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A Double Phase Problem Involving Hardy Potentials

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Abstract
In this paper, we deal with the following double phase problem

$$\begin{cases} -\mathrm{div}\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right) = \gamma\left(\dfrac{|u|^{p-2}u}{|x|^p} + a(x)\dfrac{|u|^{q-2}u}{|x|^q}\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is an open, bounded set with Lipschitz boundary, $0 \in \Omega$, $N \geq 2$, $1 < p < q < N$, weight $a(\cdot) \geq 0$, γ is a real parameter and f is a subcritical function. By variational method, we provide the existence of a non-trivial weak solution on the Musielak-Orlicz-Sobolev space $W_0^{1,\mathcal{H}}(\Omega)$, with modular function $\mathcal{H}(t,x) = t^p + a(x)t^q$. For this, we first introduce the Hardy inequalities for space $W_0^{1,\mathcal{H}}(\Omega)$, under suitable assumptions on $a(\cdot)$.

Keywords Double phase problems · Hardy potentials · Variational methods

Mathematics Subject Classification 35J62 · 35J92 · 35J20

1 Introduction

In the present paper, we study the following problem

$$\begin{cases} -\mathrm{div}\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right) = \gamma\left(\dfrac{|u|^{p-2}u}{|x|^p} + a(x)\dfrac{|u|^{q-2}u}{|x|^q}\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^N$ is an open, bounded set with Lipschitz boundary, $0 \in \Omega$, $N \geq 2$, γ is a real parameter, $1 < p < q < N$ and

$$\frac{q}{p} < 1 + \frac{1}{N}, \quad a : \overline{\Omega} \to [0, \infty) \text{ is Lipschitz continuous.} \tag{1.2}$$

Here, we assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a *Carathéodory* function verifying

(f_1) *there exists an exponent* $r \in (q, p^*)$, *with the critical Sobolev exponent* $p^* = Np/(N-p)$, *such that for any* $\varepsilon > 0$ *there exists* $\delta_\varepsilon = \delta(\varepsilon) > 0$ *and*

$$|f(x,t)| \leq q\varepsilon\, |t|^{q-1} + r\delta_\varepsilon\, |t|^{r-1}$$

holds for a.e. $x \in \Omega$ *and any* $t \in \mathbb{R}$;

(f_2) *there exist* $\theta \in (q, p^*)$, $c > 0$ *and* $t_0 \geq 0$ *such that*

$$c \leq \theta F(x,t) \leq tf(x,t)$$

for a.e. $x \in \Omega$ *and any* $|t| \geq t_0$, *where* $F(x,t) = \int_0^t f(x,\tau)d\tau$.

The existence of r in (f_1) is assured by (1.2) and $q > 1$, which yield $N(q-p) < p < qp$ so that $q < p^*$. The function $f(x,t) = \phi(x)\left(\theta t^{\theta-1} + rt^{r-1}\right)$, with $\phi \in L^\infty(\Omega)$ and $\phi > 0$ a.e. in Ω, verifies all assumptions $(f_1) - (f_2)$.

Problem (1.1) is driven by the so-called double phase operator, which switches between two different types of elliptic rates, according to the modulating function $a(\cdot)$. The functionals with double phase were introduced by Zhikov in [29–32] in order to describe models for strongly anisotropic materials and provide examples of Lavrentiev's phenomenon. Other physical applications can be found for instance on transonic flow [2], quantum physics [4] and reaction diffusion systems [8]. Also, (1.1) falls into the class of problems driven by operators with non-standard growth conditions, according to Marcellini's definition given in [18, 19]. Following this direction, Mingione et al. prove different regularity results for minimizers of double phase functionals in [3, 10, 11]. See also [7, 25] for regularity results in more generalized situations. In [9], Colasuonno and Squassina analyze the eigenvalue problem with Dirichlet boundary condition of the double phase operator. In particular, in [9,Sect. 2] they provide the basic tools to solve variational problems like (1.1), introducing the standard condition (1.2). Recently, Mizuta and Shimomura study Hardy–Sobolev inequalities in the unit ball for double phase functionals in [20]. Concerning nonlinear problems driven by the double phase operator, we refer to [13, 16, 17, 22, 24] where existence and multiplicity results are provided via variational techniques. While, in [15, 27, 28] the double phase operator interacts with a convection term depending on the gradient of the solution, causing a non-variational characterization of the problem.

Inspired by the above papers, we provide an existence result for (1.1) by variational method. The main novelty, as well as the main difficulty, of problem (1.1) is the presence of a double phase Hardy potential. Indeed, such term is responsible of the lack of compactness of the Euler-Lagrange functional related to (1.1). In order to handle the double phase potential in (1.1), our weight function $a : \overline{\Omega} \to [0, \infty)$ satisfies

(a) *$a(\lambda x) \leq a(x)$ for any $\lambda \in (0, 1]$ and any $x \in \overline{\Omega}$.*

A simple example of Lipschitz continuous function verifying (a) is given by $a(x) = |x|$. Also, we control parameter γ with the Hardy constants

$$H_m := \left(\frac{m}{N-m}\right)^{-m}, \tag{1.3}$$

when $m = p$ and $m = q$. Thus, we are ready to introduce the main result of the paper.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set with Lipschitz boundary, $0 \in \Omega$ and $N \geq 2$. Let $1 < p < q < N$ and $a(\cdot)$ satisfy (1.2) and (a). Let $(f_1) - (f_2)$ hold true. Then, for any $\gamma \in (-\infty, \min\{H_p, H_q\})$ problem (1.1) admits a non-trivial weak solution.*

The proof of Theorem 1.1 is based on the application of the classical mountain pass theorem, see for example [23]. Also, Theorem 1.1 generalizes [17,Theorem 1.3], where the authors consider problem (1.1) with $\gamma = 0$. However, our situation with $\gamma \neq 0$ is much more delicate than [17], because of the lack of compactness, as well explained in Remark 3.1.

The paper is organized as follows. In Sect. 2, we introduce the basic properties of the Musielak–Orlicz and Musielak–Orlicz–Sobolev spaces, including also the new Hardy inequalities, and we set the variational structure of problem (1.1). In Sect. 3, we prove Theorem 1.1.

