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On the Controllability of a Free-Boundary Problem for 1D Heat Equation with Local and Nonlocal Nonlinearities

V. Costa[1] · J. Límaco[2] · A. R. Lopes[1] · L. Prouvée[1]

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Abstract
This paper deals with the analysis of the internal control of a free-boundary problem for the 1D heat equation with local and nonlocal nonlinearities. We prove a local null controllability result with distributed controls, locally suported in space. The proof is based on Schauder's fixed point theorem combined with some appropriate specific estimates.

Keywords Free-boundary problems · 1D nonlinear heat equation · Carleman estimates

1 Introduction

Let $T > 0$, $0 < \kappa_1 < \kappa_2 < L_* < L_0 < B$ and $y_0 \in C^{2+\frac{1}{2}}([0, L_0])$ be given. For any function $L \in C^{1+\frac{1}{4}}([0, T])$ with $0 < L_* \leq L(t) \leq B, \ \ t \in (0, T)$ we will set

$$Q_L := \{(x, t) : x \in (0, L(t)) \text{ and } t \in (0, T)\}.$$

Dedicated to Professor Vítor Costa Silva in memoriam.

✉ J. Límaco
jlimaco@id.uff.br

V. Costa
vitor.costa@ime.uerj.br

A. R. Lopes
andre.lopes@ime.uerj.br

L. Prouvée
laurent.prouvee@ime.uerj.br

1 Instituto de Matemática e Estatistica, Universidade do Estado do Rio de Janeiro, Campus Maracanã, Rio de Janeiro, RJ 20550-900, Brazil

2 Instituto de Matemática e Estatística, Universidade Federal Fluminense, Campus Gragoatá, Niterói, RJ 24210-200, Brazil

In this paper, we will investigate the null controllability properties of a free-boundary problem for the nonlinear 1D parabolic equation of the form

$$\begin{cases} y_t - \beta\left(\int_0^{L(t)} y dx\right) y_{xx} + g(y, y_x) = v1_\omega, & (x,t) \in Q_L, \\ y(0,t) = 0;\ y(L(t),t) = 0, & t \in (0,T), \\ y(x,0) = y_0(x), & x \in (0, L_0), \end{cases} \tag{1.1}$$

with the additional boundary condition

$$L(t) = L_0 - \int_0^t \left[\beta\left(\int_0^{L(s)} y(x,s)dx\right) y_x(L(s),s)\right] ds, \text{ therefore,}$$

$$-L'(t) = \beta\left(\int_0^{L(t)} y(x,t)dx\right) y_x(L(t),t), \tag{1.2}$$

with $t \in (0,T)$ and $L_0 = L(0)$. Here $y = y(x,t)$ is the state, $v = v(x,t)$ is a control function that acts on the system at any time through a nonempty open set $\omega = (\kappa_1, \kappa_2)$, and 1_ω denotes the characteristic function of the ω. Regarding the functions β and g, we make the following assumptions:

(A1) $\beta : \mathbb{R} \to \mathbb{R}$ is a C^1 function that possesses bounded derivatives and satisfies

$$0 < \beta_0 < \beta(r) < \beta_1 < +\infty, \ \forall r \in \mathbb{R}.$$

(A2) $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a C^2 function, with bounded derivatives, such that $g(0,0) = 0$.

The main purpose of this paper is to prove the local null controllability result of (1.1). To accomplish this goal, let us recall the following classical controllability concept:

Definition 1.1 It will be said that (1.1) is null controllable at time T if there exist a control $v \in L^2(\omega \times (0,T))$, a function $L \in C^{1+\frac{1}{4}}([0,T])$ and an associated solution $y = y(x,t)$ satisfying (1.1), (1.2), and

$$y(x,T) = 0, \ x \in (0, L(T)), \tag{1.3}$$

for each $y_0 \in C^{2+\frac{1}{2}}([0, L_0])$.

Definition 1.2 It will be said that (1.1) is approximately controllable at time T if there exist a control $v \in L^2(\omega \times (0,T))$, a function $L \in C^{1+\frac{1}{4}}([0,T])$ and an associated state $y = y(x,t)$ satisfying (1.1), (1.2) and

$$\|y(\cdot,T)\|_{L^2(0,L(T))} \leq \varepsilon, \tag{1.4}$$

for any $y_0 \in C^{2+\frac{1}{2}}([0, L_0])$ and $\varepsilon > 0$.

As we have already mentioned, we are interested in the local null controllability of (1.1), that is, in other words, the system (1.1) is said to be *locally null controllable* at any time $T > 0$ if, there exists $\delta > 0$ such that, if $\|y_0\|_{C^{2+\frac{1}{2}}([0,L_0])} \le \delta$, there exists a triplet (L, v, y) with

$$\begin{cases} L \in C^{1+\frac{1}{4}}([0,T]), \quad L_* \le L(t) \le B, \\ v \in L^2(\omega \times (0,T)), \end{cases} \tag{1.5}$$

satisfying (1.1), (1.2), and (1.3).

In mathematics, the expression free-boundary problem (FBP) refers to a problem in which one or several variables must be determined in different domains in space or in space-time. In a brief definition, we can say that a FBP is a boundary value problem defined in a domain that is not given a priori, therefore, a part of the unknown. If the domains are known, the problem reduces to solve equations, usually partial differential equations or ordinary differential equations. Free-boundary problems arise in various mathematical models that encompass applications ranging from Physics to Economics, Finances and biological phenomena where there is an extra effect of the environment. This effect in general deals with a qualitative change of the environment and an appearance of a phase transition; for example, ice to water, liquid to crystal, purchases to sales (assets), active to inactive (biology).

Free-boundary problems similar to (1.1)–(1.2) are connected to several interesting applications. We mention the following works:

- Tumor growth and other phenomena from mathematical biology; see Friedman [23, 24].
- Fluid-solid interaction; see Doubova and Fernández-Cara [12], Vázquez and Zuazua [36] and Liu et al. [31].
- Gas flow through porous media; see Aronson [2], Fasano [15] and Vázquez [35].
- Solidification and related Stefan problems; see Friedman [22].
- The analysis and computation of free surfaces flows; see Hermans [26, 27], Stoker [33, 34] and Wrobel and Brebbia [38].

In the last years, there are many works addressing controllability problems of linear and semilinear PDE's. In particular, let us mention Fursikov and Imanuvilov [25], Barbu [3], Fernández-Cara and Zuazua [20], Doubova et al. [13] and Xu Liu and Xu Zhang [30] and the references therein in the context of bounded domains. In the context of the linear and semilinear PDE's, we also mention the following articles [11, 14, 16, 17, 21, 29].

For parabolic free-boundary problems, controllability questions have been considered only in a few papers; see for instance Fernández-Cara et al. in [18] and Fernández-Cara and de Sousa in [19]. In both cited papers, the common point is that the main operator is linear and the free-boundary condition is given by

$$-L'(t) = y_x(L(t), t), \ t \in (0, T). \tag{1.6}$$

In the present paper, with an extension in mind for another more realistic and interesting problems, we have considered a nonlocal term in the main part of the partial derivative operator. In this way, the free-boundary condition (1.2) becomes more general than condition (1.6). This is the main novelty in this work.

In addition, in [37] the authors studied the null controllability of a free-boundary problem for the quasi-linear 1D parabolic equation.

The nonlocal term in (1.1) appears naturally in some physical models. For example, they can arise in heat conduction in materials with memory, nuclear reactors, and population dynamics, for instance the bacteria in a container, the diffusion coefficients may depend on the total amount of individuals; see for instance [9, 39]. We also mention that, in the context of elasticity theory, terms in the form

$$\beta\Big(\int_0^{L(t)} |y(x,t)|^2 dx\Big) \quad \text{and}$$
$$\beta\Big(\int_0^{L(t)} |y_x(x,t)|^2 dx\Big)$$

appear, respectively, in Carrier and Kirchhoff equations. These equations arise in nonlinear vibration theory; see for instance [32].

Our main result is the following:

Theorem 1.1 *Assume that $T > 0$ and $0 < \kappa_1 < \kappa_2 < L_* < L_0 < B$. Under the previous assumptions on β and g, the nonlinear system (1.1) is locally null-controllable.*

For the proof of this theorem, we will first fix $\varepsilon > 0$ and prove the existence of triplets $(L_\varepsilon, y_\varepsilon, v_\varepsilon)$ that are uniformly bounded in an appropriate space and satisfy (1.1), (1.2) and (1.4). To this end, we will introduce a fixed point reformulation relying suitable linearized problems and we check that, if the initial data y_0 is sufficiently small, than the *Schauder's Fixed Point Theorem* can be applied. Finally, we take limits as $\varepsilon \to 0$ and we see that, at least for a subsequence, we get convergence to a solution of (1.1), (1.2) and (1.3).

Throughout this paper, we denote by C a generic positive constant; for example: C_1, C_2, etc. are other positive (specific) constants; when it makes sense.

The paper is organized as follows: Sect. 2 is devoted to recall some known results and prove the approximate controllability of the linearized system (2.1). The Sect. 3 deals with the proof of Theorem 1.1. We present in Sect. 4 some open questions. In Appendix A, we sketch the proof of a Carleman estimate and in Appendix B we prove some relevant lemmas.

2 Analysis of the Controllability of the Linearized System in a Non-cylindrical Domain and Regularity Property

Given $L_0 > 0$, $T > 0$, and $0 < \kappa_1 < \kappa_2 < L_* < L_0 < B$, and fixing $y_0 \in L^2(\Omega)$, assume that $L \in C^{1+\frac{1}{4}}([0,T])$ is a prescribed function satisfying

$$0 < L_* \le L(t) \le B, \ \ t \in (0,T)\,.$$

In this section we will prove that, for any $(\overline{y}, \overline{L}) \in C^{2,1}_{x,t}(\overline{Q_{\overline{L}}}) \times C^{1+\frac{1}{4}}([0, T])$, the linear system

$$\begin{cases} y_t - \beta\left(\int_0^{\overline{L}(t)} \overline{y}(x,t)dx\right) y_{xx} + a(\overline{y}, \overline{y}_x)y + b(\overline{y}, \overline{y}_x)y_x = v\widetilde{1}_\omega, & (x,t) \in Q_{\overline{L}}, \\ y(0,t) = 0;\ y(\overline{L}(t), t) = 0, & t \in (0,T), \\ y(x,0) = y_0(x), & x \in (0, L_0), \end{cases} \tag{2.1}$$

is approximate controllable and also, that we can find a pair $(v_\varepsilon, y_\varepsilon)$ satisfying $\|y_\varepsilon(\cdot, T)\|_{L^2(0,\overline{L}(T))} \leq \varepsilon$
for $\varepsilon > 0$. In (2.1) we consider $L_* < \overline{L}(t) < B$, with $\overline{L}(0) = L_0$, $\widetilde{1}_\omega \in C_0^\infty(\omega)$, with $\widetilde{1}_\omega = 1$ in $\omega_1 \subset\subset \omega$ and $Q_{\overline{L}} = \{(x,t) : x \in (0, \overline{L}(t)) \text{ and } t \in (0,T)\}$.

We can verify that, for every $v \in L^2(\omega \times (0,T))$ and every $y_0 \in L^2(0, L_0)$, there exists a unique solution y to (2.1), with $y \in L^2(0,T; H_0^1(0, \overline{L}(t)))$ and $y_t \in L^2(0,T; H^{-1}(0, \overline{L}(t)))$. Consequently,

$$y \in C^0([0,T]; L^2(0, \overline{L}(t))) .$$

To this end, an appropriate change of variable $w(\zeta, t) := y(x,t)$ allows us to rewrite (2.1) as a similar problem for a parabolic PDE of the form

$$w_t - \frac{\beta(t)}{L^2(t)} w_{\zeta\zeta} + \widetilde{a}(\zeta, t)w + \widetilde{b}(\zeta, t)w_\zeta = G(\zeta, t),$$

in a cylindrical domain, with bounded coefficients $\widetilde{a}, \widetilde{b} \in C^{1/2,1/4}_{\xi,t}(\overline{Q})$ and square-integrable right hand side $G \in L^2(Q)$, where $Q := (0,1) \times (0,T)$ (see Sect. 2.4 for the definition of ζ in *Control Regularity*).

