

Therefore, by Lemma 4.2(b) and (4.8), we get

$$\lim_{n\to\infty} J_i^{\ell,\gamma_n}([\psi^n_{-i}, \chi^n_i]) = J_i^\ell([\psi^*_{-i}, \phi_i])$$

for all $\ell \in \mathcal{L}_0$. This fact, (4.9) and Lemma 4.2(a) yield that

$$J_i^0(\psi^*) \leq J_i^0([\psi^*_{-i}, \phi_i]) = J_i^0([\psi^*_{-i}, \pi_i])$$

for any feasible strategy $\pi_i \in \Pi_i$ for which (4.5) holds. □

Let Ψ_i^ϑ be the space of ϑ-equivalence classes of strategies in Φ_i endowed with the weak* topology. Clearly, Ψ_i^ϑ is a compact metric space. The cost functionals $J_i^\ell(\boldsymbol{\varphi})$, $\ell \in \mathcal{L}_0$ and $i \in \mathcal{N}$, are well defined for any profile $\boldsymbol{\varphi} = (\varphi_1, ..., \varphi_N) \in \widehat{\Psi}^\vartheta = \prod_{j\in\mathcal{N}} \Psi_j^\vartheta$.

Remark 4.3 From Example 3.16 in [14] based on Rademacher's functions, it follows that the weak* limit of the sequence of approximate equilibria in Theorem 2.13 need not be a stationary Nash equilibrium. The same example can be used to see that the cost functionals J_i^ℓ, $\ell \in \mathcal{L}_0$ and $i \in \mathcal{N}$, may be discontinuous on $\widehat{\Psi}^\vartheta$.

Consider the two-person game. It follows from Lemma 4.2 that $J_i^\ell(\varphi_1, \varphi_2)$ is separately continuous in φ_1 and φ_2. Therefore, the functions

$$R_1(\varphi_1) := \min_{\varphi_2\in\Psi_2^\vartheta} \max_{\ell\in\mathcal{L}} (J_1^\ell(\varphi_1, \varphi_2) - \kappa_1^\ell) \quad \text{and} \quad R_2(\varphi_2) := \min_{\varphi_1\in\Psi_1^\vartheta} \max_{\ell\in\mathcal{L}} (J_2^\ell(\varphi_1, \varphi_2) - \kappa_2^\ell)$$

are upper semicontinuous on Ψ_1^ϑ and Ψ_2^ϑ, respectively.

Remark 4.4 Consider a two-person game satisfying the standard Slater condition A5. Then, it follows

$$R_1(\varphi_1) < 0 \quad \text{and} \quad R_2(\varphi_2) < 0$$

for all $\varphi_1 \in \Psi_1^\vartheta$ and $\varphi_2 \in \Psi_2^\vartheta$. Since R_1 and R_2 are upper semicontinuous on the compact spaces Ψ_1^ϑ and Ψ_2^ϑ, respectively, we conclude that

$$\max_{\varphi_1 \in \Psi_1^\vartheta} R_1(\varphi_1) < 0 \quad \text{and} \quad \max_{\varphi_2 \in \Psi_2^\vartheta} R_2(\varphi_2) < 0. \tag{4.10}$$

Obviously, φ_1 and φ_2 in inequalities (4.10) can be understood as representatives of (denoted by the same letters) classes in Ψ_1^ϑ and Ψ_2^ϑ, respectively. Then, it is apparent that A5 implies A4 for the considered two-person game.

Since in the N-person $ARAT$ game the cost functionals are continuous on $\widehat{\Psi}^\vartheta$ with the product topology [13], A5 implies A4 in this case.

Finally,we note that in the countable state space case, the weak* topology on Ψ_i^ϑ is actually the topology of point-wise convergence and all cost functionals J_i^ℓ are continuous on the compact space $\widehat{\Psi}^\theta$ with the product topology. Therefore, the standard Slater condition A4 made in the literature for these games, see [3, 4, 28, 42], is equivalent to A5.

5 Non-existence of Stationary Equilibria in Discounted Constrained Games

In this section, we consider discounted stochastic games with the given initial state distribution η. If $c_i^\ell = 0$ and $\kappa_i^\ell = 1$ for all $i \in \mathcal{N}$ and $\ell \in \mathcal{L}$, then the game in this class is trivially constrained and Assumption A3 automatically holds. Our aim is to conclude from [29] that such a game may have no stationary Nash equilibrium. For this, we need the following fact.