2 Preliminaries

The function $\mathcal{H} : \Omega \times [0, \infty) \to [0, \infty)$ defined as

$$\mathcal{H}(x, t) := t^p + a(x)t^q, \quad \text{for a.e. } x \in \Omega \text{ and for any } t \in [0, \infty),$$

with $1 < p < q$ and $0 \leq a(\cdot) \in L^1(\Omega)$, is a generalized N-function (N stands for *nice*), according to the definition in [12, 21], and satisfies the so called (Δ_2) condition, that is

$$\mathcal{H}(x, 2t) \leq t^q \mathcal{H}(x, t), \quad \text{for a.e. } x \in \Omega \text{ and for any } t \in [0, \infty).$$

Therefore, by [21] we can define the Musielak–Orlicz space $L^{\mathcal{H}}(\Omega)$ as

$$L^{\mathcal{H}}(\Omega) := \{u : \Omega \to \mathbb{R} \text{ measurable} : \ \varrho_{\mathcal{H}}(u) < \infty\},$$

endowed with the Luxemburg norm

$$\|u\|_{\mathcal{H}} := \inf\left\{\lambda > 0 : \ \varrho_{\mathcal{H}}\left(\frac{u}{\lambda}\right) \leq 1\right\},$$

where $\varrho_{\mathcal{H}}$ denotes the $\mathcal{H}$-modular function, set as

$$\varrho_{\mathcal{H}}(u) := \int_\Omega \mathcal{H}(x, |u|)dx = \int_\Omega \left(|u|^p + a(x)|u|^q\right) dx. \tag{2.1}$$

By [9, 12], the space $L^{\mathcal{H}}(\Omega)$ is a separable, uniformly convex, Banach space. While, by [17,Proposition 2.1] we have the following relation between the norm $\|\cdot\|_{\mathcal{H}}$ and the $\mathcal{H}$-modular.

Proposition 2.1 *Assume that* $u \in L^{\mathcal{H}}(\Omega)$, $\{u_j\}_j \subset L^{\mathcal{H}}(\Omega)$ *and* $c > 0$. *Then*

(i) *for* $u \neq 0$, $\|u\|_{\mathcal{H}} = c \Leftrightarrow \varrho_{\mathcal{H}}\left(\frac{u}{c}\right) = 1$;
(ii) $\|u\|_{\mathcal{H}} < 1$ $(resp. = 1, > 1) \Leftrightarrow \varrho_{\mathcal{H}}(u) < 1$ $(resp. = 1, > 1)$;
(iii) $\|u\|_{\mathcal{H}} < 1 \Rightarrow \|u\|^q_{\mathcal{H}} \leq \varrho_{\mathcal{H}}(u) \leq \|u\|^p_{\mathcal{H}}$;
(iv) $\|u\|_{\mathcal{H}} > 1 \Rightarrow \|u\|^p_{\mathcal{H}} \leq \varrho_{\mathcal{H}}(u) \leq \|u\|^q_{\mathcal{H}}$;
(v) $\lim\limits_{j\to\infty} \|u_j\|_{\mathcal{H}} = 0\,(\infty) \Leftrightarrow \lim\limits_{j\to\infty} \varrho_{\mathcal{H}}(u_j) = 0\,(\infty)$.

The related Sobolev space $W^{1,\mathcal{H}}(\Omega)$ is defined by

$$W^{1,\mathcal{H}}(\Omega) := \left\{u \in L^{\mathcal{H}}(\Omega) : \ |\nabla u| \in L^{\mathcal{H}}(\Omega)\right\},$$

endowed with the norm

$$\|u\|_{1,\mathcal{H}} := \|u\|_{\mathcal{H}} + \|\nabla u\|_{\mathcal{H}}, \tag{2.2}$$

where we write $\|\nabla u\|_{\mathcal{H}} = \||\nabla u|\|_{\mathcal{H}}$ to simplify the notation. We denote by $W^{1,\mathcal{H}}_0(\Omega)$ the completion of $C^\infty_0(\Omega)$ in $W^{1,\mathcal{H}}(\Omega)$ which can be endowed with the norm

$$\|u\| := \|\nabla u\|_{\mathcal{H}},$$

equivalent to the norm set in (2.2), thanks to [9,Proposition 2.18(iv)] whenever (1.2) holds true.

For any $m \in [1, \infty)$ we indicate with $L^m(\Omega)$ the usual Lebesgue space equipped with the norm $\|\cdot\|_m$. Then, by [9,Proposition 2.15(ii)-(iii)] we have the following embeddings.

Proposition 2.2 *Let* (1.2) *holds true. For any* $m \in [1, p^*]$ *there exists* $C_m = C(N, p, q, m, \Omega) > 0$ *such that*

$$\|u\|^m_m \leq C_m \|u\|^m$$

for any $u \in W^{1,\mathcal{H}}_0(\Omega)$. *Moreover, the embedding* $W^{1,\mathcal{H}}_0(\Omega) \hookrightarrow L^m(\Omega)$ *is compact for any* $m \in [1, p^*)$.

We denote by $L^q_a(\Omega)$ the weighted space of all measurable functions $u : \Omega \to \mathbb{R}$ with the seminorm

$$\|u\|_{q,a} := \left(\int_\Omega a(x)|u|^q dx \right)^{1/q} < \infty.$$

Using this further notation, in the next result we provide the Hardy inequalities for space $W^{1,\mathcal{H}}_0(\Omega)$. The proof of the lemma is inspired by [14,Lemma 2.1].