As usual, the controllability of (2.1) is closely related to the properties of the solutions to the associated adjoint state. In this case, the adjoint system is

$$\begin{cases} -\varphi_t - \beta(t)\varphi_{xx} + a(x,t)\varphi + b(x,t)\varphi_x = F(x,t), & (x,t) \in Q_L, \\ \varphi(0,t) = 0;\ \varphi(\overline{L}(t), t) = 0, & t \in (0,T), \\ \varphi(x,T) = \varphi^T(x), & x \in (0, \overline{L}(T)), \end{cases} \tag{2.2}$$

where $F \in L^2(Q_{\overline{L}})$ and $\varphi^T \in L^2(0, \overline{L}(T))$.

Next, we sketch the points used in the proof of the null controllability of the linearized system using an observability estimate. First, we use a global Carleman estimate satisfied by the solutions of (2.2). Second, this estimate allows us to establish an observability estimate. Third, we prove the approximate controllability of (2.1) by using the observability estimate. Finaly, we establish a regularity property for the pair control-state in a certain Hölder space.

2.1 A Carleman Estimate for the Solutions to (2.2)

In this section, we will recall some Carleman estimates satisfied by the solutions to (2.2).

Denote by $\Sigma_{\overline{L}} := \{(x,t) : x = 0 \text{ or } x = \overline{L}(t) \text{ with } 0 < t < T\}$ the lateral boundary of $Q_{\overline{L}}$.

The following technical result, due to Fursikov and Imanuvilov [25], is fundamental:

Lemma 2.1 *Let ω_0 be an arbitrary non-empty open subdomain with $\overline{\omega}_0 \subset \omega$. There exists a function $\alpha_0 \in C^1\left(\overline{Q}_{\overline{L}}\right)$ with $\alpha_{0,xx} \in C^0\left(\overline{Q}_{\overline{L}}\right)$ such that*

$$\begin{cases} \alpha_0(x,t) = 0 & \forall (x,t) \in \Sigma_{\overline{L}}, \\ |\alpha_{0,x}| > 0 & \text{in } \overline{Q}_{\overline{L}} \backslash (\omega_0 \times (0,T)), \\ \alpha_0(x,t) = 1 - \dfrac{x-b}{\overline{L}(t) - b} & \forall x \in (b, \overline{L}(t)) \text{ and } \forall t \in [0,T]. \end{cases}$$

The proof of this lemma can be found in [18] (see Lemma 2.1).

Let α_1 and γ be real functions, and let ξ and α be weights defined by

$$\begin{aligned} \alpha_1(x,t) &:= \alpha_0(x,t) + 1, \qquad \gamma(t) := t^k (T-t)^k, \\ \xi(x,t) &:= \frac{e^{\lambda \alpha_1(x,t)}}{\gamma(t)}, \qquad \alpha(x,t) := \frac{e^{2\lambda \|\alpha_1\|_\infty} - e^{\lambda \alpha_1(x,t)}}{\gamma(t)}, \end{aligned} \tag{2.3}$$

where $\lambda > 0$ and $k \geq 2$ are real numbers.

The next result is a Carleman estimate for the solutions to adjoint system (2.2).

Theorem 2.1 *Let α_0 and γ, ξ, α be as defined as in Lemma 2.1 and (2.3), respectively. There exist λ_0, s_0, and C_0, positive constants, only depending on a, a', L_*, ω, and T, such that, for any $s \geq s_0$, $\lambda \geq \lambda_0$, any $F \in L^2(Q_{\overline{L}})$, and any $\varphi^T \in L^2(0, \overline{L}(T))$, one has the following inequality:*

$$\begin{aligned} &\iint_{Q_{\overline{L}}} e^{-2s\alpha} \left[(s\xi)^{-1} (|\varphi_t|^2 + |\varphi_{xx}|^2) + (s\xi)\lambda^2 |\varphi_x|^2 + (s\xi)^3 \lambda^4 |\varphi|^2 \right] dxdt \\ &\quad + s\lambda \int_0^T \left[e^{-2s\alpha(\overline{L}(t),t)} \xi(\overline{L}(t),t) |\varphi_x(\overline{L}(t),t)|^2 + e^{-2s\alpha(0,t)} \xi(0,t) |\varphi_x(0,t)|^2 \right] dt \\ &\quad \leq C \left(\iint_{Q_{\overline{L}}} e^{-2s\alpha} |F|^2 dxdt + \iint_{\omega \times (0,T)} e^{-2s\alpha} (s\xi)^3 \lambda^4 |\varphi|^2 dxdt \right), \end{aligned} \tag{2.4}$$

where φ is the corresponding solution to (2.2).

The proof is given in Appendix.

2.2 An Observability Inequality

Consider the homogeneous adjoint system:

$$\begin{cases} -\varphi_t - \beta(t)\,\varphi_{xx} + a(x,t)\varphi + b(x,t)\varphi_x = 0, & (x,t) \in Q_{\overline{L}}, \\ \varphi(0,t) = 0;\ \varphi(\overline{L}(t),t) = 0, & t \in (0,T), \\ \varphi(x,T) = \varphi^T(x), & x \in (0,\overline{L}(T)), \end{cases} \tag{2.5}$$

where $\varphi^T \in L^2(0,\overline{L}(T))$.

Now, we will prove the observability inequality for weak solutions of the adjoint system (2.5). Observe that it is a consequence of the previous Carleman inequality.

Proposition 2.1 *There exists $C > 0$, only depending on $\|a\|_{L^\infty(Q_{\overline{L}})}$, $\|b\|_{L^\infty(Q_{\overline{L}})}$, $\|\overline{L}'\|_\infty$, L_*, B, ω, and T, such that, for any $\varphi^T \in L^2(0,\overline{L}(T))$, the associated solution to* (2.5) *satisfies*

$$\int_0^{L_0} |\varphi(x,0)|^2 dx \le C \iint_{\omega\times(0,T)} e^{-2s_0\alpha}\xi^3|\varphi|^2 dxdt. \tag{2.6}$$

Proof Let us take $\lambda = \lambda_0$ and $s = s_0$ in (2.4). Then

$$\int\int_{Q_{\overline{L}}} e^{-2s_0\alpha}\xi^3|\varphi|^2\,dx\,dt \ \le C\int\int_{\omega\times(0,T)} e^{-2s_0\alpha}\xi^3|\varphi|^2\,dx\,dt$$

and, consequently,

$$\begin{aligned} &\int_{T/4}^{3T/4}\int_0^{\overline{L}(t)} |\varphi|^2\,dx\,dt \\ &\quad \le C\int_{T/4}^{3T/4}\int_0^{\overline{L}(t)} e^{-2s_0\alpha}\xi^3|\varphi|^2\,dx\,dt \\ &\quad \le C\int\int_{\omega\times(0,T)} e^{-2s_0\alpha}\xi^3|\varphi|^2\,dx\,dt. \end{aligned} \tag{2.7}$$

On the other hand, multiplying the PDE in (2.5) by φ and integrating in $(0,\overline{L}(t))$, we get the following identity

$$\begin{aligned} &-\frac{1}{2}\frac{d}{dt}\left(\int_0^{\overline{L}(t)} |\varphi|^2\,dx\right) \\ &\quad +\frac{1}{2}\overline{L}'(t)\,|\varphi(\overline{L}(t),t)|^2 + \int_0^{\overline{L}(t)} \beta(t)|\varphi_x|^2\,dx \\ &= -\int_0^{\overline{L}(t)} a|\varphi|^2\,dx - \int_0^{\overline{L}(t)} b\varphi\varphi_x\,dx, \qquad \forall t \in (0,T). \end{aligned}$$

Since $\varphi(\overline{L}(t),t)\equiv 0$, we deduce that, for a small $\epsilon>0$,

$$-\frac{1}{2}\frac{d}{dt}\left(\int_0^{\overline{L}(t)}|\varphi(x,t)|^2\,dx\right)+\int_0^{\overline{L}(t)}\beta(t)|\varphi_x|^2\,dx$$
$$\leq c_a\int_0^{\overline{L}(t)}|\varphi|^2\,dx+\frac{c_b}{4\epsilon}\int_0^{\overline{L}(t)}|\varphi|^2\,dx+\epsilon\int_0^{\overline{L}(t)}|\varphi_x|^2\,dx,$$

and this implies that

$$-\frac{d}{dt}\left(\int_0^{\overline{L}(t)}|\varphi(x,t)|^2\,dx\right)\leq 2M\int_0^{\overline{L}(t)}|\varphi(x,t)|^2\,dx. \tag{2.8}$$

Integrating (2.8) in time, we have

$$\|\varphi(\cdot,0)\|^2_{L^2(0,L(0))}\leq e^{2MT}\|\varphi(\cdot,t)\|^2_{L^2(0,\overline{L}(t))},\quad \forall t\in(0,T)$$

and

$$\frac{T}{2}\int_0^{L(0)}|\varphi(x,0)|^2\,dx\leq\int_{T/4}^{3T/4}\int_0^{\overline{L}(t)}e^{2MT}|\varphi(x,t)|^2\,dx\,dt. \tag{2.9}$$

From (2.7) and (2.9), we find (2.6) and the proof is done. □

2.3 Approximate Controllability of the Linearized System

The approximate controllability of system (2.1) is obtained as a consequence of the observability inequality seen in Proposition 2.1.

Theorem 2.2 *For any $y_0\in L^2(0,L_0)$ and $\varepsilon>0$, there exists pairs (v_ϵ,y_ϵ), with $y_\epsilon\in C^0([0,T];L^2(0,\overline{L}(t)))$ and $v_\epsilon\in L^2(\omega\times(0,T))$, satisfying (2.1) and*

$$\|y_\epsilon(\cdot,T)\|_{L^2(0,\overline{L}(T))}\leq\varepsilon\,. \tag{2.10}$$

Furthermore, v_ϵ can be found such that

$$\|v_\epsilon\|_{L^2(\omega\times(0,T))}\leq C_1\|y_0\|_{L^2(0,L_0)}, \tag{2.11}$$

where C_1 depends on $\|a\|_{L^\infty(Q_{\overline{L}})}$, $\|b\|_{L^\infty(Q_{\overline{L}})}$, $\|\overline{L}'\|_\infty$, L_*, ω, and T.

Proof Thus, let $y_0\in L^2(0,L_0)$ and $\varepsilon>0$ be given and let us introduce the functional $J_\varepsilon(\cdot;a,b,L)$ with

$$J_\varepsilon(\varphi^T;a,b,\overline{L})=\frac{1}{2}\iint_{\omega\times(0,T)}e^{-2s_0\alpha}\xi^3|\varphi|^2dxdt$$
$$+\varepsilon\|\varphi^T\|_{L^2(0,\overline{L}(T))}+(\varphi(\cdot,0),y_0)_{L^2(0,L_0)}\,,$$

for all $\varphi^T \in L^2(0, \overline{L}(T))$.

Here, φ is the solution of (2.2) associated to φ^T. Using (2.6), it is relatively easy to check that $J_\varepsilon(\cdot; a, b, \overline{L})$ is strictly convex, continuous, and coercive in $L^2(0, \overline{L}(T))$, so it possesses a unique minimum $\widehat{\varphi}_\varepsilon^T \in L^2(0, \overline{L}(T))$, whose associated solution is denoted by $\widehat{\varphi}_\varepsilon$.