Proposition 5.1 *Let* **A1** *and* **A2** *be satisfied and in addition let* $p(\cdot|x, \boldsymbol{a}) \ll \eta$ *for all* $(x, \boldsymbol{a}) \in \mathbb{K}$. *If* $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_N) \in \Phi$ *is a stationary Nash equilibrium in the discounted stochastic game with the initial state distribution* η, *i.e.,*

$$J_i^0(\boldsymbol{\varphi}) \le J_i^0([\boldsymbol{\varphi_{-i}}, \pi_i]) \tag{5.1}$$

for all $i \in \mathcal{N}$ *and* $\pi_i \in \Pi_i$, *then there exists a stationary Nash equilibrium* $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_N)$ *in the unconstrained stochastic game for all initial states, i.e.,*

$$J_i^0(\boldsymbol{\psi})(x) \le J_i^0([\boldsymbol{\psi_{-i}}, \pi_i])(x) \tag{5.2}$$

for all $i \in \mathcal{N}$, $\pi_i \in \Pi_i$ *and* $x \in X$. *Moreover,* $\varphi_i(da_i|x) = \psi_i(da_i|x)$ *for* η*-a.e.* $x \in X$ *and for all* $i \in \mathcal{N}$.

We start with necessary notation. Let $\boldsymbol{\phi} = (\phi_1, ..., \phi_N) \in \Phi$. Then

$$\phi(d\boldsymbol{a}|x) := \phi_1(da_1|x) \otimes \phi_2(da_2|x) \otimes \cdots \otimes \phi_N(da_N|x)$$

is the product measure on A determined by $\phi_i(da_i|x)$, $i = 1, 2, ..., N$. Recall that by $\phi_{-i}(d\boldsymbol{a_{-i}}|x)$ we denote the projection of $\phi(d\boldsymbol{a}|x)$ on A_{-i}. We put

$$c_i^0(x, \boldsymbol{\phi}) := \int_A c_i^0(x, \boldsymbol{a})\phi(d\boldsymbol{a}|x) \quad \text{and} \quad p(dy|x, \boldsymbol{\phi}) := \int_A p(dy|x, \boldsymbol{a})\phi(d\boldsymbol{a}|x).$$

If $\sigma_i \in \Phi_i$, then

$$c_i^0(x, [\boldsymbol{\phi_{-i}}, \sigma_i]) := \int_{A_i}\int_{A_{-i}} c_i^0(x, [\boldsymbol{a_{-i}}, a_i])\phi_{-i}(d\boldsymbol{a_{-i}}|x)\sigma_i(da_i|x),$$
$$p(dy|x, [\boldsymbol{\phi_{-i}}, \sigma_i]) := \int_{A_i}\int_{A_{-i}} p(dy|x, [\boldsymbol{a_{-i}}, a_i])\phi_{-i}(d\boldsymbol{a_{-i}}|x)\sigma_i(da_i|x).$$

If $\nu_i \in \Pr(A_i)$, then

$$c_i^0(x, [\boldsymbol{\phi_{-i}}, \nu_i]) := c_i^0(x, [\boldsymbol{\phi_{-i}}, \sigma_i]) \quad \text{and} \quad p(dy|x, [\boldsymbol{\phi_{-i}}, \nu_i]) := p(dy|x, [\boldsymbol{\phi_{-i}}, \sigma_i])$$

with $\sigma_i(da_i|x) = \nu_i(da_i)$ for all $x \in X$.

