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# Existence and Non-existence of Global Solutions for a Nonlocal Choquard–Kirchhoff Diffusion Equations in $\mathbb{R}^N$

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**Abstract**
In this paper, we investigate the local existence, global existence, and blow-up of solutions to the Cauchy problem for Choquard–Kirchhoff-type equations involving the fractional $p$-Laplacian. As a particular case, we study the following initial value problem

$$\begin{cases} u_t + M(\|u\|^p)\,[(-\Delta)^s_p u + V(x)|u|^{p-2}u] = \left(\int_{\mathbb{R}^N} \frac{|u|^q}{|x-y|^\mu}\,dy\right)|u|^{q-2}u & \text{in } \mathbb{R}^N \times (0,+\infty), \\ u(x,0) = u_0(x), & \text{in } \mathbb{R}^N, \end{cases}$$

where

$$\|u\| = \left([u]^p_{s,p} + \int_{\mathbb{R}^N} V(x)|u|^p\,dx\right)^{1/p},$$

$s \in (0,1)$, $N > ps$, $p, q > 2$, $(-\Delta)^s_p$ is the fractional $p$-Laplacian, $u_0 : \mathbb{R}^N \to [0,+\infty)$ is the initial function, $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function given by $M(\sigma) = \sigma^{\theta-1}$, $\theta \in [1, N/(N-sp))$ and $V : \mathbb{R}^N \to \mathbb{R}^+$ is the potential function. Under some appropriate conditions, the well-posedness of nonnegative solutions for the above Cauchy problem is established by employing the Galerkin method. Moreover, the asymptotic behavior of global solutions is investigated under some assumptions on the initial data. We also establish upper and lower bounds for the blow-up time.

**Keywords** Choquard–Kirchhoff diffusion equations · Fractional $p$-Laplacian · Global existence · Blow-up · Galerkin method

**Mathematics Subject Classification** 35R11 · 35B40 · 35K57 · 35B41

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## 1 Introduction and the Main Results

The aim of this paper is to discuss the global well-posedness, asymptotic behavior and blow-up phenomena to the following fractional Choquard–Kirchhoff-type parabolic equations

$$\begin{cases} u_t + M(\|u\|^p)\,[(-\Delta)^s_p u + V(x)|u|^{p-2}u] = (\mathcal{K}_\mu * |u|^q)|u|^{q-2}u & \text{in } \mathbb{R}^N \times (0,+\infty), \\ u(x,0) = u_0(x), & \text{in } \quad \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $p, q > 2$, $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function given by $M(\sigma) = \sigma^{\theta-1}$, $\theta \in [1, N/(N-sp))$, $s \in (0,1)$, $N > sp$, $\mu \in (0, N)$, and $V : \mathbb{R}^N \to \mathbb{R}^+$ is a scalar potential. Hereafter $\mathcal{K}_\mu(x) = |x|^{-\mu}$,

$$\|u\| = \left([u]^p_{s,p} + \int_{\mathbb{R}^N} V(x)|u|^p\,dx\right)^{1/p}, \quad [u]_{s,p} = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dxdy\right)^{1/p}.$$

$(-\Delta_p)^s$ is the fractional $p$-Laplacian which, up to a normalization constant, is defined for each $x \in \mathbb{R}^N$ as

$$(-\Delta)^s_p \varphi(x) = 2 \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{|\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dy \tag{1.2}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. Here $B_\epsilon(x)$ denotes the ball in $\mathbb{R}^N$ centered at $x$ with radius $\epsilon$.

In the last few decades, great attention has been paid to study problems involving fractional Laplacian. The interest in studying such problems was stimulated by their applications in continuum mechanics, minimal surfaces, conversation laws, population dynamics, image processing, finance, and many others, see for example [1–3] and the references therein. In particle the study of parabolic equations involving fractional $p$-Laplacian like (1.1) are important in many field of sciences, such as biology, geophysics, as well as in Riemannian geometry in the so-called scalar curvature problem on the sphere $\mathbb{S}^N$, for more details see [4–6]. From the mathematical point of view, the difficulty in solving problem like (1.1) is due to the lack of compactness which caused by the invariant action of the conformal group, or one of its subgroups, we refer to [7] for details. When $\mu = p = q = 2$ and $V \equiv M \equiv 1$, the stationary equation related to (1.1) becomes the well-known Choquard or nonlinear Schrödinger-Newton equation

$$-\Delta u + u = \left(\mathcal{K}_2 * |u|^2\right) u \quad \text{in } \mathbb{R}^N. \tag{E$_1$}$$

In the case when $N = 3$, equation $(E_1)$ was first introduced in 1954 by Pekar [8]. In 1996, Penrose [9] used equation $(E_1)$ in a different context as a model in self-gravitating matter. The literature on equations of the type $(E_1)$ is very large and rich, so here we just list some papers where the authors studied the existence and multiplicity of solutions for $(E_1)$, see for example [10–13] and the references therein. If $s \to 1^-$,

$p = 2$ and $V \equiv 0$, the homogeneous equation related to (1.1) reduces to the following equation

$$u_t - M(\|u\|^2)\Delta u = 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty), \; u_t(0) = u_0. \tag{E$_1'$}$$

As far as we know, the first and the only result on the global existence and nonexistence of solutions of $(E_1')$ was obtained by Gobbino [14] in a more abstract setting. In this work the author classified the existence and nonexistence of solutions of $(E_1')$ into the following cases :

- If $u_0 \in D(A^{1/2})$ and $M\left(\|\nabla u_0\|_2^2\right) \neq 0$, then there exists at least one global solution.
- If $u_0 \in D(A^{\beta})$ with $\beta$ small enough, then there is no solution.

Equation $(E_1')$ is related to the parabolic analogue of the Kirchhoff equation

$$u_{tt} - M(\|u\|^2)\Delta u = 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty). \tag{E$_2'$}$$

This equation was first introduced by Gustav Robert Kirchhoff in 1876, which describes the movement of an elastic string that is constrained at the extrema, taking into account a possible growth of the tension of the vibrating string in view of its extension.

Recently, many authors investigated the existence of local and global solutions for this abstract initial value problem

$$u''(t) + M\left(\|A^{\frac{1}{2}}u(t)\|^2\right)Au(t) = 0, \quad u(0) = u_0, \; u'(0) = u_1, \tag{P}$$

where $A$ is a nonnegative self-adjoint linear operator on a Hilbert space $H$ with dense domain $D(A)$. Powers of the operator $A$ are just defined by the spectral operator, that is

$$A^s u := \sum_{k=0}^{\infty} \lambda_k^{2s} u_k e_k, \quad \forall s \geq 0, \quad \forall u \in D(A^s). \tag{1.3}$$

The local and global existence of solutions of $(P)$ has been proved under different assumptions on the initial data $(u_0, u_1)$ and on the nonlinear function $M$, see for example the works by Ghisi and Gobbino [15–17] and the references cited therein. It is important to point out here when $p = 2$ and $s \in (0, 1)$ the fractional Laplace operator $(-\Delta)^s$ defined in (1.2) and the spectral operator $A^s$ given in (1.3) are completely different, we refer the reader to the monograph [18] for a comparison between these operators.

In the whole space $\mathbb{R}^N$, Papadopoulos and Stavarakakis [19] investigated the global existence and blow-up of solutions for the following nonlocal quasilinear hyperbolic problem of Kirchhoff type

$$u_{tt} - \phi(x)\|\nabla u\|^2\Delta u + \delta u_t = |u|^{\alpha}u \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+, \tag{E$_3'$}$$

where $N \geq 3, \delta \geq 0$ and $\rho(x) = (\phi(x))^{-1}$ is a positive function lying in $L^{N/2}(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$. [20,21]