Let us now introduce the control $v_\varepsilon = e^{-2s_0\alpha}\xi^3\widehat{\varphi}_\varepsilon\widetilde{1}_\omega$ and denote by y_ε the solution to (2.1) associated to v_ε, that is

$$\begin{cases} y_{\varepsilon,t} - \beta\left(\displaystyle\int_0^{\overline{L}(t)} \overline{y}(x,t)dx\right) y_{\varepsilon,xx} + a(\overline{y}, \overline{y}_x)y_\varepsilon + b(\overline{y}, \overline{y}_x)y_{\varepsilon,x} = v_\varepsilon\widetilde{1}_\omega, & (x,t) \in Q_{\overline{L}}, \\ y_\varepsilon(x,t) = 0;\ y_\varepsilon(\overline{L}(t), t) = 0, & t \in (0,T), \\ y_\varepsilon(x,0) = y_0(x), & x \in (0, L_0), \end{cases} \tag{2.12}$$

Then, either $\widehat{\varphi}_\varepsilon^T = 0$ or we can differentiate the functional at $\widehat{\varphi}_\varepsilon^T$ and obtain a necessary condition to reach a minimum at $\widehat{\varphi}_\varepsilon^T$:

$$\begin{cases} \displaystyle\iint_{\omega\times(0,T)} e^{-2s_0\alpha}\xi^3\widehat{\varphi}_\varepsilon\varphi dxdt + \varepsilon\left(\frac{\widehat{\varphi}_\varepsilon^T}{\|\widehat{\varphi}_\varepsilon^T\|_{L^2(0,L_0)}}, \varphi^T\right)_{L^2(0,L(T))} + (\widehat{\varphi}_\varepsilon(\cdot,0), y_0)_{L^2(0,L_0)} = 0 \\ \forall\varphi^T \in L^2(0, \overline{L}(T)) \end{cases} \tag{2.13}$$

Furthermore, from the inequality $J_\varepsilon(\widehat{\varphi}_\varepsilon^T) \le J_\varepsilon(0) = 0$ and (2.6), we deduce that

$$\begin{aligned} &\frac{1}{2}\iint_{\omega\times(0,T)} e^{-2s_0\alpha}\xi^3|\widehat{\varphi}_\varepsilon|^2dxdt + \varepsilon\|\widehat{\varphi}_\varepsilon^T\|_{L^2(0,\overline{L}(T))} \le -(\widehat{\varphi}_\varepsilon(0), y_0)_{L^2(0,\overline{L}(T))} \\ &\quad\le \frac{1}{4}\iint_{\omega\times(0,T)} e^{-2s_0\alpha}\xi^3|\widehat{\varphi}_\varepsilon|^2dxdt + \|y_0\|^2_{L^2(0,L_0)} \end{aligned}$$

and, consequently,

$$\begin{aligned} \|e^{s_0\alpha}\xi^{-3/2}v_\varepsilon\|^2_{L^2(\omega\times(0,T))} &\le \iint_{\omega\times(0,T)} e^{-2s_0\alpha}\xi^3|\widehat{\varphi}_\varepsilon|^2dxdt \\ &\quad+\varepsilon\|\widehat{\varphi}_\varepsilon^T\|^2_{L^2(0,L(T))} \le 4C\|y_0\|^2_{L^2(0,L_0)}. \end{aligned} \tag{2.14}$$

So,

$$\|e^{s_0\alpha}\xi^{-3/2}v_\varepsilon\|_{L^2(\omega\times(0,T))} \le 2C\|y_0\|_{L^2(0,L_0)}. \tag{2.15}$$

It is not difficulty to check that $v_\varepsilon = e^{-2s_0\alpha}\xi^3\widehat{\varphi}_\varepsilon\widetilde{1}_\omega$ is a solution of

$$\begin{cases} -\varphi_{\varepsilon,t} - \beta(t)\,\varphi_{\varepsilon,xx} + a(x,t)\varphi_\varepsilon + b(x,t)\varphi_{\varepsilon,x} = 0, & (x,t) \in Q_L, \\ \varphi_\varepsilon(0,t) = 0;\ \varphi_\varepsilon(L(t),t) = 0, & t \in (0,T), \\ \varphi_\varepsilon(x,T) = -\frac{1}{\varepsilon}y_\varepsilon(x,T), & x \in (0, L(T)). \end{cases};\tag{2.16}$$

Multiplying both sides of the first equation of (2.12) by $\widehat{\varphi}_\varepsilon$ and integrating it on $Q_{\overline{L}}$, we obtain

$$\frac{1}{\varepsilon}\int_0^{\overline{L}(T)} |y_\varepsilon(T)|^2 dx + \iint_{\omega\times(0,T)} e^{-2s_0\alpha}\xi^3|\widehat{\varphi}_\varepsilon|^2 dxdt = -(\widehat{\varphi}_\varepsilon(\cdot,0), y_0)_{L^2(0,L_0)},$$

which implies that

$$\frac{1}{\varepsilon}\int_0^{\overline{L}(T)} |y_\varepsilon(T)|^2 dx + \iint_{\omega\times(0,T)} e^{-2s_0\alpha}\xi^3|\widehat{\varphi}_\varepsilon|^2 dxdt \le C\|y_0\|^2_{L^2(0,L_0)} \tag{2.17}$$

and therefore

$$\|y_\varepsilon\|_{L^2(0,\overline{L}(T))} \le \varepsilon\, C\|y_0\|_{L^2(0,L_0)}\,. \tag{2.18}$$

As $\varepsilon \to 0$, $y_\varepsilon(T) \to 0$ in $L^2(0,\overline{L}(T))$ and therefore v_ε is an approximated null-control for (2.1).

The proof of Theorem 2.2 is completed. □

An immediate consequence of Theorem 2.2 is the following null controllability:

Corollary 2.1 *For any* $y_0 \in L^2(0,L_0)$, *there exists pairs* (v,y), *with* $y \in C^0([0,T];L^2(0,\overline{L}(t)))$ *and* $v \in L^2(\omega\times(0,T))$, *satisfying (2.1) and* $y(x,T)=0$, $\forall x \in (0,\overline{L}(T))$. *Furthermore, v can be found such that*

$$\|v\|_{L^2(\omega\times(0,T))} \le C_2\|y_0\|_{L^2(0,L_0)}, \tag{2.19}$$

where C_2 *depends on* $\|a\|_{L^\infty(Q_{\overline{L}})}$, $\|b\|_{L^\infty(Q_{\overline{L}})}$, $\|\overline{L}'\|_\infty$, L_*, ω, *and* T.

2.4 A Regularity Property

Let (v,y) be a control-state pair furnished by Corollary 2.1. We will see in this section that, for some $\theta \in [0,1)$ only depending on $\|\overline{L}'\|_\infty$, L_*, L_*, ω and T, one has

$$y \in C^{2+\theta,1+\theta/2}_{x,t}\left(\overline{Q_{\overline{L}}}\right) \quad\text{and}\quad v \in C^{\theta,\frac{\theta}{2}}_{x,t}(\overline{Q_{\overline{L}}}) \tag{2.20}$$

where $C^{m+\theta,(m+\theta)/2}_{x,t}\left(\overline{Q_{\overline{L}}}\right)$ is the space of functions $u : \overline{Q_{\overline{L}}} \to \mathbb{R}$ such that $D_t^r D_x^s u(x,t)$ is continuous in $\overline{Q_{\overline{L}}}$ for $2r+s < m+\theta$, with m a non-negative integer, and the norm is given by

$$\|u\|_{C^{m+\theta,(m+\theta)/2}_{x,t}(\overline{Q_{\overline{L}}})} = \sum_{2r+s\le m} \|D_t^r D_x^s u(x,t)\|_\infty$$
$$+ \sum_{2r+s=m}\left(\sup_{(x,t),(x',t')\in\overline{Q_{\overline{L}}}} \frac{|D_t^r D_x^s u(x,t) - D_t^r D_x^s u(x',t')|}{|x-x'|^\theta + |t-t'|^{\theta/2}}\right) < +\infty.$$

In the sequel, we recall some relevant Lemmas (their proofs will be given in Appendix B):

Lemma 2.2 *Assume that* $\beta \in C_b^1(\mathbb{R})$, $g \in C_b^2(\mathbb{R}^2)$, $\overline{L} \in C^{1+\frac{1}{4}}([0,T])$, *and* $\overline{y} \in C_{x,t}^{2,1}(\overline{Q_{\overline{L}}})$.

(i) *If* $\beta(t) = \beta\left(\int_0^{\overline{L}(t)} \overline{y}(x,t)\,dx\right)$, *then* $\beta \in C^\theta([0,T])$, *for all* $0 \le \theta < 1$;

(ii) *If* $a(x,t) = \int_0^1 \frac{\partial g}{\partial s}(\lambda\overline{y}(x,t), \lambda\overline{y}_x(x,t))\,d\lambda$ *and* $b(x,t) = \int_0^1 \frac{\partial g}{\partial p}(\lambda\overline{y}(x,t), \lambda\overline{y}_x(x,t))\,d\lambda$, *then one has* $a, b \in C_{x,t}^{\theta,\theta/2}(\overline{Q_{\overline{L}}})$, *for all* $0 \le \theta < 1$;

(iii) *If* $\widetilde{a}(\xi,t) = a(L(t)\xi,t)$, *then* $\widetilde{a} \in C_{\xi,t}^{\theta,\theta/2}(\overline{Q})$, *for all* $0 \le \theta < 1$. *If we consider the function* $\widetilde{b}(\xi,t) = b(L(t)\xi,t) - \frac{\xi L'}{L}$, *with* $L \in C^{1+\gamma/2}([0,T])$, *then* $\widetilde{b} \in C_{\xi,t}^{\gamma,\gamma/2}(\overline{Q})$, *for all* $0 \le \gamma < 1$.

Lemma 2.3 *If* $\overline{L} \in C^{1+\frac{1}{4}}([0,T])$, $\overline{y} \in C_{x,t}^{2,1}(\overline{Q_{\overline{L}}})$ *and* $\beta \in C_b^1(\mathbb{R})$, *then the function* $L : [0,T] \to \mathbb{R}$ *given by*

$$L(t) = L_0 - \int_0^t \left[\beta\left(\int_0^{\overline{L}(s)} y(x,s)dx\right) y_x(\overline{L}(s),s)\right] ds$$

is such that $L \in C^{1+\theta}([0,T])$, *with* $0 < \theta < 1$.

Now, we will apply a standard technique that leads to the construction of a control-state with the required regularity (similar ideas were used in [30]). In this way, let us detail the following steps:

Step 1 Control regularity

From Sect. 2.3, $v_\varepsilon := e^{-2s_0\alpha}\xi^3\widehat{\varphi}_\varepsilon\widetilde{1}_\omega$, where $\widehat{\varphi}_\varepsilon$ is solution to

$$\begin{cases} -\widehat{\varphi}_{\varepsilon,t} - \beta(t)\widehat{\varphi}_{\varepsilon,xx} + a(x,t)\widehat{\varphi}_\varepsilon + b(x,t)\widehat{\varphi}_{\varepsilon,x} = 0, & (x,t) \in Q_L, \\ \widehat{\varphi}_\varepsilon(0,t) = 0; \widehat{\varphi}_\varepsilon(L(t),t) = 0, & t \in (0,T), \\ \widehat{\varphi}_\varepsilon(x,T) = \widehat{\varphi}_\varepsilon^T(x), & x \in (0,L_0), \end{cases} . \tag{2.21}$$

Let us introduce:

$$\widetilde{\alpha}(x,t) = \min_{\substack{x\in(0,L(t))\\0<t<T}} \left\{e^{2\lambda_0\|\alpha_1\|_\infty} - e^{\lambda_0\alpha_1(x,t)}\right\} \text{ and } \overline{\alpha}(x,t) = \max_{\substack{x\in(0,L(t))\\0<t<T}} \left\{e^{2\lambda_0\|\alpha_1\|_\infty} - e^{\lambda_0\alpha_1(x,t)}\right\},$$

where α_1 was given in (2.3). Let δ be a real number such that $0 < \delta \le 1/4$ with $2\widetilde{\alpha} - (1+\delta)\overline{\alpha} > 0$.
Thus, one has

$$\alpha \le \frac{\overline{\alpha}}{\gamma} \le (1+\delta)\frac{\overline{\alpha}}{\gamma}.$$

Let us consider $z_\varepsilon = e^{-s_0(1+\delta)\frac{\overline{\alpha}}{\gamma}} \dfrac{1}{\gamma^3}\widehat{\varphi}_\varepsilon$, such that z_ε satisfies:

$$\begin{cases} -z_{\varepsilon,t} - \beta(t) z_{\varepsilon,xx} + a(x,t) z_\varepsilon + b(x,t) z_{\varepsilon,x} = F_\varepsilon, & (x,t) \in Q_L, \\ z_\varepsilon(0,t) = 0; \, z_\varepsilon(L(t),t) = 0, & t \in (0,T), \\ z_\varepsilon(x,T) = 0, & x \in (0,L_0), \end{cases} , \tag{2.22}$$

where $F_\varepsilon = \left(e^{-s_0(1+\delta)\frac{\overline{\alpha}}{\gamma}} \frac{1}{\gamma^3}\right)_t \widehat{\varphi}_\varepsilon$. Observe that

$$|F_\varepsilon|^2 = \left| \left(e^{-s_0(1+\delta)\frac{\overline{\alpha}}{\gamma}} \frac{1}{\gamma^3}\right)_t \widehat{\varphi}_\varepsilon \right|^2 = \left| e^{-s_0(1+\delta)\frac{\overline{\alpha}}{\gamma}} \left(\frac{\gamma'}{\gamma^2}\frac{1}{\gamma^3} - \frac{3\gamma'}{\gamma^4}\right) \widehat{\varphi}_\varepsilon \right|^2,$$

