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Global Solutions and Blow-Up for the Wave Equation with Variable Coefficients: I. Interior Supercritical Source

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Abstract
In this paper, we consider the variable coefficient wave equation with damping and supercritical source terms. The goal of this work is devoted to prove the local and global existence, and classify decay rate of energy depending on the growth near zero on the damping term. Moreover, we prove the blow-up of the weak solution with positive initial energy as well as nonpositive initial energy.

Keywords Wave equation with variable coefficients · supercritical source · Existence of solutions · Energy decay rates · Blow-up

Mathematics Subject Classification 35L05 · 35L15 · 35A01 · 35B40 · 35B44

1 Introduction

In this paper, we are concerned with the local and global existence, energy decay rates and finite time blow-up of the solution for the following wave equation

$$\begin{cases} u_{tt} - \mu(t)Lu + g(u_t) = f(u) & \text{in} \quad \Omega \times (0, +\infty), \\ u = 0 & \text{on} \quad \Gamma \times (0, +\infty), \\ u(x, 0) = u_0(x), \qquad u_t(x, 0) = u_1(x), & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $f(u) = |u|^\gamma u$ and $Lu = div(A(x)\nabla u) = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)$. Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 3)$ with smooth boundary Γ.

The damping-source interplay in system (1.1) arise naturally in many contexts, for instance, in classical mechanics, fluid dynamics and quantum field theory (cf. [29,45]).

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The interaction between two competitive force, that is damping term and source term, make the problem attractive from the mathematical point of view.

For the present polynomial nonlinear source term case such as $|u|^\gamma u$, the stability of (1.1) has been studied by many authors (see [7,8,14–18,20–26,28,30,31,39–42,47] and a list on references therein), where the source term is subcritical or critical. However, very few results addressed wave equations influenced by supercritical sources (cf. [1–5,11,12,46]). [1–5] are the first papers that introduced super-supercritical sources and answered the open questions related to local existence, global existence versus blow-up of solution, uniqueness, and continuous dependence on data. [46] proved the local and global existence, uniqueness and Hadamard well-posedness for the wave equation when source terms can be supercritical or super-supercritical. However, the author do not considered the energy decay and blow-up of the solutions. [11] considered a system of nonlinear wave equations with supercritical sources and damping terms. They proved global existence and exponential and algebraic uniform decay rates of energy moreover, blow-up result for weak solutions with nonnegative initial energy. But as far as I know, the only problem with considering supercritical source is the constant coefficients case, that is, $A = I$ and dimension $n = 3$.

In the case of variable coefficients, that is $A \neq I$, stability of the wave equation was considered in [9,15,18,22,37,52]. The wave equations with variable coefficients arise in mathematical modeling of inhomogeneous media in solid mechanics, electromagnetic, fluid flows through porous media. The variable coefficients problem has been widely studied (see [6,32–34,36,48–51] and a list of references therein). However, there were very few results considered the source term. For instance, [27] proved the energy decay of the variable-coefficient wave equation with nonlinear acoustic boundary conditions and source term. Recently, [19] studied the uniform energy decay rates of the wave equation with variable coefficients applying the Riemannian geometry method and modified multiplier method. But, above mentioned references were considered sub-critical source. There is none, to my knowledge, for the variable coefficients problem having both damping and source terms taking into account supercritical source.

Our main motivation and the techniques are constituted by three dimensional case [1–5], in which the source term can be supercritical on variable coefficient problem. The variable coefficient part of the elliptic operator L does not bring in major additional challenges, since it's strongly elliptic and the maximal monotonicity hold. The difference from previous literatures is that we take into account the supercritical source for $n \geq 3$ and blow-up of solutions with positive initial energy as well as nonpositive initial energy.

In order to overcome difficulties to prove above statements, first, we refine the energy space and a constant used in potential well method, because we do not guarantee $H_0^1(\Omega) \hookrightarrow L^{\gamma+2}(\Omega)$ since the source term is supercritical. Also we have a hypothesis on damping term for proving existence of solutions (see Remark 2.1). Second, we rely on the Faedo-Galerkin method combined with suitable truncations-approximation since the supercritical source lacks the globally Lipschitz condition. Third, we refine the key point constants used to prove blow-up result. So, this paper has improved and generalized previous literatures.

The goal of this paper is to prove the existence result using the Faedo-Galerkin method and truncated approximation method, and classify the energy decay rate applying the method developed in [38]. Moreover, we prove the blow-up of the weak solution with positive initial energy as well as nonpositive initial energy. This paper is organized as follows: In Sect. 2, we recall the notation, hypotheses and some necessary preliminaries and introduce our main result. In Sect. 3, we prove the local existence of weak solutions, and show the global existence of weak solution in each two conditions in Sect. 4. In Sect. 5, we prove the uniform decay rate under suitable conditions on the initial data and damping term by the differential geometric approach. In Sect. 6, we prove the blow-up of the weak solution with positive initial energy as well as nonpositive initial energy by using contradiction method.

2 Preliminaries

We begin this section by introducing some notations and our main results. Throughout this paper, we define the Hilbert space $H_0^1(\Omega) = \{u \in H^1(\Omega); u = 0 \text{ on } \Gamma\}$ with the norm $||u||_{H_0^1(\Omega)} = ||\nabla u||_{L^2(\Omega)}$ and $\mathcal{H} = \{u \in H_0^1(\Omega); u \in L^{\gamma+2}(\Omega)\}$ with the norm $||u||_{\mathcal{H}} = ||u||_{H_0^1(\Omega)} + ||u||_{L^{\gamma+2}(\Omega)}$. $||\cdot||_p$ is denoted by the $L^p(\Omega)$ norm and $\langle u, v\rangle = \int_\Omega u(x)v(x)dx$.

(H_1) Hypothesis on A.

The matrix $A = \big(a_{ij}(x)\big)$, where $a_{ij} \in C^1(\overline{\Omega})$, is symmetric and there exists a positive constant a_0 such that for all $x \in \overline{\Omega}$ and $\omega = (\omega_1, \cdots, \omega_n)$ we have

$$\sum_{i,j=1}^{n} a_{ij}(x)\omega_j\omega_i \geq a_0|\omega|^2. \tag{2.1}$$

(H_2) Hypothesis on ¯.

Let $\mu \in W^{1,\infty}(0,\infty) \cap W^{1,1}(0,\infty)$ satisfying following conditions:

$$\mu(t) \geq \mu_0 > 0 \quad \text{and} \quad \mu'(t) < 0 \quad \text{a.e. in} \quad [0,\infty), \tag{2.2}$$

where μ_0 is a positive constant.

(H_3) Hypothesis on g.

Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing C^1 function such that $g(0) = 0$ and suppose that there exist positive constants c_1, c_2, ρ and a strictly increasing and odd function β of C^1 class on $[-1, 1]$ such that

$$|\beta(s)| \leq |g(s)| \leq |\beta^{-1}(s)| \quad \text{if} \quad |s| \leq 1, \tag{2.3}$$

$$c_1|s|^{\rho+1} \leq |g(s)| \leq c_2|s|^{\rho+1} \quad \text{if} \quad |s| > 1, \tag{2.4}$$

where β^{-1} denotes the inverse function of β.

(H_4) Hypothesis on and æ.

Let γ and ρ be positive constants satisfying the following condition:

$$\frac{2}{n-2} < \gamma \leq \frac{n+2}{n-2} \quad \text{and} \quad \rho \geq \frac{2(n-2)\gamma - 4}{n+2-(n-2)\gamma}. \tag{2.5}$$

By using the hypothesis (H_1), we verify that the bilinear form $a(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ defined by

$$a(u(t), v(t)) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) \frac{\partial u(t)}{\partial x_j} \frac{\partial v(t)}{\partial x_i} dx = \int_\Omega A \nabla u(t) \nabla v(t) dx$$

is symmetric and continuous. On the other hand, from (2.1) for $\omega = \nabla u$, we get

$$a(u(t), u(t)) \geq a_0 \int_\Omega \sum_{i=1}^{n} \Big| \frac{\partial u}{\partial x_i} \Big|^2 dx = a_0 ||\nabla u(t)||_2^2. \tag{2.6}$$

Remark 2.1 In view of the critical Sobolev imbedding $H_0^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$, the map $f(u) = |u|^\gamma u$ is not locally Lipschitz from $H_0^1(\Omega)$ into $L^2(\Omega)$ for the supercritical (or super-supercritical) values $\frac{2}{n-2} < \gamma \leq \frac{n+2}{n-2}$. However, by the hypothesis on ρ $\left(\rho \geq \frac{2(n-2)\gamma - 4}{n+2-(n-2)\gamma}\right)$, $f(u)$ is locally Lipschitz from $H_0^1(\Omega)$ into $L^{\frac{\rho+2}{\rho+1}}(\Omega)$.

Definition 2.1 (Weak solution) $u(x,t)$ is called a weak solution of (1.1) on $\Omega \times (0, T)$ if $u \in C(0, T; \mathcal{H}) \cap C^1(0, T; L^2(\Omega))$, $u_t \in L^{\rho+2}(0, T; \Omega)$ and satisfies (1.1) in the distribution sense, i.e.,

$$\int_0^T \langle u_{tt}, \phi \rangle dt + \int_0^T \mu(t) a(u, \phi) dt + \int_0^T \langle g(u_t), \phi \rangle dt = \int_0^T \langle f(u), \phi \rangle dt,$$

for any $\phi \in C(0, T; \mathcal{H}) \cap C^1(0, T; L^2(\Omega))$, $\phi \in L^{\rho+2}(0, T; \Omega)$ and $u(x, 0) = u_0(x) \in \mathcal{H}$, $u_t(x, 0) = u_1(x) \in L^2(\Omega)$.

Remark 2.2 One easily check that $\mathcal{H} = H_0^1(\Omega)$ when $\frac{2}{n-2} < \gamma \leq \frac{4}{n-2}$. Moreover, if $n = 3$, then we can replace $\mathcal{H}$ by $H_0^1(\Omega)$ when $2 < \gamma \leq 4$, since $H_0^1(\Omega) \hookrightarrow L^{\gamma+2}(\Omega)$. (see Figure 1).

The energy associated to the problem (1.1) is given by

$$E(t) = \frac{1}{2} ||u_t(t)||_2^2 + \frac{1}{2} \mu(t) a(u(t), u(t)) - \frac{1}{\gamma+2} ||u(t)||_{\gamma+2}^{\gamma+2}.$$

We now state our main results.

Theorem 2.1 *(Local existence) Suppose that* $(H_1) - (H_4)$ *hold. Then given the initial data* $(u_0, u_1) \in \mathcal{H} \times L^2(\Omega)$, *there exist* $T > 0$ *and a weak solution of problem (1.1). Moreover, the following energy identity holds for all* $0 \leq t \leq T$:

$$E(t) + \int_0^t \int_\Omega g(u_s) u_s dx ds - \frac{1}{2} \int_0^t \mu'(s) a(u, u) ds = E(0). \tag{2.7}$$

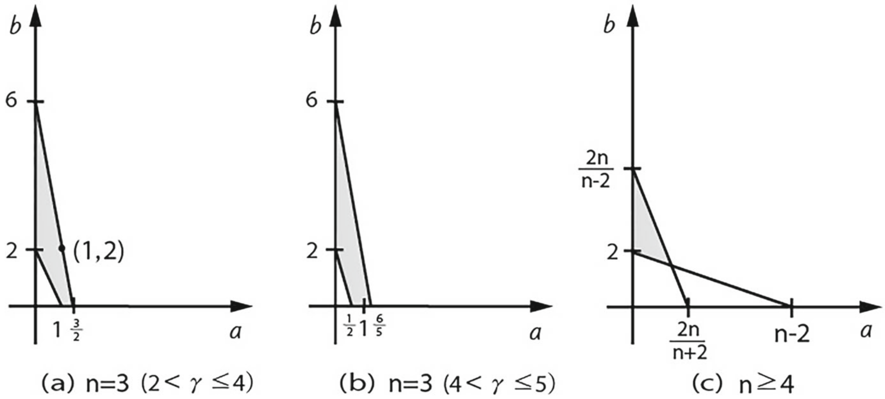

Fig. 1 The admissible range of parameters a and b with respect to the imbedding $H_0^1(\Omega) \hookrightarrow L^{a\gamma+b}(\Omega)$

Remark 2.3 Theorem 2.1 implies that if f is supercritical, then we have a Hadamard wellposedness i.e., continuous dependence with respect to initial data by the same arguments as [4] and [46]. More precisely, in dimension $n = 3$, continuous dependence on initial data was proved in [4]. When $n \geq 4$, one studied in [46], for $\frac{2}{n-2} < \gamma \leq \frac{4}{n-2}$. However, for the case $\frac{4}{n-2} < \gamma \leq \frac{n+2}{n-2}$, that is, f is super-supercritical, no longer considers the Hadamard wellposeness. Even though there are a few results in [46], but it was proved under the restricted initial condition, restricted regularity and restricted dimension. This matter remains a challenging open problem.