It is worth motioning that the global existence and blow-up of solutions for the parabolic equations of Kirchhoff type involving the classical Laplacian operator have been widely studied by many authors. For instance, we refer to [22–25] and the references therein for the setting of bounded domains. Concerning the global existence, asymptotic behavior and blow-up of solutions for hyperbolic equations of Kirchhoff type involving the fractional Laplacian in the case of bounded domain, we refer the reader to some recent results obtained in [26,27]. However, to the author's best knowledge there is no result on the global existence and blow-up of solutions for parabolic and hyperbolic equations of Kirchhoff type involving fractional Laplacian operator in whole $\mathbb{R}^N$. Recently, Fiscella and Valdinoci [28] proposed a fractional counterpart to the Kirchhoff operator which models the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string. We recall that Kirchhoff problems, with Kirchhoff function $M$, are said to be *non-degenerate* if $M(0) > 0$, and *degenerate* if $M(0) = 0$. Xiang, Rădulescu and Zhang [5] considered the following diffusion model of Kirchhoff-type

$$\begin{cases} u_t + M\left([u]_s^2\right)(-\Delta)^s u = |u|^{p-2}u & \text{in } \Omega \times \mathbb{R}^+, \\ u(x,t) = 0 & \text{in } \left(\mathbb{R}^N \backslash \Omega\right) \times \mathbb{R}^+, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{E$_2$}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ and $(-\Delta)^s$ is the fractional Laplacian with $0 < s < 1 < p < \infty$. Under some appropriate conditions the authors obtained a nonnegative local weak solution of $(E_2)$ by using the Galerkin method. Moreover, they proved also an estimate for the lower and upper bounds of the blow-up time. Leter, by combining the Galerkin method with potential well theory, Ding and Zhou [29] investigated the global existence and blow-up of solutions for problem $(E_2)$. Pucci et al. [30] studied the following anomalous diffusion model of Kirchhoff type

$$\begin{cases} u_t + M\left([u]_{s,p}^p\right)(-\Delta)_p^s u = f(x,t) & \text{in } \Omega \times \mathbb{R}^+, \\ u(x,t) = 0 & \text{in } \left(\mathbb{R}^N \backslash \Omega\right) \times \mathbb{R}^+, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{E$_3$}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $f \in L^2_{loc}(\mathbb{R}_0^+, L^2(\Omega))$ and $(-\Delta)_p^s$ is the fractional $p$-Laplacian. By using the sub-differential approach, the authors established the well-posedness of solutions for problem $(E_3)$. Moreover, the large time behavior and extinction of solutions are also investigated. With the help of potential well theory, Fu and Pucci [31], studied the existence of global weak solutions and established the vacuum isolating and blow-up of strong solutions for the following class of problem

$$\begin{cases} u_t + (-\Delta)^s u = |u|^{p-2}u, \quad x \in \Omega, \ t > 0, \\ u(x,t) = 0, \quad x \in \mathbb{R}^N \backslash \Omega, \ t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega \end{cases} \tag{M$_2$}$$

where $s \in (0,1)$, $N > 2s$ and $2 < p \leq 2_s^* = 2N/(N-2s)$.

To the best of our knowledge, there is no result on global existence and asymptotic behavior as well as blow-up of solutions in finite time for initial problem (1.1). There is no doubt we encounter serious difficulties because of the lack of compactness, Hardy–Littlewood–Sobolev nonlinearity as well as the degenerate nature of the Kirchhoff coefficient.

Inspired by the above works, especially by [5,14,29], we study the well-posedness and asymptotic behavior as well as blow-up of solutions in finite time for initial problem (1.1). Moreover, we give an estimate for the lower and upper bounds of the blow-up time. We stress that these results are new even in the case of classical Laplacian where $M \equiv 1$.

In order to present the main results of this paper, let us recall some results related to the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ (see [18,32]). Let $0 < s < 1 < p < \infty$ be real numbers. The fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined as follows :

$$W^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) \; : \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dxdy < \infty \right\},$$

equipped with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} = \left( \|u\|_p^p + \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dxdy \right)^{1/p}.$$

As it is well known $(W^{s,p}(\mathbb{R}^N), \|\,.\,\|_{W^{s,p}(\mathbb{R}^N)})$ is a reflexive separable Banach space and the embedding $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is continuous for any $q \in [p, p_s^*]$, namely there exists a positive constant $S_q$ such that

$$\|u\|_q \leq S_q \|u\|_{W^{s,p}(\mathbb{R}^N)}.$$

Here $p_s^*$ is the critical exponent defined as $p_s^* = \frac{Np}{N-sp}$, see [32] for more details. In order to obtain the existence of weak solutions for (1.1), we consider the subspace of $W^{s,p}(\mathbb{R}^N)$

$$W = \left\{ u \in W^{s,p}(\mathbb{R}^N) \; : \int_{\mathbb{R}^N} V(x)|u|^p\,dx < \infty \right\},$$

endowed with the norm

$$\|u\| = \left( \|V^{1/p}u\|_p^p + \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dxdy \right)^{1/p}.$$

Throughout the paper we assume the following hypotheses on $V$ :

The function $V : \mathbb{R}^N \to \mathbb{R}_+$ is measurable and there exists $V_0 > 0$ such that $\inf_{\mathbb{R}^N} V(x) \geq V_0$. $(V_1)$

$$\frac{1}{V} \in L^{\frac{2}{p-2}}(\mathbb{R}^N). \tag{V$_2$}$$

A typical example of $V$ is given by $V(x) = \exp(\rho|x|^2)$ for all $\rho > 0$.

The energy functional associated with initial value problem (1.1) is given by

$$E(u) = \frac{1}{\theta p}\|u\|^{\theta p} - \frac{1}{2q}\int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u|^q)|u|^q \, dx, \quad u \in W. \tag{1.4}$$

We define the Nehari functional by

$$I(u) = \|u\|^{\theta p} - \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u|^q)|u|^q \, dx, \quad u \in W. \tag{1.5}$$

In view of (1.4) and (1.5), it is easy to see that

$$E(u) = \left(\frac{1}{\theta p} - \frac{1}{2q}\right)\|u\|^{\theta p} + \frac{1}{2q} I(u). \tag{1.6}$$

Let

$$d = \inf_{u \in \mathcal{N}} E(u), \tag{1.7}$$

where $\mathcal{N}$ is the Nehari manifold, which is defined by

$$\mathcal{N} = \{u \in W \backslash \{0\} : I(u) = 0\}. \tag{1.8}$$

We shall show in Lemma 3.3 below that

$$d \geq \left(\frac{2q - \theta p}{2\theta q p}\right)\left(\frac{\Lambda_{rq}^{2q}}{C(N, \mu, r)}\right)^{\theta p/(2q-\theta p)}, \tag{1.9}$$

where $\Lambda_\nu$ denotes the best constant in the embedding $W \hookrightarrow L^\nu(\mathbb{R}^N)$ for all $\nu \in [2, p_s^*]$, i.e,

$$\Lambda_\nu = \inf\left\{\frac{\|u\|}{\|u\|_\nu} : u \in W \backslash \{0\}\right\}. \tag{1.10}$$

Clearly, $\Lambda_\nu > 0$. Before stating the main results of this paper, let us give the definition of the weak solutions for initial value problem (1.1).

**Definition 1.1** We say that $u \in L^\infty(0, T; W)$ is a positive weak solution of problem (1.1) if, $u_t \in L^2(0, T; L^2(\mathbb{R}^N))$ and the following equalities hold

(1) $\int_{\mathbb{R}^N} u_t v\,dx + \|u\|^{p(\theta-1)}\langle u, v\rangle = \int_{\mathbb{R}^N} (\mathcal{K}_\mu * u^q)u^{q-1}uv\,dx$ for each $v \in W$ and a.e time $0 \le t \le T$, and
(2) $u(0) = u_0$.

where

$$\langle u, v\rangle = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}\,dxdy + \int_{\mathbb{R}^N} V|u|^{p-2}uv\,dx.$$

Our first result reads as follows :

**Theorem 1.1** *Suppose that* $0 \le u_0 \in W$ *and the following condition holds :*

$$2 < qr < 2q < \min\left\{\theta p + \frac{4p_s^* - 2r\theta p}{rp_s^*}, \frac{(2-r)p_s^*}{2}\right\}, \quad r = \frac{2N}{2N-\mu}, \quad p_s^* = \frac{Np}{N-sp}. \tag{1.11}$$

*Then there exists* $T > 0$ *such that problem (1.1) admits at least a nontrivial, nonnegative weak solution for all* $t \in (0, T]$. *Moreover, for a.e.* $t \in [0, T]$,

$$\int_0^t \|u_s(s)\|_2^2\,ds + E(u(t)) \le E(u_0). \tag{1.12}$$

***Remark 1*** From the regularity of weak solutions stated in Definition 1.1 and [33, Proposition 1.2] we infer that $u \in C([0, T], L^2(\mathbb{R}^N))$. Therefore, the initial condition (2) in Definition 1.1 exists and makes sense.

Before introducing the second result, let us give the following definition :

**Definition 1.2** (Maximal existence time) Let $u(t)$ be a solution of problem (1.1). We define the maximal existence time $T_{\max}$ of $u$ as follows :

$$T_{\max} = \sup\{t > 0 \,:\, u = u(t) \text{ exists on } [0, T]\}.$$

(1) If $T_{\max} < \infty$ we say that the solution of (1.1) blows up and $T_{\max}$ is the blow up time.
(2) If $T_{\max} = \infty$, we say that the solution is global.

The proof of the following Theorem 1.2 relies on the potential well method which was introduced by Sattinger in [34]; see [25,35,36] and the references therein for some results on global existence of solutions.