Then, we obtain

$$\begin{aligned} |F_\varepsilon|^2 &\leq C \left| e^{-2s_0\frac{\overline{\alpha}}{\gamma}} \left(\frac{1}{\gamma^5} e^{-s_0\delta\frac{\overline{\alpha}}{\gamma}} \right) \widehat{\varphi}_\varepsilon \right|^2 \\ &\leq C \left| e^{-2s_0\alpha} \xi^3 \left(e^{-s_0\delta\frac{\overline{\alpha}}{\gamma}} \frac{1}{\gamma^2} \right) \widehat{\varphi}_\varepsilon \right|^2 \\ &\leq C_1 e^{-2s_0\alpha} \xi^3 |\widehat{\varphi}_\varepsilon|^2, \end{aligned}$$

whence

$$\iint_{Q_L} |F_\varepsilon|^2 dx dt \leq C_1 \iint_{Q_L} e^{-2s_0\alpha} \xi^3 |\widehat{\varphi}_\varepsilon|^2 dx dt.$$

From Carleman inequality and (2.15), we have

$$\iint_{Q_L} |F_\varepsilon|^2 dx dt \leq C_2 \iint_{\omega\times(0,T)} e^{-2s_0\alpha} \xi^3 |^2\widehat{\varphi}_\varepsilon|^2 dx dt \leq C\|y_0\|^2_{L^2(0,L_0)}.$$

Let us now consider the following change of variables:

$$\zeta = \frac{x}{L(t)}, \quad \widetilde{a}(\zeta,t) = a(L(t)\zeta,t), \quad \widetilde{b}(\zeta,t) = b(L(t)\zeta,t) - \zeta\frac{L'(t)}{L(t)},$$
$$\eta_\varepsilon(\zeta,t) = z_\varepsilon(x,t) \quad \text{and} \quad G_\varepsilon(\zeta,t) = F_\varepsilon(x,t).$$

We have $\eta_\varepsilon(\zeta,t)$ is well defined in $Q := (0,1)\times(0,T)$ and, moreover,

$$\begin{cases} -\eta_{\varepsilon,t} - \dfrac{\beta(t)}{L^2(t)}\eta_{\varepsilon,\zeta\zeta} + \widetilde{a}(\zeta,t)\eta_\varepsilon + \widetilde{b}(\zeta,t)\eta_{\varepsilon,\zeta} = G_\varepsilon(\zeta,t), & (\zeta,t) \in Q, \\ \eta_\varepsilon(0,t) = 0; \, \eta_\varepsilon(1,t) = 0, & t \in (0,T), \\ \eta_\varepsilon(\zeta,T) = 0, & \zeta \in (0,1), \end{cases} . \tag{2.23}$$

Since $G_\varepsilon(\zeta,t) \in L^2((0,1)\times(0,T))$, then $\eta_\varepsilon(\zeta,t) \in W_2^{2,1}((0,1)\times(0,T))$, and

$$\|\eta_\varepsilon\|_{W_2^{2,1}(Q)} \leq C\|G_\varepsilon\|_{L^2(Q)} \leq C\|F_\varepsilon\|_{L^2(Q_L)}.$$

Since $W_2^{2,1}((0,1)\times(0,T)) \hookrightarrow C_{\zeta,t}^{1/2,1/4}([0,1]\times[0,T])$ is a continuous embedding, then

$$\|\eta_\varepsilon\|_{C_{\zeta,t}^{1/2,1/4}(\overline{Q})} \le C\|F_\varepsilon\|_{L^2(Q_L)} \le C\|y_0\|_{L^2(0,L_0)}.$$

Therefore, from the change of variables, $z_\varepsilon \in C_{x,t}^{1/2,1/4}\left(\overline{Q}\right)$, and $\|z_\varepsilon\|_{C_{x,t}^{1/2,1/4}(\overline{Q})} \le C\|y_0\|_{L^2(0,L_0)}$, thus, $v_\varepsilon = e^{-2s_0\alpha}\xi^3\gamma^3 e^{s_0(1+\delta)\frac{\overline{\alpha}}{\gamma}} z_\varepsilon \widetilde{1}_\omega \in C_{x,t}^{1/2,1/4}\left(\overline{Q_L}\right)$. Observe that:

$$e^{-2s_0\alpha}\xi^3\gamma^3 e^{s_0(1+\delta)\frac{\overline{\alpha}}{\gamma}} \le \xi^3\gamma^3 e^{\frac{-s_0}{\gamma}(2\widetilde{\alpha}-(1+\delta)\overline{\alpha})},$$

where the right-hand side of the last inequality is bounded, and consequently,

$$\|v_\varepsilon\|_{C_{x,t}^{1/2,1/4}(\overline{Q_L})} \le C\|y_0\|_{L^2(0,L_0)}.$$

Step 2 State regularity

If $\beta \in C^1(\mathbb{R})$, $L \in C^{1+\frac{1}{4}}([0,T])$, and $\overline{y} \in C_{x,t}^{2,1}\left(\overline{Q_L}\right)$, then, from Lemma 2.2, the functions

$$\beta(t) = \beta\left(\int_0^{L(t)} \overline{y}(s,t)ds\right), \quad a(x,t) = \int_0^1 \frac{\partial g}{\partial s}\left(\lambda\overline{y}(x,t), \lambda\overline{y}_x(x,t)\right)d\lambda,$$

and

$$b(x,t) = \int_0^1 \frac{\partial g}{\partial p}\left(\lambda\overline{y}(x,t), \lambda\overline{y}_x(x,t)\right)d\lambda$$

satisfy

$$\beta(t) \in C^\theta([0,T]), \text{ and } a(x,t), b(x,t) \in C_{x,t}^{\theta,\theta/2}\left(\overline{Q_L}\right), \text{ for all } 0 \le \theta < 1, \tag{2.24}$$

therefore, y_ε is solution to

$$\begin{cases} y_{\varepsilon,t} - \beta(t)\, y_{\varepsilon,xx} + a(x,t)y_\varepsilon + b(x,t)y_{\varepsilon,x} = v_\varepsilon\widetilde{1}_\omega, & (x,t) \in Q_L, \\ y_\varepsilon(0,t) = 0;\ y_\varepsilon(L(t),t) = 0, & t \in (0,T), \\ y_\varepsilon(x,0) = y_0(x), & x \in (0,L_0), \end{cases} \tag{2.25}$$

Since $L \in C^{1+1/4}([0,T])$, from (2.24) and change of variables $\zeta = \dfrac{x}{L(t)}$, $\widetilde{a}(\zeta,t) = a(L(t)\zeta,t)$, and $\widetilde{b}(\zeta,t) = b(L(t)\zeta,t) - \zeta\dfrac{L'(t)}{L(t)}$, we have

$$v_\varepsilon \in C_{x,t}^{1/2,1/4}\left(\overline{Q_L}\right), \quad \widetilde{a}, \widetilde{b} \in C_{\zeta,t}^{1/2,1/4}\left(\overline{Q}\right), \text{ and } \quad \frac{\beta(t)}{(L(t))^2} \in C^{1/4}([0,T]),$$

thus, w_ε is solution to

$$\begin{cases} w_{\varepsilon,t} - \dfrac{\beta(t)}{L^2(t)} w_{\varepsilon,\zeta\zeta} + \widetilde{a}(\zeta,t) w_\varepsilon + \widetilde{b}(\zeta,t) w_{\varepsilon,\zeta} = \widetilde{v}_\varepsilon(\zeta,t), & (\zeta,t) \in Q, \\ w_\varepsilon(0,t) = 0;\ w_\varepsilon(1,t) = 0, & t \in (0,T), \\ w_\varepsilon(\zeta,0) = y_0(L(0)\zeta, 0), & \zeta \in (0,1), \end{cases}.$$

From Theorem 5.2, Chap. IV, p. 320 [28], the function $w_\varepsilon \in C^{2+1/2,1+1/4}_{\zeta,t}\left(\overline{Q}\right)$ and

$$\|w_\varepsilon\|_{C^{2+1/2,1+1/4}_{\zeta,t}(\overline{Q})} \leq C\left(\|\tilde{v}_\varepsilon\|_{C^{1/2,1/4}_{\zeta,t}(\overline{Q})} + \|y_0\|_{C^{2+1/2}(0,1)}\right)$$

Therefore, for $\theta = 1/2$ fixed, we have $y_\varepsilon \in C^{2+1/2,1+1/4}_{x,t}\left(\overline{Q_{\overline{L}}}\right)$ and

$$\|y_\varepsilon\|_{C^{2+1/2,1+1/4}_{x,t}(\overline{Q_{\overline{L}}})} \leq C\left(\|y_0\|_{L^2(0,L_0)} + \|y_0\|_{C^{2+1/2}(0,L_0)}\right) \tag{2.26}$$

3 Main Result

This section is devoted to prove the main result, namely, TheoremReferences [11, 14, 16, 17, 21 and 29] were provided in the reference list; however, these were not mentioned or cited in the manuscript. As a rule, if a citation is present in the text, then it should be present in the list. Please provide the location of where to insert the reference citation in the main body text. Kindly ensure that all references are cited in ascending numerical order. 1.1. It will be a consequence of Theorem 2.2 and a fixed-point argument (Fig. 1).

For this purpose, let $(\overline{y}, \overline{L}) \in C^{2,1}_{x,t}(\overline{Q_{\overline{L}}}) \times C^{1+\frac{1}{4}}([0,T])$ be given, with $L_* \leq \overline{L}(t) \leq B$, $\overline{L}(0) = L_0$ and $y_0 \in C^{2+\frac{1}{2}}([0, L_0])$.

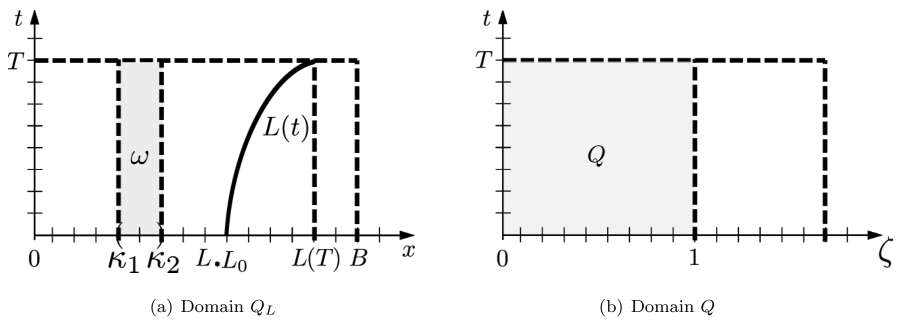

Fig. 1 **a** Domain Q_L; **b** Domain Q

Next, we recall the problem given by (2.12) with an additional condition

$$\begin{cases} y_{\varepsilon,t} - \beta\left(\int_0^{\overline{L}(t)} \overline{y}(x,t)dx\right) y_{\varepsilon,xx} + a(\overline{y},\overline{y}_x)y_\varepsilon + b(\overline{y},\overline{y}_x)y_{\varepsilon,x} = v_\varepsilon \widetilde{1}_\omega, & (x,t)\in Q_{\overline{L}}, \\ y_\varepsilon(0,t)=0;\ y_\varepsilon(\overline{L}(t),t)=0, & t\in(0,T), \\ y_\varepsilon(x,0)=y_0(x), & x\in(0,L_0), \\ \|y_\varepsilon(\cdot,T)\|_{L^2(0,\overline{L}(T))} \le \varepsilon & \end{cases} \tag{3.1}$$

Let us introduce the sets

$$\mathsf{Y} = \left\{\overline{y}\in C^{2,1}_{x,t}\left(\overline{Q_{\overline{L}}}\right) : \|\overline{y}\|_{C^{2,1}_{x,t}\left(\overline{Q_{\overline{L}}}\right)} \le R\right\} \quad \text{and}$$

$$\mathsf{Z} = \left\{\overline{L}\in C^{1+\frac14}([0,T]) : L_* \le \overline{L}(t) \le B,\ \overline{L}(0)=L_0,\ \|\overline{L}\|_{C^{1+\frac14}([0,T])} \le R\right\}.$$

where the constant $R > 0$ will be determined later.