Theorem 2.2 *(Global existence) Suppose that* $(H_1) - (H_4)$ *hold and the initial data* $(u_0, u_1) \in \mathcal{H} \times L^2(\Omega)$. *If one of the assumptions hold:* $\rho \geq \gamma$ *or*

$$E(0) < d_0 \quad \textit{and} \quad a(u_0, u_0) < \lambda_0^2, \tag{2.8}$$

where

$$\lambda_0 = \left(\frac{\mu_0}{K_0^{\gamma+2}}\right)^{1/\gamma} \quad \textit{and} \quad d_0 = \frac{\gamma\mu_0}{2(\gamma+2)}\lambda_0^2, \quad K_0 = \sup_{u\in\mathcal{H}, u\neq 0}\left(\frac{||u||_{\gamma+2}}{[a(u,u)]^{1/2}}\right).$$

Then the weak solution $u(x,t)$ *of (1.1) is global.*

Theorem 2.3 *(Energy decay rates) Suppose that the hypotheses in Theorem 2.1 and (2.8) with* $\rho \leq \gamma$. *Then we have following energy decay rates:*

(i) ***Case 1*** *:* β *is linear. Then we have*

$$E(t) \leq C_1 e^{-\omega t},$$

where ω *is a positive constant.*

(ii) ***Case 2*** *: β has polynomial growth near zero, that is, $\beta(s) = s^{\rho+1}$. Then we have*

$$E(t) \leq \frac{C_2}{(1+t)^{\frac{2}{\rho}}}.$$

(iii) ***Case 3*** *: β does not necessarily have polynomial growth near zero. Then we have*

$$E(t) \leq C_3\Big(F^{-1}\Big(\frac{1}{t}\Big)\Big)^2,$$

where $F(s) = s\beta(s)$ and C_i $(i = 1, 2, 3)$ are positive constants that depends only on $E(0)$.

Theorem 2.4 *(Blow-up) Suppose that hypotheses $(H_1) - (H_4)$ hold and, in addition, that $\rho < \gamma$. Moreover, assume that*

$$(u_0, u_1) \in \{(u_0, u_1) \in \mathcal{H} \times L^2(\Omega); a(u_0, u_0) > \lambda_0^2, \ -1 < E(0) < d_0\}$$

and

$$\beta^{-1}(1) \leq \Big(\frac{(\gamma+2)(\mu_0\gamma\lambda_0^2 - 2(\gamma+2)E_1)^2}{8(\gamma+1)meas(\Omega)(\mu_0\lambda_0^2 - 2E_1)}\Big)^{\frac{\gamma+1}{\gamma+2}}, \tag{2.9}$$

where

$$E_1 = \begin{cases} 0 & if \quad E(0) < 0, \\ positive\ constant\ satisfying \quad E(0) < E_1 < d_0 \quad and \quad E_1 < E(0) + 1 & if \quad E(0) \geq 0. \end{cases}$$

Then the weak solution of the problem (1.1) blows up in finite time.

Remark 2.4 The inequality $\rho \geq \gamma$ always holds true under the following condition: (see Figure 2)

$$\begin{cases} 4 \leq \gamma \leq 5 & \text{if} \quad n = 3, \\ 2 \leq \gamma \leq 3 & \text{if} \quad n = 4, \\ \frac{4}{n-2} \leq \gamma \leq \frac{n+2}{n-2} & \text{if} \quad n \geq 5. \end{cases}$$

In other words, if f is super-supercritical, then the inequality $\rho \geq \gamma$ always holds under the assumption $\rho \geq \frac{2(n-2)\gamma-4}{n+2-(n-2)\gamma}$.

Remark 2.5 We summarize our results.

(1) Local existence is obtained for the region I, II and III in Figure 2.
(2) Hadamard wellposedness is satisfied for the region I and II in Figure 2.

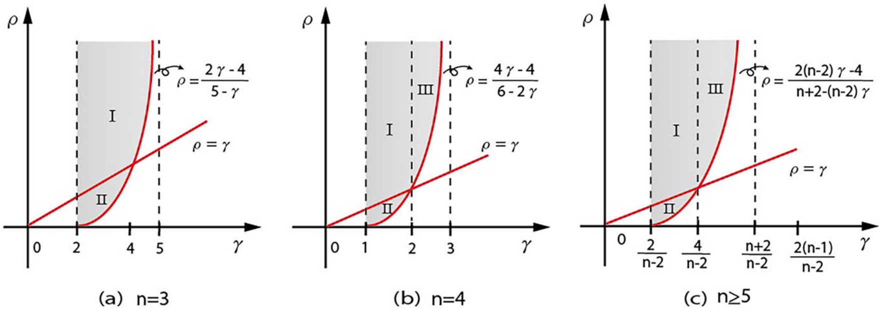

Fig. 2 The admissible range of the damping parameter ρ and the exponent of the source γ

(3) Global existence is obtained for the region I and III, or the region II with the condition (2.8) in Figure 2.
(4) Energy decay rate is obtained for the region II in Figure 2 with the condition (2.8).
(5) For the region II in Figure 2 with the condition (2.9), we obtain the blow-up in finite time.

3 Local Existence

3.1 Globally Lipschitz Source

We first deal with the case where the source f is globally Lipschitz from $H_0^1(\Omega)$ to $L^2(\Omega)$. In this case, we have the following result.

Proposition 3.1 *Assume that* $(H_1) - (H_3)$ *hold. In addition, assume that* $(u_0, u_1) \in \mathcal{H} \times L^2(\Omega)$ *and* $f : H_0^1(\Omega) \to L^2(\Omega)$ *is globally Lipschitz continuous satisfying* $c_3|s|^{\gamma+1} \leq |f(s)| \leq c_4|s|^{\gamma+1}$, *where* c_3, c_4 *are for some positive constants. Then problem (1.1) has a unique global solution* $u \in C(0, T; \mathcal{H}) \cap C^1(0, T; L^2(\Omega))$ *for arbitrary* $T > 0$.

Proof We construct an approximate solution by using the Faedo-Galerkin method. Let $\{w_j\}_{j\in\mathbb{N}}$ be a basis in $H_0^1(\Omega)$ and define $V_m = span\{w_1, w_2, \cdots, w_m\}$. Let u_0^m and u_1^m be sequences of V_m such that $u_0^m \to u_0$ strongly in $H_0^1(\Omega)$ and $u_1^m \to u_1$ strongly in $L^2(\Omega)$. We search for a function, for each $m \in \mathbb{N}$,

$$u^m(t) = \sum_{j=1}^{m} \delta^{jm}(t) w_j$$

satisfying the approximate equation

$$\begin{cases} \langle u_{tt}^m, w\rangle + \mu(t)a(u^m, w) + \langle g(u_t^m), w\rangle = \langle f(u^m), w\rangle \quad \text{for all} \quad w \in V_m, \\ u_0^m = \sum_{j=1}^m \langle u_0, w_j\rangle w_j, \quad u_1^m = \sum_{j=1}^m \langle u_1, w_j\rangle w_j. \end{cases} \tag{3.1}$$

Since (3.1) is a normal system of ordinary differential equations, there exist u^m, solutions to problem (3.1). A solution u to problem (1.1) on some internal $[0, t_m)$, $t_m \in (0, T]$ will be obtain as the limit of u^m as $m \to \infty$. Next, we show that $t_m = T$ and the local solution is uniformly bounded independent of m and t. For this purpose, let us replace w by u_t^m in (3.1) we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}||u_t^m||_2^2 + \frac{1}{2}\mu(t)a(u^m, u^m) + \frac{1}{\gamma+2}||u^m||_{\gamma+2}^{\gamma+2}\right] + \int_\Omega g(u_t^m)\, u_t^m dx \\
&= \frac{1}{2}\mu'(t)a(u^m, u^m) + \int_\Omega f(u^m)\, u_t^m dx + \int_\Omega |u^m|^\gamma u^m u_t^m dx \\
&\le \frac{1}{2}\mu'(t)a(u^m, u^m) + \left(1 + \frac{1}{c_3}\right)\int_\Omega |f(u^m)|\, |u_t^m| dx.
\end{aligned} \tag{3.2}$$

We will now estimate $\int_\Omega g(u_t^m)\, u_t^m dx$ and $\int_\Omega f(u^m)\, u_t^m dx$. From the hypothesis on g, we have

$$\begin{aligned}
\int_\Omega g(u_t^m)\, u_t^m dx &= \int_{|u_t^m|\le 1} g(u_t^m)\, u_t^m dx + \int_{|u_t^m|>1} g(u_t^m)\, u_t^m dx \\
&\ge \int_{|u_t^m|>1} g(u_t^m)\, u_t^m dx \\
&\ge c_1 \int_\Omega |u_t^m|^{\rho+2} dx - c_1 \int_{|u_t^m|\le 1} |u_t^m|^{\rho+2} dx \\
&\ge c_1 ||u_t^m||_{\rho+2}^{\rho+2} - c_1 meas(\Omega).
\end{aligned} \tag{3.3}$$

Under the assumption that f is globally Lipschitz from $H_0^1(\Omega)$ into $L^2(\Omega)$ we have

$$||f(u^m)||_2 \le ||f(u^m) - f(0)||_2 + ||f(0)||_2 \le L_f ||\nabla u||_2 + ||f(0)||_2 \le C_4(||\nabla u||_2 + 1),$$

where L_f is the Lipschitz constant and C_4 is for some positive constant, so that by Hölder's and Young's inequalities and from the fact (2.6) we deduce that

$$\int_\Omega |f(u^m)|\, |u_t^m| dx \le \frac{1}{2}||f(u^m)||_2^2 + \frac{1}{2}||u_t^m||_2^2 \le C_4\left(\frac{1}{a_0}a(u^m, u^m) + 1\right) + \frac{1}{2}||u_t^m||_2^2. \tag{3.4}$$

Replacing (3.3) and (3.4) in (3.2) we get

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}||u_t^m||_2^2 + \frac{1}{2}\mu(t)a(u^m, u^m) + \frac{1}{\gamma+2}||u^m||_{\gamma+2}^{\gamma+2}\right] + c_1||u_t^m||_{\rho+2}^{\rho+2} \\
&\le c_1 meas(\Omega) + C_4\left(1 + \frac{1}{c_3}\right) + \frac{1}{2}\left(1 + \frac{1}{c_3}\right)||u_t^m||_2^2 + \left(\frac{1}{2}\mu'(t) + \frac{C_4}{a_0}\left(1 + \frac{1}{c_3}\right)\right)a(u^m, u^m).
\end{aligned} \tag{3.5}$$

Integrating (3.5) over $(0, t)$ with $t \in (0, t_m)$ we have

$$\begin{aligned}
&\frac{1}{2}||u_t^m||_2^2 + \frac{1}{2}\mu(t)a(u^m, u^m) + \frac{1}{\gamma+2}||u^m||_{\gamma+2}^{\gamma+2} + c_1\int_0^t ||u_s^m(s)||_{\rho+2}^{\rho+2}ds \\
&\le \Big(c_1 meas(\Omega) + C_4\Big(1+\frac{1}{c_3}\Big)\Big)T + \frac{1}{2}||u_1||_2^2 + \frac{1}{2}\mu(0)a(u_0, u_0) + \frac{1}{\gamma+2}||u_0||_{\gamma+2}^{\gamma+2} \\
&\quad + \frac{1}{2}\Big(1+\frac{1}{c_3}\Big)\int_0^t ||u_s^m||_2^2 ds + \Big(\frac{1}{2}||\mu'||_{L^\infty(0,T)} + \frac{C_4}{a_0}\Big(1+\frac{1}{c_3}\Big)\Big)\int_0^t a(u^m(s), u^m(s))ds.
\end{aligned} \tag{3.6}$$

Therefore, by Gronwall's lemma we obtain

$$||u_t^m||_2^2 + a(u^m, u^m) + ||u^m||_{\gamma+2}^{\gamma+2} + \int_0^t ||u_s^m(s)||_{\rho+2}^{\rho+2}ds \le C_5, \tag{3.7}$$

where C_5 is a positive constant which is independent of m and t. The estimate (3.7) implies that

$$u^m \quad \text{is uniformly bounded in} \quad L^\infty(0, T; \mathcal{H})) \tag{3.8}$$

and

$$u_t^m \quad \text{is uniformly bounded in} \quad L^\infty(0, T; L^2(\Omega)). \tag{3.9}$$

We note that from (3.9), taking the hypotheses on g into account we also obtain

$$\int_0^t \int_\Omega |g(u_s^m(s))|^2 dxds \le C_6, \tag{3.10}$$

where C_6 is a positive constant independent of m and t.