**Theorem 1.2** *Assume that* $0 \le u_0 \in W$ *and the following conditions hold true :*

$$\theta p < 2q < p_s^*, \tag{1.13}$$

*and*

$$E(u_0) < d \quad and \quad I(u_0) > 0. \tag{1.14}$$

*Then the positive weak solution $u = u(t)$ of initial value problem (1.1) exists globally. Moreover,*

$$\int_0^t \|u_s(s)\|_2^2 \, ds + E(u(t)) \le E(u_0), \quad a.e.\ t \in [0, +\infty). \tag{1.15}$$

***Remark 2*** Let us discuss the initial value for which the condition (1.14) is satisfied.

(1) If $0 < E(u_0) < d$ and $I(u_0) \ge 0$ then we have $I(u_0) > 0$. Indeed, assume that $I(u_0) = 0$. From (1.6) we observe that $u_0 \ne 0$. Thus, $u_0 \in \mathcal{N}$, then by the definition of $d$ in (1.7), we infer that $E(u_0) \ge d$, contradiction.
(2) If $E(u_0) = 0$ and $I(u_0) \ge 0$ then we have $u_0 = 0$. In fact, if we assume $u_0 \ne 0$. Then from (1.6) it follows that $I(u_0) = 0$. This gives $u_0 \in \mathcal{N}$ and $0 = E(u_0) \ge d > 0$, contradiction by (1.9).
(3) By (1.6), clearly the case $E(u_0) < 0$ and $I(u_0) \ge 0$ can not occur.

Furthermore, we have the following corollary to Theorem 1.2.

**Corollary 1.3** *Let $0 \le u_0 \in W$ and (1.13) holds. Assume that*

$$E(u_0) \le d \quad and \quad I(u_0) \ge 0. \tag{1.16}$$

*Then initial value problem (1.1) admits a global weak solution.*

The following theorem shows the asymptotic behavior of global solutions of initial value problem (1.1).

**Theorem 1.4** *Suppose the assumptions made in Theorem 1.2 are satisfied. Assume that $\epsilon \in (0, s)$ and*

$$E(u_0) < \left(\frac{2q - \theta p}{2q\theta p}\right) \left(\frac{\Lambda_{rq}^{2q}}{C(N, \mu, r)}\right)^{\frac{\theta p}{2q - \theta p}}. \tag{1.17}$$

*Then, there holds*

$$\|u(t)\|_{W^{s-\epsilon, p}(\mathbb{R}^N)} \le \omega \left(\frac{\theta p}{1 + \chi(\theta p - 1)t}\right)^{\frac{\beta(1-\eta)}{\theta p - 1}}, \quad \forall \eta, \beta \in (0, 1), \quad \forall t \ge 0. \tag{1.18}$$

*where* $\omega = \Lambda_{p_s^*}^{(\beta-1)(1-\eta)} \left(\frac{2q\theta p}{2q - \theta p}\right)^{\frac{\eta + (1-\eta)(1-\beta)}{\theta p}}$ *and* $\chi = \left(1 - C(N, \mu, r)\Lambda_{rq}^{-2q} \left(\frac{2q\theta p}{2q - \theta p} E(u_0)\right)^{\frac{2q - \theta p}{\theta p}}\right)$.

The following theorem shows that the local solution obtained in Theorem 1.1 can not be extended globally in time.

**Theorem 1.5** *Suppose that* $0 \leq u_0 \in W$, $\theta \in (2, N/(N-sp))$ *and the following conditions hold :*

$$\theta p < qr < 2q < \min\left\{\theta p + \frac{4p_s^* - 2r\theta p}{rp_s^*}, \frac{(2-r)p_s^*}{2}\right\},$$

$$r = \frac{2N}{2N-\mu}, \quad p_s^* = \frac{Np}{N-sp}, \tag{1.19}$$

$$E(u_0) < 0. \tag{1.20}$$

*Then the local weak solution* $u(t)$ *obtained in Theorem 1.1 belows up in finite time* $T$, *where* $T$ *satisfies that*

$$\frac{2^{\frac{2\theta^2 p^2}{\theta p - 2q(1-\gamma)}} \Lambda_{p_s^*}^{\frac{2q(1-\gamma)\theta p}{\theta p - 2q(1-\gamma)}}}{(\alpha - 1)\|u_0\|_2^{2(\alpha-1)}} \leq T \leq \frac{8\|u_0\|_2^2(\theta p - 1)}{-\theta p E(u_0)(\theta p - 4)^2}.$$

*with* $\alpha = \frac{\theta p q \gamma}{\theta p - 2q(1-\gamma)} > 1$.

***Remark 3*** Notice that, due to the fact that $2r\theta p < 4\theta p < 4p_s^*$ the condition (1.19) exists. Hence, from (1.19) we observe that the condition (1.11) still holds. Therefore, in view of Theorem 1.1 there exists a local weak solution for problem (1.1) satisfies the energy inequality (1.12).

***Remark 4*** The results presented in this paper can be easily extended to more general Choquard–Kirchhoff-type equations. For example, with the same technique, we can deal with nonlinearities $M : [0, +\infty) \to [0, +\infty)$ of class $C^1$ such that

$$c_1 r^{\theta - 1} \leq M(r) \leq c_2 r^{\theta - 1}$$

for suitable positive constants $c_1$ and $c_2$. However, this generality only complicates proofs without bringing any new idea.

The remaining part of the paper is organized as follows. Section 2, contains some preliminary results, which are required in the proof of the main results. In section 3, we shall present some properties involving the functional $E$ restricted to the Nehari manifold $\mathcal{N}$. In section 4, we establish local existence of solutions by using the Galerkin method. In section 5, under some conditions on the initial data, we show the global existence of solutions for (1.1). Furthermore, we give a decay estimate for these global solutions in fractional Sobolev spaces for large time. Finally, in section 6, finite time blow-up of weak solutions of (1.1) is proved, in addition to this we give an estimate for the upper and lower bounds of the blow-up time.

Throughout the paper $c, c_i, C, C_i$, $i = 1, 2, \ldots$ denote positive constants which may vary from line to line, but are independent of terms that take part in any limit process and we use the notation $\| \,.\, \|_p$ for the standard $L^p(\mathbb{R}^N)-$norm. Furthermore we use "$\rightarrow$", "$\rightharpoonup$" and "$\rightharpoonup^*$" to denote the strong convergence, weak convergence and weak star convergence respectively.

## 2 Functional Framework

In this section, we give some technical results that will be used in the next sections.

**Lemma 2.1** *Let* $(V_1)$ *and* $(V_2)$ *hold. Then the embedding* $W \hookrightarrow L^q(\mathbb{R}^N)$ *is continuous for all* $q \in [2, p_s^*]$.

***Proof*** We have

$$\int_{\mathbb{R}^N} |u|^2\,dx = \int_{\mathbb{R}^N} \frac{1}{V^{\frac{2}{p}}(x)} V^{\frac{2}{p}}(x)|u|^2\,dx.$$

Since $p > 2$ from condition $(V_2)$ and the Hölder inequality we deduce

$$\int_{\mathbb{R}^N} |u|^2\,dx \le \left(\int_{\mathbb{R}^N} \frac{1}{V^{\frac{2}{p-2}}(x)}\,dx\right)^{(p-2)/2} \left(\int_{\mathbb{R}^N} V(x)|u|^p\,dx\right)^{2/p} \le C\|u\|^2.$$

On the other hand, by $(V_1)$ we have $W \hookrightarrow W^{s,p}(\mathbb{R}^N) \hookrightarrow L^{p_s^*}(\mathbb{R}^N)$. Then $2 < q < p_s^*$ and $\theta \in (0, 1)$ satisfy the following equation

$$\frac{1}{q} = \frac{\theta}{p_s^*} + \frac{1-\theta}{2}.$$

By the Hölder inequality, we have

$$\|u\|_q \le \|u\|_2^{1-\theta}\|u\|_{p_s^*}^{\theta} \le C\|u\|.$$

Hence this completes the proof. □

From now on, $B_R(0)$ denotes the ball in $\mathbb{R}^N$ of center zero and radius $R > 0$.

**Lemma 2.2** *Assume that* $(V_1)$ *and* $(V_2)$ *hold. Then the embedding* $W \hookrightarrow L^q(\mathbb{R}^N)$ *is compact for all* $q \in [2, p_s^*)$.