Let v_i, $i = 1, 2, ..., N$, be bounded measurable functions on X. For each $x \in X$, by $\Gamma_x(v_1, ..., v_N)$ we denote the one-step N-person game, where the payoff (cost) function for player $i \in \mathcal{N}$ is

$$(1-\alpha)c_i^0(x, \boldsymbol{a}) + \alpha \int_X v_i(y)p(dy|x, \boldsymbol{a}), \quad \text{where} \quad \boldsymbol{a} = (a_1, ..., a_N) \in A.$$

Proof of Proposition 5.1 From (5.1), it follows that for each set $S \in \mathcal{F}$, we have

$$\begin{aligned} J_i^0(\boldsymbol{\varphi}) &= \int_X \left((1-\alpha)c_i^0(x, \boldsymbol{\varphi}) + \alpha \int_X J_i^0(\boldsymbol{\varphi})(y)p(dy|x, \boldsymbol{\varphi})\right)\eta(dx) \\ &\le \int_S \min_{\nu_i \in \Pr(A_i)} \left((1-\alpha)c_i^0(x, [\boldsymbol{\varphi_{-i}}, \nu_i]) + \alpha \int_X J_i^0(\boldsymbol{\varphi})(y)p(dy|x, [\boldsymbol{\varphi_{-i}}, \nu_i])\right)\eta(dx) \\ &\quad + \int_{X\setminus S} \left((1-\alpha)c_i^0(x, \boldsymbol{\varphi}) + \alpha \int_X J_i^0(\boldsymbol{\varphi})(y)p(dy|x, \boldsymbol{\varphi})\right)\eta(dx) \end{aligned}$$

Hence, for each $S \in \mathcal{F}$,

$$\int_S \left((1-\alpha)c_i^0(x, \boldsymbol{\varphi}) + \alpha \int_X J_i^0(\boldsymbol{\varphi})(y)p(dy|x, \boldsymbol{\varphi})\right)\eta(dx) \le$$
$$\int_S \min_{\nu_i \in \Pr(A_i)} \left((1-\alpha)c_i^0(x, [\boldsymbol{\varphi_{-i}}, \nu_i]) + \alpha \int_X J_i^0(\boldsymbol{\varphi})(y)p(dy|x, [\boldsymbol{\varphi_{-i}}, \nu_i])\right)\eta(dx).$$

Thus, for every $i \in \mathcal{N}$, there exists $S_i \in \mathcal{F}$ such that $\eta(S_i) = 1$ and for all $x \in S_i$, we have

$$(1-\alpha)c_i^0(x, \boldsymbol{\varphi}) + \alpha \int_X J_i^0(\boldsymbol{\varphi})(y)p(dy|x, \boldsymbol{\varphi}) \leq$$
$$\min_{\nu_i \in \Pr(A_i)} \left((1-\alpha)c_i^0(x, [\boldsymbol{\varphi_{-i}}, \nu_i]) + \alpha \int_X J_i^0(\boldsymbol{\varphi})(y)p(dy|x, [\boldsymbol{\varphi_{-i}}, \nu_i]) \right). \quad (5.3)$$

Let $\widehat{S} := S_1 \cap S_2 \cdots \cap S_N$. Now consider the game $\Gamma_x(v_1, ..., v_N)$, where $v_i(y) = J_i^0(\boldsymbol{\varphi})(y)$, $y \in X$. By Lemma 5 in [36], there exists $\boldsymbol{\phi} \in \Phi$ such that $\boldsymbol{\phi}(d\boldsymbol{a}|x) = (\phi_1(da_1|x), ..., \phi_N(da_N|x))$ is a Nash equilibrium in the game $\Gamma_x(v_1, ..., v_N)$ for all $x \in X \setminus \widehat{S}$. For every $i \in \mathcal{N}$, define $\psi_i(da_i|x) := \varphi_i(da_i|x)$, if $x \in \widehat{S}$, and $\psi_i(da_i|x) := \phi_i(da_i|x)$, if $x \in X \setminus \widehat{S}$. Then, using (5.3), we conclude that $\boldsymbol{\psi}(d\boldsymbol{a}|x) = (\psi_1(da_1|x), ..., \psi_N(da_N|x))$ is a Nash equilibrium in the game $\Gamma_x(v_1, ..., v_N)$ for all $x \in X$. Define $v_i^0(y) := v_i(y) = J_i^0(\boldsymbol{\varphi})(y)$ for each $y \in \widehat{S}$ and

$$v_i^0(y) := (1-\alpha)c_i^0(y, \boldsymbol{\psi}) + \alpha \int_X J_i^0(\boldsymbol{\varphi})(z)p(dz|y, \boldsymbol{\psi})$$