From (3.7)-(3.10), there exists a subsequence of $\{u^m\}$, which we still denote by $\{u^m\}$, such that

$$u^m \to u \quad \text{weak star in} \quad L^\infty(0, T; \mathcal{H}), \tag{3.11}$$

$$u_t^m \to u_t \quad \text{weak star in} \quad L^\infty(0, T; L^2(\Omega)), \tag{3.12}$$

$$u_{tt}^m \to u_{tt} \quad \text{weak star in} \quad L^\infty(0, T; H^{-1}(\Omega)), \tag{3.13}$$

$$g(u_t^m) \to \psi \quad \text{weakly in} \quad L^2(0, T; L^2(\Omega)). \tag{3.14}$$

Since $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, we have, thanks to Aubin-Lions Theorem that

$$u^m \to u \quad \text{strongly in} \quad L^\infty(0, T; L^2(\Omega)),$$

and consequently, by making use of Lions lemma (cf. [35]), we deduce

$$f(u^m) \to f(u) \quad \text{weakly in} \quad L^\infty(0, T; L^2(\Omega)). \tag{3.15}$$

Convergences (3.11)-(3.15) permit us to pass to the limit in the (3.1). Since $\{w_j\}$ is a basis of $H_0^1(\Omega)$ and V_m is dense in $H_0^1(\Omega)$, after passing to the limit we obtain

$$\int_0^T \int_\Omega u_{tt}, v dx\theta(t)dt + \int_0^T \mu(t)a(u,v)\theta(t)dt + \int_0^T \int_\Omega \psi v dx\theta(t)dt \\ = \int_0^T \int_\Omega f(u) v dx\theta(t)dt. \tag{3.16}$$

for all $\theta \in D(0,T)$ and $v \in H_0^1(\Omega)$.

From the (3.16) and taking $v \in D(\Omega)$, we conclude that

$$u_{tt} - \mu(t)Lu + \psi = f(u) \quad \text{in} \quad D'(\Omega \times (0,T)). \tag{3.17}$$

Our goal is to show that $\psi = g(u_t)$. Indeed, considering $w = u^m$ in (3.1) and then integrating over $(0,T)$, we have

$$\int_0^T \langle u_{tt}^m, u^m \rangle dt + \int_0^T \mu(t)a(u^m, u^m)dt + \int_0^T \langle g(u_t^m), u^m \rangle dt = \int_0^T \langle f(u^m), u^m \rangle dt.$$

Then from convergences (3.11)-(3.15) we obtain

$$\lim_{m\to\infty} \int_0^T \mu(t)a(u^m, u^m)dt = -\int_0^T \langle u_{tt}, u \rangle dt - \int_0^T \langle \psi, u \rangle dt + \int_0^T \langle f(u), u \rangle dt. \tag{3.18}$$

By combining (3.17) and (3.18), we have

$$\lim_{m\to\infty} \int_0^T \mu(t)a(u^m, u^m)dt = \int_0^T \mu(t)a(u,u)dt,$$

which implies that

$$u^m \to u \quad \text{strongly in} \quad L^2(0,T; H_0^1(\Omega)). \tag{3.19}$$

Next, considering $w = u_t^m$ in (3.1) and then integrating over $(0,T)$, we have

$$\int_0^T \langle u_{tt}^m, u_t^m \rangle dt + \int_0^T \mu(t)a(u^m, u_t^m)dt + \int_0^T \langle g(u_t^m), u_t^m \rangle dt = \int_0^T \langle f(u^m), u_t^m \rangle dt.$$

From (3.12), (3.13), (3.15), (3.17) and (3.19), we arrive at

$$\lim_{m\to\infty} \int_0^T \langle g(u_t^m), u_t^m \rangle dt = \int_0^T \langle \psi, u_t \rangle dt. \tag{3.20}$$

On the other hand, since g is a nondecreasing monotone function, we get

$$\int_0^T \langle g(u_t^m) - g(\varphi), u_t^m - \varphi \rangle dt \geq 0$$

for all $\varphi \in L^2(\Omega)$. Thus, it implies that

$$\int_0^T \langle g(u_t^m), \varphi \rangle dt + \int_0^T \langle g(\varphi), u_t^m - \varphi \rangle dt \leq \int_0^T \langle g(u_t^m), u_t^m \rangle dt.$$

By considering (3.12), (3.14) and (3.20), we obtain

$$\int_0^T \langle \psi - g(\varphi), u_t - \varphi \rangle dt \geq 0, \tag{3.21}$$

which implies that $\psi = g(u_t)$.

We now show the uniqueness of the solution. Let u^1 and u^2 be two solutions of problem (1.1). Then $z = u^1 - u^2$ verifies

$$\langle z_{tt}, w \rangle + \mu(t) a(z, w) + \langle g(u_t^1) - g(u_t^2), w \rangle = \langle f(u^1) - f(u^2), w \rangle,$$

for all $w \in \mathcal{H}$. By replacing $w = z_t$ in above identity and observing that g is monotonously nondecreasing and $f : H_0^1(\Omega) \to L^2(\Omega)$ is globally Lipschitz, it holds that

$$\frac{d}{dt}\left[\frac{1}{2}||z_t||_2^2 + \frac{1}{2}\mu(t) a(z, z)\right] \leq C_7 a(z, z),$$

Where C_7 is for some positive constant. By integrating from 0 to t and using Gronwall's Lemma, we conclude that $||z_t||_2 = a(z, z) = 0$. □

3.2 Locally Lipschitz Source

In this subsection, we loosen the globally Lipschitz condition on the source by allowing f to be locally Lipschitz continuous. More precisely, we have the following result.

Proposition 3.2 *Assume that* $(H_1) - (H_4)$ *hold. In addition, assume that* $(u_0, u_1) \in \mathcal{H} \times L^2(\Omega)$ *and* $f : H_0^1(\Omega) \to L^2(\Omega)$ *is locally Lipschitz continuous satisfying* $c_3|s|^{\gamma+1} \leq |f(s)| \leq c_4|s|^{\gamma+1}$, *where* c_3, c_4 *are for some positive constants. Then problem (1.1) has a unique local solution* $u \in C(0, T; \mathcal{H}) \cap C^1(0, T; L^2(\Omega))$ *for some* $T > 0$.

Proof Define

$$f_K(u) = \begin{cases} f(u) & \text{if } \ [a(u,u)]^{\frac{1}{2}} \leq K, \\ f\left(\frac{Ku}{[a(u,u)]^{\frac{1}{2}}}\right) & \text{if } \ [a(u,u)]^{\frac{1}{2}} > K, \end{cases}$$

where K is a positive constant. With this truncated f_K, we consider the following problem:

$$\begin{cases} u_{tt} - \mu(t)Lu + g(u_t) = f_K(u) & \text{in} \quad \Omega \times (0, +\infty), \\ u = 0 & \text{on} \quad \Gamma \times (0, +\infty), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & \text{in} \quad \Omega. \end{cases} \tag{3.22}$$

Since $f_K : H_0^1(\Omega) \to L^2(\Omega)$ is globally Lipschitz with Lipschitz constant $L_f(K)$ for each K (see [10]), then by Proposition 3.1, the truncated problem (3.22) has a unique global solution $u_K \in C(0, T; \mathcal{H}) \cap C^1(0, T; L^2(\Omega))$ for arbitrary $T > 0$. For simplifying the notation in the rest of the proof, we shall express u_K as u.

Multiplying (3.22) by u_t and integrating on $\Omega \times (0, t)$, where $0 < t < T$ we obtain by using the fact $\mu'(s) < 0$ for all $s > 0$,

$$\begin{aligned} &\frac{1}{2}(||u_t||_2^2 + \mu(t)a(u,u)) + \frac{1}{\gamma+2}||u||_{\gamma+2}^{\gamma+2} + \int_0^t \int_\Omega g(u_s(x,s))u_s(x,s)dxds \\ &= \frac{1}{2}(||u_1||_2^2 + \mu(0)a(u_0,u_0)) + \frac{1}{\gamma+2}||u_0||_{\gamma+2}^{\gamma+2} + \frac{1}{2}\int_0^t \mu'(s)a(u(x,s),u(x,s))ds \\ &\quad + \int_0^t \int_\Omega f_K(u(x,s))u_s(x,s)dxds + \int_0^t \int_\Omega |u(x,s)|^\gamma u(x,s)\, u_s(x,s)dxds \\ &\le \frac{1}{2}(||u_1||_2^2 + \mu(0)a(u_0,u_0)) + \frac{1}{\gamma+2}||u_0||_{\gamma+2}^{\gamma+2} \\ &\quad + \left(1 + \frac{1}{c_3}\right)\int_0^t \int_\Omega |f_K(u(x,s))|\, |u_s(x,s)|dxds. \end{aligned} \tag{3.23}$$

We note that $f_K : H_0^1(\Omega) \to L^{\frac{\rho+2}{\rho+1}}(\Omega)$ is globally Lipschitz with Lipschitz constant $L_f(K)$ (see [10,13]). Hence we estimate the last term on the right-hand side of (3.23) as follows:

$$\begin{aligned} &\left(1 + \frac{1}{c_3}\right)\int_0^t \int_\Omega |f_K(u(x,s))|\, |u_s(x,s)|dxds \\ &\le \left(1 + \frac{1}{c_3}\right)\int_0^t ||f_K(u(s))||_{\frac{\rho+2}{\rho+1}} ||u_s||_{\rho+2}ds \\ &\le \epsilon \int_0^t ||u_s(s)||_{\rho+2}^{\rho+2}ds + C(\epsilon)\int_0^t ||f_K(u(s))||_{\frac{\rho+2}{\rho+1}}^{\frac{\rho+2}{\rho+1}}ds \\ &\le \epsilon \int_0^t ||u_s(s)||_{\rho+2}^{\rho+2}ds + C(\epsilon)\Big(\frac{2}{\sqrt{a_0}}L_f(K)\Big)^{\frac{\rho+2}{\rho+1}} \int_0^t a(u(s),u(s))ds \\ &\quad + tC(\epsilon)\Big(\Big(\frac{2}{\sqrt{a_0}}L_f(K)\Big)^{\frac{\rho+2}{\rho+1}} + 2^{-(\rho+1)}|f(0)|^{\frac{\rho+2}{\rho+1}}meas(\Omega)\Big). \end{aligned} \tag{3.24}$$

From the hypothesis on g, we have

$$\int_0^t \int_\Omega g(u_s(x,s))u_s(x,s)dxds \geq c_1 \int_0^t ||u_s(s)||_{\rho+2}^{\rho+2}ds - tc_1 meas(\Omega). \quad (3.25)$$

By replacing (3.24) and (3.25) in (3.23) and choosing $\epsilon \leq c_1$, we get

$$\begin{aligned} &||u_t||_2^2 + a(u,u) + ||u||_{\gamma+2}^{\gamma+2} + \int_0^t ||u_s(s)||_{\rho+2}^{\rho+2}ds \\ &\quad \leq C_8 + C_1(L_f(K))T + C_2(L_f(K)) \int_0^t ||u_s(s)||_2^2 + a(u(s),u(s))ds \end{aligned} \quad (3.26)$$

for all $t \in [0, T]$, where

$$\begin{aligned} C_8 &= \frac{1}{2\alpha}(||u_1||_2^2 + \mu(0)a(u_0,u_0)) + ||u_0||_{\gamma+2}^{\gamma+2}, \\ C_1(L_f(K)) &= \frac{1}{\alpha}C(\epsilon)\Big(\Big(\frac{2}{\sqrt{a_0}}L_f(K)\Big)^{\frac{\rho+2}{\rho+1}} + 2^{-(\rho+1)}|f(0)|^{\frac{\rho+2}{\rho+1}}meas(\Omega)\Big) + \frac{1}{\alpha}c_1 meas(\Omega), \\ C_2(L_f(K)) &= \frac{1}{\alpha}C(\epsilon)\Big(\frac{2}{\sqrt{a_0}}L_f(K)\Big)^{\frac{\rho+2}{\rho+1}}, \end{aligned}$$

for $\alpha = \min\{\frac{\mu_0}{2}, \frac{1}{\gamma+2}, c_1 - \epsilon\}$. Thus by Gronwall's inequality, (3.26) becomes

$$\begin{aligned} &||u_t||_2^2 + a(u,u) + ||u||_{\gamma+2}^{\gamma+2} + \int_0^t ||u_s(s)||_{\rho+2}^{\rho+2}ds \\ &\quad \leq (C_8 + C_1(L_f(K))T)e^{C_2(L_f(K))t} \quad \text{for all} \quad t \in [0,T]. \end{aligned}$$