***Proof*** It is well know that the embedding $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^q(B_R(0))$ is compact for all $q \in [2, p_s^*)$. Therefore $W \hookrightarrow L^q(B_R(0))$ is compact for all $q \in [2, p_s^*)$. Since $\frac{1}{V} \in L^{\frac{2}{p-2}}(\mathbb{R}^N)$, for any $\varepsilon > 0$ there exists $R_1 > 0$ such that

$$\left(\int_{\mathbb{R}^N \setminus B_R(0)} \frac{1}{V^{\frac{2}{p-2}}(x)}\,dx\right)^{(p-2)/p} < \varepsilon, \quad \text{for all } R \ge R_1. \tag{2.1}$$

Let $\{u_m\}_m$ be a bounded sequence in $W$. Then, for any $\varepsilon > 0$ there exists $m_0 > 0$ such that

$$\int_{B_{R_1}(0)} |u_m - u|^2\, dx < \varepsilon, \quad \text{for all } m \geq m_0. \tag{2.2}$$

Combining (2.1) and (2.2), for all $m \geq m_0$ we have

$$\begin{aligned}\int_{\mathbb{R}^N} |u_m - u|^2\, dx &= \int_{B_{R_1}(0)} |u_m - u|^2\, dx + \int_{\mathbb{R}^N \setminus B_{R_1}(0)} \frac{1}{V^{\frac{2}{p}}(x)} V^{\frac{2}{p}}(x) |u_m - u|^2\, dx \\ &\leq \varepsilon + C_1 \left( \int_{\mathbb{R}^N \setminus B_{R_1}(0)} \frac{1}{V^{\frac{2}{p-2}}(x)}\, dx \right)^{p-2/p} \leq (1 + C_1)\varepsilon.\end{aligned}$$

This combined with $2 < q < p_s^*$ yields

$$\|u_m - u\|_q \leq \|u_m - u\|_{p_s^*}^{\theta} \|u_m - u\|_2^{1-\theta} \leq C_2 \varepsilon^{1-\theta}, \ \forall \theta \in (0, 1), \ \forall m \geq m_0.$$

Thus $\|u_m - u\|_q \to 0$, as $m \to \infty$. Hence the proof is now complete. □

Throughout this paper, we shall assume that $q$ satisfies

$$2 < q < \frac{p(N - \frac{\mu}{2})}{N - sp}. \tag{1.3}$$

In what follows we recall the so-called Hardy–Littlewood–Sobolev inequality, see [37, Theorem 4.3]. Hereafter, $W'$ denotes the dual space of $W$.

**Theorem 2.3** *Assume that* $1 < r, j < \infty$, $0 < \mu < N$ *and*

$$\frac{1}{r} + \frac{1}{j} + \frac{\mu}{N} = 2.$$

*Then there exists* $C(N, \mu, j, r) > 0$ *such that*

$$\int_{\mathbb{R}^{2N}} \frac{|u(y)|.|u(x)|}{|x - y|^{\mu}}\, dxdy \leq C(N, \mu, j, r) \|u\|_j \|u\|_r.$$

From this Theorem, one can observe that the operator $H : W \to W'$ defined by

$$\langle H(u), v \rangle = \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u|^q)|u|^{q-2} uv\, dx.$$

for all $u, v \in W$ is well defined. Indeed, clearly, for all $u \in W$, $H(u)$ is a linear map. Using the Hardy–Littlewood–Sobolev inequality for $j = r = \frac{2N}{2N-\mu}$ we obtain

$$\int_{\mathbb{R}^{2N}} \frac{|u(y)|^q.|u(x)|^{q-1}.|v(x)|}{|x - y|^{\mu}}\, dxdy \leq C(N, \mu, r) \| |u|^q \|_r \| |u|^{q-1} |v| \|_r.$$

Note that by (1.3) we have $2 < qr < p_s^*$. In view of the Hölder inequality and the continuous embedding $W \hookrightarrow L^\kappa(\mathbb{R}^N)$ for all $\kappa \in [2, p_s^*]$ we deuce

$$\int_{\mathbb{R}^{2N}} \frac{|u(y)|^q.|u(x)|^{q-1}.|v(x)|}{|x-y|^\mu}\,dxdy \le C(N,\mu,r)\|u\|^{2q-1}\|v\|.$$

Therefore, for $v \in W$ with $\|v\| \le 1$ we obtain

$$\|H(u)\|_{W'} \le C(N,\mu,r)\|u\|^{2q-1}.$$

**Lemma 2.4** *[38, Proposition 4.7.12] Let $\kappa \in (1,\infty)$. Assume $\{w_m\}$ is a bounded sequence in $L^\kappa(\mathbb{R}^N)$ and converges almost everywhere to $w$. Then $w_m \rightharpoonup w$ in $L^\kappa(\mathbb{R}^N)$.*

**Lemma 2.5** *(Brezis-Lieb lemma) Let $\kappa \in (1,\infty)$. Assume $\{w_m\}$ is a bounded sequence in $L^\kappa(\mathbb{R}^N)$ and converges almost everywhere to $w$. Then for any $t \in [1,\kappa]$*

*(1)* $\lim\limits_{m\to\infty} \int_{\mathbb{R}^N} ||w_m|^{t-2}w_m h - |w_m - w|^{t-2}(w_m - w)h - |w|^{t-2}wh|^{\frac{\kappa}{t}}\,dx = 0,$ *for all* $h \in L^\kappa(\mathbb{R}^N)$,

*(2)* $\lim\limits_{m\to\infty} \int_{\mathbb{R}^N} ||w_m|^t - |w_m - w|^t - |w|^t|^{\frac{\kappa}{t}}\,dx = 0.$

***Proof*** Using the Young inequality, for any $\varepsilon > 0$ there exist $C(\varepsilon), C_1 > 0$ such that for all $a, b \in \mathbb{R}$ we have

$$\Big||a+b|^{t-2}(a+b)h - |a|^{t-2}ah\Big|^{\frac{\kappa}{t}} \le C_1|h|^\kappa + C(\varepsilon)|b|^\kappa + \varepsilon|a|^\kappa.$$

Taking $a = w_m - w$ and $b = w$ and using the above inequality, yields

$$\begin{aligned} f_{m,\varepsilon} &= (||w_m|^{t-2}w_m h - |w_m - w|^{t-2}(w_m - w)h - |w|^{t-2}wh|^{\frac{\kappa}{t}} - \varepsilon|w_m - w|^\kappa)^+ \\ &\le (1 + C(\varepsilon))|w|^\kappa + C_2|h|^\kappa. \end{aligned}$$

Note that, according to the above assumptions we have $(1 + C(\varepsilon))|w|^\kappa + C_2|h|^\kappa \in L^1(\mathbb{R}^N)$ and $f_{m,\varepsilon} \to 0$ a.e in $\mathbb{R}^N$. Thus the Lebesgue dominated convergence theorem implies

$$\int_{\mathbb{R}^N} f_{m,\varepsilon}\,dx \to 0 \text{ as } m \to \infty.$$

Therefore, we get

$$||w_m|^{t-2}w_m h - |w_m - w|^{t-2}(w_m - w)h - |w|^{t-2}wh|^{\frac{\kappa}{t}} \le f_{m,\varepsilon} + \varepsilon|w_m - w|^\kappa,$$

which implies

$$\limsup_{m\to\infty} \int_{\mathbb{R}^N} ||w_m|^{t-2}w_m h - |w_m - w|^{t-2}(w_m - w)h - |w|^{t-2}wh|^{\frac{\kappa}{t}}\,dx \le c\varepsilon.$$

where $c = sup_m\|w_m - w\|_\kappa^\kappa < \infty$. Further letting $\varepsilon \to 0$, we obtain the desired result. In a similar manner, we conclude that (2) holds. □

Throughout the paper without further mentioning, we put $r = \frac{2N}{2N-\mu}$.

**Lemma 2.6** *Let $\{u_m\}_m$ be a bounded sequence in $L^{rq}(\mathbb{R}^N)$ such that $u_m \to u$ a.e in $\mathbb{R}^N$ as $m \to \infty$. Then for any $w \in L^{rq}(\mathbb{R}^N)$ we have*

$$\lim_{m\to\infty} \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u_m|^q)|u_m|^{q-2}u_m w\, dx \to \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u|^q)|u|^{q-2}uw\, dx.$$

***Proof*** Denote $v_m = u_m - u$ and observe that

$$\begin{aligned}
&\int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u_m|^q)|u_m|^{q-2}u_m w\, dx \\
&\quad = \int_{\mathbb{R}^N} [\mathcal{K}_\mu * (|u_m|^q - |v_m|^q)](|u_m|^{q-2}u_m w - |v_m|^{q-2}v_m w)\, dx \\
&\qquad + \int_{\mathbb{R}^N} [\mathcal{K}_\mu * (|u_m|^q - |v_m|^q)]|v_m|^{q-2}v_m w\, dx \\
&\qquad + \int_{\mathbb{R}^N} [\mathcal{K}_\mu * (|u_m|^{q-2}u_m w - |v_m|^{q-2}v_m w)]|v_m|^q\, dx \\
&\qquad + \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |v_m|^q)|v_m|^{q-2}v_m w\, dx
\end{aligned} \tag{1.4}$$

Applying Lemma 2.5 with $t = q$ and $\kappa = rq$, we find

$$|u_m|^q - |v_m|^q \to |u|^q \ \text{ in } L^r(\mathbb{R}^N),$$

and

$$|u_m|^{q-2}u_m w - |v_m|^{q-2}v_m w \to |u|^{q-2}uw \ \text{ in } L^r(\mathbb{R}^N).$$