for each $y \in X \setminus \widehat{S}$. Then, $\eta(X \setminus \widehat{S}) = 0$ and our assumption $p(\cdot|x, \boldsymbol{a}) \ll \eta(\cdot)$, $(x, \boldsymbol{a}) \in \mathbb{K}$, imply that $\Gamma_x(v_1^0, ..., v_N^0) = \Gamma_x(v_1, ..., v_N)$ for all $x \in X$. Therefore, for all $x \in X$, $\psi(d\boldsymbol{a}|x)$ is a Nash equilibrium in the game $\Gamma_x(v_1^0, ..., v_N^0)$ and

$$v_i^0(x) = (1-\alpha)c_i^0(x, \boldsymbol{\psi}) + \alpha \int_X v_i^0(y)(y)p(dy|x, \boldsymbol{\psi}).$$

Using these facts and the Bellman equations for discounted dynamic programming [8, 24], we conclude that (5.2) holds. □

Remark 5.2 Levy and McLennan [29] gave an example of a discounted stochastic game with no constraints having no stationary Nash equilibrium. This is an 8-person stochastic game with finite action sets for the players and $X = [0, 1]$ as the state space. The definitions of payoff functions and transition probabilities in their game are rather complicated and are not given here. We only mention that the transition probabilities are absolutely continuous with respect to the probability measure $\eta_1 = (\lambda_1 + \delta_1)/2$, where λ_1 is the Lebesgue measure on [0, 1] and δ_1 is the Dirac measure concentrated at the point 1. Assume that η_1 is the initial state distribution in this game. If this game had a stationary Nash equilibrium, then by Proposition 5.1, it would have a stationary Nash equilibrium for all initial states. From [29], it follows that it is impossible.[1]

6 Remarks on Games with Unbounded Costs

Our results can be extended to a class of games with unbounded cost functions c_i^ℓ under some uniform integrability condition introduced in [16]. The method for doing

[1] We thank John Yehuda Levy for pointing out this fact.

this relies on truncations of the costs and using an approximation by bounded games. This was done in our paper [28] in the countable state space case. In a special situation, described below and inspired by the work of Wessels [40] on dynamic programming, a reduction to the bounded case can be obtained by the well-known data transformation as described in Remark 2.5 in [12] or Sect. 10 in [17]. Following Wessels [40], we make the following assumptions.

Assumption W

(i) There exist a measurable function $\omega : X \to [1, \infty)$ and $c_0 > 0$ such that $|c_i^\ell(x, \boldsymbol{a})| \le c_0\omega(x)$ for all $x \in X$, $\boldsymbol{a} \in A$, $i \in \mathcal{N}$ and $\ell \in \mathcal{L}_0$.
(ii) There exists $\beta > 1$ such that $\alpha\beta < 1$ and

$$\int_X \omega(y)p(dy|x, \boldsymbol{a}) \le \beta\omega(x)$$

for all $x \in X$, $\boldsymbol{a} \in A$.
(iii) If $\boldsymbol{a}^n \to \boldsymbol{a}$ as $n \to \infty$, then

$$\int_X |\delta(x, y, \boldsymbol{a}^n) - \delta(x, y, \boldsymbol{a})|\omega(y)\mu(dy) \to 0.$$

To describe the equivalent model with bounded costs we extend the state space X by adding an isolated *absorbing state* 0^*. All the costs at this absorbing state are *zero*. Let $c_i^{\ell,\omega}(x, \boldsymbol{a}) := \frac{c_i^\ell(x,\boldsymbol{a})}{\omega(x)}$, and

$$p^\omega(B|x, \boldsymbol{a}) := \frac{\int_B \omega(y)p(dy|x, \boldsymbol{a})}{\beta\omega(x)}, \quad B \in \mathcal{F},\ x \in X,\ \boldsymbol{a} \in A,$$
$$p^\omega(0^*|x, \boldsymbol{a}) := 1 - \frac{\int_X \omega(y)p(dy|x, \boldsymbol{a})}{\beta\omega(x)}, \quad x \in X,\ \boldsymbol{a} \in A.$$

Now define the new initial state distribution as

$$\eta_0(B) := \frac{\int_B \omega(x)\eta(dx)}{\eta\omega}, \quad \text{where} \quad \eta\omega = \int_X \omega(x)\eta(dx).$$