Lemma 2.1 *Let* (1.2) *and* (a) *hold true. Then, for any* $u \in W^{1,\mathcal{H}}_0(\Omega)$ *we have*

$$H_p \|u\|^p_{H_p} \le \|\nabla u\|^p_p, \qquad \text{with } \ \|u\|_{H_p} := \int_\Omega \frac{|u|^p}{|x|^p} dx$$
$$H_q \|u\|^q_{H_{q,a}} \le \|\nabla u\|^q_{q,a}, \quad \text{with } \ \|u\|_{H_{q,a}} := \int_\Omega a(x) \frac{|u|^q}{|x|^q} dx,$$

where H_p *and* H_q *are given in* (1.3)*.*

Proof By [14,Lemma 2.1], (2.1) and Proposition 2.1, we know that

$$\|u\|^p_{H_p} \le \left(\frac{p}{N-p} \right)^p \|\nabla u\|^p_p,$$

for any $u \in W^{1,\mathcal{H}}_0(\Omega)$. Now, taking inspiration from [14,Lemma 2.1], let $u \in C^\infty_0(\Omega)$. Then, we have

$$|u(x)|^q = -\int_1^\infty \frac{d}{d\lambda} |u(\lambda x)|^q d\lambda = -q \int_1^\infty |u(\lambda x)|^{q-2} u(\lambda x) \nabla u(\lambda x) \cdot x \, d\lambda$$

a.e. in $\mathbb{R}^N$. Hence, by Hölder inequality, (a) and trivially extending $a(\cdot)$ in the whole space $\mathbb{R}^N$

$$\begin{aligned}
\int_\Omega a(x) \frac{|u(x)|^q}{|x|^q} dx &= \int_{\mathbb{R}^N} a(x) \frac{|u(x)|^q}{|x|^q} dx \\
&= -q \int_1^\infty \int_{\mathbb{R}^N} a(x) \frac{|u(\lambda x)|^{q-2} u(\lambda x)}{|x|^{q-1}} \nabla u(\lambda x) \cdot \frac{x}{|x|} dx \, d\lambda \\
&= -q \int_1^\infty \int_{\mathbb{R}^N} \frac{1}{\lambda^{N+1-q}} \, a\left(\frac{y}{\lambda}\right) \frac{|u(y)|^{q-2} u(y)}{|y|^{q-1}} \nabla u(y) \cdot \frac{y}{|y|} dy \, d\lambda \\
&\le q \int_1^\infty \frac{d\lambda}{\lambda^{N+1-q}} \int_{\mathbb{R}^N} a(y) \frac{|u(y)|^{q-1}}{|y|^{q-1}} |\nabla u(y)| dy \\
&\le \frac{q}{N-q} \left(\int_\Omega a(y) \frac{|u(y)|^q}{|y|^q} dy \right)^{(q-1)/q} \left(\int_\Omega a(y) |\nabla u(y)|^q dy \right)^{1/q}.
\end{aligned}$$

From this, we obtain

$$\|u\|_{H_{q,a}}^q \le \left(\frac{q}{N-q}\right)^q \|\nabla u\|_{q,a}^q,$$

which holds true for any $u \in W_0^{1,\mathcal{H}}(\Omega)$ by density, (2.1) and Proposition 2.1. □

We are now ready to introduce the variational setting for problem (1.1). We say that a function $u \in W_0^{1,\mathcal{H}}(\Omega)$ is a weak solution of (1.1) if

$$\begin{aligned}&\int_\Omega \left(|\nabla u|^{p-2} + a(x)|\nabla u|^{q-2}\right)\nabla u \cdot \nabla \varphi dx \\ &= \gamma \int_\Omega \left(\frac{|u|^{p-2}u}{|x|^p} + a(x)\frac{|u|^{q-2}u}{|x|^q}\right)\varphi dx + \int_\Omega f(x,u)\varphi dx,\end{aligned}$$

for any $\varphi \in W_0^{1,\mathcal{H}}(\Omega)$. Clearly, the weak solutions of (1.1) are exactly the critical points of the Euler-Lagrange functional $J_\gamma : W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$, given by

$$J_\gamma(u) := \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_{q,a}^q - \gamma\left(\frac{1}{p}\|u\|_{H_p}^p + \frac{1}{q}\|u\|_{H_{q,a}}^q\right) - \int_\Omega F(x,u)dx,$$

which is well defined and of class C^1 on $W_0^{1,\mathcal{H}}(\Omega)$.

3 Proof of Theorem 1.1

Throughout the section we assume that $\Omega \subset \mathbb{R}^N$ is an open, bounded set with Lipschitz boundary, $0 \in \Omega$, $N \ge 2$, $1 < p < q < N$, (1.2) and (a) hold true, without further mentioning. Also, we denote with $t^+ = \max\{t, 0\}$ and $t^- = \max\{-t, 0\}$ respectively the positive and negative parts of a number $t \in \mathbb{R}$.

We recall that functional $J_\gamma : W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ fulfills the Palais-Smale condition (PS) if any sequence $\{u_j\}_j \subset W_0^{1,\mathcal{H}}(\Omega)$ satisfying

$$\{J_\gamma(u_j)\}_j \text{ is bounded and } J_\gamma'(u_j) \to 0 \text{ in } \left(W_0^{1,\mathcal{H}}(\Omega)\right)^* \text{ as } j \to \infty, \tag{3.1}$$

possesses a convergent subsequence in $W_0^{1,\mathcal{H}}(\Omega)$.

The verification of the (PS) condition for J_γ is fairly delicate, considering the contribution of the double phase Hardy potential. Indeed, even if $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^p(\Omega, |x|^{-p})$ and $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^q(\Omega, a(x)|x|^{-q})$ by Lemma 2.1, these embeddings are not compact. For this, we exploit a suitable tricky step analysis combined with the celebrated Brézis and Lieb lemma in [6,Theorem 1], which can be applied in $W_0^{1,\mathcal{H}}(\Omega)$ if we first prove the convergence $\nabla u_j(x) \to \nabla u(x)$ a.e. in Ω, as $j \to \infty$.

Proposition 3.1 *Let $(f_1)-(f_2)$ hold true. Then, for any $\gamma \in (-\infty, \min\{H_p, H_q\})$ the functional J_γ verifies the (PS) condition.*

Proof Let us fix $\gamma \in (-\infty, \min\{H_p, H_q\})$ and let $\{u_j\}_j \subset W_0^{1,\mathcal{H}}(\Omega)$ be a sequence satisfying (3.1).