If we choose $T = \min\{\frac{1}{C_1(L_f(K))}, \frac{1}{C_2(L_f(K))}\ln 2\}$, then

$$||u_t||_2^2 + a(u,u) + ||u||_{\gamma+2}^{\gamma+2} + \int_0^t ||u_s(s)||_{\rho+2}^{\rho+2}ds \leq 2(C_8+1) \leq K^2 \quad \text{for all} \quad t \in [0,T], \quad (3.27)$$

provided we choose $K^2 \geq 2(C_8+1)$. Consequently, (3.27) gives us that $[a(u,u)]^{\frac{1}{2}} \leq K$ for all $t \in [0, T]$. Therefore, by the definition of f_K, we have that $f_K(u) = f(u)$ on $[0, T]$. By the uniqueness of solutions, the solution of the truncated problem (3.22) accords with the solution of the original, non-truncated problem (1.1) for $t \in [0, T]$, which means that the proof of Proposition 3.2 is completed. □

3.3 Completion of the Proof for the Local Existence

In order to establish the existence of solutions, we need to extend the result in Proposition 3.2 where the source f is locally Lipschitz from $H_0^1(\Omega)$ into $L^{\frac{\rho+2}{\rho+1}}(\Omega)$. For the construction of the Lipschitz approximation for the source, we employ another truncated function introduced in [43]. Let $\eta_n \in C_0^\infty(\mathbb{R})$ be a cut off function such that

$$\begin{cases} 0 \le \eta_n \le 1, \\ \eta_n(s) = 1, \quad \text{if} \quad |s| \le n, \\ \eta_n(s) = 0, \quad \text{if} \quad |s| \ge 2n, \end{cases}$$

and $|\eta_n'(s)| \le \frac{C}{n}$ for some constant C independent from n and define

$$f_n(u) = f(u)\eta_n(u). \tag{3.28}$$

Then the truncated function f_n is satisfied the following lemma. The proof of this lemma is a routine series of estimates as in [1,44], so we omit it here.

Lemma 3.1 *The following statements hold.*

(1) $f_n : H_0^1(\Omega) \to L^2(\Omega)$ is globally Lipschitz continuous.

(2) $f_n : H_0^{1-\epsilon}(\Omega) \to L^{\frac{\rho+2}{\rho+1}}(\Omega)$ is locally Lipschitz continuous with Lipschitz constant independent if n.

With the truncated source f_n defined in (3.28), by Proposition 3.2 and Lemma 3.1, we have a unique local solution $u^n \in C(0, T; \mathcal{H}) \cap C^1(0, T; L^2(\Omega))$ satisfying the following approximation of (1.1)

$$\begin{cases} u_{tt} - \mu(t)Lu + g(u_t) = f_n(u) & \text{in} \quad \Omega \times (0, +\infty), \\ u = 0 & \text{on} \quad \Gamma \times (0, +\infty), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & \text{in} \quad \Omega. \end{cases} \tag{3.29}$$

From Lemma 3.1, the life span T of each solution u^n is independent of n. Also we known that T depends on K, where $K^2 \ge 2(C_8 + 1)$, however, since $||u_1^n||_2^2 + a(u_0^n, u_0^n) + ||u_0^n||_{\gamma+2}^{\gamma+2} \to ||u_1||_2^2 + a(u_0, u_0) + ||u_0||_{\gamma+2}^{\gamma+2}$, we can choose K sufficiently large so that K is independent of n. By (3.27),

$$||u_t^n||_2^2 + a(u^n, u^n) + ||u^n||_{\gamma+2}^{\gamma+2} \le K^2 \tag{3.30}$$

for all $t \in [0, T]$. Therefore, there exists a function u and a subsequence of $\{u^n\}$, which we still denote by $\{u^n\}$, such that

$$u^n \to u \quad \text{weak star in} \quad L^\infty(0, T; \mathcal{H}), \tag{3.31}$$

$$u_t^n \to u_t \quad \text{weak star in} \quad L^\infty(0, T; L^2(\Omega)). \tag{3.32}$$

By (3.30), (3.31) and (3.32), we infer

$$||u_t||_2^2 + a(u,u) + ||u||_{\gamma+2}^{\gamma+2} \le K^2 \tag{3.33}$$

for all $t \in [0, T]$. Moreover, by Aubin-Lions Theorem, we have

$$u^n \to u \quad \text{strongly in} \quad L^\infty(0,T; H^{1-\epsilon}(\Omega)), \tag{3.34}$$

for $0 < \epsilon < 1$. Since u^n is a solution of (3.29), it holds that

$$\begin{aligned} &\int_0^T \int_\Omega u_{tt}^n, \phi dxdt + \int_0^T \mu(t) a(u^n, v) dt \\ &+ \int_0^T \int_\Omega g(u_t^n)\phi dxdt = \int_0^T \int_\Omega f_n(u^n)\phi dxdt, \end{aligned} \tag{3.35}$$

for any $\phi \in C(0,T;\mathcal{H}) \cap C^1(0,T;L^2(\Omega))$, $\phi \in L^{\rho+2}(0,T;\Omega)$.

Now we will show that

$$\lim_{n\to\infty} \int_0^T \int_\Omega f_n(u^n)\phi dxdt = \int_0^T \int_\Omega f(u)\phi dxdt. \tag{3.36}$$

Indeed, we have

$$\begin{aligned} &\Big|\int_0^T \int_\Omega (f_n(u^n) - f(u))\phi dxdt\Big| \\ &\le \int_0^T \int_\Omega |f_n(u^n) - f_n(u)|\, |\phi| dxdt + \int_0^T \int_\Omega |f_n(u) - f(u)|\, |\phi| dxdt. \end{aligned} \tag{3.37}$$

By (2) in Lemma 3.1 and (3.34), we obtain

$$\begin{aligned} &\int_0^T \int_\Omega |f_n(u^n) - f_n(u)|\, |\phi| dxdt \\ &\le \Big(\int_0^T \int_\Omega |f_n(u^n) - f_n(u)|^{\frac{\rho+2}{\rho+1}} dxdt\Big)^{\frac{\rho+1}{\rho+2}} \Big(\int_0^T \int_\Omega |\phi|^{\rho+2} dxdt\Big)^{\frac{1}{\rho+2}} \\ &\le C(K)||\phi||_{L^{\rho+2}(0,T;\Omega)} \Big(\int_0^T ||u^n - u||_{H^{1-\epsilon}(\Omega)}^{\frac{\rho+2}{\rho+1}} dt\Big)^{\frac{\rho+1}{\rho+2}} \to 0. \end{aligned} \tag{3.38}$$

Since $\eta_n(u(x)) \to 1$ a.e. in Ω, we have $f_n(u) \to f(u)$ a.e. Then we also have $|f_n(u) - f(u)|^{\frac{\rho+2}{\rho+1}} \le 2^{\frac{\rho+2}{\rho+1}} |f(u)|^{\frac{\rho+2}{\rho+1}}$ and $f(u) \in L^{\frac{\rho+2}{\rho+1}}(\Omega)$, for $u \in H_0^1(\Omega)$. Thus by the Lebesgue Dominated Convergence Theorem, we have

$$\begin{aligned} &\int_0^T \int_\Omega |f_n(u) - f(u)|\, |\phi| dx dt \\ &\le \Big(\int_0^T \int_\Omega |f_n(u) - f(u)|^{\frac{\rho+2}{\rho+1}} dx dt\Big)^{\frac{\rho+1}{\rho+2}} \Big(\int_0^T \int_\Omega |\phi|^{\rho+2} dx dt\Big)^{\frac{1}{\rho+2}} \\ &\le ||\phi||_{L^{\rho+2}(0,T;\Omega)} \Big(\int_0^T \int_\Omega |f(u)|^{\frac{\rho+2}{\rho+1}} |\eta_n(u) - 1|^{\frac{\rho+2}{\rho+1}} dx dt\Big)^{\frac{\rho+1}{\rho+2}} \to 0. \end{aligned} \tag{3.39}$$

From convergences (3.38) and (3.39), (3.37) gives us (3.36).

On the other hand, by using similar arguments from (3.18) to (3.21), we get

$$g(u_t^n) \to g(u_t) \quad \text{weakly in} \quad L^2(0, T; L^2(\Omega)). \tag{3.40}$$

Convergences (3.32), (3.33), (3.36) and (3.40) permit us to pass to the limit in (3.35) and conclude the following result.

Proposition 3.3 *Assume that* $(H_1) - (H_4)$ *hold. In addition, assume that* $(u_0, u_1) \in \mathcal{H} \times L^2(\Omega)$ *and* $f : H_0^1(\Omega) \to L^{\frac{\rho+2}{\rho+1}}(\Omega)$ *is locally Lipschitz continuous. Then problem (1.1) has a local solution* $u \in C(0, T; \mathcal{H}) \cap C^1(0, T; L^2(\Omega))$ *for some* $T > 0$.

Let $f(u) = |u|^\gamma u$, then $f : H_0^1(\Omega) \to L^{\frac{\rho+2}{\rho+1}}(\Omega)$ is locally Lipschitz continuous (see Remark 2.1). Thus by Proposition 3.3, the proof of the local existence statement in Theorem 2.1 is completed.

3.4 Energy Identity

It is well known that to prove the uniqueness of weak solutions, we will justify the energy identity (2.7). The energy identity can be derived formally by multiplying (1.1) by u_t. But, such a calculation is not justified, since u_t is not sufficiently regular to be the test function in as required in Definition 2.1. To overcome this problem, we employ the operator $T^\epsilon = (I - \epsilon L)^{-1}$ to smooth function in space, which is mentioned in [13]. We recall important properties of T^ϵ which play an essential role when establishing the energy identity.

Lemma 3.2 [13] *Let* $u^\epsilon = T^\epsilon u$. *Then following statements hold.*

1. *If* $u \in L^2(\Omega)$, *then* $||u^\epsilon||_2 \le ||u||_2$ *and* $u^\epsilon \to u$ *in* $L^2(\Omega)$ *as* $\epsilon \to 0$.
2. *If* $u \in H_0^1(\Omega)$, *then* $||\nabla u^\epsilon||_2 \le ||\nabla u||_2$ *and* $u^\epsilon \to u$ *in* $H_0^1(\Omega)$ *as* $\epsilon \to 0$.
3. *If* $u \in L^p(\Omega)$ *with* $1 < p < \infty$, *then* $||u^\epsilon||_p \le ||u||_p$ *and* $u^\epsilon \to u$ *in* $L^p(\Omega)$ *as* $\epsilon \to 0$.

We will now justify the energy identity (2.7). We play the operator T^ϵ on every term of (1.1) and multiply by u_t^ϵ. Then we obtain by integrating in space and time

$$\int_0^t \int_\Omega u_{ss}^\epsilon u_s^\epsilon dxds + \int_0^t \mu(s)a(u^\epsilon, u_s^\epsilon)ds$$
$$+ \int_0^t \int_\Omega T^\epsilon(g(u_s))u_s^\epsilon dxds = \int_0^t \int_\Omega T^\epsilon(f(u))u_s^\epsilon dxds. \tag{3.41}$$

Since $u \in H_0^1(\Omega)$ and $u_t \in L^2(\Omega)$, we have by Lemma 3.2, $u^\epsilon \to u$ in $H_0^1(\Omega)$ and $u_t^\epsilon \to u_t$ in $L^2(\Omega)$. Therefore using this convergences, we have

$$\lim_{\epsilon\to 0}\Big(\int_0^t \int_\Omega u_{ss}^\epsilon u_s^\epsilon dxds + \int_0^t \mu(s)a(u^\epsilon, u_s^\epsilon)ds\Big) = \frac{1}{2}\big(||u_t||_2^2 + a(u,u) - ||u_1||_2^2 - a(u_0,u_0)\big). \tag{3.42}$$

Since $u_t, g(u_t) \in L^2(\Omega)$, we easily check that

$$\lim_{\epsilon\to 0}\int_0^t \int_\Omega T^\epsilon(g(u_s))u_s^\epsilon dxds = \int_0^t \int_\Omega (g(u_s))u_s^\epsilon dxds. \tag{3.43}$$

Recall that $u_t \in L^{\rho+2}(\Omega)$ and $f(u) \in L^{\frac{\rho+2}{\rho+1}}(\Omega)$. By Lemma 3.2, we have $u_t^\epsilon \to u_t$ in $L^{\rho+2}(\Omega)$ and $T^\epsilon(f(u)) \to f(u)$ in $L^{\frac{\rho+2}{\rho+1}}(\Omega)$. Thus by Lebesgue Dominated Convergence Theorem, we obtain

$$\lim_{\epsilon\to 0}\int_0^t \int_\Omega T^\epsilon(f(u))u_s^\epsilon dxds = \int_0^t \int_\Omega (f(u))u_s^\epsilon dxds. \tag{3.44}$$

Convergences (3.42)-(3.44) permit us to pass to the limit in (3.41), consequently, the energy identity (2.7) holds.