Here, we assume that $\eta\omega < \infty$. Then, we obtain primitive data for a bounded constrained stochastic game, in which the discount factor is $\alpha\beta$. We denote the expected discounted costs in the bounded game under consideration by $\mathcal{J}_i^\ell(\boldsymbol{\pi})$. It is easy to see that

$$\mathcal{J}_i^\ell(\boldsymbol{\pi}) = \frac{J_i^\ell(\boldsymbol{\pi})}{\eta\omega}, \quad \text{for all} \quad i \in \mathcal{N}, \ell \in \mathcal{L}_0,\ \boldsymbol{\pi} \in \Pi.$$

Theorems 2.3 and 2.13 can be established for the bounded game described above with minor modifications. For example, one has to define new constraint constants as

$\kappa_i^\ell/\eta\omega$, $i \in \mathcal{N}$, $\ell \in \mathcal{L}$. Using the above transformation, we can immediately deduce similar results for games with unbounded cost functions satisfying Assumption **W**.

Acknowledgements We thank two reviewers for very helpful reports.

Funding We acknowledge the financial support from the National Science Centre, Poland: Grant 2016/23/B/ST/00425.

Declarations

Conflict of interest The authors have not disclosed any competing interests.

7 Appendix

In this section, we prove a lemma which plays an important role in the proofs of our theorems.

Let player $i \in \mathcal{N}$ be fixed. We also fix $\gamma > 0$, the partition $\mathcal{P}^\gamma = \{X_n^\gamma : n \in \mathbb{N}_0\}$ of the state space X, the cost functions $c_i^{\ell,\gamma}$ and the transition function p^γ in the game $\mathcal{G}^\gamma$. We fix $\boldsymbol{\varphi_{-i}} \in \Phi_{-i}^\gamma = \prod_{j\in\mathcal{N}\setminus\{i\}} \Phi_j^\gamma$.

A piecewise constant Markov strategy for player i is a sequence $\pi_i = (f^t)_{t\in\mathbb{N}}$, where $f^t \in \Phi_i^\gamma$ for all $t \in \mathbb{N}$.

Lemma 7.1 *For fixed $\boldsymbol{\varphi} \in \Phi^\gamma$ and each $\phi_i \in \Phi_i$ there exists a piecewise constant Markov strategy $\pi_i = (f^t)_{t\in\mathbb{N}}$ for player i such that*

$$J_i^{\ell,\gamma}([\boldsymbol{\varphi_{-i}}, \phi_i]) = J_i^{\ell,\gamma}([\boldsymbol{\varphi_{-i}}, \pi_i]) \quad \textit{for all} \quad \ell \in \mathcal{L}_0.$$

For a proof we need some auxiliary results. Let $d \in \mathbb{N}$.

Lemma 7.2 *Assume that $Y \in \mathcal{F}$ and ρ_0 is a probability measure on X such that $\rho_0(Y) = 1$. Let $v = (v_0, \ldots, v_{d-1})$, where every $v_j : X \to \mathbb{R}$ is a bounded measurable function. Then, there exist points $y_0, \ldots, y_d \in Y$ and non-negative numbers $\beta_0, \ldots, \beta_d$ such that $\sum_{j=0}^d \beta_j = 1$ and*

$$\int_Y v(x)\rho_0(dx) = \sum_{j=0}^d \beta_j v(y_j). \tag{7.1}$$

Proof Consider the distribution function of v defined by: $\zeta_v(B) := \rho_0(v^{-1}(B))$, where B is any Borel set in $\mathbb{R}^d$. Using Theorem 16.13 on page 229 in [9] and Lemma 3 on page 74 in [18], we obtain

$$\int_Y v(x)\rho_0(dx) = \int_{\mathbb{R}^d} z\zeta_v(z)dz \in co\{v(y) : y \in X\}.$$

Applying Carathéodory's theorem, we find points $y_0, ..., y_d \in Y$ and numbers $\beta_0, ..., \beta_d \geq 0$ such that $\sum_{j=0}^{d} \beta_j = 1$ and (7.1) holds. □

We use $\mathcal{C}(A_i)$ to denote the space of all real-valued continuous functions on A_i and $\Pr(A_i)$ for the space of all probability measures on A_i.