We first show that $\{u_j\}_j$ is bounded in $W_0^{1,\mathcal{H}}(\Omega)$, arguing by contradiction. Then, going to a subsequence, still denoted by $\{u_j\}_j$, we have $\lim_{j\to\infty} \|u_j\| = \infty$ and $\|u_j\| \geq 1$ for any $j \geq n$, with $n \in \mathbb{N}$ sufficiently large. Thus, according to (f_2) and Lemma 2.1, we get

$$\begin{aligned} J_\gamma(u_j) - \frac{1}{\theta}\langle J'_\gamma(u_j), u_j\rangle &= \left(\frac{1}{p}-\frac{1}{\theta}\right)\|\nabla u_j\|_p^p + \left(\frac{1}{q}-\frac{1}{\theta}\right)\|\nabla u_j\|_{q,a}^q - \gamma\left(\frac{1}{p}-\frac{1}{\theta}\right)\|u_j\|_{H_p}^p \\ &\quad - \gamma\left(\frac{1}{q}-\frac{1}{\theta}\right)\|u_j\|_{H_{q,a}}^q - \int_\Omega \left[F(x,u_j) - \frac{1}{\theta} f(x,u_j)u_j\right] dx \\ &\geq \left(\frac{1}{p}-\frac{1}{\theta}\right)\left(1-\frac{\gamma^+}{H_p}\right)\|\nabla u_j\|_p^p + \left(\frac{1}{q}-\frac{1}{\theta}\right)\left(1-\frac{\gamma^+}{H_q}\right)\|\nabla u_j\|_{q,a}^q \\ &\quad - \int_{\Omega_{t_0}} \left[F(x,u_j) - \frac{1}{\theta} f(x,u_j)u_j\right]^+ dx \\ &\geq \left(\frac{1}{q}-\frac{1}{\theta}\right)\left(1-\frac{\gamma^+}{\min\{H_p,H_q\}}\right)\varrho_{\mathcal{H}}(\nabla u_j) - D, \end{aligned} \tag{3.2}$$

since $\theta > q > p$ by (f_2), where

$$\Omega_{t_0} = \left\{x \in \Omega : \ |u_j(x)| \leq t_0\right\} \quad \text{and} \quad D = |\Omega| \sup_{x\in\Omega, |t|\leq t_0} \left[F(x,t) - \frac{1}{\theta} f(x,t)t\right]^+ < \infty,$$

with the last inequality which is consequence of (f_1). Thus, by (3.1) there exist c_1, $c_2 > 0$ such that (3.2) and Proposition 2.1 yield at once that as $j \to \infty$,

$$c_1 + c_2\|u_j\| + o(1) \geq \left(\frac{1}{q}-\frac{1}{\theta}\right)\left(1-\frac{\gamma^+}{\min\{H_p,H_q\}}\right)\|u_j\|^p - D$$

giving the desired contradiction, since $\theta > q > p > 1$ and $\gamma < \min\{H_p, H_q\}$.

Hence, $\{u_j\}_j$ is bounded in $W_0^{1,\mathcal{H}}(\Omega)$. By Propositions 2.1–2.2, Lemma 2.1, [5,Theorem 4.9] and the reflexivity of $W_0^{1,\mathcal{H}}(\Omega)$, there exist a subsequence, still denoted by $\{u_j\}_j$, and $u \in W_0^{1,\mathcal{H}}(\Omega)$ such that

$$\begin{aligned} u_j \rightharpoonup u \text{ in } W_0^{1,\mathcal{H}}(\Omega), &\qquad \nabla u_j \rightharpoonup \nabla u \text{ in } \left[L^{\mathcal{H}}(\Omega)\right]^N, \\ u_j \rightharpoonup u \text{ in } L^p(\Omega, |x|^{-p}), &\quad u_j \rightharpoonup u \text{ in } L^q(\Omega \setminus A, a(x)|x|^{-q}), \\ \|u_j - u\|_{H_p}^p + \|u_j - u\|_{H_{q,a}}^q \to \ell, &\quad u_j \to u \text{ in } L^m(\Omega), \\ u_j(x) \to u(x) \text{ a.e. in } \Omega, &\quad |u_j(x)| \leq h(x) \text{ a.e. in } \Omega, \end{aligned} \tag{3.3}$$

as $j \to \infty$, with $m \in [1, p^*)$, $h \in L^q(\Omega)$ and A is the nodal set of weight $a(\cdot)$, given by

$$A := \{x \in \Omega : \ a(x) = 0\}.$$

Indeed, since $a(\cdot)$ is a Lipschitz continuous function by (1.2), then $\Omega \setminus A$ is an open subset of $\mathbb{R}^N$. Also, $h \in L^q(\Omega)$ by Proposition 2.2 and [5,Theorem 4.9], since $q < p^*$ by (1.2).

Now, we claim that

$$\nabla u_j(x) \to \nabla u(x) \text{ a.e. in } \Omega, \ \text{ as } j \to \infty. \tag{3.4}$$

Let $\varphi \in C^\infty(\mathbb{R}^N)$ be a cut-off function with $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in $B(0, 1/2)$ and $\varphi \equiv 0$ in $B(0, 1)$. Then, we define $\psi_R(x) = 1 - \varphi(x/R)$ for any $R > 0$, so that $\psi_R \in C^\infty(\mathbb{R}^N)$ with $0 \leq \psi_R \leq 1$, $\psi_R \equiv 1$ in $\mathbb{R}^N \setminus B(0, R)$, $\psi_R \equiv 0$ in $B(0, R/2)$ and the sequence $\{\psi_R u_j\}_j$ is bounded in $W_0^{1,\mathcal{H}}(\Omega)$, thanks to Proposition 2.1. By simple calculation, for any $j \in \mathbb{N}$ we have

$$\begin{aligned}\langle J'_\gamma(u_j), \psi_R(u_j - u)\rangle = &\int_\Omega \psi_R \left(|\nabla u_j|^{p-2}\nabla u_j + a(x)|\nabla u_j|^{q-2}\nabla u_j\right) \cdot (\nabla u_j - \nabla u)dx \\ &+ \int_\Omega \left(|\nabla u_j|^{p-2}\nabla u_j + a(x)|\nabla u_j|^{q-2}\nabla u_j\right) \cdot \nabla\psi_R(u_j - u)dx \\ &- \gamma \int_\Omega \psi_R \left(\frac{|u_j|^{p-2}u_j}{|x|^p} + a(x)\frac{|u_j|^{q-2}u_j}{|x|^q}\right)(u_j - u)dx \\ &- \int_\Omega \psi_R f(x, u_j)(u_j - u)dx.\end{aligned} \tag{3.5}$$