Now, we set the mapping

$$\begin{array}{rcl} \mathsf{H}_\varepsilon : \mathsf{Y}\times\mathsf{Z} & \longrightarrow & C^{2,1}_{x,t}(\overline{Q_{\overline{L}}}) \times C^{1+\frac14}([0,T]) \\ (\overline{y},\overline{L}) & \longmapsto & (y_\varepsilon, L_\varepsilon) \end{array},$$

where y_ε satisfies (3.1) for $v_\varepsilon = e^{-2s_0\alpha}\xi^3\widehat{\varphi}_\varepsilon\widetilde{1}_\omega$, $\widehat{\varphi}_\varepsilon$ is the unique minimum of $J_\varepsilon(\cdot, g(\overline{y},\overline{y}_x),\overline{L})$ and

$$L_\varepsilon(t) = L_0 - \int_0^t \left[\beta\left(\int_0^{\overline{L}(s)} \overline{y}(x,s)\ dx\right) y_{\varepsilon,x}(\overline{L}(s),s)\right] ds. \tag{3.2}$$

Our goal is to prove that H_ε satisfies the hypothesis of the Schauder's Fixed Point Theorem. We can verify from the results in Sect. 2 that the mapping H_ε is well defined.

For $\|y_0\|_{C^{2+\frac12}([0,L_0])}$ sufficiently small, from (2.26) we have that

$$\|y_\varepsilon\|_{C^{2,1}_{x,t}(\overline{Q_{\overline{L}}})} \le C\left(\|y_0\|_{L^2(0,L_0)} + \|y_0\|_{C^{2+\frac12}([0,L_0])}\right) \le C_1\|y_0\|_{C^{2+\frac12}([0,L_0])} \le R_1\,.$$

Furthermore, we have the following estimate:

$$\begin{aligned} \left|\int_0^{\overline{L}(s)} \overline{y}(x,s)\ dx\right| &\le \int_0^{\overline{L}(s)} \|\overline{y}\|_{C^0(\overline{Q_{\overline{L}}})}\ dx \\ &\le \|\overline{y}\|_{C^0(\overline{Q_{\overline{L}}})}\|\overline{L}\|_{C^0([0,T])} \\ &\le C\|y_0\|_{C^{2+\frac12}([0,L_0])}. \end{aligned}$$

From the last estimate, one has

$$
\begin{aligned}
|L_\varepsilon(t) - L_0| &\le \int_0^t |\beta| \left| \int_0^{\overline{L}(s)} \overline{y}(x,s)\, dx \right| |y_{\varepsilon,x}(\overline{L}(s),s)|\, ds \\
&\le C_2 \|y_0\|_{C^{2+\frac{1}{2}}([0,L_0])} T.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
|L'_\varepsilon(t)| &= \left| \beta \left(\int_0^{\overline{L}(s)} \overline{y}(x,s)\, dx \right) y_{\varepsilon,x}(\overline{L}(t),t) \right| \\
&\le C_3 \|y_0\|_{C^{2+\frac{1}{2}}([0,L_0])}.
\end{aligned}
$$

In view of the Lemma 2.3, we obtain

$$
\|L_\varepsilon\|_{C^{1+\frac{1}{4}}([0,T])} \le C_4 \|y_0\|_{C^{2+\frac{1}{2}}([0,L_0])}.
$$

Now, we take $R = \min \left\{ \frac{R_1}{C_1}, \frac{L_0 - L_*}{C_2 T}, \frac{B - L_0}{C_2 T}, \frac{R_1}{C_4} \right\}$
Therefore, for $\|y_0\|_{C^{2+\frac{1}{2}}([0,L_0])} \le R$ one has

$$
\|y_\varepsilon\|_{C^{2,1}_{x,t}(\overline{Q_{\overline{L}}})} \le R, \quad \|L_\varepsilon\|_{C^{1+\frac{1}{4}}([0,T])} \le R \quad \text{and} \quad L_* \le L_\varepsilon(t) \le B.
$$

Thus, we verify that H_ε maps $\mathsf{Y} \times \mathsf{Z}$ into itself, that is

$$
\mathsf{H}_\varepsilon(\mathsf{Y} \times \mathsf{Z}) \subset \mathsf{Y} \times \mathsf{Z}.
$$

Furthermore,

$$
y_\varepsilon \in C^{2+\frac{1}{2},1+\frac{1}{4}}_{x,t}(\overline{Q_{\overline{L}}}) \hookrightarrow C^{2,1}_{x,t}(\overline{Q_{\overline{L}}}) \quad \text{compactly embedded.}
$$

From Lemma 2.3, we have that

$$
L_\varepsilon \in C^{1+\sigma}([0,T]) \hookrightarrow C^{1+\frac{1}{4}}([0,T]) \text{ compactly embedded, for } \frac{1}{4} < \sigma < 1.
$$

Therefore H_ε maps $\mathsf{Y} \times \mathsf{Z}$ into a compact set of $C^{2,1}_{x,t}(\overline{Q_{\overline{L}}}) \times C^{1+\frac{1}{4}}([0,T])$.

In view of the previous properties of H_ε, there exists $\delta > 0$ (independent of ε) such that, if $\|y_0\|_{C^{2+\frac{1}{2}}([0,L_0])} \le \delta$, we can apply Schauder's Fixed Point Theorem to the mapping $\mathsf{H}_\varepsilon : \mathsf{Y} \times \mathsf{Z} \mapsto C^{2,1}_{x,t}(\overline{Q_{\overline{L}}}) \times C^{1+\frac{1}{4}}([0,T])$.

Let $(y_\varepsilon, L_\varepsilon)$ be a fixed point of H_ε for each $\varepsilon > 0$. Then, it is clear that $(y_\varepsilon, L_\varepsilon)$, together with v_ε, satisfies (1.1),(1.2), (2.10) and (2.11).

So, we can extract subsequences indexed by ε satisfying

$$\begin{cases} y_\varepsilon \to y & \text{in} \quad C^{2,1}_{x,t}(\overline{Q_L}) \\ L_\varepsilon \to L & \text{in} \quad C^1([0,T]) \\ v_\varepsilon \rightharpoonup v & \text{in} \quad L^2(\omega \times (0,T)). \end{cases} \tag{3.3}$$

From (3.3), we can take limits in system (3.1) and deduce that y is the state associated to control v and (1.1)–(1.2) is locally null controllable.

Hence, Theorem 1.1 is proved.

4 Open Questions

As a first comment, an interest question concerns the global null controllability to (1.1)–(1.2), which does not seem to be simple. To prove a global result, we would have to use a global inverse mapping theorem, but this requires much more complicated estimates, which do not seem to be accesible.

Other important topics arise from our current research:

- In the system (1.1), we can replace the nonlocal nonlinearity $\beta\left(\int_0^{L(t)} y dx\right)$ by $\beta\left(\int_0^{L(t)} y_x dx\right)$, in order to analyze whether it is possible to prove results about null controllability.
- An interesting case deals with the null controllability of the degenerate system

$$\begin{cases} y_t - \left(\beta\left(x, \displaystyle\int_0^{L(t)} y dx\right) y_x\right)_x + g(y, y_x) = v 1_\omega, & (x,t) \in Q_L, \\ y(0,t) = 0; \; y(L(t),t) = 0, & t \in (0,T), \\ y(x,0) = y_0(x), & x \in (0, L_0), \end{cases} \tag{4.1}$$

with the additional boundary condition

$$-L'(t) = \beta\left(x, \int_0^{L(t)} y(x,t) dx\right) y_x(L(t),t),$$

where β is a separated variables function given by $\beta(x,r) = \ell(r)a(x)$ and β defines an operator which degenerates at $x = 0$ and has a nonlocal term. More precisely, the function a behaves x^α, with $\alpha \in (0,1)$.

On the controllability of degenerate parabolic equations, for an instance, we mention the following works: Cannarsa et al. [4–8], Alabau-Boussouira et al. [1] and Demarque et al. [10].

- Another interesting case is found when the control function acts on the free boundary, as we can see in the system below:

$$\begin{cases} y_t - \beta\left(\int_0^{L(t)} y dx\right) y_{xx} + g(y, y_x) = 0, & (x,t) \in Q_L, \\ y(0,t) = 0; \; y(L(t),t) = v(t), & t \in (0,T), \\ y(x,0) = y_0(x), & x \in (0, L_0), \end{cases} \tag{4.2}$$

  together with (1.2) and (1.3).
  However, this control problem needs a deeper analysis.

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Declarations

Competing interest The authors have not disclosed any competing interests.

Appendix A: Proof of Theorem 2.1

Let ψ be new variable defined by $\varphi(x,t) := e^{s\alpha(x,t)}\psi(x,t)$. It yields:

$$\varphi_t = s\alpha_t e^{s\alpha}\psi + e^{s\alpha}\psi_t, \quad \varphi_x = s\alpha_x e^{s\alpha}\psi + e^{s\alpha}\psi_x \quad \text{and}$$
$$\varphi_{xx} = e^{s\alpha}\left(\psi_{xx} + s^2\alpha_x^2\psi + 2s\alpha_x\psi_x + s\alpha_{xx}\psi\right).$$

The functions α and ξ yield:

$$\begin{aligned} \alpha_x &= -\xi_x = -\lambda\alpha_{0,x}\xi, \quad \alpha_{xx} = -\lambda^2\alpha_{0,x}^2\xi - \lambda\alpha_{0,xx}\xi, \\ \alpha_t &= -k\frac{T-2t}{(T-t)^{k+1}t^{k+1}}\left(e^{2\lambda\|\alpha_1\|_\infty} - e^{\lambda\alpha_1(x,t)}\right) - \lambda\alpha_{0,t}\xi, \\ \alpha_{x,t} &= k\lambda\xi\left(\frac{(T-2t)}{t(T-t)}\alpha_{0,x} - \frac{\alpha_{0,tx}}{k} - \lambda\frac{\alpha_{0,t}\alpha_{0,x}}{k}\right) \end{aligned}$$

and, for λ sufficiently large, one has:

$$\begin{aligned} |\alpha_x| &\leq C\lambda\xi, \quad |\alpha_{xx}| \leq C\lambda^2\xi, \\ |\alpha_t| &\leq C\xi^{1+\frac{1}{k}} + C\lambda\xi \leq C\lambda\xi^{\frac{3}{2}} \quad \text{and} \\ |\alpha_{x,t}| &\leq C\lambda\xi^2. \end{aligned}$$

On the other hand, we have that $\psi(x,0) \equiv 0$ in $(0, L(0))$ and $\psi(x,T) \equiv 0$ in $(0, \overline{L}(T))$.
Moreover, replacing φ by $e^{s\alpha}\psi$ in the PDE (2.2), it yields:

$$e^{-2s\alpha}(s\alpha_t\psi + \psi_t) + e^{-2s\alpha}\Big[\beta(t)\psi_{xx} + \beta(t)s^2\alpha_x^2\psi + 2\beta(t)s\alpha_x\psi_x$$

$$+\beta(t)\, s\alpha_{xx}\psi\Big] + e^{s\alpha}(a\psi) + e^{s\alpha}\Big[s\alpha_x(b\psi) + (b\psi_x)\Big] = F(x,t),$$

therefore,

$$\begin{aligned}&\Big[\psi_t + 2\beta(t)\, s\alpha_x\psi_x\Big] + \Big[\beta(t)\,\psi_{xx} + \beta(t)\, s^2\alpha_x^2\psi\Big]\\ &\quad = -e^{-s\alpha}F(x,t) - s\alpha_t\psi - \beta(t)\, s\alpha_{xx}\psi - a\psi - s\alpha_x b\psi - b\psi_x.\end{aligned}$$

Considering the following notation

$$\begin{cases} U\psi := \psi_t + 2\beta(t)\, s\alpha_x\psi_x = (U\psi)_1 + (U\psi)_2 \\ V\psi := \beta(t)\,\psi_{xx} + \beta(t)\, s^2\alpha_x^2\psi = (V\psi)_1 + (V\psi)_2 \end{cases},$$

it yields:

$$U\psi + V\psi = G(x,t),$$

where

$$\begin{aligned}G(x,t) = &-e^{-s\alpha(x,t)}F(x,t) - s\alpha_t(x,t)\psi(x,t) - \beta(t)s\alpha_{xx}(x,t)\psi(x,t)\\ &-a(x,t)\psi(x,t) - s\alpha_x(x,t)b(x,t)\psi(x,t) - b(x,t)\psi_x(x,t).\end{aligned}$$