Lemma 7.3 *Let ρ be a probability measure on X. For each $\ell \in \mathcal{L}_0$ assume that $u^\ell : X \times A_i \to \mathbb{R}$ is a bounded function such that $u^\ell(x, a_i) = u_n^\ell(a_i)$ for all $x \in X_n^\gamma$, $a_i \in A_i$, where $u_n^\ell \in \mathcal{C}(A_i)$, $n \in \mathbb{N}_0$. Then, for any $\phi_i \in \Phi_i$ there exists $f \in \Phi_i^\gamma$ such that*

$$\int_X \int_{A_i} u^\ell(x, a_i)\phi_i(da_i|x)\rho(dx) = \int_X \int_{A_i} u^\ell(x, a_i) f(da_i|x)\rho(dx) \quad \textit{for all } \ell \in \mathcal{L}_0. \tag{7.2}$$

Proof Assume first that $\rho(X_n^\gamma) > 0$ and define $\rho_0(B) = \frac{\rho(B\cap X_n^\gamma)}{\rho(X_n^\gamma)}$, $B \in \mathcal{F}$. Applying Lemma 7.2 with $d = L + 1$ and $v = (u^0, ..., u^L)$, we infer that there exist points $y_0(n), ..., y_{L+1}(n)$ in X_n^γ and $\beta_0(n), ..., \beta_{L+1}(n) \geq 0$ such that $\sum_{j=0}^{L+1} \beta_j(n) = 1$ and

$$\frac{1}{\rho(X_n^\gamma)} \int_{X_n^\gamma} \int_{A_i} u^\ell(x, a_i)\phi_i(da_i|x)\rho(dx) = \frac{1}{\rho(X_n^\gamma)} \int_{X_n^\gamma} \int_{A_i} u_n^\ell(a_i)\phi_i(da_i|x)\rho(dx)$$
$$= \sum_{j=0}^{L+1} \beta_j(n) \int_{A_i} u_n^\ell(a_i)\phi_i(da_i|y_j(n)) \;\; \text{for all } \ell \in \mathcal{L}_0.$$

For each $x \in X_n^\gamma$, define $f(da_i|x) := \nu_n(da_i)$, where $\nu_n \in \Pr(A_i)$ is given as

$$\nu_n(da_i) := \sum_{j=0}^{L+1} \beta_j(n)\phi_i(da_i|y_j(n)).$$

If $\rho(X_n^\gamma) = 0$, then $f(da_i|x)$ is defined for all $x \in X_n^\gamma$ by $f(da_i|x) = \nu_n(da_i)$ where ν_n is any fixed measure in $\Pr(A_i)$. Note that, we have

$$\int_{X_n^\gamma} \int_{A_i} u^\ell(x, a_i)\phi_i(da_i|x)\rho(dx) = \int_{A_i} u_n^\ell(a_i)\nu_n(da_i)\rho(X_n^\gamma)$$
$$= \int_{X_n^\gamma} \int_{A_i} u^\ell(x, a_i) f(da_i|x)\rho(dx),$$

for all $\ell \in \mathcal{L}_0,\ n \in \mathbb{N}_0$. Hence,

$$\sum_{n\in\mathbb{N}_0}\int_{X_n^\gamma}\int_{A_i} u^\ell(x,a_i)\phi_i(da_i|x)\rho(dx) = \sum_{n\in\mathbb{N}_0}\int_{X_n^\gamma}\int_{A_i} u^\ell(x,a_i)f(da_i|x)\rho(dx),$$

for all $\ell \in \mathcal{L}_0$, which implies (7.2). □

Since $i \in \mathcal{N}$, $\gamma > 0$, $\boldsymbol{\varphi_{-i}} \in \Phi^\gamma_{-i}$ and $\phi_i \in \Phi_i$ are fixed, the notation for the proof of Lemma 7.1 can be simplified.