The Hardy–Littlewood–Sobolev inequality ensures that

$$\left\{\begin{array}{c} \mathcal{K}_\mu * (|u_m|^q - |v_m|^q) \to \mathcal{K}_\mu * |u|^q \\ \mathcal{K}_\mu * (|u_m|^{q-2}u_m w - |v_m|^{q-2}v_m w) \to \mathcal{K}_\mu * (|u|^{q-2}uw) \end{array}\right. \text{ in } L^{\frac{2N}{\mu}}(\mathbb{R}^N). \tag{1.5}$$

In view of Lemma 2.4, we have

$$|u_m|^{q-2}u_m w \rightharpoonup |u|^{q-2}uw, \quad |v_m|^q \rightharpoonup 0, \quad |v_m|^{q-2}v_m w \rightharpoonup 0 \ \text{ in } \ L^r(\mathbb{R}^N). \tag{1.6}$$

Combining (1.5)–(1.6), we find

$$\left\{\begin{array}{l} \lim\limits_{m\to\infty} \int_{\mathbb{R}^N} [\mathcal{K}_\mu * (|u_m|^q - |v_m|^q)](|u_m|^{q-2}u_m w - |v_m|^{q-2}v_m w)\, dx = \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u|^q)|u|^{q-2}uw\, dx, \\ \lim\limits_{m\to\infty} \int_{\mathbb{R}^N} [\mathcal{K}_\mu * (|u_m|^q - |v_m|^q)]|v_m|^{q-2}v_m w\, dx = 0, \\ \lim\limits_{m\to\infty} \int_{\mathbb{R}^N} [\mathcal{K}_\mu * (|u_m|^{q-2}u_m w - |v_m|^{q-2}w_m w)]|v_m|^q\, dx = 0. \end{array}\right. \tag{1.7}$$

By Hölder's inequality and the Hardy–Littlewood–Sobolev inequality, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^N}(\mathcal{K}_\mu * |v_m|^q)|v_m|^{q-2}v_m w\,dx\right| &\le C(N,\mu,r)\|v_m\|_{rq}^q \||v_m|^{q-2}u_m w\|_r\\ &\le C\||v_m|^{q-2}v_m w\|_r. \end{aligned}\tag{1.8}$$

On the other hand, using Lemma 2.4 we infer that $|v_m|^{(q-1)r} \rightharpoonup 0$ in $L^{\frac{q}{q-1}}(\mathbb{R}^N)$, so

$$\left(\int_{\mathbb{R}^N}|v_m|^{(q-1)r}|w|^r\right)^{1/r} \to 0.$$

Hence, from (1.8) we obtain

$$\lim_{m\to\infty}\int_{\mathbb{R}^N}(\mathcal{K}_\mu * |v_m|^q)|v_m|^{q-2}v_m w\,dx = 0. \tag{1.9}$$

Using (1.7) and (1.9) and passing to the limit in (1.4) as $m \to \infty$, we reach the conclusion. □

## 3 Properties Involving the Functional *E* Restricted to $\mathcal{N}$

In this section, we provide some properties involving the functional $E$ restricted to the Nehari manifold $\mathcal{N}$. These properties turn out to be very useful in discussing the global existence and blow-up of solutions for initial value problem (1.1).

For $\lambda > 0$, we consider the function $g$ defined by

$$g(\lambda) = E(\lambda u) = \frac{\lambda^{\theta p}}{\theta p}\|u\|^{\theta p} - \frac{\lambda^{2q}}{2q}\int_{\mathbb{R}^N}(\mathcal{K}_\mu * |u|^q)|u|^q\,dx. \tag{3.1}$$

**Lemma 3.1** *Let $u \in W\backslash\{0\}$ and $\theta p < 2q$. Then we have*

(1) $\lim_{\lambda\to 0^+} g(\lambda) = 0$ *and* $\lim_{\lambda\to+\infty} g(\lambda) = -\infty$.
(2) *There is a unique $\lambda^* = \lambda^*(u) > 0$ such that $g'(\lambda^*) = 0$.*
(3) *$g(\lambda)$ is increasing on $(0,\lambda^*)$, decreasing on $(\lambda^*,+\infty)$ and attains the maximum at $\lambda^*$.*
(4) *$I(\lambda u) > 0$ for $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ for $\lambda^* < \lambda < +\infty$ and $I(\lambda^* u) = 0$.*

***Proof*** For $u \in W\backslash\{0\}$, by definition of $g(\lambda) = E(\lambda u)$, it is clear that the first statement holds due to $\theta p < 2q$. Now, by differentiating $g(\lambda)$ we obtain

$$\frac{d}{d\lambda}g(\lambda) = \lambda^{\theta p-1}\left(\|u\|^{\theta p} - \lambda^{2q-\theta p}\int_{\mathbb{R}^N}(\mathcal{K}_\mu * |u|^q)|u|^q\,dx\right). \tag{3.2}$$

Therefore, by taking

$$\lambda^* = \lambda^*(u) = \left(\frac{\|u\|^{\theta p}}{\int_{\mathbb{R}^N}(\mathcal{K}_\mu * |u|^q)|u|^q\,dx}\right)^{1/(2q-\theta p)},$$

the second and third statements can be shown easily. In order to show the fourth statement, one can check that

$$I(\lambda u) = \lambda g'(\lambda).$$

The proof is now complete. □

**Lemma 3.2** *Let $u \in W\backslash\{0\}$ and $\theta p < 2q$. Then*

(1) *If $\|u\| < \left(\frac{\Lambda_{rq}^{2q}}{C(N,\mu,r)}\right)^{1/(2q-\theta p)}$, then $I(u) > 0$.*

(2) *If $I(u) \leq 0$, then $\|u\| \geq \left(\frac{\Lambda_{rq}^{2q}}{C(N,\mu,r)}\right)^{1/(2q-\theta p)}$.*

***Proof*** By the definition of $I$, we have

$$I(u) = \|u\|^{\theta p} - \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u|^q)|u|^q \, dx$$

Using the Hardy–Littlewood–Sobolev inequality, we obtain

$$\begin{aligned} I(u) &\geq \|u\|^{\theta p} - C(N, \mu, r)\Lambda_{rq}^{-2q}\|u\|^{2q} \\ &= \|u\|^{\theta p}\left(1 - C(N, \mu, r)\Lambda_{rq}^{-2q}\|u\|^{2q-\theta p}\right) \end{aligned}$$

Hence the proof follows from the above inequality. □

The next lemma shows that the stationary problem related to (1.1) admits a ground state solution.

**Lemma 3.3** *Assume that $\theta p < q$ holds. Then $d \geq \left(\frac{2q-\theta p}{2\theta qp}\right)\left(\frac{\Lambda_{rq}^{2q}}{C(N,\mu,r)}\right)^{\theta p/(2q-\theta p)}$ and there exists $u \in \mathcal{N}$ such that $d = E(u)$.*

***Proof*** For any $u \in \mathcal{N}$, from Lemma 3.2 we know that $\|u\|^{\theta p} \geq \left(\frac{\Lambda_{rq}^{2q}}{C(N,\mu,r)}\right)^{\theta p/(2q-\theta p)}$. Then by (1.6) with $I(u) = 0$, it follows that

$$E(u) = \left(\frac{2q-\theta p}{2\theta qp}\right)\|u\|^{\theta p} \geq \left(\frac{2q-\theta p}{2\theta qp}\right)\left(\frac{\Lambda_{rq}^{2q}}{C(N,\mu,r)}\right)^{\theta p/(2q-\theta p)} > 0.$$

By the definition of $d$ we obtain $d \geq \left(\frac{2q-\theta p}{2\theta qp}\right)\left(\frac{\Lambda_{rq}^{2q}}{C(N,\mu,r)}\right)^{\theta p/(2q-\theta p)}$. It remains to prove the second part of the lemma. Let $\{u_n\}_n$ be a minimizing sequence of $d$, i.e, $\{u_n\}_n \subset \mathcal{N}$ and $E(u_n) \to d$ as $n \to \infty$. From (1.6) we have

$$E(u_n) = \left(\frac{2q-\theta p}{2\theta qp}\right)\|u_n\|^{\theta p},$$

which implies that $\{u_n\}_n$ is bounded in $W$. Going if necessary to a subsequence, we assume that

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } \quad W, \\ u_n \to u & \text{strongly in } \; L^\nu(\mathbb{R}^N), \; \forall \nu \in [2, p_s^*), \\ u_n \to u & \text{a.e. in } \quad \mathbb{R}^N. \end{cases} \tag{3.3}$$

Now we claim that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u_n|^q)|u_n|^q \, dx = \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u|^q)|u|^q \, dx. \tag{3.4}$$