Of course, all integrals in (3.5) are zero whenever $\overline{\Omega} \subset B(0, R/2)$, since $\psi_R \equiv 0$ in $B(0, R/2)$. Thus, let us consider $R > 0$ sufficiently small such that

$$\left[\mathbb{R}^N \setminus B(0, R/2)\right] \cap \overline{\Omega} \neq \emptyset. \tag{3.6}$$

By Hölder inequality, (3.3), the facts that $\psi_R \in C^\infty(\mathbb{R}^N)$, $a(\cdot)$ is continuous in $\overline{\Omega}$ and $\{u_j\}_j$ is bounded in $W_0^{1,\mathcal{H}}(\Omega)$, we get

$$\begin{aligned}&\int_\Omega \left(|\nabla u_j|^{p-2}\nabla u_j + a(x)|\nabla u_j|^{q-2}\nabla u_j\right) \cdot \nabla\psi_R(u_j - u)dx \\ &\leq C\left(\|\nabla u_j\|_p^{p-1}\|u_j - u\|_p + \|\nabla u_j\|_{q,a}^{q-1}\|u_j - u\|_{q,a}\right) \leq \widetilde{C}\left(\|u_j - u\|_p + \|u_j - u\|_q\right) \to 0,\end{aligned} \tag{3.7}$$

as $j \to \infty$, for suitable $C, \widetilde{C} > 0$. Similarly, by considering also (f_1) with $\varepsilon = 1$, we obtain

$$\left| \int_\Omega \psi_R f(x, u_j)(u_j - u)dx \right| \le \int_\Omega \left(q|u_j|^{q-1} + r\delta_1 |u_j|^{r-1} \right) |u_j - u| dx \\ \le C \left(\|u_j - u\|_q + \|u_j - u\|_r \right) \to 0 \tag{3.8}$$

as $j \to \infty$, for a suitable $C > 0$. Furthermore, by (3.3) and [1,Proposition A.8], considering that $a(\cdot) > 0$ in $\Omega \setminus A$, we have

$$|u_j|^{p-2}u_j \rightharpoonup |u|^{p-2}u \ \text{ in } \ L^{p'}(\Omega, |x|^{-p}),$$
$$|u_j|^{q-2}u_j \rightharpoonup |u|^{q-2}u \ \text{ in } \ L^{q'}(\Omega \setminus A, a(x)|x|^{-q})$$

so that

$$\begin{aligned} \lim_{j\to\infty} \int_\Omega \psi_R \frac{|u_j|^{p-2}u_j}{|x|^p} u dx &= \int_\Omega \psi_R \frac{|u|^p}{|x|^p} dx, \\ \lim_{j\to\infty} \int_\Omega \psi_R\, a(x) \frac{|u_j|^{q-2}u_j}{|x|^q} u dx &= \lim_{j\to\infty} \int_{\Omega\setminus A} \psi_R\, a(x) \frac{|u_j|^{q-2}u_j}{|x|^q} u dx \\ &= \int_{\Omega\setminus A} \psi_R\, a(x) \frac{|u|^q}{|x|^q} dx \\ &= \int_\Omega \psi_R\, a(x) \frac{|u|^q}{|x|^q} dx. \end{aligned} \tag{3.9}$$

While, by (3.3) it follows that

$$\psi_R(x) \frac{|u_j(x)|^p}{|x|^p} \le \left(\frac{2}{p}\right)^p |u_j(x)|^p \le \left(\frac{2}{p}\right)^p h^p(x) \quad \text{a.e in } \Omega \setminus B(0, R/2),$$

so that, since $\psi_R \equiv 0$ in $B(0, R/2)$, the dominated convergence theorem gives

$$\begin{aligned} \lim_{j\to\infty} \int_\Omega \psi_R \frac{|u_j|^p}{|x|^p} dx = \lim_{j\to\infty} \int_{\Omega\setminus B(0,R/2)} \psi_R \frac{|u_j|^p}{|x|^p} dx &= \int_{\Omega\setminus B(0,R/2)} \psi_R \frac{|u|^p}{|x|^p} dx \\ &= \int_\Omega \psi_R \frac{|u|^p}{|x|^p} dx. \end{aligned} \tag{3.10}$$

Similarly, by using also (1.2), for a suitable constant $L > 0$ we get

$$\psi_R(x) a(x) \frac{|u_j(x)|^q}{|x|^q} \le L \left(\frac{2}{q}\right)^q h^q(x) \quad \text{a.e in } \Omega \setminus B(0, R/2),$$

which yields joint with the dominated convergence theorem

$$\lim_{j\to\infty} \int_\Omega \psi_R\, a(x) \frac{|u_j|^q}{|x|^q} dx = \int_\Omega \psi_R\, a(x) \frac{|u|^q}{|x|^q} dx. \tag{3.11}$$

Thus, by (3.1), (3.5), (3.7)–(3.11), we obtain

$$\lim_{j\to\infty}\int_\Omega \psi_R\left(|\nabla u_j|^{p-2}\nabla u_j + a(x)|\nabla u_j|^{q-2}\nabla u_j\right)\cdot(\nabla u_j - \nabla u)dx = 0.$$

By Hölder inequality and being $\psi_R \le 1$, we see that functional

$$G : g \in \left[L^{\mathcal{H}}(\Omega)\right]^N \mapsto \int_\Omega \psi_R\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right)\cdot g\,dx$$

is linear and bounded. Hence, by (3.3) we have

$$\lim_{j\to\infty}\int_\Omega \psi_R\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right)\cdot(\nabla u_j - \nabla u)dx = 0,$$

so that, denoting $\Omega_R := \{x \in \Omega : \ |x| > R\}$ for any $R > 0$, we get

$$\begin{aligned}
&\lim_{j\to\infty}\int_{\Omega_R}\Big[|\nabla u_j|^{p-2}\nabla u_j - |\nabla u|^{p-2}\nabla u \\
&\quad + a(x)\left(|\nabla u_j|^{q-2}\nabla u_j - |\nabla u|^{q-2}\nabla u\right)\Big]\cdot(\nabla u_j - \nabla u)dx \\
&\le \lim_{j\to\infty}\int_\Omega \psi_R\Big[|\nabla u_j|^{p-2}\nabla u_j - |\nabla u|^{p-2}\nabla u \\
&\quad + a(x)\left(|\nabla u_j|^{q-2}\nabla u_j - |\nabla u|^{q-2}\nabla u\right)\Big]\cdot(\nabla u_j - \nabla u)dx \\
&= 0
\end{aligned} \tag{3.12}$$