4 Global Existence

In order to prove the global existence of solutions of (1.1), it suffices to show that $||u_t||_2^2 + a(u,u) + ||u||_{\gamma+2}^{\gamma+2}$ is bounded independent of t. We now consider the following two cases:

4.1 $\rho \geq \gamma$

Using the energy identity (2.7), we have

$$\frac{d}{dt}\left[\frac{1}{2}||u_t||_2^2 + \frac{1}{2}\mu(t)a(u,u) + \frac{1}{\gamma+2}||u||_{\gamma+2}^{\gamma+2}\right] + \int_\Omega g(u_t)u_t dx$$
$$= \frac{1}{2}\mu'(t)a(u,u) + 2\int_\Omega |u|^\gamma u u_t dx. \tag{4.1}$$

By the same argument as (3.3), we have

$$\int_\Omega g(u_t)\, u_t dx \geq c_1 ||u_t||_{\rho+2}^{\rho+2} - c_1 meas(\Omega). \tag{4.2}$$

Using the Hölder and Young inequalities with $\frac{\gamma+1}{\gamma+2} + \frac{1}{\gamma+2} = 1$ and the imbedding $L^{\rho+2}(\Omega) \hookrightarrow L^{\gamma+2}(\Omega)$, we deduce that

$$\begin{aligned} 2\int_\Omega |u|^\gamma u u_t dx &\leq C(\epsilon_1)||u||_{\gamma+2}^{\gamma+2} + \epsilon_1 C_{\rho+2}^{\gamma+2} ||u_t||_{\rho+2}^{\gamma+2} \\ &\leq C(\epsilon_1)||u||_{\gamma+2}^{\gamma+2} + \epsilon_1 2^{\rho+1} C_{\rho+2}^{\gamma+2}\big(1 + ||u_t||_{\rho+2}^{\rho+2}\big), \end{aligned} \tag{4.3}$$

where $C_{\rho+2}$ is an imbedding constant. By replacing (4.2) and (4.3) in (4.1) and using (2.2), we get

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}||u_t||_2^2 + \frac{1}{2}\mu(t)a(u,u) + \frac{1}{\gamma+2}||u||_{\gamma+2}^{\gamma+2}\right] \\ &\leq C(\epsilon_1)||u||_{\gamma+2}^{\gamma+2} + (c_1 meas(\Omega) + \epsilon_1 2^{\rho+1} C_{\rho+2}^{\gamma+2}) + (\epsilon_1 2^{\rho+1} C_{\rho+2}^{\gamma+2} - c_1)||u_t||_{\rho+2}^{\rho+2}. \end{aligned} \tag{4.4}$$

Let

$$\widetilde{E}(t) = \frac{1}{2}||u_t||_2^2 + \frac{1}{2}\mu(t)a(u,u) + \frac{1}{\gamma+2}||u||_{\gamma+2}^{\gamma+2}$$

and choosing $\epsilon_1 = \frac{c_1}{2^{\rho+1}C_{\rho+2}^{\gamma+2}}$, then we rewrite (4.4) as

$$\widetilde{E}'(t) \leq C_9 + C_{10}\widetilde{E}(t),$$

where C_9 and C_{10} are positive constants. Now applying Gronwall's inequality, we have that $\widetilde{E}(t) \leq (C_{11}\widetilde{E}(0) + C_{12})e^{C_{11}t}$, where C_{11} and C_{12} are positive constants. Consequently, since $\widetilde{E}(0)$ is bounded we conclude that $||u_t||_2^2 + a(u,u) + ||u||_{\gamma+2}^{\gamma+2}$ is bounded.

4.2 $E(0) < d_0$ and $a(u_0, u_0) < \lambda_0^2$

First of all, we will find a stable region. We set

$$0 < K_0 := \sup_{u\in\mathcal{H}, u\neq 0}\left(\frac{||u||_{\gamma+2}}{[a(u,u)]^{1/2}}\right) < \infty$$

and the functional

$$J(u) = \frac{\mu_0}{2}a(u,u) - \frac{1}{\gamma+2}||u||_{\gamma+2}^{\gamma+2}, \qquad u \in \mathcal{H}. \tag{4.5}$$

We also define the function, for $\lambda > 0$,

$$j(\lambda) = \frac{\mu_0}{2}\lambda^2 - \frac{1}{\gamma+2}K_0^{\gamma+2}\lambda^{\gamma+2}, \tag{4.6}$$

then

$$\lambda_0 = \Big(\frac{\mu_0}{K_0^{\gamma+2}}\Big)^{1/\gamma}$$

is the absolute maximum point of j and

$$j(\lambda_0) = \frac{\gamma\mu_0}{2(\gamma+2)}\lambda_0^2 = d_0.$$

The energy associated to the problem (1.1) is given by

$$E(t) = \frac{1}{2}||u_t(t)||_2^2 + \frac{1}{2}\mu(t)a(u(t), u(t)) - \frac{1}{\gamma+2}||u(t)||_{\gamma+2}^{\gamma+2}, \tag{4.7}$$

for $u \in \mathcal{H}$. By (2.2) and (4.5)-(4.7), we deduce

$$E(t) \geq J(u(t)) \geq \frac{\mu_0}{2}a(u(t), u(t)) - \frac{K_0^{\gamma+2}}{\gamma+2}[a(u(t), u(t))]^{(\gamma+2)/2} = j([a(u(t), u(t))]^{1/2}). \tag{4.8}$$

Lemma 4.1 *Let u be a weak solution for problem (1.1). Suppose that*

$$E(0) < d_0 \quad \textit{and} \quad a(u_0, u_0) < \lambda_0^2.$$

Then

$$a(u(t), u(t)) < \lambda_0^2 \quad \textit{for all} \quad t \geq 0.$$

Proof It is easy to verify that j is increasing for $0 < \lambda < \lambda_0$, decreasing for $\lambda > \lambda_0$, $j(\lambda) \to -\infty$ as $\lambda \to +\infty$. Then since $d_0 > E(0) \geq j([a(u_0, u_0)]^{1/2}) \geq j(0) = 0$, there exist $\lambda_0' < \lambda_0 < \tilde{\lambda_0}$, which verify

$$j(\lambda_0') = j(\tilde{\lambda_0}) = E(0). \tag{4.9}$$

Considering that $E(t)$ is nonincreasing, we have

$$E(t) \leq E(0) \quad \text{for all} \quad t \geq 0. \tag{4.10}$$

From (4.8) and (4.9), we deduce that

$$j([a(u_0, u_0)]^{1/2}) \leq E(0) = j(\lambda_0'). \tag{4.11}$$

Since $[a(u_0, u_0)]^{1/2} < \lambda_0$, $\lambda_0' < \lambda_0$ and j is increasing in $[0, \lambda_0)$, from (4.11) it holds that

$$[a(u_0, u_0)]^{1/2} \leq \lambda_0'. \tag{4.12}$$

Next, we will prove that

$$[a(u(t), u(t))]^{1/2} \leq \lambda_0' \quad \text{for all} \quad t \geq 0. \tag{4.13}$$

We argue by contradiction. Suppose that (4.13) does not hold. Then there exists time t^* which verifies

$$[a(u(t^*), u(t^*))]^{1/2} > \lambda_0'. \tag{4.14}$$

If $[a(u(t^*), u(t^*))]^{1/2} < \lambda_0$, from (4.8), (4.9) and (4.14) we can write

$$E(t^*) \geq j([a(u(t^*), u(t^*))]^{1/2}) > j(\lambda_0') = E(0),$$

which contradicts (4.10).

If $[a(u(t^*), u(t^*))]^{1/2} \geq \lambda_0$, then we have, in view of (4.12), that there exists $\bar{\lambda_0}$ which verifies

$$[a(u_0, u_0)]^{1/2} \leq \lambda_0' < \bar{\lambda_0} < \lambda_0 \leq [a(u(t^*), u(t^*))]^{1/2}. \tag{4.15}$$

Consequently, from the continuity of the function $[a(u(\cdot), u(\cdot))]^{1/2}$ there exists $\bar{t} \in (0, t^*)$ verifying

$$[a(u(\bar{t}), u(\bar{t}))]^{1/2} = \bar{\lambda_0}. \tag{4.16}$$

Then from (4.8), (4.9), (4.15) and (4.16), we get

$$E(\bar{t}) \geq j([a(u(\bar{t}), u(\bar{t}))]^{1/2}) = j(\bar{\lambda_0}) > j(\lambda_0') = E(0),$$

which also contradicts (4.10). This completes the proof of Lemma 4.1. □

From (4.8) and Lemma 4.1, we arrive at

$$E(t) \geq J(u(t)) > a(u(t), u(t))\Big(\frac{\mu_0}{2} - \frac{K_0^{\gamma+2}}{\gamma+2}\lambda_0^{\gamma}\Big) = \mu_0\Big(\frac{1}{2} - \frac{1}{\gamma+2}\Big)a(u(t), u(t)) \tag{4.17}$$

and, consequently,

$$J(u) \geq 0 \;\; (J(u) = 0 \;\text{ iff }\; u = 0) \quad \text{and} \quad a(u(t), u(t)) \leq \frac{2(\gamma+2)}{\mu_0 \gamma} E(t). \tag{4.18}$$

By virtue of (4.17), we get

$$J(u(t)) > \frac{\mu_0 \gamma}{2(\gamma+2)} a(u(t), u(t)). \tag{4.19}$$

Hence

$$\frac{1}{2}||u_t(t)||_2^2 + \frac{\mu_0 \gamma}{2(\gamma+2)} a(u(t), u(t)) < \frac{1}{2}||u_t(t)||_2^2 + J(u(t)) \leq E(t) \leq E(0).$$

Therefore, there exists a positive constant C_{13} independent of t such that

$$||u_t(t)||_2^2 + a(u(t), u(t)) \leq C_{13} E(0). \tag{4.20}$$

Moreover, if we define the functional $I(u(t))$ by

$$I(u(t)) = \mu_0 a(u(t), u(t)) - ||u(t)||_{\gamma+2}^{\gamma+2},$$

then from the relationship $I(u(t)) = (\gamma+2)J(u(t)) - \frac{\mu_0\gamma}{2} a(u(t), u(t))$ and the strict inequality (4.19), we obtain

$$I(u(t)) > 0 \quad \text{for all} \quad t \geq 0. \tag{4.21}$$

Consequently, from (4.20) and (4.21) we have

$$||u_t(t)||_2^2 + a(u(t), u(t)) + ||u(t)||_{\gamma+2}^{\gamma+2} \leq (1+\mu_0) C_{13} E(0).$$

This it the completion of the proof of the global existence of solutions of (1.1).

5 Energy Decay Rates

In this section we prove the uniform decay rates of problem (1.1). In the following section, the symbol C is a generic positive constant, which may be different in various occurrences. We define the energy associated to problem (1.1):

$$E(t) = \frac{1}{2}||u_t||_2^2 + \frac{1}{2}\mu(t) a(u, u) - \frac{1}{\gamma+2}||u||_{\gamma+2}^{\gamma+2}.$$

Then

$$E'(t) = \frac{1}{2}\mu'(t) a(u, u) - \int_\Omega g(u_t) u_t dx \leq 0,$$

it follows that $E(t)$ is a nonincreasing function.

First of all, we recall technical lemmas which will play an essential role when establishing the energy decay rates.