By a direct computation we have

$$\begin{aligned} &\left| \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u_n|^q)|u_n|^q \, dx - \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u|^q)|u|^q \, dx \right| \\ &\le \int_{\mathbb{R}^N} (\mathcal{K}_\mu * \left| |u_n|^q - |u|^q \right|)|u_n|^q \, dx \\ &\quad + \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u|^q) \left| |u_n|^q - |u|^q \right| \, dx. \end{aligned}$$

Since $2 < qr < p_s^*$, by using the Hardy–Littlewood–Sobolev inequality, we have

$$\begin{aligned} &\left| \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u_n|^q)|u_n|^q \, dx - \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u|^q)|u|^q \, dx \right| \\ &\le 2C(N, \mu, r)\|u_n - u\|_{qr} \|u_n\|_{qr}^{2q-1} \\ &\le C_1 \|u_n - u\|_{qr} \to 0 \text{ as } n \to \infty. \end{aligned}$$

Since $I(u_n) = 0$, it follows from Lemma 3.2 that $\|u_n\|^{\theta p} \ge \left( \frac{\Lambda_{rq}^{2q}}{C(N,\mu,r)} \right)^{\theta p/(2q-\theta p)}$. By the definition of $I$ we have

$$\left( \frac{\Lambda_{rq}^{2q}}{C(N, \mu, r)} \right)^{\theta p/(2q-\theta p)} \le \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u_n|^q)|u_n|^q \, dx.$$

So it follows from (3.4) and the above inequality that

$$0 < \left( \frac{\Lambda_{rq}^{2q}}{C(N, \mu, r)} \right)^{\theta p/(2q-\theta p)} \le \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u|^q)|u|^q \, dx.$$

□

This ensures that $u \neq 0$. Combining (3.3)–(3.4) and the weak lower semicontinuity of the norm, yield

$$E(u) \leq \lim_{n\to\infty} E(u_n) \leq d, \tag{3.5}$$

and

$$I(u) \leq \lim_{n\to\infty} I(u_n) = 0.$$

To finish the proof we need to show that $I(u) = 0$. Indeed, arguing by contradiction, by supposing that $I(u) < 0$. Since $u \in W\backslash\{0\}$ it follows from Lemma 3.1 that there exists $\lambda^* \in (0, 1)$ such that $I(\lambda^* u) = 0$. Hence,

$$\begin{aligned} d \leq E(\lambda^* u) &= (\lambda^*)^\theta \left(\frac{1}{\theta p} - \frac{1}{2q}\right) \|u\|^{\theta p} \\ &\leq (\lambda^*)^\theta \liminf_{n\to\infty} \left(\frac{1}{\theta p} - \frac{1}{2q}\right) \|u_n\|^{\theta p} < \liminf_{n\to\infty} E(u_n) = d, \end{aligned}$$

which is absurd. Hence $I(u) = 0$. Therefore, $u \in \mathcal{N}$ and the conclusion follows immediately from (3.5).

## 4 Local Existence

This section is devoted to the proof of Theorem (1.1). In the sequel we will use the Galerkin method.

In what follows $(.,.)$ denotes the inner product in $L^2(\mathbb{R}^N)$ and $v^+ = \max\{v, 0\}$. Since $W$ is separable and $W$ dense in $L^2(\mathbb{R}^N)$, we have a base $\mathcal{V} = \{w_i, i \in \mathbb{N}\}$ in $W$, and also in $L^2(\mathbb{R}^N)$ such that $(w_i, w_j) = \delta_{i,j}$, $i, j = 1, 2, \ldots$. For $m \in \mathbb{N}^*$ we look for the approximate solution $u_m(x,t) = \sum_{i=1}^{m} g_{im}(t) w_i$ satisfying the following identity :

$$\begin{aligned} &\int_{\mathbb{R}^N} u_{mt} w_j \, dx + \|u_m\|^{p(\theta-1)} \langle u_m, w_j \rangle \\ &= \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u_m^+|^q) |u_m^+|^{q-2} u_m^+ w_j \, dx, \;\; j = 1, 2, \ldots, m, \end{aligned} \tag{4.1}$$

and

$$u_m(0) = u_{0m}, \tag{4.2}$$

where

$$\langle u, w_j\rangle = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(w_j(x)-w_j(y))}{|x-y|^{N+sp}}\,dxdy + \int_{\mathbb{R}^N} V|u|^{p-2}uw_j\,dx.$$

Since $\mathcal{V}=\{w_i, i\in\mathbb{N}\}$ is dense in $W$ and $u_0\in W$, there exists $\{\eta_{im}, i=1,2,\dots,m\}$ such that

$$u_m(0)=\sum_{i=1}^{m}\eta_{im}w_i \to u_0 \quad \text{strongly in } W. \tag{4.3}$$

Then (4.1)–(4.2) is reduced to the following initial value problem for a system of nonlinear differential equations on $g_{jm}(t)$ :

$$\begin{cases} g'_{jm}(t)=G_j(g), & j=1,\dots,m,\\ g_{jm}(0)=\eta_{jm}, & j=1,\dots,m, \end{cases} \tag{4.4}$$

where

$$G_j(g)=-\|u_m\|^{p(\theta-1)}\langle u_m, w_j\rangle + \int_{\mathbb{R}^N}(\mathcal{K}_\mu * |u_m^+|^q)|u_m^+|^{q-2}u_m w_j\,dx.$$

By the Picard iteration method, there exists $t_{0,m}>0$ depending on $|\eta_{jm}|$ such that problem (4.4) admits a unique solution $g_{jm}\in C^1([0,t_{0,m}])$.

Multiplying (4.1) by $g_{jm}(t)$ and summing over $j=1,2,\dots,m$, yields

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|_2^2 + \|u_m(t)\|^{\theta p} = \int_{\mathbb{R}^N}(\mathcal{K}_\mu * |u_m^+|^q)|u_m^+|^q\,dx, \quad \forall t\in[0,t_{0,m}]. \tag{4.5}$$

By using the Hardy–Littlewood–Sobolev inequality, we obtain

$$\int_{\mathbb{R}^N}(\mathcal{K}_\mu * |u_m^+|^q)|u_m^+|^q\,dx \le C(N,\mu,r)\|u_n\|_{rq}^{2q}.$$

From (1.11) we have $2<qr<p_s^*$, by using the interpolation inequality and the continuous embedding $W\hookrightarrow L^{p_s^*}(\mathbb{R}^N)$, we find

$$\int_{\mathbb{R}^N}(\mathcal{K}_\mu * |u_m^+|^q)|u_m^+|^q\,dx \le \Lambda_{p_s^*}^{-2q(1-\gamma)}C(N,\mu,r)\|u_n\|_2^{2q\gamma}\|u_m\|^{2q(1-\gamma)}, \tag{4.6}$$

where $\gamma\in(0,1)$ satisfies

$$\frac{1}{qr}=\frac{\gamma}{2}+\frac{1-\gamma}{p_s^*}.$$

Since $2 < qr < 2q < \min\left\{\theta p + \frac{4p_s^* - 2r\theta p}{rp_s^*}, \frac{(2-r)p_s^*}{2}\right\}$, we have

$$2q(1-\gamma) < \theta p,$$

and

$$\alpha = \frac{\theta pq\gamma}{\theta p - 2q(1-\gamma)} > 1.$$

Using the Young inequality in (4.6), for any $\varepsilon \in (0, 1)$ we obtain

$$\int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u_m^+|^q)|u_m^+|^q \, dx \le C(\varepsilon) \left(\|u_m(t)\|_2^2\right)^\alpha + \varepsilon \|u_m(t)\|^{\theta p}, \tag{4.7}$$

Combining (4.5) and (4.7), we get

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|_2^2 + (1-\varepsilon)\|u_m(t)\|^{\theta p} \le C(\varepsilon)\left(\|u_m(t)\|_2^2\right)^\alpha, \quad \forall t \in [0, t_{0,m}].$$

Taking $\varepsilon = \frac{1}{2}$, yields

$$\frac{d}{dt}\|u_m(t)\|_2^2 \le 2C_1 \left(\|u_m(t)\|_2^2\right)^\alpha, \quad \forall t \in [0, t_{0,m}]. \tag{4.8}$$

Since $\alpha > 1$, according to (4.8), we obtain

$$\|u_m(t)\|_2^2 \le \left(C_2^{1-\alpha} - 2(\alpha-1)C_1 t\right)^{\frac{1}{1-\alpha}}. \tag{4.9}$$

only if $t < T^* = \frac{C_2^{1-\alpha}}{2(\alpha-1)C_1}$, where $C_2 = \sup_m \int_{\mathbb{R}^N} u_m^2(x, 0)\, dx \in (0, +\infty)$. It follows that

$$\|u_m(t)\|_2^2 \le 2^{\frac{1}{\gamma-1}} C_2, \quad \forall t \le \min\left\{t_{0,m}, T^*/2\right\}. \tag{4.10}$$