Let $\varphi_{-i}(d\boldsymbol{a_{-i}}|x)$ be the product measure on A_{-i} induced by $\varphi_j(da_j|x)$ with $j \neq i$. For $\ell \in \mathcal{L}_0$, $x \in X$ and $a_i \in A_i$, we put

$$c^\ell(x,a_i) := \int_{A_{-i}} c_i^{\ell,\gamma}(x,[\boldsymbol{a_{-i}},a_i])\varphi_{-i}(d\boldsymbol{a_{-i}}|x),$$
$$q(dy|x,a_i) := \int_{A_{-i}} p^\gamma(dy|x,[\boldsymbol{a_{-i}},a_i])\varphi_{-i}(d\boldsymbol{a_{-i}}|x).$$

Next, we put

$$c^\ell_{\phi_i}(x) := \int_{A_i} c^\ell(x,a_i)\phi_i(da_i|x),$$

and, for any bounded measurable function $w : X \to \mathbb{R}$,

$$Q_{\phi_i}w(x) := \int_{A_i} w(y)q(dy|x,a_i)\phi_i(da_i|x).$$

Similarly, we define $c^\ell_g(x)$ and $Q_g w(x)$ for any $g \in \Phi_i^\gamma$. Next, if $g^1, g^2, ..., g^T \in \Phi_i^\gamma$, then

$$\eta w = \int_X w(x)\eta(dx) \quad \text{and} \quad Q_{g^1}Q_{g^2}\cdots Q_{g^T}w(x) = Q_{g^1}(Q_{g^2}\cdots Q_{g^T}w)(x)$$

and

$$\eta Q_{g^1}Q_{g^2}\cdots Q_{g^T}w := \int_X Q_{g^1}Q_{g^2}\cdots Q_{g^T}w(x)\eta(dx).$$

Note that $\eta Q_{g^1}Q_{g^2}\cdots Q_{g^T}$ is the probability distribution of the state x_{T+1} of the process, when player i uses a Markov strategy $(g^t)_{t\in\mathbb{N}}$.

We now introduce new notation for expected costs. Recalling that $\phi_i \in \Phi_i$, we put

$$I^\ell(\phi_i)(x) := J_i^{\ell,\gamma}([\boldsymbol{\varphi_{-i}},\phi_i])(x) \quad \text{and} \quad I^{\ell,\eta}(\phi_i) := \int_X I^\ell(\phi_i)(x)\eta(dx), \quad \ell \in \mathcal{L}_0.$$

If $\pi_i = (g^t)_{t\in\mathbb{N}}$ is a piecewise constant strategy for player i, then $I_T^{\ell,\eta}(\pi_i) = I_T^{\ell,\eta}(g^1, ..., g^T)$ denotes the expected discounted cost in the T-step game $\mathcal{G}^\gamma$ under assumption that the other players use $\boldsymbol{\varphi_{-i}}$. Then, the cost over the infinite time horizon is

$$I^{\ell,\eta}(\pi_i) = \lim_{T\to\infty} I_T^{\ell,\eta}(\pi_i).$$

Proof of Lemma 7.1 We show by induction that for given $\phi_i \in \Phi_i$ there exists $\pi_i = (f^t)_{t\in\mathbb{N}}$ with $f^t \in \Phi_i^\gamma$ for all $t \in \mathbb{N}$ such that for all $T \in \mathbb{N}$, we have

$$I^{\ell,\eta}(\phi_i) = I^{\ell,\eta}(f^1, ..., f^T) + \alpha^T \eta Q_{f^1} \cdots Q_{f^T}((1-\alpha)c^\ell_{\phi_i} + \alpha Q_{\phi_i} I^\ell(\phi_i)). \tag{7.3}$$

We shall use the following equation

$$I^\ell(\phi_i)(x) = (1-\alpha)c^\ell_{\phi_i}(x) + \alpha Q_{\phi_i} I^\ell(\phi_i)(x), \quad \text{for each} \quad x \in X.$$

Assume that $T = 1$. Then,

$$\begin{aligned} I^{\ell,\eta}(\phi_i) &= \eta((1-\alpha)c^\ell_{\phi_i} + \alpha Q_{\phi_i} I^\ell(\phi_i)) \\ &= \int_X \int_{A_i} \left((1-\alpha)c^\ell(x, a_i) + \alpha \int_X I^\ell(\phi_i)(y) q(dy|x, a_i)\right) \phi_i(da_i|x)\eta(dx). \end{aligned}$$