Therefore,

$$\begin{aligned}&\|U\psi\|^2_{L^2(Q_{\overline{L}})} + \|V\psi\|^2_{L^2(Q_{\overline{L}})} + 2(U\psi, V\psi)_{L^2(Q_{\overline{L}})}\\
&\quad = \|e^{-s\alpha}F + s\alpha_t\psi + \beta s\alpha_{xx}\psi + a\psi + s\alpha_x b\psi + b\psi_x\|^2_{L^2(Q_{\overline{L}})}\\
&\quad \le C\Big(\|e^{-2s\alpha}F\|^2_{L^2(Q_{\overline{L}})} + \|s\alpha_t\psi\|^2_{L^2(Q_{\overline{L}})} + \|\beta s\alpha_{xx}\psi\|^2_{L^2(Q_{\overline{L}})}\\
&\qquad + \|a\psi\|^2_{L^2(Q_{\overline{L}})} + \|s\alpha_x b\psi\|^2_{L^2(Q_{\overline{L}})} + \|b\psi_x\|^2_{L^2(Q_{\overline{L}})}\Big)\\
&\quad \le C\Big(\iint_{Q_{\overline{L}}} e^{-2s\alpha}|F|^2 dxdt + \iint_{Q_{\overline{L}}} s^2|\alpha_t|^2|\psi|^2 dxdt\\
&\qquad + \iint_{Q_{\overline{L}}} \beta s^2|\alpha_{xx}|^2|\psi|^2 dxdt + \iint_{Q_{\overline{L}}} |a|^2_{L^\infty(Q_{\overline{L}})}|\psi|^2 dxdt\\
&\qquad + \iint_{Q_{\overline{L}}} |b|^2_{L^\infty(Q_{\overline{L}})} s^2|\alpha_x|^2|\psi|^2 dxdt + \iint_{Q_{\overline{L}}} |b|^2_{L^\infty(Q_{\overline{L}})}|\psi_x|^2 dxdt\Big)\\
&\quad \le C\Big(\iint_{Q_{\overline{L}}} e^{-2s\alpha}|F|^2 dxdt + \iint_{Q_{\overline{L}}} s^2\widetilde{C}\lambda^2\xi^3|\psi|^2 dxdt\\
&\qquad + C_\beta\iint_{Q_{\overline{L}}} s^2\widetilde{C}\lambda^4\xi^2|\psi|^2 dxdt + |a|^2_{L^\infty(Q_{\overline{L}})}\iint_{Q_{\overline{L}}} |\psi|^2 dxdt\end{aligned}$$

$$+|b|^2_{L^\infty(Q_{\overline{L}})} \iint_{Q_{\overline{L}}} s^2\lambda^2|\alpha_{0,x}|^2\xi^2|\psi|^2 dxdt + |b|^2_{L^\infty(Q_{\overline{L}})} \iint_{Q_{\overline{L}}} |\psi_x|^2 dxdt\Big)$$
$$\leq C\Big(\iint_{Q_{\overline{L}}} e^{-2s\alpha}|F|^2 dxdt + \iint_{Q_{\overline{L}}} s^2\lambda^4\xi^3|\psi|^2 dxdt + \iint_{Q_{\overline{L}}} |\psi_x|^2 dxdt\Big). \tag{4.3}$$

Let us compute $(U\psi, V\psi)_{L^2(Q_{\overline{L}})}$ and use the fact that $\iint_{Q_{\overline{L}}} = \int_0^T \int_0^{\overline{L}(t)}$.

$$\begin{aligned}
I_1 &= ((U\psi)_1, (V\psi)_1)_{L^2(Q_{\overline{L}})} = \int_0^T \int_0^{\overline{L}(t)} \beta\psi_t\psi_{xx} dxdt \\
&= \int_0^T \beta\left(\psi_t\psi_x\Big|_0^{\overline{L}(t)} - \int_0^{\overline{L}(t)} \psi_x\psi_{xt} ds\right) dt \\
&= -\frac{1}{2}\int_0^T \beta \int_0^{\overline{L}(t)} \frac{d}{dt}[\psi_x]^2 dxdt \\
&= -\frac{1}{2}\int_0^T \left[\beta\frac{d}{dt}\int_0^{\overline{L}(t)} (\psi_x)^2 dx - \beta\left[\psi_x(\overline{L}(t), t)\right]^2 \overline{L}'(t)\right] dt \\
&= \frac{1}{2}\int_0^T \int_0^{\overline{L}(t)} \beta'(\psi_x)^2 dxdt + \frac{1}{2}\int_0^T \beta\left[\psi_x(\overline{L}(t), t)\right]^2 \overline{L}'(t) dt; \\
I_2 &= ((U\psi)_1, (V\psi)_2)_{L^2(Q_{\overline{L}})} = \int_0^T \int_0^{\overline{L}(t)} s^2\beta\psi_t\alpha_x^2\psi dxdt \\
&= \int_0^T \int_0^{\overline{L}(t)} s^2\beta\psi_t\alpha_x^2\psi dxdt \\
&= \frac{s^2}{2}\int_0^T \beta\int_0^{\overline{L}(t)} \frac{d}{dt}(\psi)^2\alpha_x^2 dxdt \\
&= -\frac{s^2}{2}\int_0^T \int_0^{\overline{L}(t)} \left(\beta'\alpha_x^2 + 2\beta\alpha_x\alpha_{xt}\right)\psi^2 dxdt; \\
I_3 &= ((U\psi)_2, (V\psi)_1)_{L^2(Q_{\overline{L}})} = \int_0^T \int_0^{\overline{L}(t)} 2s\beta^2\alpha_x\psi_x\psi_{xx} dxdt \\
&= \int_0^T \int_0^{\overline{L}(t)} s\beta^2\alpha_x\frac{d}{dx}(\psi_x)^2 dxdt \\
&= -\int_0^T \int_0^{\overline{L}(t)} s\beta^2\alpha_{xx}\psi_x^2 dxdt + \int_0^T s\psi_x^2(\overline{L}(t), t)\beta^2\alpha_x(\overline{L}(t), t) dt \\
&\quad - \int_0^T s\psi_x^2(0, t)\beta^2\alpha_x(0, t) dt; \\
I_4 &= ((U\psi)_2, (V\psi)_2)_{L^2(Q_{\overline{L}})} = \int_0^T \int_0^{\overline{L}(t)} 2s^3\beta^2\alpha_x^3\psi\psi_x dxdt
\end{aligned}$$

$$= s^3 \int_0^T \int_0^{\overline{L}(t)} \beta^2 \alpha_x^3 \frac{d}{dx}(\psi)^2 dxdt = -3s^3 \int_0^T \int_0^{\overline{L}(t)} \beta^2 \alpha_x^2 \alpha_{xx} \psi^2 dxdt.$$

Therefore,

$$\begin{aligned}(U\psi, V\psi)_{L^2(Q_{\overline{L}})} = &-\int_0^T \int_0^{\overline{L}(t)} s\beta^2 \alpha_{xx} \psi_x^2 dxdt - 3s^3 \int_0^T \int_0^{\overline{L}(t)} \beta^2 \alpha_x^2 \alpha_{xx} \psi^2 dxdt \\ &-\frac{s^2}{2} \int_0^T \int_0^{\overline{L}(t)} 2\beta \alpha_x \alpha_{xt} \psi^2 dxdt + \int_0^T s\psi_x^2(\overline{L}(t), t)\beta^2 \alpha_x(L(t), t)dt \\ &+\frac{1}{2} \int_0^T \beta \left[\psi_x(\overline{L}(t), t)\right]^2 \overline{L}'(t)dt - \int_0^T s\psi_x^2(0, t)\beta^2 \alpha_x(0, t)dt \\ &+\frac{1}{2} \int_0^T \int_0^{\overline{L}(t)} \beta'(\psi_x)^2 dxdt - \frac{s^2}{2} \int_0^T \int_0^{\overline{L}(t)} \beta' \alpha_x^2 \psi^2 dxdt.\end{aligned}$$

Now, computing $\|U\psi\|^2_{L^2(Q_{\overline{L}})} + \|V\psi\|^2_{L^2(Q_{\overline{L}})} + 2(U\psi, V\psi)_{L^2(Q_{\overline{L}})}$, we have:

$$\begin{aligned}(U\psi, V\psi)_{L^2(Q_{\overline{L}})} &= I_1 + I_2 + I_3 + I_4 \\ &\geq C\left(\frac{1}{2}\int_0^T \beta\psi_x^2(\overline{L}(t), t)\overline{L}'(t)dt + \frac{1}{2}\int_0^T \beta' \int_0^{\overline{L}(t)} \psi_x^2 dxdt\right) \\ &\quad +\left(\frac{-s^2}{2}\iint_{Q_{\overline{L}}} \beta'\alpha_x^2\psi^2 dxdt - C_2 s^2\lambda^2 \iint_{Q_{\overline{L}}} \xi^3|\psi|^2 dxdt\right) \\ &\quad +\left(\frac{sC_3}{2}\iint_{Q_{\overline{L}}} \lambda^2\xi|\psi_x|^2 dxdt\right. \\ &\quad -s\iint_{\omega_0\times(0,T)} \lambda^2 C_3\xi|\psi_x|^2 dxdt + s\lambda C_3\int_0^T \left[\xi(\overline{L}(t), t)\psi_x^2(\overline{L}(t), t)\right. \\ &\quad \left.\left. +\xi(0,t)\psi_x^2(0,t)\right]dt\right) + \left(C_4\iint_{Q_{\overline{L}}} s^3\lambda^4\xi^3\psi^2 dxdt\right. \\ &\quad \left. -\overline{C}\iint_{\omega_0\times(0,T)} s^3\lambda^4\xi^3\psi^2 dxdt\right); \end{aligned} \tag{4.4}$$

$$\begin{aligned}\|U\psi\|^2_{L^2(Q_{\overline{L}})} &= \iint_{Q_{\overline{L}}} |\psi_t + 2\beta s\alpha_x\psi_x|^2 dxdt = \iint_{Q_{\overline{L}}} \left|\psi_t + 2\beta s(\lambda\alpha_{0,x}\xi)\psi_x\right|^2 dxdt \\ &\geq C\left(\iint_{Q_{\overline{L}}} |\psi_t|^2 dxdt - \iint_{Q_{\overline{L}}} s^2\lambda^2\xi^2|\psi_x|^2 dxdt\right) \\ &\geq \widetilde{C}\left(\iint_{Q_{\overline{L}}} (s\xi)^{-1}|\psi_t|^2 dxdt - \iint_{Q_{\overline{L}}} (s\xi)\lambda^2|\psi_x|^2 dxdt\right); \end{aligned} \tag{4.5}$$

$$\begin{aligned}\|V\psi\|^2_{L^2(Q_{\overline{L}})} &= \iint_{Q_{\overline{L}}} |\beta\psi_{xx} + \beta s^2\alpha_x^2\psi|^2 dxdt \\ &= \iint_{Q_{\overline{L}}} \left|\beta\psi_{xx} - (-\beta s^2\lambda^2\alpha_{0,x}^2\xi^2\psi)\right|^2 dxdt \\ &\geq C\left(\iint_{Q_{\overline{L}}} |\psi_{xx}|^2 dxdt - \iint_{Q_{\overline{L}}} s^4\lambda^4\xi^4|\psi|^2 dxdt\right)\end{aligned}$$

$$\geq \widetilde{C}\bigg(\iint_{Q_{\overline{L}}} (s\xi)^{-1}|\psi_{xx}|^2 dxdt - \iint_{Q_{\overline{L}}} s^3\lambda^4\xi^3|\psi|^2 dxdt \bigg). \tag{4.6}$$