since $\psi_R \equiv 1$ in $\mathbb{R}^N \setminus B(0, R)$. Now, we recall the well known Simon inequalities, see [26], such that

$$|\xi - \eta|^m \le \begin{cases} \kappa_m\,(|\xi|^{m-2}\xi - |\eta|^{m-2}\eta)\cdot(\xi - \eta), & \text{if } m \ge 2, \\ \kappa_m\left[(|\xi|^{m-2}\xi - |\eta|^{m-2}\eta)\cdot(\xi-\eta)\right]^{m/2}(|\xi|^m + |\eta|^m)^{(2-m)/2}, & \text{if } \\ 1 < m < 2, \end{cases} \tag{3.13}$$

for any $\xi, \eta \in \mathbb{R}^N$, with $\kappa_m > 0$ a suitable constant. Therefore, if $p \ge 2$ by (3.13) we have

$$\begin{aligned}
&\int_{\Omega_R}|\nabla u_j - \nabla u|^p dx \\
&\le \kappa_p\int_{\Omega_R}\left(|\nabla u_j|^{p-2}\nabla u_j - |\nabla u|^{p-2}\nabla u\right)\cdot(\nabla u_j - \nabla u)dx.
\end{aligned} \tag{3.14}$$

While, if $1 < p < 2$ by (3.13) and the Hölder inequality we obtain

$$\begin{aligned}&\int_{\Omega_R} |\nabla u_j - \nabla u|^p dx\\ &\leq \kappa_p \int_{\Omega_R} \left[\left(|\nabla u_j|^{p-2}\nabla u_j - |\nabla u|^{p-2}\nabla u\right)\cdot(\nabla u_j - \nabla u)\right]^{p/2}\left(|\nabla u_j|^p + |\nabla u|^p\right)^{(2-p)/2} dx\\ &\leq \kappa_p \left[\int_{\Omega_R} \left(|\nabla u_j|^{p-2}\nabla u_j - |\nabla u|^{p-2}\nabla u\right)\cdot(\nabla u_j - \nabla u)dx\right]^{p/2}\left(\|\nabla u_j\|_p^p + \|\nabla u\|_p^p\right)^{(2-p)/2}\\ &\leq \widetilde{\kappa_p}\left[\int_{\Omega_R} \left(|\nabla u_j|^{p-2}\nabla u_j - |\nabla u|^{p-2}\nabla u\right)\cdot(\nabla u_j - \nabla u)dx\right]^{p/2}\end{aligned} \tag{3.15}$$

where the last inequality follows by the boundedness of $\{u_j\}_j$ in $W_0^{1,\mathcal{H}}(\Omega)$ and Proposition 2.1, with a suitable new $\widetilde{\kappa_p} > 0$. Also, by convexity and since $a(x) \geq 0$ a.e. in Ω by (1.2), we have

$$a(x)\left(|\nabla u_j|^{q-2}\nabla u_j - |\nabla u|^{q-2}\nabla u\right)\cdot(\nabla u_j - \nabla u) \geq 0 \text{ a.e. in } \Omega. \tag{3.16}$$

Thus, combining (3.12), (3.14)–(3.16) we prove that $\nabla u_j \to \nabla u$ in $[L^p(\Omega_R)]^N$ as $j \to \infty$, whenever $R > 0$ satisfies (3.6). However, when $\overline{\Omega} \subset B(0, R/2)$ we have $\Omega_R = \emptyset$. Thus, for any $R > 0$ the sequence $\nabla u_j \to \nabla u$ in $[L^p(\Omega_R)]^N$ as $j \to \infty$, and by diagonalization we prove claim (3.4).

Since the sequence $\{|\nabla u_j|^{p-2}\nabla u_j\}_j$ is bounded in $L^{p'}(\Omega)$, by (3.4) we get

$$\lim_{j\to\infty} \int_{\Omega} |\nabla u_j|^{p-2}\nabla u_j \cdot \nabla u \, dx = \|\nabla u\|_p^p. \tag{3.17}$$

While, since $\{|\nabla u_j|^{q-2}\nabla u_j\}_j$ is bounded in $L^{q'}(\Omega \setminus A, a(x))$, by (3.4) and [1,Proposition A.8]

$$\begin{aligned}\lim_{j\to\infty} \int_{\Omega} a(x)|\nabla u_j|^{q-2}\nabla u_j \cdot \nabla u \, dx &= \lim_{j\to\infty} \int_{\Omega\setminus A} a(x)|\nabla u_j|^{q-2}\nabla u_j \cdot \nabla u \, dx\\ &= \|\nabla u\|_{q,a}^q.\end{aligned} \tag{3.18}$$

Also, arguing as in (3.8) and (3.9), we can prove

$$\begin{aligned}&\lim_{j\to\infty} \int_{\Omega} f(x, u_j)(u_j - u)dx = 0,\\ &\lim_{j\to\infty} \int_{\Omega} \left(\frac{|u_j|^{p-2}u_j}{|x|^p}u + a(x)\frac{|u_j|^{q-2}u_j}{|x|^q}u\right) dx = \|u\|_{H_p}^p + \|u\|_{H_{q,a}}^q.\end{aligned} \tag{3.19}$$