Applying Lemma 7.3 with $\rho = \eta$ and

$$u^\ell(x, a_i) = (1-\alpha)c^\ell(x, a_i) + \alpha \int_X I^\ell(\phi_i)(y) q(dy|x, a_i) \tag{7.4}$$

we obtain $f^1 \in \Phi_i^\gamma$ such that

$$\int_X \int_{A_i} u^\ell(x, a_i)\phi_i(da_i|x)\eta(dx) = \int_X \int_{A_i} u^\ell(x, a_i) f^1(da_i|x)\eta(dx) \quad \text{for all} \quad \ell \in \mathcal{L}_0.$$

Then, we get

$$\begin{aligned} I^{\ell,\eta}(\phi_i) &= \eta I^\ell(\phi_i) = \eta((1-\alpha)c^\ell_{\phi_i} + \alpha Q_{\phi_i} I^\ell(\phi_i)) \\ &= \eta((1-\alpha)c^\ell_{f^1} + \alpha Q_{f^1} I^\ell(\phi_i)) = \eta(1-\alpha)c^\ell_{f^1} + \alpha\eta Q_{f^1} I^\ell(\phi_i) \\ &= I_1^{\ell,\eta}(f^1) + \alpha\eta Q_{f^1}((1-\alpha)c^\ell_{\phi_i} + \alpha Q_{\phi_i} I^\ell(\phi_i)) \quad \text{fro all} \quad \ell \in \mathcal{L}_0. \end{aligned}$$

We have obtained (7.3) for $T = 1$. Assume now that (7.3) holds for $T = m$ with some $m \geq 1$. Then we have for some $f^1, ..., f^m \in \Phi_i^\gamma$ that

$$I^{\ell,\eta}(\phi_i) = I^{\ell,\eta}(f^1, ..., f^m) + \alpha^m \eta Q_{f^1} \cdots Q_{f^m}((1-\alpha)c^\ell_{\phi_i} + \alpha Q_{\phi_i} I^\ell(\phi_i))$$

for all $\ell \in \mathcal{L}_0$. Applying Lemma 7.3 with $u^\ell(x, a_i)$ given by (7.4) and $\rho = \eta Q_{f^1} \cdots Q_{f^m}$, we obtain $f^{m+1} \in \Phi_i^\gamma$ such that

$$
\begin{aligned}
&\eta Q_{f^1} \cdots Q_{f^m}((1-\alpha)c^\ell_{\phi_i} + \alpha Q_{\phi_i} I^\ell(\phi_i)) \\
&\quad = \eta Q_{f^1} \cdots Q_{f^m}((1-\alpha)c^\ell_{f^{m+1}} + \alpha Q_{f^{m+1}} I^\ell(\phi_i)) \\
&\quad = \eta Q_{f^1} \cdots Q_{f^m}(1-\alpha)c^\ell_{f^{m+1}} + \alpha\eta Q_{f^1} \cdots Q_{f^m} Q_{f^{m+1}}((1-\alpha)c^\ell_{\phi_i} + \alpha Q_{\phi_i} I^\ell(\phi_i)).
\end{aligned}
$$

Thus for all $\ell \in \mathcal{L}_0$ we get

$$
\begin{aligned}
I^{\ell,\eta}(\phi_i) &= I^{\ell,\eta}(f^1, ..., f^m) + \alpha^m \eta Q_{f^1} \cdots Q_{f^m}(1-\alpha)c^\ell_{f^{m+1}} \\
&\quad + \alpha^{m+1}\eta Q_{f^1} \cdots Q_{f^m} Q_{f^{m+1}}((1-\alpha)c^\ell_{\phi_i} + \alpha Q_{\phi_i} I^\ell(\phi_i)) \\
&= I^{\ell,\eta}(f^1, ..., f^{m+1}) + \alpha^{m+1}\eta Q_{f^1} \cdots Q_{f^m} Q_{f^{m+1}}((1-\alpha)c^\ell_{\phi_i} + \alpha Q_{\phi_i} I^\ell(\phi_i)).
\end{aligned}
$$

This finishes the induction step. Taking the limit in (7.3) as $T \to \infty$, we obtain

$$
I^{\ell,\eta}(\phi_i) = I^{\ell,\eta}(\pi_i) \quad \text{with} \quad \pi_i = (f^1, f^2, ...)
$$

for all $\ell \in \mathcal{L}_0$. Going back to our original notation, we deduce that this is the assertion of Lemma 7.1. □

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