Now, we claim that (4.10) holds for all $t \in [0, T^*/2]$. Indeed, if $T^*/2 \le t_{0,m}$, then there is nothing to prove. Otherwise, if $t_{0,m} < T^*/2$, then $\|u(t_{0,m})\|_2^2 \le 2^{\frac{1}{\gamma-1}} C_2$. Thus, we can replace $u_{0m}$ in (4.2) by $u_m(x, t_{0,m})$ and extend the solution to the interval $[0, T^*/2]$ by repeating the above process. Thus, we obtain

$$\|u_m(t)\|_2^2 \le 2^{\frac{1}{\gamma-1}} C_2, \quad \forall t \in [0, T] \quad \left(T = T^*/2\right). \tag{4.11}$$

Now, multiplying the $j^{th}$ equations in (4.1) by $g'_{jm}(t)$ and summing over $j$ from 1 to $m$, afterward, integrating over $(0, t)$ yields

$$\int_0^t \|u_{ms}(s)\|_2^2 \, ds + E(u_m(t)) = E(u_{0m}), \quad t \in [0, T]. \tag{4.12}$$

From (4.3) and the continuity of the functional $E$ on $W$, we have

$$|E(u_{0m})| \leq c, \quad \forall m \in \mathbb{N}^*. \tag{4.13}$$

Remembering that

$$E(u_m(t)) = \frac{1}{\theta p}\|u_m(t)\|^{\theta p} - \frac{1}{2q}\int_{\mathbb{R}^N} \mathcal{K}_\mu * (|u_m^+|^q)|u_m^+|^q\, dx.$$

Thus, combining (4.7), (4.11), (4.12) and (4.13) we get

$$\int_0^t \|u_{ms}(s)\|_2^2\, ds + \left(\frac{1}{\theta p} - \varepsilon\right)\|u_m(t)\|^{\theta p} \leq c, \quad \forall t \in [0, T].$$

Taking $\varepsilon = \frac{1}{2\theta p}$, we conclude that

$$\int_0^t \|u_{ms}(s)\|_2^2\, ds \leq c_1, \quad \forall t \in [0, T], \tag{4.14}$$

and

$$\|u_m(t)\| \leq c_2, \quad \forall t \in [0, T]. \tag{4.15}$$

Then, by (4.14) and (4.15), there exist $u$ and a subsequence of $\{u_m\}_m$, still denoted by $\{u_m\}_m$ such that, as $m \to \infty$

$$\begin{cases} u_m \rightharpoonup^* u \ \text{ in } L^\infty(0, T; W)), \\ u_{mt} \rightharpoonup u_t \ \text{ in } L^2(0, T; L^2(\mathbb{R}^N)), \end{cases} \tag{4.16}$$

Then it follows from Aubin-Lions compactness theorem, see [39, Corollary 4] that

$$u_m \to u \ \text{ in } \ C([0, T], L^\nu(\mathbb{R}^N)), \ \ \forall \nu \in [2, p_s^*). \tag{4.17}$$

This implies

$$u_m(x, t) \to u(x, t) \ \text{ a.e. in } \ \mathbb{R}^N, \ \forall t \geq 0. \tag{4.18}$$

Now, by using [40, proposition 1.3] we prove that $\{\|u_m(t)\|^{p(\theta-1)}\}_m$ is relatively compact in $L^1(0, T)$. Indeed, by (4.15) we have $\|u_m(t)\|^{p(\theta-1)} \leq c$, for all $m$ and $t$. This implies that $\int_0^T \|u_m(t)\|^{p(\theta-1)}\, dt \leq cT$ for all $m$. On the other hand, for any $\varepsilon > 0$, there exits $\delta = \frac{\varepsilon}{c}$ such that for any measurable subset $A$ with $|A| < \delta$, there holds

$$\int_A \|u_m(t)\|^{p(\theta-1)}\, dt \leq c|A| < \varepsilon.$$

Consequently, $\{\|u_m(t)\|^{p(\theta-1)}\}_m$ is relatively compact in $L^1(0,T)$. Therefore, there exist $\beta(t) \in L^1(0,T)$ and a subsequence of $\{\|u_m(t)\|^{p(\theta-1)}\}_m$, still denoted by $\{\|u_m(t)\|^{p(\theta-1)}\}_m$ such that

$$\|u_m(t)\|^{p(\theta-1)} \to \beta(t), \quad \text{a.e. } t \in [0,T]. \tag{4.19}$$

From (4.16) and [41, Lemma 3.1.7], we infer that

$$u_m(0) \to u(0) \;\; \text{weakly in } L^2(\mathbb{R}^N). \tag{4.20}$$

On the other hand, from (4.3) we have $u_m(0) \to u_0$ strongly in $W$. Thus, this combined with (4.20) yields

$$u(0) = u_0. \tag{4.21}$$

In the next step, we show that the equality (1) in Definition 1.1 holds.

First, we claim that

$$u_m^+ \to u^+ \;\; \text{in} \;\; C([0,T], L^\nu(\mathbb{R}^N)), \;\; \forall \nu \in [2, p_s^*). \tag{4.22}$$

To this end, for any $\nu \in [2, p_s^*)$ we have

$$\begin{aligned}\int_{\mathbb{R}^N} |u_m^+ - u^+|^\nu \, dx &= \int_{\mathbb{R}^N} \left| \frac{|u_m| - |u|}{2} - \frac{(u_m - u)}{2} \right|^\nu dx \\ &\le \int_{\mathbb{R}^N} |u_m - u|^\nu \, dx \to 0 \text{ as } m \to \infty.\end{aligned}$$

Therefore, the claim holds. Combining (3.4) with (4.22), yields

$$\int_{\mathbb{R}^N} (\mathcal{K}_\mu * (u_m^+)^q)(u_m^+)^q \, dx \to \int_{\mathbb{R}^N} (\mathcal{K}_\mu * (u^+)^q)(u^+)^q \, dx, \;\; \text{as } m \to \infty. \tag{4.23}$$

On the other hand, by (4.22) and Lemma 2.6 we get

$$\begin{aligned}&\int_{\mathbb{R}^N} (\mathcal{K}_\mu * (u_m^+)^{q-1})(u_m^+)^{q-1} w \, dx \\ &\to \int_{\mathbb{R}^N} (\mathcal{K}_\mu * (u^+)^{q-1})(u^+)^{q-1} w \, dx, \;\; \text{as } m \to \infty, \quad \forall w \in L^{rq}(\mathbb{R}^N).\end{aligned} \tag{4.24}$$

Now, let us consider the linear operator $L : W \to W'$ defined by

$$\langle L(u), v \rangle = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+sp}} \, dxdy.$$

From here on we set $p' = \frac{p-1}{p}$ is the Hölder conjugate of $p$. In view of (4.15) we derive

$$\left\{ \frac{|u_m(x,t)-u_m(y,t)|^{p-2}(u_m(x,t)-u_m(y,t))}{|x-y|^{\frac{N+sp}{p'}}} \right\}_m \quad \text{is bounded in } L^{p'}(\mathbb{R}^{2N}),\ \forall t \in [0,T],$$

and

$$\frac{|u_m(x,t)-u_m(y,t)|^{p-2}(u_m(x,t)-u_m(y,t))}{|x-y|^{\frac{N+sp}{p'}}} \to \frac{|u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t))}{|x-y|^{\frac{N+sp}{p'}}} \quad \text{a.e. in } \mathbb{R}^{2N},\quad \forall t \geq 0.$$

Invoking Lemma (2.4), we deduce

$$\frac{|u_m(x,t)-u_m(y,t)|^{p-2}(u_m(x,t)-u_m(y,t))}{|x-y|^{\frac{N+sp}{p'}}} \rightharpoonup \frac{|u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t))}{|x-y|^{\frac{N+sp}{p'}}} \quad \text{weakly in } L^{p'}(\mathbb{R}^{2N}). \tag{4.25}$$

Thus, for any $v \in W$ we have

$$\frac{v(x)-v(y)}{|x-y|^{\frac{N+sp}{p}}} \in L^p(\mathbb{R}^{2N}),$$

which implies, as $m \to \infty$

$$\langle L(u_m(t)), v\rangle \to \langle L(u(t)), v\rangle,\quad \forall v \in W,\quad \forall t \in [0,T]. \tag{4.26}$$

Similarly, as $\{V^{\frac{p-1}{p}}(x)|u_m|^{p-2}u_m\}_m$ is bounded in $L^{p'}(\mathbb{R}^N)$ and

$$V^{\frac{p-1}{p}}(x)|u_m|^{p-2}u_m \to V^{\frac{p-1}{p}}(x)|u|^{p-2}u \quad \text{a.e. in } \mathbb{R}^N,\ t \geq 0,$$

we obtain

$$V^{\frac{p-1}{p}}(x)|u_m|^{p-2}u_m \rightharpoonup V^{\frac{p-1}{p}}(x)|u|^{p-2}u \quad \text{weakly in } \ L^{p'}(\mathbb{R}^N),$$