Furthermore, using (3.3), (3.4) and the Brézis and Lieb lemma in [6,Theorem 1], we obtain

$$\begin{aligned} &\|\nabla u_j\|_p^p - \|\nabla u_j - \nabla u\|_p^p = \|\nabla u\|_p^p + o(1),\\ &\|\nabla u_j\|_{q,a}^q - \|\nabla u_j - \nabla u\|_{q,a}^q = \|\nabla u\|_{q,a}^q + o(1),\\ &\|u_j\|_{H_p}^p - \|u_j - u\|_{H_p}^p = \|u\|_{H_p}^p + o(1),\\ &\|u_j\|_{H_{q,a}}^q - \|u_j - u\|_{H_{q,a}}^q = \|u\|_{H_{q,a}}^q + o(1) \end{aligned} \tag{3.20}$$

as $j \to \infty$. Thus, by (3.1), (3.17), (3.18) and (3.19), we get

$$\begin{aligned} o(1)=\left\langle J_\gamma'(u_j), u_j - u\right\rangle &= \int_\Omega \left(|\nabla u_j|^{p-2}\nabla u_j + a(x)|\nabla u_j|^{q-2}\nabla u_j\right)\cdot(\nabla u_j - \nabla u)dx\\ &\quad - \gamma\int_\Omega\left(\frac{|u_j|^{p-2}u_j}{|x|^p} + a(x)\frac{|u_j|^{q-2}u_j}{|x|^q}\right)(u_j-u)dx\\ &\quad - \int_\Omega f(x,u_j)(u_j-u)dx\\ &= \|\nabla u_j\|_p^p - \|\nabla u\|_p^p + \|\nabla u_j\|_{q,a}^p - \|\nabla u\|_{q,a}^p\\ &\quad - \gamma\left(\|u_j\|_{H_p}^p - \|u\|_{H_p}^p + \|u_j\|_{H_{q,a}}^q - \|u\|_{H_{q,a}}^q\right) + o(1) \end{aligned} \tag{3.21}$$

as $j \to \infty$. Hence, by (3.20) it follows that

$$\begin{aligned} \|\nabla u_j - \nabla u\|_p^p + \|\nabla u_j - \nabla u\|_{q,a}^q &= \gamma\left(\|u_j - u\|_{H_p}^p + \|u_j - u\|_{H_{q,a}}^q\right) + o(1)\\ &= \gamma\ell + o(1) \end{aligned} \tag{3.22}$$

as $j \to \infty$. Now, assume for contradiction that $\ell > 0$. Then, from Lemma 2.1, (3.22) and the fact that $\gamma < \min\{H_p, H_q\}$, we have

$$\begin{aligned} &\lim_{j\to\infty}\|\nabla u_j - \nabla u\|_p^p + \lim_{j\to\infty}\|\nabla u_j - \nabla u\|_{q,a}^q\\ &\le \gamma^+\left(\lim_{j\to\infty}\|u_j - u\|_{H_p}^p + \lim_{j\to\infty}\|u_j - u\|_{H_{q,a}}^q\right)\\ &< \min\{H_p, H_q\}\left(\lim_{j\to\infty}\|u_j - u\|_{H_p}^p + \lim_{j\to\infty}\|u_j - u\|_{H_{q,a}}^q\right)\\ &\le \lim_{j\to\infty}\|\nabla u_j - \nabla u\|_p^p + \lim_{j\to\infty}\|\nabla u_j - \nabla u\|_{q,a}^q \end{aligned}$$

which is impossible. Therefore $\ell = 0$, so that by (3.22) we have $\nabla u_j \to \nabla u$ in $\left[L^p(\Omega) \cap L_a^q(\Omega)\right]^N$ as $j \to \infty$, implying that $u_j \to u$ in $W_0^{1,\mathcal{H}}(\Omega)$ thanks to (2.1) and Proposition 2.1. This concludes the proof.

□

Now, we complete the proof of Theorem 1.1, proving first that functional J_γ satisfies the geometric features of the mountain pass theorem.

Lemma 3.1 *Let (f_1) holds true. Then, for any $\gamma \in (-\infty, \min\{H_p, H_q\})$ there exist $\rho = \rho(\gamma) \in (0, 1]$ and $\alpha = \alpha(\rho) > 0$ such that $J_\gamma(u) \geq \alpha$ for any $u \in W_0^{1,\mathcal{H}}(\Omega)$, with $\|u\| = \rho$.*

Proof Let us fix $\gamma \in (-\infty, \min\{H_p, H_q\})$. By (f_1), for any $\varepsilon > 0$ we have a $\delta_\varepsilon > 0$ such that

$$|F(x,t)| \leq \varepsilon|t|^q + \delta_\varepsilon|t|^r, \quad \text{for a.e. } x \in \Omega \text{ and any } t \in \mathbb{R}. \tag{3.23}$$

Thus, by (3.23), Lemma 2.1, Propositions 2.1 and 2.2, for any $u \in W_0^{1,\mathcal{H}}(\Omega)$ with $\|u\| \leq 1$, we obtain

$$\begin{aligned}
J_\gamma(u) &\geq \frac{1}{p}\left(1 - \frac{\gamma^+}{H_p}\right)\|\nabla u\|_p^p + \frac{1}{q}\left(1 - \frac{\gamma^+}{H_q}\right)\|\nabla u\|_{q,a}^q - \varepsilon\|u\|_q^q - \delta_\varepsilon\|u\|_r^r \\
&\geq \frac{1}{q}\left(1 - \frac{\gamma^+}{\min\{H_p, H_q\}}\right)\varrho_{\mathcal{H}}(\nabla u) - \varepsilon C_q\|u\|^q - \delta_\varepsilon C_r\|u\|^r \\
&\geq \left[\frac{1}{q}\left(1 - \frac{\gamma^+}{\min\{H_p, H_q\}}\right) - \varepsilon C_q\right]\|u\|^q - \delta_\varepsilon C_r\|u\|^r,
\end{aligned}$$

since $q > p$ and $\gamma < \min\{H_p, H_q\}$. Therefore, choosing $\varepsilon > 0$ sufficiently small so that

$$\sigma_\varepsilon = \frac{1}{q}\left(1 - \frac{\gamma^+}{\min\{H_p, H_q\}}\right) - \varepsilon C_q > 0,$$

for any $u \in W_0^{1,\mathcal{H}}(\Omega)$ with $\|u\| = \rho \in \left(0, \min\{1, [\sigma_\varepsilon/(2\delta_\varepsilon C_r)]^{1/(r-q)}\}\right]$, we get

$$J_\gamma(u) \geq \left(\sigma_\varepsilon - \delta_\varepsilon C_r \rho^{r-q}\right)\rho^q := \alpha > 0.$$