Combining the estimates (4.3) and (4.4), they yield:

$$\begin{aligned}
&C\bigg(\iint_{Q_{\overline{L}}} e^{-2s\alpha}|F|^2 dxdt + \iint_{Q_{\overline{L}}} s^2\lambda^4\xi^3|\psi|^2 dxdt + \iint_{Q_{\overline{L}}} |\psi_x|^2 dxdt \bigg) \\
&\geq \|U\psi\|^2_{L^2(Q_{\overline{L}})} + \|V\psi\|^2_{L^2(Q_{\overline{L}})} + 2(U\psi, V\psi)_{L^2(Q_{\overline{L}})} \\
&\quad \|U\psi\|^2_{L^2(Q_{\overline{L}})} + \|V\psi\|^2_{L^2(Q_{\overline{L}})} + \bigg(\int_0^T \beta\psi_x^2(\overline{L}(t),t)\overline{L}'(t)dt \\
&\quad + \int_0^T \beta' \int_0^{\overline{L}(t)} \psi_x^2 dxdt - s^2 \iint_{Q_{\overline{L}}} \beta'\alpha_x^2\psi^2 dxdt \\
&\quad -s^2\lambda^2 \iint_{Q_{\overline{L}}} \xi^3|\psi|^2 dxdt + s\iint_{Q_{\overline{L}}} \lambda^2\xi|\psi_x|^2 dxdt \\
&\quad +s\iint_{\omega_0\times(0,T)} \lambda^2\xi|\psi_x|^2 dxdt + s\lambda\int_0^T \Big[\xi(\overline{L}(t),t)\psi_x^2(\overline{L}(t),t) \\
&\quad - \xi(0,t)\psi_x^2(0,t)\Big]dt + \iint_{Q_{\overline{L}}} s^3\lambda^4\xi^3\psi^2 dxdt - \iint_{\omega_0\times(0,T)} s^3\lambda^4\xi^3\psi^2 dxdt \bigg).
\end{aligned} \tag{4.7}$$

Also, given $\varepsilon > 0$, we have:

$$\begin{aligned}
\iint_{\omega_0\times(0,T)} (s\xi)\lambda^2|\psi_x|^2 dxdt \ &\leq \varepsilon \iint_{Q_{\overline{L}}} (s\xi)^{-1}|\psi_{xx}|^2 dxdt \\
&\quad + C_\varepsilon \iint_{\omega_0\times(0,T)} (s\xi)^3\lambda^4|\psi|^2 dxdt
\end{aligned} \tag{4.8}$$

In fact,

$$\begin{aligned}
&\bigg| \iint_{\omega_0\times(0,T)} (s\xi)\lambda^2\psi_x^2 dxdt \bigg| = \bigg| \iint_{\omega_0\times(0,T)} s\lambda^2(\xi\psi_x)_x\psi dxdt \bigg| \\
&\leq \bigg| \iint_{\omega_0\times(0,T)} s\lambda^2\xi_x\psi_x\psi dxdt \bigg| + \bigg| \iint_{\omega_0\times(0,T)} s\lambda^2\xi\psi_{xx}\psi dxdt \bigg| \\
&\leq C\bigg| \iint_{\omega_0\times(0,T)} (s\xi)(\lambda\psi_x)(\lambda^2\psi) dxdt \bigg| + \bigg| \iint_{\omega_0\times(0,T)} (s\xi)\psi_{xx}(\lambda^2\psi) dxdt \bigg| \\
&\leq C\bigg(\varepsilon_1 \iint_{\omega_0\times(0,T)} (s\xi)|\lambda\psi_x|^2 dxdt + \frac{1}{4\varepsilon_1} \iint_{\omega_0\times(0,T)} (s\xi)|\lambda^2\psi|^2 dxdt \bigg) \\
&\quad + \bigg(\varepsilon_2 \iint_{\omega_0\times(0,T)} (s\xi)^{-1}|\psi_{xx}|^2 dxdt + \frac{1}{4\varepsilon_2} \iint_{\omega_0\times(0,T)} (s\xi)^3|\lambda^2\psi|^2 dxdt \bigg).
\end{aligned}$$

Therefore,

$$\iint_{\omega_0\times(0,T)} (s\xi)\lambda^2|\psi_x|^2 dxdt \le \frac{\varepsilon_2}{1-C\varepsilon_1}\iint_{\omega_0\times(0,T)} (s\xi)^{-1}|\psi_{xx}|^2 dxdt$$
$$+C_{\varepsilon_1\varepsilon_2}\iint_{\omega_0\times(0,T)} (s\xi)^3\lambda^4|\psi|^2 dxdt$$
$$\le \frac{\varepsilon_2}{1-C\varepsilon_1}\iint_{Q_{\overline{L}}} (s\xi)^{-1}|\psi_{xx}|^2 dxdt$$
$$+C_{\varepsilon_1\varepsilon_2}\iint_{\omega_0\times(0,T)} (s\xi)^3\lambda^4|\psi|^2 dxdt.$$

Combining the inequalities (4.5)–(4.8), they yield:

$$\iint_{Q_{\overline{L}}} (s\xi)^{-1}\left[|\psi_t|^2+|\psi_{xx}|^2\right] dxdt + \iint_{Q_{\overline{L}}}\left[(s\xi)\lambda^2|\psi_x|^2+(s\xi)^3\lambda^4|\psi|^2\right] dxdt$$
$$+(s\lambda)\int_0^T\left[\xi(\overline{L}(t),t)|\psi_x(\overline{L}(t),t)|^2+\xi(0,t)|\psi_x(0,t)|^2\right] dt$$
$$+\int_0^T \beta'\int_0^{\overline{L}(t)} |\psi_x|^2 dxdt$$
$$\le C\left(\iint_{Q_{\overline{L}}} e^{-2s\alpha}|F|^2 dxdt + \iint_{\omega\times(0,T)} s^3\lambda^4\xi^3|\psi|^2 dxdt\right.$$
$$\left. + \iint_{Q_{\overline{L}}} |\psi_x|^2 dxdt + \iint_{Q_{\overline{L}}} \beta'(s\xi)^2\lambda^2|\psi|^2 dxdt\right).$$

Therefore,

$$\iint_{Q_{\overline{L}}}\left[(s\xi)^{-1}(|\psi_t|^2+|\psi_{xx}|^2)+(s\xi)\lambda^2|\psi_x|^2+(s\xi)^3\lambda^4|\psi|^2\right] dxdt$$
$$+s\lambda\int_0^T\left[\xi(\overline{L}(t),t)|\psi_x(\overline{L}(t),t)|^2+\xi(0,t)|\psi_x(0,t)|^2\right] dt$$
$$\le C\left(\iint_{Q_{\overline{L}}} e^{-2s\alpha}|F|^2 dxdt + \iint_{\omega\times(0,T)} (s\xi)^3\lambda^4|\psi|^2 dxdt\right).$$

Coming back to the original variable φ, we have (2.4).

Appendix B: Proof of Lemma 2.2 and 2.3

Proof of Lemma 2.2 (i) For all $0 \le \theta < 1$, we have:

$$\frac{|\beta(t_1)-\beta(t_2)|}{|t_1-t_2|^\theta} = \frac{1}{|t_1-t_2|^\theta}\cdot\left|\beta\left(\int_0^{\overline{L}(t_1)} \overline{y}(x,t_1)\,dx\right)-\beta\left(\int_0^{\overline{L}(t_2)} \overline{y}(x,t_2)\,dx\right)\right|$$

$$
\begin{aligned}
&\leq \frac{C}{|t_1-t_2|^\theta}\cdot\Big|\int_0^{\overline{L}(t_1)}\overline{y}(x,t_1)\,dx-\int_0^{\overline{L}(t_2)}\overline{y}(x,t_2)\,dx\Big|\\
&\leq \frac{C}{|t_1-t_2|^\theta}\cdot\Big|\int_{\overline{L}(t_2)}^{\overline{L}(t_1)}\overline{y}(x,t_1)\,dx+\int_0^{\overline{L}(t_2)}\overline{y}(x,t_1)-\overline{y}(x,t_2)\,dx\Big|\\
&\leq \frac{C}{|t_1-t_2|^\theta}\cdot(|\overline{L}(t_1)-\overline{L}(t_2)|+|t_1-t_2|)\\
&\leq C\frac{|t_1-t_2|}{|t_1-t_2|^\theta}<\infty.
\end{aligned}
$$

Then, $\beta\in C^\theta([0,T])$, for all $0\leq\theta<1$.
(ii) For all $0\leq\theta<1$, one has:

$$
\begin{aligned}
&\frac{|a(x_1,t_1)-a(x_2,t_2)|}{(|x_1-x_2|+|t_1-t_2|^{1/2})^\theta}\\
&\leq \frac{\int_0^1\Big|\frac{\partial g}{\partial s}(\lambda\overline{y}(x_1,t_1),\lambda\overline{y}_x(x_1,t_1))-\frac{\partial g}{\partial s}(\lambda\overline{y}(x_2,t_2),\lambda\overline{y}_x(x_2,t_2))\Big|\,d\lambda}{(|x_1-x_2|+|t_1-t_2|^{1/2})^\theta}\\
&\leq C\cdot\frac{(|\overline{y}(x_1,t_1)-\overline{y}(x_2,t_2)|+|\overline{y}_x(x_1,t_1)-\overline{y}_x(x_2,t_2)|)}{(|x_1-x_2|+|t_1-t_2|^{1/2})^\theta}\\
&\leq C\cdot\frac{(|x_1-x_2|+|t_1-t_2|)}{(|x_1-x_2|+|t_1-t_2|^{1/2})^\theta}<\infty.
\end{aligned}
$$

Then, $a\in C^{\theta,\theta/2}_{x,t}(\overline{Q}_L)$, for all $0\leq\theta<1$. The proof is analogous to $b(x,t)$.
(iii) For all $0\leq\theta<1$, one has:

$$
\begin{aligned}
\frac{|\widetilde{a}(\xi_1,t_1)-\widetilde{a}(\xi_2,t_2)|}{(|\xi_1-\xi_2|+|t_1-t_2|^{1/2})^\theta}&=\frac{|a(L(t_1)\xi_1,t_1)-a(L(t_2)\xi_2,t_2)|}{(|\xi_1-\xi_2|+|t_1-t_2|^{1/2})^\theta}\\
&=\frac{|a(L(t_1)\xi_1,t_1)-a(L(t_2)\xi_2,t_2)|}{(|L(t_1)\xi_1-L(t_2)\xi_2|+|t_1-t_2|^{1/2})^\theta}\cdot\\
&\quad\frac{(|L(t_1)\xi_1-L(t_2)\xi_2|+|t_1-t_2|^{1/2})^\theta}{(|\xi_1-\xi_2|+|t_1-t_2|^{1/2})^\theta},
\end{aligned}
$$

On the other hand, from (ii), we have

$$
\frac{|a(L(t_1)\xi_1,t_1)-a(L(t_2)\xi_2,t_2)|}{(|L(t_1)\xi_1-L(t_2)\xi_2|+|t_1-t_2|^{1/2})^\theta}\leq C.
$$

Then, one has,

$$
\begin{aligned}
\frac{|\widetilde{a}(\xi_1,t_1)-\widetilde{a}(\xi_2,t_2)|}{(|\xi_1-\xi_2|+|t_1-t_2|^{1/2})^\theta}&\leq C\cdot\frac{(L(t_1)|\xi_1-\xi_2|+|\xi_2|.|L(t_1)-L(t_2)|+|t_1-t_2|^{1/2})^\theta}{(|\xi_1-\xi_2|+|t_1-t_2|^{1/2})^\theta}\\
&\leq C_1\cdot\frac{(|\xi_1-\xi_2|+|L(t_1)-L(t_2)|+|t_1-t_2|^{1/2})^\theta}{(|\xi_1-\xi_2|+|t_1-t_2|^{1/2})^\theta}
\end{aligned}
$$

$$< \infty, \quad \forall \theta \in [0,1).$$

Then, $\widetilde{a} \in C^{\theta,\theta/2}_{\xi,t}(\overline{Q})$, for all $0 \le \theta < 1$.

In a similar way, we obtain $\widehat{b}(\xi,t) = b(L(t)\xi,t) \in C^{\theta,\theta/2}_{\xi,t}(\overline{Q})$. Let us analyse the function $\overline{b}(\xi,t) = \xi \dfrac{L'(t)}{L(t)}$, for $L \in C^{1+\gamma/2}([0,T])$.