Lemma 5.1 [38] *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a nonincreasing function and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ a strictly increasing function of class C^1 such that*

$$\phi(0) = 0 \quad \text{and} \quad \phi(t) \to +\infty \quad \text{as} \quad t \to +\infty.$$

Assume that there exists $\sigma \geq 0$ and $\omega > 0$ such that

$$\int_S^{+\infty} E^{1+\sigma}(t)\phi'(t)dt \leq \frac{1}{\omega}E^\sigma(0)E(S)$$

for all $S \geq 0$. Then E has the following decay property:

$$\begin{aligned}
&\text{if} \quad \sigma = 0, \quad \text{then} \quad E(t) \leq E(0)e^{1-\omega\phi(t)}, \quad \text{for all} \quad t \geq 0,\\
&\text{if} \quad \sigma > 0, \quad \text{then} \quad E(t) \leq E(0)\Big(\frac{1+\sigma}{1+\omega\sigma\phi(t)}\Big)^{\frac{1}{\sigma}}, \quad \text{for all} \quad t \geq 0.
\end{aligned}$$

Lemma 5.2 [38] *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a nonincreasing function and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ a strictly increasing function of class C^1 such that*

$$\phi(0) = 0 \quad \text{and} \quad \phi(t) \to +\infty \quad \text{as} \quad t \to +\infty.$$

Assume that there exists $\sigma > 0$, $\sigma' \geq 0$ and $C > 0$ such that

$$\int_S^{+\infty} E^{1+\sigma}(t)\phi'(t)dt \leq CE^{1+\sigma}(S) + \frac{C}{(1+\phi(S))^{\sigma'}}E^\sigma(0)E(S), \qquad 0 \leq S < +\infty.$$

Then, there exists $C > 0$ such that

$$E(t) \leq E(0)\frac{C}{(1+\phi(t))^{(1+\sigma')/\sigma}}, \qquad \forall t > 0.$$

Let us now multiply equation (1.1) by $E^p(t)\phi'(t)u$, $p \geq 0$ and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a concave nondecreasing function of class C^2, such that $\phi(t) \to +\infty$ as $t \to +\infty$, and then integrate the obtained result over $\Omega \times [S, T]$. Then we have

$$
\begin{aligned}
0 &= \int_S^T E^p(t)\phi'(t)\int_\Omega u\Big(u_{tt}-\mu(t)Lu+g(u_t)-|u|^\gamma u\Big)dxdt \\
&= \Big[E^p(t)\phi'(t)\int_\Omega u_t u dx\Big]_S^T - \int_S^T \big(pE^{p-1}(t)E'(t)\phi'(t)+E^p(t)\phi''(t)\big)\int_\Omega u_t u dxdt \\
&\quad - \int_S^T E^p(t)\phi'(t)||u_t||_2^2 dt + \int_S^T E^p(t)\phi'(t)\mu(t)a(u,u)dt \\
&\quad + \int_S^T E^p(t)\phi'(t)\int_\Omega g(u_t)udxdt \\
&\quad - \int_S^T E^p(t)\phi'(t)||u||_{\gamma+2}^{\gamma+2}dt.
\end{aligned} \tag{5.1}
$$

By the definition of $E(t)$, we can rewrite (5.1) as

$$
\begin{aligned}
2&\int_S^T E^{p+1}(t)\phi'(t)dt \\
&= -\Big[E^p(t)\phi'(t)\int_\Omega u_t u dx\Big]_S^T + \int_S^T \big(pE^{p-1}(t)E'(t)\phi'(t)+E^p(t)\phi''(t)\big)\int_\Omega u_t u dxdt \\
&\quad - \int_S^T E^p(t)\phi'(t)\int_\Omega g(u_t)udxdt + \frac{\gamma}{\gamma+2}\int_S^T E^p(t)\phi'(t)||u||_{\gamma+2}^{\gamma+2}dt \\
&\quad + 2\int_S^T E^p(t)\phi'(t)||u_t||_2^2 dt.
\end{aligned} \tag{5.2}
$$

Now we are going to estimate terms on the right hand side of (5.2).

Estimate for $-\Big[E^p(t)\phi'(t)\int_\Omega u_t u dx\Big]_S^T$;
Using Young's and Poincaré's inequalities, (2.6) and (4.18), we obtain

$$
\Big|\int_\Omega u_t u dx\Big| \le CE(t), \tag{5.3}
$$

consequently,

$$
-\Big[E^p(t)\phi'(t)\int_\Omega u_t u dx\Big]_S^T \le CE^{p+1}(S). \tag{5.4}
$$

Estimate for $\int_S^T \big(pE^{p-1}(t)E'(t)\phi'(t)+E^p(t)\phi''(t)\big)\int_\Omega u_t u dxdt$;
From (5.3), we have

$$
\begin{aligned}
&\int_S^T \big(pE^{p-1}(t)E'(t)\phi'(t)+E^p(t)\phi''(t)\big)\int_\Omega u_t u dxdt \\
&\le CE^p(S)\int_S^T -E'(t)dt + CE^{p+1}(S)\int_S^T -\phi''(t)dt \\
&\le CE^{p+1}(S).
\end{aligned} \tag{5.5}
$$

Estimate for $-\int_S^T E^p(t)\phi'(t)\int_\Omega g(u_t)udxdt$;
We will consider the following divided domain:

$$\int_\Omega g(u_t)udx = \int_{|u_t|\le 1} g(u_t)udx + \int_{|u_t|>1} g(u_t)udx$$

Using Young's and Poincaré's inequalities, (2.6) and (4.18), we have

$$\begin{aligned}\int_{|u_t|\le 1} g(u_t)udx &\le \frac{1}{2}\int_{|u_t|\le 1}|u|^2dx+\frac{1}{2}\int_{|u_t|\le 1}|g(u_t)|^2dx \le \frac{1}{2}||u||_2^2+\frac{1}{2}\int_{|u_t|\le 1}|g(u_t)|^2dx\\ &\le CE(t)+\frac{1}{2}\int_{|u_t|\le 1}|g(u_t)|^2dx.\end{aligned}$$

On the other hand, by the Young inequality with $\frac{\rho+1}{\rho+2}+\frac{1}{\rho+2}=1$ and using the assumption $\rho\ge\gamma$ we have

$$\begin{aligned}\int_{|u_t|>1} g(u_t)udx &\le \int_{|u_t|>1}|u|^{\rho+2}dx+\frac{\rho+1}{\rho+2}\int_{|u_t|>1}|g(u_t)|^{\frac{\rho+2}{\rho+1}}dx\\ &\le \frac{1}{\rho+2}||u||_{\rho+2}^{\rho+2}+\frac{\rho+1}{\rho+2}\int_{|u_t|>1}|g(u_t)|^{\frac{\rho+2}{\rho+1}}dx\\ &\le C(1+||u||_{\gamma+2})^{\rho+2}+\frac{\rho+1}{\rho+2}\int_{|u_t|>1}|g(u_t)|^{\frac{\rho+2}{\rho+1}}dx\\ &\le 2^{\gamma+1}C(1+||u||_{\gamma+2}^{\gamma+2})+\frac{\rho+1}{\rho+2}\int_{|u_t|>1}|g(u_t)|^{\frac{\rho+2}{\rho+1}}dx.\end{aligned}$$

Hence, from (4.18) and (4.21) we get

$$\begin{aligned}\Big|\int_S^T E^p(t)\phi'(t)\int_\Omega g(u_t)udxdt\Big| &\le CE^{p+1}(S)+\frac{1}{2}\int_S^T E^p(t)\phi'(t)\int_{|u_t|\le 1}|g(u_t)|^2dxdt\\ &\quad+\frac{\rho+1}{\rho+2}\int_S^T E^p(t)\phi'(t)\int_{|u_t|>1}|g(u_t)|^{\frac{\rho+2}{\rho+1}}dxdt.\end{aligned}\tag{5.6}$$

Estimate for $\frac{\gamma}{\gamma+2}\int_S^T E^p(t)\phi'(t)||u||_{\gamma+2}^{\gamma+2}dt$;
Using (4.18) and (4.21) we have

$$\frac{\gamma}{\gamma+2}\int_S^T E^p(t)\phi'(t)||u||_{\gamma+2}^{\gamma+2}dt\le CE^{p+1}(S).\tag{5.7}$$

Estimate for $2\int_S^T E^p(t)\phi'(t)||u_t||_2^2dt$;

From the definition of $J(u)$ and $E(t)$ and using (4.18), we have

$$2\int_S^T E^p(t)\phi'(t)||u_t||_2^2 dt \le CE^{p+1}(S). \tag{5.8}$$

By replacing (5.4)-(5.8) in (5.2), we obtain

$$\begin{aligned}\int_S^T E^{p+1}(t)\phi'(t)dt \le CE^{p+1}(S) + C\int_S^T E^p(t)\phi'(t)\int_{|u_t|\le 1}|g(u_t)|^2 dxdt \\ + C\int_S^T E^p(t)\phi'(t)\int_{|u_t|>1}|g(u_t)|^{\frac{\rho+2}{\rho+1}} dxdt.\end{aligned} \tag{5.9}$$

Now we are going to estimate the last two terms with respect to g on the right hand side of (5.9).

5.1 Case 1 : β is Linear

Since β is linear, we can rewrite the hypothesis of g as follows:

$$\begin{aligned}&c_5|s| \le |g(s)| \le c_6|s| \quad \text{if} \quad |s| \le 1,\\ &c_1|s| \le c_1|s|^{\rho+1} \le |g(s)| \le c_2|s|^{\rho+1} \quad \text{if} \quad |s| > 1,\end{aligned}$$

for some positive constants c_3, c_4. Hence we get

$$\int_{|u_t|\le 1}|g(u_t)|^2 dx \le c_6\int_{|u_t|\le 1} g(u_t)u_t dx \le -c_6 E'(t) \tag{5.10}$$

and

$$\int_{|u_t|>1}|g(u_t)|^{\frac{\rho+2}{\rho+1}} dx = \int_{|u_t|>1}|g(u_t)|^{\frac{1}{\rho+1}}|g(u_t)|dx \le c_2^{\frac{1}{\rho+1}}\int_{|u_t|>1} u_t g(u_t)dx \le -c_2^{\frac{1}{\rho+1}}E'(t). \tag{5.11}$$

Combining (5.9), (5.10) and (5.11), it follows that

$$\int_S^T E^{p+1}(t)\phi'(t)dt \le CE^p(0)E(S),$$

which implies by Lemma 5.1 with $p = 0$

$$E(t) \le E(0)e^{1-\frac{\phi(t)}{C}}.$$

Let us set $\phi(t) := mt$, where m is for some positive constant, then $\phi(t)$ satisfies all the required properties and we obtain that the energy decays exponentially to zero.

5.2 Case 2 : β has Polynomial Growth Near Zero

Assume that $\beta(s) = s^{\rho+1}$. Let $p = \frac{\rho}{2}$, then we rewrite (5.9) as

$$\begin{aligned}\int_S^T E^{\frac{\rho}{2}+1}(t)\phi'(t)dt \le CE(S) + C\int_S^T E^{\frac{\rho}{2}}(t)\phi'(t)\int_{|u_t|\le 1}|g(u_t)|^2 dxdt \\ + C\int_S^T E^{\frac{\rho}{2}}(t)\phi'(t)\int_{|u_t|>1}|g(u_t)|^{\frac{\rho+2}{\rho+1}} dxdt.\end{aligned} \tag{5.12}$$

By the hypothesis of g and the Hölder inequality with $\frac{2}{\rho+2} + \frac{\rho}{\rho+2} = 1$, we have

$$\int_{|u_t|\le 1}|g(u_t)|^2 dx \le \int_{|u_t|\le 1}(u_t g(u_t))^{\frac{2}{\rho+2}} dx \le C\left(\int_\Omega u_t q(u_t)dx\right)^{\frac{2}{\rho+2}} \le C\big(-E'(t)\big)^{\frac{2}{\rho+2}}$$

Hence

$$\int_S^T E^{\frac{\rho}{2}}(t)\phi'(t)\int_{|u_t|\le 1}|g(u_t)|^2 dxdt \le \epsilon_2\int_S^T E^{\frac{\rho}{2}+1}(t)\phi'(t)dt + C(\epsilon_2)E(S). \tag{5.13}$$

Similarly as (5.11) we have

$$\int_S^T E^{\frac{\rho}{2}}(t)\phi'(t)\int_{|u_t|>1}|g(u_t)|^{\frac{\rho+2}{\rho+1}} dxdt \le CE(S). \tag{5.14}$$

By replacing (5.13) and (5.14) in (5.12) and choosing ϵ_2 sufficiently small, we get

$$\int_S^T E^{\frac{\rho}{2}+1}(t)\phi'(t)dt \le CE(S),$$

which implies by Lemma 5.1 and choosing $\phi(t) = mt$,

$$E(t) \le \frac{CE(0)}{(1+t)^{\frac{2}{\rho}}}.$$

5.3 Case 3: β Does Not Necessarily Have Polynomial Growth Near Zero

We will use the method of partitions of domain modified the arguments in [38]. Let

$$\Omega_1 = \{x \in \Omega;\ |u_t(x)| \le 1\} \quad \text{and} \quad \Omega_2 = \{x \in \Omega;\ |u_t(x)| > 1\}.$$

For every $t \geq 1$, we consider the following partitions of domain depending $\phi'(t)$:

$$\Omega^1 = \{x \in \Omega; |u_t| \leq \phi'(t)\}, \quad \Omega^2 = \{x \in \Omega; \phi'(t) < |u_t| \leq 1\}, \quad \Omega^3 = \{x \in \Omega; |u_t| > 1\}$$

if $\phi'(t) \leq 1$, or

$$\Omega^4 = \{x \in \Omega; |u_t| \leq 1 < \phi'(t)\}, \Omega^5 = \{x \in \Omega; 1 < |u_t| \leq \phi'(t)\}, \Omega^6 = \{x \in \Omega; |u_t| > \phi'(t) > 1\}$$

if $\phi'(t) > 1$. Then $\Omega_1 = \Omega^1 \cup \Omega^2$ (or Ω^4) and $\Omega_2 = \Omega^3$ (or $\Omega^5 \cup \Omega^6$).