This completes the proof. □

Lemma 3.2 *Let $(f_1) - (f_2)$ hold true. Then, for any $\gamma \in \mathbb{R}$ there exists $e \in W_0^{1,\mathcal{H}}(\Omega)$ such that $J_\gamma(e) < 0$ and $\|e\| > 1$.*

Proof Let us fix $\gamma \in \mathbb{R}$. By (f_1) and (f_2), there exist $d_1 > 0$ and $d_2 \geq 0$ such that

$$F(x,t) \geq d_1|t|^\theta - d_2 \quad \text{for a.e. } x \in \Omega \text{ and any } t \in \mathbb{R}. \tag{3.24}$$

Thus, if $\varphi \in W_0^{1,\mathcal{H}}(\Omega)$ with $\|\varphi\| = 1$, then by Proposition 2.1 also $\varrho_{\mathcal{H}}(\nabla\varphi) = 1$, so that by (3.24), for any $t \geq 1$ we have

$$J_\gamma(t\varphi) \leq \frac{t^q}{p} - t^p\frac{\gamma^-}{p}\|\varphi\|_{H_p}^p - t^q\frac{\gamma^-}{q}\|\varphi\|_{H_{q,a}}^q - t^\theta d_1\|\varphi\|_\theta^\theta - d_2|\Omega|.$$

Since $\theta > q > p$ by (f_2), passing to the limit as $t \to \infty$ we get $J_\gamma(t\varphi) \to -\infty$. Thus, the assertion follows by taking $e = t_\infty\varphi$, with t_∞ sufficiently large. □

Proof of Theorem 1.1 Since $J_\gamma(0) = 0$, by Proposition 3.1, Lemmas 3.1–3.2 and the mountain pass theorem, we prove the existence of a non-trivial weak solution of (1.1). □

We conclude this section with a result of independent interest, which shows how (3.4) allows us to cover the complete situation in Theorem 1.1, with $1 < p < q < N$ and $\gamma \in (-\infty, \min\{H_p, H_q\})$. For this, let $L_\gamma : W_0^{1,\mathcal{H}}(\Omega) \to \left(W_0^{1,\mathcal{H}}(\Omega)\right)^*$ be an operator such that

$$\begin{aligned}\langle L_\gamma(u), v\rangle &:= \int_\Omega \left(|\nabla u|^{p-2} + a(x)|\nabla u|^{q-2}\right) \nabla u \cdot \nabla v dx \\ &\quad -\gamma \int_\Omega \left(\frac{|u|^{p-2}u}{|x|^p} v + a(x)\frac{|u|^{q-2}u}{|x|^q} v\right) dx,\end{aligned}$$

for any $u, v \in W_0^{1,\mathcal{H}}(\Omega)$.

Lemma 3.3 *Let $2 \le p < q < N$ and $\gamma \in (-\infty, \min\{H_p, H_q\}/\max\{\kappa_p, \kappa_q\})$, with κ_p and κ_q given by (3.13). Then, the operator L_γ is a mapping of (S) type, that is if $u_j \rightharpoonup u$ in $W_0^{1,\mathcal{H}}(\Omega)$ and*

$$\lim_{j\to\infty} \langle L_\gamma(u_j) - L_\gamma(u), u_j - u\rangle = 0, \tag{3.25}$$

then $u_j \to u$ in $W_0^{1,\mathcal{H}}(\Omega)$.

Proof Let us fix $2 \le p < q < N$ and $\gamma \in (-\infty, \min\{H_p, H_q\}/\max\{\kappa_p, \kappa_q\})$. Let $\{u_j\}_j$ be a sequence in $W_0^{1,\mathcal{H}}(\Omega)$ such that $u_j \rightharpoonup u$ in $W_0^{1,\mathcal{H}}(\Omega)$ and (3.25) holds true. Then, up to a subsequence $\{u_j\}_j$ is bounded in $W_0^{1,\mathcal{H}}(\Omega)$ and by Lemma 2.1 and [5,Theorem 4.9], we obtain

$$\|u_j - u\|_{H_p}^p + \|u_j - u\|_{H_{q,a}}^q \to \ell, \qquad u_j(x) \to u(x) \text{ a.e. in } \Omega,$$

as $j \to \infty$. Thus, by [6,Theorem 1] we get

$$\begin{aligned}&\|u_j\|_{H_p}^p - \|u_j - u\|_{H_p}^p = \|u\|_{H_p}^p + o(1),\\ &\|u_j\|_{H_{q,a}}^q - \|u_j - u\|_{H_{q,a}}^q = \|u\|_{H_{q,a}}^q + o(1)\end{aligned} \tag{3.26}$$

as $j \to \infty$. While, by (3.13) we have

$$\begin{aligned}&\int_\Omega \left[\left(|\nabla u_j|^{p-2}\nabla u_j - |\nabla u|^{p-2}\nabla u\right) + a(x)\left(|\nabla u_j|^{q-2}\nabla u_j - |\nabla u|^{q-2}\nabla u\right)\right] \cdot (\nabla u_j - \nabla u)dx\\ &\ge \frac{1}{\max\{\kappa_p, \kappa_q\}} \left(\|u_j - u\|_p^p + \|u_j - u\|_{q,a}^q\right)\end{aligned} \tag{3.27}$$

for any $j \in \mathbb{N}$. Hence, combining (3.25)–(3.27), as $j \to \infty$

$$\frac{1}{\max\{\kappa_p, \kappa_q\}} \|\nabla u_j - \nabla u\|_p^p + \|\nabla u_j - \nabla u\|_{q,a}^q$$
$$= \gamma \left(\|u_j - u\|_{H_p}^p + \|u_j - u\|_{H_{q,a}}^q \right) + o(1) = \gamma \ell + o(1),$$

which recalls (3.22), up to a constant. From this point, we can argue as in the end of the proof of Proposition 3.1, proving that $u_j \to u$ in $W_0^{1,\mathcal{H}}(\Omega)$. □

Remark 3.1 When $2 \le p < q < N$ and $\gamma \in (-\infty, K_{p,q} \min\{H_p, H_q\})$, with

$$K_{p,q} := \min \left\{ 1, \frac{1}{\max\{\kappa_p, \kappa_q\}} \right\},$$

we can prove Proposition 3.1 arguing as in [17,Lemma 5.1] and using Lemma 3.3 instead of (3.4).

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