For all $0 \le \gamma < 1$,

$$\begin{aligned}
&\frac{|\overline{b}(\xi_1,t_1) - \overline{b}(\xi_2,t_2)|}{(|\xi_1-\xi_2| + |t_1-t_2|^{1/2})^\gamma} \\
&= \frac{1}{(|\xi_1-\xi_2| + |t_1-t_2|^{1/2})^\gamma} \cdot \left|\xi_1 \frac{L'(t_1)}{L(t_1)} - \xi_2 \frac{L'(t_2)}{L(t_2)}\right| \\
&\le \frac{C}{(|\xi_1-\xi_2| + |t_1-t_2|^{1/2})^\gamma} \cdot \left(|\xi_1-\xi_2| \cdot \left|\frac{L'(t_1)}{L(t_1)}\right| + |\xi_2| \cdot \left|\frac{L'(t_1)}{L(t_1)} - \frac{L'(t_2)}{L(t_2)}\right|\right) \\
&\le C_1 \cdot \frac{\left(|\xi_1-\xi_2| + |L'(t_1) - L'(t_2)| \dfrac{1}{|L(t_1)|} + |L'(t_2)| \cdot \dfrac{|L(t_2) - L(t_1)|}{|L(t_1)L(t_2)|}\right)}{(|\xi_1-\xi_2| + |t_1-t_2|^{1/2})^\gamma} \\
&\le C_2 \cdot \frac{\left(|\xi_1-\xi_2| + |L'(t_1) - L'(t_2)| + |L(t_2) - L(t_1)|\right)}{(|\xi_1-\xi_2| + |t_1-t_2|^{1/2})^\gamma} < \infty,
\end{aligned}$$

since $L \in C^{1+\gamma/2}([0,T])$.

Then, $\widetilde{b} \in C^{\gamma,\gamma/2}_{\xi,t}(\overline{Q})$, for all $0 \le \gamma < 1$. □

Proof of Lemma 2.3 For $0 \le \theta < 1$, one has:

$$\begin{aligned}
&\frac{|L'(t_1) - L'(t_2)|}{|t_1-t_2|^\theta} \\
&= \frac{1}{|t_1-t_2|^\theta} \cdot \left[\beta\left(\int_0^{\overline{L}(t_1)} y(x,t_1)dx\right) y_x(\overline{L}(t_1),t_1)\right. \\
&\quad \left. -\beta\left(\int_0^{\overline{L}(t_2)} y(x,t_2)dx\right) y_x(\overline{L}(t_2),t_2)\right] \\
&= \frac{1}{|t_1-t_2|^\theta} \cdot \left[\beta\left(\int_0^{\overline{L}(t_1)} y(x,t_1)dx\right) - \beta\left(\int_0^{\overline{L}(t_2)} y(x,t_2)dx\right)\right] y_x(\overline{L}(t_1),t_1) \\
&\quad + \frac{1}{|t_1-t_2|^\theta} \cdot \beta\left(\int_0^{\overline{L}(t_2)} y(x,t_2)dx\right) \cdot \left[y_x(\overline{L}(t_1),t_1) - y_x(\overline{L}(t_2),t_2)\right] \\
&= I_1 + I_2
\end{aligned}$$

For $0 \le \theta < 1$, we obtain the following estimates for I_1 and I_2:

$$|I_1| \le \frac{C}{|t_1-t_2|^\theta} \cdot \left|\int_0^{\overline{L}(t_1)} y(x,t_1)\,dx - \int_0^{\overline{L}(t_2)} y(x,t_2)\,dx\right|$$

$$
\begin{aligned}
&\leq \frac{C}{|t_1-t_2|^\theta}\cdot\left|\int_{\overline{L}(t_2)}^{\overline{L}(t_1)}\overline{y}(x,t_1)\,dx+\int_0^{\overline{L}(t_2)}\overline{y}(x,t_1)-\overline{y}(x,t_2)\,dx\right|\\
&\leq \frac{C_1}{|t_1-t_2|^\theta}\cdot|\overline{L}(t_1)-\overline{L}(t_2)|+c\int_0^{\overline{L}(t_2)}\frac{|\overline{y}(x,t_1)-\overline{y}(x,t_2)|}{|t_1-t_2|^\theta}\,dx\\
&\leq C_2\left(\sup_{t_1,t_2\in[0,T]}\frac{|\overline{L}(t_1)-\overline{L}(t_2)|}{|t_1-t_2|^\theta}+\sup_{t_1,t_2\in[0,T]}\frac{|\overline{y}(x,t_1)-\overline{y}(x,t_2)|}{|t_1-t_2|^\theta}\right)<\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
|I_2| &\leq \frac{C}{|t_1-t_2|^\theta}\cdot|y_x(\overline{L}(t_1),t_1)-y_x(\overline{L}(t_2),t_2)|\\
&\leq \frac{C}{|t_1-t_2|^\theta}\cdot(|\overline{L}(t_1)-\overline{L}(t_2)|+|t_1-t_2|)<\infty.
\end{aligned}
$$

□

References

1. Alabau-Boussouira, F., Cannarsa, P., Fragnelli, G.: Carleman estimates for degenerate parabolic operators with applications to null controllability. J. Evol. Equ. **2**, 161–204 (2006)
2. Aronson, D.G.: Some properties of the interface for a gas flow in porous media , In: Fasano, A., Primicerio, M. (eds.) Free Boundary Problems: Theory and Applications, Research Notes Math., No. 78, vol. I, Pitman, London (1983)
3. Barbu, V.: Exact controllability of the superlinear heat equations. Appl. Math. Optim. **42**, 73–89 (2000)
4. Cannarsa, P., Fragnelli, G.: Null controllability of semilinear degenerate parabolic equations in bounded domains. Electron. J. Differ. Equ. **136**, 1–20 (2006)
5. Cannarsa, P., Fragnelli, G., Rocchetti, D.: Null controllability of degenerate parabolic operators with drift. Netw. Heterog. Med. **2**, 695–715 (2007)
6. Cannarsa, P., Martinez, P., Vancostenoble, J.: Carleman estimates for a class of degenerate parabolic operators. SIAM J. Control. Optim. **47**, 1–19 (2008)
7. Cannarsa, P., Martinez, P., Vancostenoble, J.: Null controllability of degenerate heat equations. Adv. Differ. Equ. **10**, 153–190 (2005)
8. Cannarsa, P., Martinez, P., Vancostenoble, J.: Persistent regional null controllability for a class of degenerate parabolic equations. Commun. Pure Appl. Anal. **3**, 607–635 (2004)
9. Chipot, M., Valente, V., Caffarelli, G.: Remarks on a nonlocal problem involving the Dirichlet energy. Rend. Semin. Mat. Univ. Padova **110**, 199–220 (2003)
10. Demarque, R., Límaco, J., Viana, L.: Local null controllability for degenerate parabolic equations with nonlocal term. Nonlinear Anal. **43**, 523–547 (2018)
11. De Menezes, S.B., Límaco, J., Medeiros, L.A.: Remarks on null controllability for semilinear heat equation in moving domains. Eletron. J. Qual. Theory Differ. Equ. **16**, 1–32 (2003)
12. Doubova, A., Fernández-Cara, E.: Some control results for simplified one-dimensional of fluid-solid interaction. Math. Models Methods Appl. Sci. **15**(5), 783–824 (2005)
13. Doubova, A., Fernández-Cara, E., González-Burgos, M., Zuazua, E.: On the controllability of parabolic systems with a nonlinear term involving the state and the gradient. SIAM J. Control. Optim. **41**(3), 798–819 (2002)
14. Fabre, C., Puel, J.P., Zuazua, E.: Approximate controllability of the semilinear heat equation. Proc. R. Soc. Edinb. Sect. A **125**, 31–61 (1995)
15. Fasano, A.: Mathematical models of some diffusive process with free boundaries , In: MAT, Series A: Mathematical Conferences, Seminars and Papers, 11, Universidad Austral, Rosario (2005)

16. Fernández-Cara, E., Guerrero, S.: Global Carleman Inequalities for parabolic systems and applications to controllability. SIAM J. Control Optim. **45**(4), 1395–1446 (2006)
17. Fernández-Cara, E., Límaco, J., Hernández, F.: Local Null Controllability of a 1D Stefan Problem. Bull Braz. Math, Soc (2018)
18. Fernández-Cara, E., Límaco, J., de Menezes, S.B.: On the controllability of a free-boundary problem for the 1D heat equation. Syst. Control Lett. **87**, 29–35 (2016)
19. Fernández-Cara, E., de Sousa, I.T.: Local null controllability of a free-boundary problem for the semilinear 1D heat equation. Bull Braz. Math. Soc. New Ser. **48**, 303–315 (2017)
20. Fernández-Cara, E., Zuazua, E.: Null and approximate controllability for weakly blowing up semilinear heat equations. Ann. Inst. H. Poincaré Anal. Non Linéaire **17**, 583–616 (2000)
21. Fernández-Cara, E., Zuazua, E.: The cost of approximate controllability for heat equations: the linear case. Adv. Differ. Equ. **5**(4–6), 465–514 (2000)
22. Friedman, A.: Variational Principles and Free-Boundary Problems. Wiley, New York (1982)
23. Friedman, A. (ed.): Tutorials in mathematical biosciences III, Cell cycle, proliferation, and cancer. Lecture Notes in Mathematics, Mathematical Biosciences Subseries, vol. 1872. Springer, Berlin (2006)
24. Friedman, A.: PDE problems arising in mathematical biology. Netw. Heterog. Med. **7**(4), 691–703 (2012)
25. Fursikov, A.V., Imanuvilov, O.Y.: Controllability of Evolution Equations, Lecture Note Series 34. Research Institute of Mathematics. Seoul National University, Seoul (1996)
26. Hermans, A.J.: Water Waves and Ship Hydrodynamics, An Introduction, 2nd edn. Springer, Dordrecht (2011)
27. Hermans, A.J.: Water waves and ship hydrodynamics, an introduction. Commun. Partial Differ. Equ. **28**(9–10), 1705–1738 (2003)
28. Ladyzhenskaya, O.A., Solonnikov, V.A., Uralćeva, N.N.: Linear and Quasilinear Equations of Parabolic Type, Translations of Mathematical Monographs, vol. 23. American Mathematical Society, Providence (1968)
29. Límaco, J., Medeiros, L.A., Zuazua, E.: Existence, uniqueness and controllability for parabolic equations in non-cylindrical domains. Matemática Contemporânea **23**(part II), 49–70 (2002)
30. Liu, X., Zhang, X.: Local controllability of multidimensional quasi-linear parabolic equations. SIAM J. Control. Optim. **50**(4), 2046–2064 (2012)
31. Liu, Y., Takahashi, T., Tucsnak, M.: Single input controllability of a simplified fluid-structure interaction model. ESAIM Control Optim. Calc. Var. **19**(1), 20–42 (2013)
32. Medeiros, L.A., Límaco, J., De Menezes, S.B.: Vibrations of elastic strings: mathematical aspects—part one. J. Comput. Anal. Appl. **4**(2), 91–127 (2002)
33. Stoker, J.J.: Water waves: the mathematical theory with applications, In: Pure and Applied Mathematics, vol. IV. Fluid flow, Interscience Publishers Inc., New York, Interscience Publishers Ltd., London (1957)
34. Stoker, J.J.: Water waves: the mathematical theory with applications. Commun. Partial Differ. Equ. **28**(9–10), 1705–1738 (2003)
35. Vázquez, J.L.: The Porous Medium Equations, Mathematical Theory, Oxford Mathematical Monographs, the Clarendon Press. Oxford University Press, Oxford (2007)
36. Vázquez, J.L., Zuazua, E.: Large time behavior for a simplified 1D model of fluid-solid interaction. Commun. Partial Differ. Equ. **28**(9–10), 1705–1738 (2003)
37. Wang, L., Lan, Y., Lei, P.: Local null controllability of a free-boundary problem for the quasi-linear 1D parabolic equation. J. Math. Anal. Appl. **506**, 125676 (2022)
38. Wrobel, L.C., Brebbia, C.A. (eds.): Computational modelling of free and moving boundary problems, vol. I. Fluid flow, Proceedings of the First International Conference held in Southampton, July 2–4, 1991. Computational Mechanics Publications, Southampton, copublished with Walter de Gruyter and Co., Berlin (1991)
39. Zheng, S., Chipot, M.: Asymptotic behavior of solutions to nonlinear parabolic equations with nonlocal terms. Asymptot. Anal. **45**, 301–312 (2005)