(I) Part on Ω^i, $i = 3, 5, 6$.

By the same argument as (5.11), we get

$$\int_S^T E^p(t)\phi'(t) \int_{\Omega^i} |g(u_t)|^{\frac{\rho+2}{\rho+1}} dxdt \leq CE^{p+1}(S)$$

consequently,

$$\int_S^T E^p(t)\phi'(t) \int_{\Omega_2} |g(u_t)|^{\frac{\rho+2}{\rho+1}} dxdt \leq CE^{p+1}(S) \tag{5.15}$$

(II) Part on Ω^2.

Using the fact β is increasing and (2.3), we obtain

$$\int_S^T E^p(t)\phi'(t) \int_{\Omega^2} |g(u_t)|^2 dxdt \leq \int_S^T E^p(t) \int_{\Omega^2} |u'(t)|\, |g(u_t)|^2 dxdt \leq \beta^{-1}(1)E^{p+1}(S). \tag{5.16}$$

(III) Part on Ω^i, $i = 1, 4$.

Using the fact $E(t)$ is nonincreasing, β^{-1} is increasing and (2.3), we have

$$\begin{aligned} \int_S^T E^p(t)\phi'(t) \int_{\Omega^i} |g(u_t)|^2 dxdt &\leq \int_S^T E^p(t)\phi'(t) \int_{\Omega^i} (\beta^{-1}(|u'(t)|))^2 dxdt \\ &\leq meas(\Omega)E^p(S) \int_S^T \phi'(t)(\beta^{-1}(\phi'(t)))^2 dt \end{aligned} \tag{5.17}$$

By (5.16) and (5.17), it follows that

$$\int_S^T E^p(t)\phi'(t) \int_{\Omega_1} |g(u_t)|^2 dxdt \leq CE^{p+1}(S) + CE^p(S) \int_S^T \phi'(t)(\beta^{-1}(\phi'(t)))^2 dt. \tag{5.18}$$

Therefore by replacing (5.15) and (5.18) in (5.9), we deduce that

$$\int_S^T E^{p+1}(t)\phi'(t)dt \le CE^{p+1}(S) + CE^p(S)\int_S^T \phi'(t)(\beta^{-1}(\phi'(t)))^2 dt. \quad (5.19)$$

To estimate the last term of the right hand side of (5.19), we need the following additional assumption over ϕ (see [38], p.434):

$$\int_1^\infty \phi'(t)(\beta^{-1}(\phi'(t)))^2 dt \quad \text{converges.}$$

Then we have from (5.19)

$$\int_S^T E^{p+1}(t)\phi'(t)dt \le CE^{p+1}(S) + CE^p(S)\int_{\phi(S)}^{+\infty}\Big(\beta^{-1}(\frac{1}{(\phi^{-1})'(s)})\Big)^2 ds. \quad (5.20)$$

Define $\psi(t) = 1 + \int_1^t \frac{1}{\beta(\frac{1}{s})}ds$, $t \ge 1$. Then ψ is strictly increasing and convex (cf. [38], [42]). We now take $\phi(t) = \psi^{-1}(t)$, then we can rewrite (5.20) as

$$\int_S^T E^{p+1}(t)\phi'(t)dt \le CE^{p+1}(S) + \frac{C}{\phi(S)}E^p(S),$$

which implies, by applying Lemma 5.2 with $p = 1$,

$$E(t) \le \frac{C}{\phi^2(t)} \qquad \forall t > 0.$$

Let s_0 be a number such that $\beta(\frac{1}{s_0}) \le 1$. Since β is nondecreasing, we have

$$\psi(s) \le 1 + (s-1)\frac{1}{\beta(\frac{1}{s})} \le \frac{1}{F(\frac{1}{s})} \qquad \forall s \ge s_0,$$

where $F(s) = s\beta(s)$, consequently, having in mind that $\phi = \psi^{-1}$, the last inequality yields

$$s \le \phi\Big(\frac{1}{F(\frac{1}{s})}\Big) = \phi(t) \qquad \text{with} \qquad t = \frac{1}{F(\frac{1}{s})}.$$

Then we conclude that

$$\frac{1}{\phi(t)} \le F^{-1}(\frac{1}{t}).$$

Therefore the proof of Theorem 2.3 is completed.

6 Blow-Up

This section is devoted to prove the blow-up result. First of all, we introduce a following lemma that is essential role for proving the blow-up.

Lemma 6.1 *Under the hypotheses given in Theorem 2.4 the weak solution to problem (1.1) verifies*

$$a(u(t), u(t)) > \lambda_0^2 \quad \textit{for all} \quad 0 < t < T_{\max}.$$

Proof We recall the function, for $\lambda > 0$,

$$j(\lambda) = \frac{\mu_0}{2}\lambda^2 - \frac{1}{\gamma+2}K_0^{\gamma+2}\lambda^{\gamma+2},$$

where $K_0 = \sup_{u\in\mathcal{H},u\neq 0}\left(\frac{||u||_{\gamma+2}}{[a(u,u)]^{1/2}}\right)$. Then

$$\lambda_0 = \left(\frac{\mu_0}{K_0^{\gamma+2}}\right)^{1/\gamma}$$

is the absolute maximum point of j and

$$j(\lambda_0) = \frac{\gamma\mu_0}{2(\gamma+2)}\lambda_0^2 = d_0.$$

The energy associated to problem (1.1) is given by

$$E(t) = \frac{1}{2}||u_t(t)||_2^2 + \frac{1}{2}\mu(t)a(u(t), u(t)) - \frac{1}{\gamma+2}||u(t)||_{\gamma+2}^{\gamma+2}.$$

We observe that from the definition of j, we have

$$E(t) \geq \frac{\mu_0}{2}a(u(t), u(t)) - \frac{1}{\gamma+2}||u(t)||_{\gamma+2}^{\gamma+2} \geq j([a(u(t), u(t))]^{1/2}) \quad \text{for all} \quad t \geq 0. \tag{6.1}$$

Note that j is increasing for $0 < \lambda < \lambda_0$, decreasing for $\lambda > \lambda_0$, $j(\lambda) \to -\infty$ as $\lambda \to +\infty$.

We will now consider the initial energy $E(0)$ divided into two cases: $E(0) \geq 0$ and $E(0) < 0$.

Case 1 : $E(0) \geq 0$.

There exist $\lambda_1' < \lambda_0 < \lambda_1$ such that

$$j(\lambda_1) = j(\lambda_1') = E(0). \tag{6.2}$$

By considering that $E(t)$ is nonincreasing, we have

$$E(t) \leq E(0) \quad \text{for all} \quad t > 0. \tag{6.3}$$

From (6.1) and (6.2) we deduce

$$j([a(u_0, u_0)]^{1/2}) \leq E(0) = j(\lambda_1). \tag{6.4}$$

Since $[a(u_0, u_0)]^{1/2} > \lambda_0$, $\lambda_0 < \lambda_1$ and $j(\lambda)$ is decreasing for $\lambda_0 < \lambda$, from (6.4) we get

$$[a(u_0, u_0)]^{1/2} \geq \lambda_1. \tag{6.5}$$

Now we will prove that

$$[a(u(t), u(t))]^{1/2} \geq \lambda_1 \quad \text{for all} \quad 0 < t < T_{\max} \tag{6.6}$$

by using the contradiction method. Suppose that (6.6) does not hold. Then there exists $t^* \in (0, T_{\max})$ which verifies

$$[a(u(t^*), u(t^*))]^{1/2} < \lambda_1. \tag{6.7}$$

If $[a(u(t^*), u(t^*))]^{1/2} > \lambda_0$, from (6.1), (6.2) and (6.7) we can write

$$E(t^*) \geq j([a(u(t^*), u(t^*))]^{1/2}) > j(\lambda_1) = E(0),$$

which contradicts (6.3).

If $[a(u(t^*), u(t^*))]^{1/2} \leq \lambda_0$, we have, in view of (6.5), that there exists $\bar{\lambda}$ which verifies

$$[a(u(t^*), u(t^*))]^{1/2} \leq \lambda_0 < \bar{\lambda} < \lambda_1 \leq [a(u_0, u_0)]^{1/2}. \tag{6.8}$$

Consequently, from the continuity of the function $[a(u(\cdot), u(\cdot))]^{1/2}$ there exists $\bar{t} \in (0, t^*)$ verifying $[a(u(\bar{t}), u(\bar{t}))]^{1/2} = \bar{\lambda}$. Then from the last identity and taking (6.1), (6.2) and (6.8) into account we deduce

$$E(\bar{t}) \geq j([a(u(\bar{t}), u(\bar{t}))]^{1/2}) = j(\bar{\lambda}) > j(\lambda_1) = E(0),$$

which also contradicts (6.3).

Case 2 : $E(0) < 0$.

There is $\lambda_2 > \lambda_0$ such that

$$j(\lambda_2) = E(0),$$

consequently, by (6.1) we have

$$j([a(u_0, u_0)]^{1/2}) \leq E(0) = j(\lambda_2).$$

From the fact $j(\lambda)$ is decreasing for $\lambda_0 < \lambda$, we get

$$[a(u_0, u_0)]^{1/2} \geq \lambda_2.$$

By the same argument as Case 1, we obtain

$$[a(u(t), u(t))]^{1/2} \geq \lambda_2 \quad \text{for all} \quad 0 < t < T_{\max}.$$

Thus the proof of Lemma 6.1 is completed. □

Now we will prove the blow-up result. In order to prove that $T_{\max}$ is necessarily finite, we argue by contradiction. Assume that the weak solution $u(t)$ can be extended to the whole interval $[0, \infty)$.

Let E_1 be a real number such that

$$E_1 = \begin{cases} 0 & \text{if} \quad E(0) < 0, \\ \text{positive constant satisfying} \quad E(0) < E_1 < d_0 \quad \text{and} \quad E_1 < E(0) + 1 & \text{if} \quad E(0) \geq 0. \end{cases}$$

By setting $H(t) := E_1 - E(t)$, we have

$$H'(t) = -E'(t) \geq 0, \tag{6.9}$$

which implies that $H(t)$ is nondecreasing, consequently,

$$0 < H_0 := E_1 - E(0) < 1 \tag{6.10}$$

and from Lemma 6.1, (2.2) and the definition of d_0,

$$\begin{aligned} H_0 \leq H(t) &\leq E_1 - \frac{\mu_0}{2} a(u(t), u(t)) + \frac{1}{\gamma+2} ||u(t)||_{\gamma+2}^{\gamma+2} \\ &< d_0 - \frac{\mu_0}{2} \lambda_0^2 + \frac{1}{\gamma+2} ||u(t)||_{\gamma+2}^{\gamma+2} \\ &\leq \frac{1}{\gamma+2} ||u(t)||_{\gamma+2}^{\gamma+2}. \end{aligned} \tag{6.11}$$

We define

$$M(t) = H^{1-\overline{\chi}}(t) + \tau N(t), \quad N(t) = \int_\Omega u_t u dx, \tag{6.12}$$

where $\overline{\chi}$ and τ are small positive constants to be chosen later. Then we have

$$M'(t) = (1 - \overline{\chi}) H^{-\overline{\chi}}(t) H'(t) + \tau N'(t). \tag{6.13}$$

We are now going to analyze the last term on the right-hand side of (6.13).

Lemma 6.2

$$N'(t) \geq C_{14}\big(||u_t||_2^2 + ||u||_{\gamma+2}^{\gamma+2} + H(t) - H'(t)H_0^{\overline{\chi}-\chi}H^{-\overline{\chi}}(t)\big) + \mu_0\Big(\frac{\theta}{2}-1\Big)a(u,u) - \theta E_1 - \frac{\zeta}{\epsilon_3}, \tag{6.14}$$

where C_{14} *is for some positive constant,* $0 < \chi < \frac{\gamma-\rho}{(\rho+2)(\gamma+2)}$, $\theta = \gamma + 2 - \epsilon_3$ *with* $0 < \epsilon_3 < \min\{1, \gamma\}$ *and* $\zeta = \frac{(\gamma+1)meas(\Omega)\big(\beta^{-1}(1)\big)^{\frac{\gamma+2}{\gamma+1}}}{\gamma+2}$.

Proof Using Eq. (1.1), we obtain

$$\begin{aligned} N'(t) &= ||u_t||_2^2 - \mu(t)a(u,u) + ||u||_{\gamma+2}^{\gamma+2} - \int_\Omega g(u_t)u dx \\ &\geq \Big(1+\frac{\theta}{2}\Big)||u_t||_2^2 + \mu_0\Big(\frac{\theta}{2}-1\Big)a(u,u) + \Big(1-\frac{\theta}{\gamma+2}\Big)||u||_{\gamma+2}^{\gamma+2} + \theta H(t) - \theta E_1 \\ &\quad - \int_\Omega g(u_t)u dx, \end{aligned} \tag{6.15}$$

where $\theta = \gamma + 2 - \epsilon_3$ with $0 < \epsilon_3 < \min\{1, \gamma\}$.

We will estimate the last term on the right-hand side of (6.15). We note that

$$\Big|\int_\Omega g(u_t)u dx\Big| \leq \int_\Omega |g(u_t)|\,|u| dx = \int_{|u_t|\leq 1} |g(u_t)|\,|u| dx + \int_{|u_t|>1} |g(u_t)|\,|u| dx.$$

By using (2.3) and the imbedding $L^{\gamma+2}(\Omega) \hookrightarrow L^{\rho+2}(\Omega)$, we have

$$\begin{aligned} \int_{|u_t|\leq 1} |g(u_t)|\,|u| dx &\leq \Big(\int_{|u_t|\leq 1} |g(u_t)|^{\frac{\rho+2}{\rho+1}} dx\Big)^{\frac{\rho+1}{\rho+2}} \Big(\int_{|u_t|\leq 1} |u|^{\rho+2} dx\Big)^{\frac{1}{\rho+2}} \\ &\leq \Big(\int_{|u_t|\leq 1} |\beta^{-1}(1)|^{\frac{\rho+2}{\rho+1}} dx\Big)^{\frac{\rho+1}{\rho+2}} ||u||_{\rho+2} \\ &\leq \beta^{-1}(1)\big(meas(\Omega)\big)^{\frac{\gamma+1}{\gamma+2}} ||u|||_{\gamma+2} \\ &\leq \frac{(\gamma+1)meas(\Omega)\big(\beta^{-1}(1)\big)^{\frac{\gamma+2}{\gamma+1}}}{\epsilon_3(\gamma+2)} + \frac{\epsilon_3^{\gamma+1}}{\gamma+2}||u||_{\gamma+2}^{\gamma+2}. \end{aligned} \tag{6.16}$$

On the other hand, by using (2.4), we obtain

$$\begin{aligned}\int_{|u_t|>1} |g(u_t)|\, |u| dx &\le c_2 \int_{|u_t|>1} |u_t|^{\rho+1} |u| dx \\ &\le c_2 \Big(\int_{|u_t|>1} |u_t|^{\rho+2} dx\Big)^{\frac{\rho+1}{\rho+2}} ||u||_{\rho+2} \\ &\le \Big(C(\epsilon_4) \int_{|u_t|>1} |u_t|^{\rho+2} dx + \epsilon_4 ||u||_{\gamma+2}^{(\gamma+2)(\chi+\frac{1}{\gamma+2})(\rho+2)}\Big) ||u||_{\gamma+2}^{-(\gamma+2)\chi},\end{aligned} \tag{6.17}$$

where $0 < \chi < \frac{\gamma-\rho}{(\rho+2)(\gamma+2)}$ and $C(\epsilon_4)$, ϵ_4 are for some positive constants. Moreover $\chi < \frac{\gamma-\rho}{(\rho+2)(\gamma+2)}$ implies that $(\chi + \frac{1}{\gamma+2})(\rho + 2) < 1$. Hence we get

$$||u||_{\gamma+2}^{(\gamma+2)(\chi+\frac{1}{\gamma+2})(\rho+2)} \le \begin{cases} ||u||_{\gamma+2}^{\gamma+2} & \text{if } ||u||_{\gamma+2}^{\gamma+2} > 1, \\ H_0^{-1} H_0 & \text{if } ||u||_{\gamma+2}^{\gamma+2} \le 1. \end{cases}$$

From (6.10) and (6.11) we have

$$||u||_{\gamma+2}^{(\gamma+2)(\chi+\frac{1}{\gamma+2})(\rho+2)} \le H_0^{-1} ||u||_{\gamma+2}^{\gamma+2}$$

and, consequently, from (2.4), (6.9), (6.11) and (6.17),

$$\begin{aligned}\int_{|u_t|>1} |g(u_t)|\, |u| dx &\le \Big(C(\epsilon_4) \int_{|u_t|>1} |u_t|^{\rho+2} dx + \epsilon_4 H_0^{-1} ||u||_{\gamma+2}^{\gamma+2}\Big) ||u||_{\gamma+2}^{-(\gamma+2)\chi} \\ &\le \Big(C(\epsilon_4) H'(t) + \epsilon_4 H_0^{-1} ||u||_{\gamma+2}^{\gamma+2}\Big) H^{-\chi}(t) \\ &\le C(\epsilon_4) H'(t) H_0^{\overline{\chi}-\chi} H^{-\overline{\chi}}(t) + \epsilon_4 H_0^{-(\chi+1)} ||u||_{\gamma+2}^{\gamma+2},\end{aligned} \tag{6.18}$$

for $0 < \overline{\chi} < \chi$. From (6.16) and (6.18), we get that

$$\Big|\int_\Omega g(u_t) u dx\Big| \le C(\epsilon_4) H'(t) H_0^{\overline{\chi}-\chi} H^{-\overline{\chi}}(t) + \Big(\frac{\epsilon_3^{\gamma+1}}{\gamma+2} + \epsilon_4 H_0^{-(\chi+1)}\Big) ||u||_{\gamma+2}^{\gamma+2} + \frac{\zeta}{\epsilon_3}, \tag{6.19}$$

where $\zeta = \frac{(\gamma+1) meas(\Omega) \big(\beta^{-1}(1)\big)^{\frac{\gamma+2}{\gamma+1}}}{\gamma+2}$.

By replacing (6.19) in (6.15) and choosing ϵ_4 small enough we obtain

$$\begin{aligned} N'(t) \ge C_{15}\big(||u_t||_2^2 + ||u||_{\gamma+2}^{\gamma+2} + H(t) - H'(t) H_0^{\overline{\chi}-\chi} H^{-\overline{\chi}}(t)\big) \\ + \mu_0 \Big(\frac{\theta}{2} - 1\Big) a(u, u) - \theta E_1 - \frac{\zeta}{\epsilon_2},\end{aligned}$$

where C_{15} is a positive constant. Therefore (6.14) follows. □

The following Lemma estimates the last three terms on the right-hand side of (6.14).

Lemma 6.3

$$\mu_0\Big(\frac{\theta}{2}-1\Big)a(u,u)-\theta E_1-\frac{\zeta}{\epsilon_3}>0 \quad for \quad \frac{\ell-\sqrt{\ell^2-4\eta\zeta}}{2\eta}\le\epsilon_3\le\frac{\ell+\sqrt{\ell^2-4\eta\zeta}}{2\eta}, \tag{6.20}$$

where $\eta=\frac{\mu_0\lambda_0^2}{2}-E_1$ *and* $\ell=\frac{\gamma\mu_0\lambda_0^2}{2}-(\gamma+2)E_1$.

Proof From Lemma 5.1 and the definition of θ, we have

$$\begin{aligned}\mu_0\Big(\frac{\theta}{2}-1\Big)a(u,u)-\theta E_1-\frac{\zeta}{\epsilon_3}&>\mu_0\Big(\frac{\theta}{2}-1\Big)\lambda_0^2-\theta E_1-\frac{\zeta}{\epsilon_3}\\&=\Big(E_1-\frac{\mu_0\lambda_0^2}{2}\Big)\epsilon_3-\frac{\zeta}{\epsilon_3}+\frac{\gamma\mu_0\lambda_0^2}{2}-(\gamma+2)E_1=-\frac{\eta\epsilon_3^2-\ell\epsilon_3+\zeta}{\epsilon_3}:=P(\epsilon_3).\end{aligned} \tag{6.21}$$

We note that

$$\eta=\frac{\mu_0\lambda_0^2}{2}-E_1>\frac{\mu_0\lambda_0^2}{2}-d_0=\frac{1}{\gamma+2}K_0^{\gamma+2}\lambda_0^{\gamma+2}>0$$

and

$$\ell=\frac{\gamma\mu_0\lambda_0^2}{2}-(\gamma+2)E_1>\frac{\gamma\mu_0\lambda_0^2}{2}-(\gamma+2)d_0=0.$$

Since (2.9) holds, we get

$$\ell^2-4\eta\zeta\ge 0.$$

Therefore, $P(\epsilon_3)$ represents a curve connecting horizontal axis points $\frac{\ell-\sqrt{\ell^2-4\eta\zeta}}{2\eta}$ and $\frac{\ell+\sqrt{\ell^2-4\eta\zeta}}{2\eta}$, and

$$P(\epsilon_3)\ge 0 \quad \text{for} \quad \frac{\ell-\sqrt{\ell^2-4\eta\zeta}}{2\eta}\le\epsilon_3\le\frac{\ell+\sqrt{\ell^2-4\eta\zeta}}{2\eta}.$$

Thus we obtain

$$\mu_0\Big(\frac{\theta}{2}-1\Big)a(u,u)-\theta E_1-\frac{\zeta}{\epsilon_3}>0 \quad \text{for} \quad \frac{\ell-\sqrt{\ell^2-4\eta\zeta}}{2\eta}\le\epsilon_3\le\frac{\ell+\sqrt{\ell^2-4\eta\zeta}}{2\eta}.$$

□

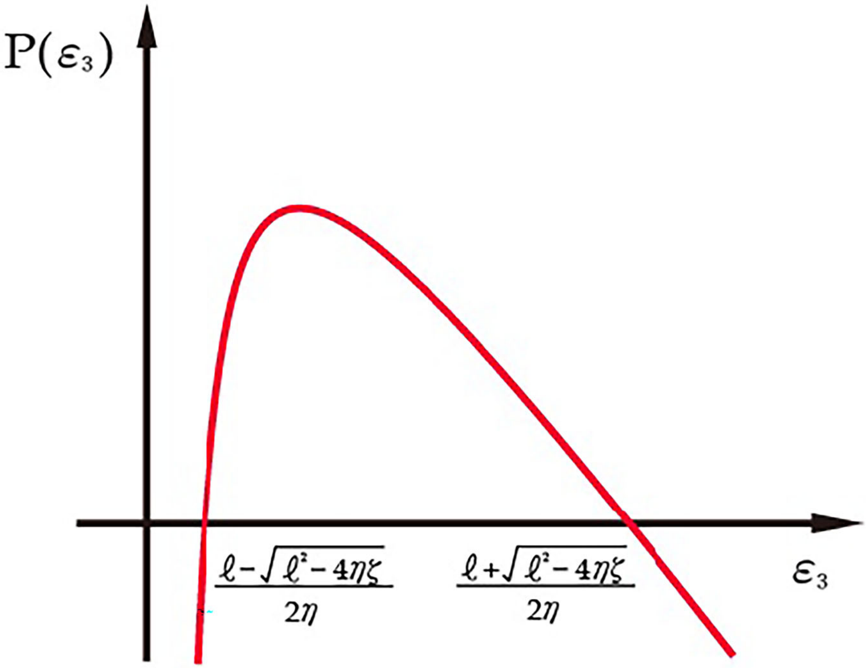

Fig. 3 The figure of $P(\epsilon_2)$

Combining (6.13), (6.14), (6.20) and then choosing $0 < \overline{\chi} < \min\{\frac{1}{2}, \chi\}$ and τ small enough, we obtain

$$M'(t) \geq C_{16}\big(||u_t||_2^2 + ||u||_{\gamma+2}^{\gamma+2} + H(t)\big),$$

where C_{16} is a positive constant, which implies that $M(t)$ is a positive increasing function. By same arguments as p.333 in [16], we have

$$M'(t) \geq C_{17}M^{\frac{1}{1-\overline{\chi}}}(t) \quad \text{for all} \quad t \geq 0,$$

where C_{17} is a positive constant and $1 < \frac{1}{1-\overline{\chi}} < 2$. Hence we conclude that $M(t)$ blows up in finite time and u also blows up in finite time. Thus this is a contradiction, consequently, the proof of Theorem 2.4 is completed.

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