For any $w \in W$, we have

$$V^{\frac{1}{p}}w \in L^p(\mathbb{R}^N).$$

Hence, as $m \to \infty$

$$\int_{\mathbb{R}^N} V(x)|u_m|^{p-2}u_m w\,dx \to \int_{\mathbb{R}^N} V(x)|u|^{p-2}uw\,dx, \quad \forall w \in W. \tag{4.27}$$

Combining (4.26) with (4.19) and using the Lebesgue dominated convergence theorem, yields

$$\int_0^T \|u_m(t)\|^{\theta(p-1)}\langle L(u_m(t)), v\rangle\,dt \to \int_0^T \beta(t)\langle L(u(t)), v\rangle\,dt, \quad \forall v \in W. \tag{4.28}$$

Integrating (4.1) with respect to $t$, we obtain

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^N} u_{mt}w_j\,dxdt + \int_0^T \|u_m\|^{p(\theta-1)}\langle u_m, w_j\rangle\,dt\\ &\quad= \int_0^T\int_{\mathbb{R}^N}(\mathcal{K}_\mu * |u_m^+|^q)|u_m^+|^{q-2}u_m^+ w_j\,dxdt, \ \ j = 1, 2, \ldots, m.\end{aligned} \tag{4.29}$$

In view of (4.17), (4.24) and (4.28), for $j$ fixed, we can pass to the limit in (4.29) to get

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^N} u_t w_j\,dxdt + \int_0^T \beta(t)\langle u, w_j\rangle\,dt\\ &\quad= \int_0^T\int_{\mathbb{R}^N}(\mathcal{K}_\mu * |u^+|^q)|u^+|^{q-2}u^+ w_j\,dxdt, \ \ j = 1, 2, \ldots, m.\end{aligned} \tag{4.30}$$

Since $\mathcal{V} = \{w_i,\ i \in \mathbb{N}\}$ is dense in $W$, we infer from (4.30) that

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^N} u_t v\,dxdt + \int_0^T \beta(t)\langle u, v\rangle\,dt\\ &\quad= \int_0^T\int_{\mathbb{R}^N}(\mathcal{K}_\mu * (u^+)^q)(u^+)^{q-1} v\,dxdt, \ \forall v \in W,\end{aligned} \tag{4.31}$$

which implies

$$\begin{aligned}&\int_{\mathbb{R}^N} u_t(t)v\,dx + \beta(t)\langle u(t), v\rangle\\ &\quad= \int_{\mathbb{R}^N}(\mathcal{K}_\mu * (u^+(t))^q)(u^+(t))^{q-1} v\,dx, \ \ \text{a.e. } t \in [0, T], \ \forall v \in W.\end{aligned} \tag{4.32}$$

If $\beta(t) = 0$. Then by (4.19), we deduce that the equality (1) in the Definition 1.1 clearly holds.

Assume $\beta(t) > 0$ for all $t \in [0, T]$. Taking $u = v$ in (4.31) and using [33, proposition 2.1], we get

$$\|u(T)\|_2^2 - \|u_0\|_2^2 + \int_0^T \beta(t)\|u(t)\|^p\,dt = \int_0^T \int_{\mathbb{R}^N} (\mathcal{K}_\mu * (u^+)^q)(u^+)^q\,dxdt. \tag{4.33}$$

On the other hand, multiplying both sides of equations in (4.1) by $g_{jm}(t)$, and summing with respect to $i$, afterward integrating over $(0, T)$ yields

$$\begin{aligned} &\|u_m(T)\|_2^2 - \|u_{0m}\|_2^2 + \int_0^T \|u_m\|^{(\theta-1)p}\langle u_m, u_m\rangle\,dt \\ &= \int_0^T \int_{\mathbb{R}^N} (\mathcal{K}_\mu * (u_m^+)^q)(u_m^+)^q\,dxdt. \end{aligned} \tag{4.34}$$

Combining (4.33) with (4.34), and using (4.17) we obtain

$$\begin{aligned} &\lim_{m\to} \int_0^T \|u_m\|^{p(\theta-1)}\langle u_m, u_m\rangle\,dt \\ &= -\|u(T)\|_2^2 - \|u_0\|_2^2 + \int_0^T \int_{\mathbb{R}^N} (\mathcal{K}_\mu * (u^+)^q)(u^+)^q\,dxdt \\ &= \int_0^T \beta(t)\|u(t)\|^p\,dt \\ &= \lim_{m\to\infty} \int_0^T \|u_m\|^{p(\theta-1)}\langle u_m, u\rangle\,dt. \end{aligned}$$

Thus, we deduce that

$$\lim_{m\to\infty} \int_0^T \|u_m\|^{p(\theta-1)}\langle u_m, u_m - u\rangle\,dt = 0. \tag{4.35}$$

Hence, there exists a subsequence, still denoted by $\{u_m\}$ such that for a.e. $t \in [0, T]$

$$\lim_{m\to\infty} \left(\|u_m(t)\|^{p(\theta-1)}\langle u_m(t), u_m(t) - u(t)\rangle\right) = 0.$$

By (4.19), as $m \to \infty$ we know that

$$\|u_m(t)\|^{p(\theta-1)} \to \beta(t) > 0, \quad \text{a.e. } t \in [0, T], \tag{4.36}$$

which implies

$$\lim_{m\to\infty} \|u_m(t)\|^p = \|u(t)\|^p \quad \text{a.e. } t \in [0, T].$$

Obviously, $\lim_{m\to\infty} \|u_m(t)\|^{p(\theta-1)} = \|u(t)\|^{p(\theta-1)}$ a.e. $t \in [0, T]$. Therefore, we obtain

$$\beta(t) = \|u(t)\|^{p(\theta-1)} \quad \text{a.e. } t \in [0, T]. \tag{4.37}$$

Inserting (4.37) in (4.32), yields

$$\begin{aligned} &\int_{\mathbb{R}^N} u_t(t) v\, dx + \|u(t)\|^{p(\theta-1)} \langle u(t), v\rangle \\ &\quad = \int_{\mathbb{R}^N} (\mathcal{K}_\mu * (u^+(t))^q)(u^+(t))^{q-1} v\, dx, \quad \text{a.e. } t \in [0, T],\ \forall v \in W. \end{aligned} \tag{4.38}$$

It remains to show that inequality (1.12) holds. From (3.5), (4.16), (4.24), (4.19) and $u_{0m} \to u_0$ in $W$, we infer

$$\begin{aligned} &\int_0^t \|u_s(s)\|_2^2\, ds + E(u(t)) \le \liminf_{m\to\infty} \left( \int_0^t \|u_{ms}(s)\|_2^2\, ds + E(u_m(t)) \right) \\ &\quad = \lim_{m\to\infty} E(u_{0m}) = E(u_0). \end{aligned}$$

**Corollary 4.1** *If $u_0(x) \ge 0$ a.e. in $\mathbb{R}^N$. Then the function $u(x, t) \ge 0$ a.e. in $\mathbb{R}^N$ and for any $t \in [0, T]$.*

***Proof*** Let $u$ be a weak solution to initial value problem (1.1). Clearly

$$u^- = \max\{-u, 0\} \in L^\infty(0, T; W).$$

Taking $v = -u^-$ in (4.38), we obtain

$$\begin{aligned} &-\int_{\mathbb{R}^N} u_t(t) u^-(t)\, dx + \|u(t)\|^{p(\theta-1)} \langle u(t), -u^-(t)\rangle \\ &\quad = -\int_{\mathbb{R}^N} (\mathcal{K}_\mu * (u^+(t))^q)(u^+(t))^{q-1} u^-(t)\, dx, \end{aligned} \tag{4.39}$$

Observe that for a.e. $x, y \in \mathbb{R}^N$,

$$\begin{aligned} &|u(x) - u(y)|^{p-2}(u(x) - u(y))(-u^-(x) + u^-(y)) \\ &\quad = |u(x) - u(y)|^{p-2}(u^-(x) - u^-(y))^2 \\ &\qquad + |u(x) - u(y)|^{p-2} u^+(y) u^-(x) + |u(x) - u(y)|^{p-2} u^-(y) u^+(x) \\ &\quad \ge |u^-(x) - u^-(y)|^p, \end{aligned}$$

and

$$-V(x)|u(x)|^{p-2} u(x) u^-(x) = V(x)|u(x)|^{p-2}(u^-(x))^2 \ge V(x)|u^-|^p.$$

Furthermore, from the definition of $u^+$ and $u^-$, we get

$$\int_{\mathbb{R}^N} (\mathcal{K}_\mu * (u^+(t))^q)(u^+(t))^{q-1} u^-(t)\, dx = 0.$$

Combining these facts with (4.39), yields

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} (u^-(t))^2\, dx + \|u^-(t)\|^{p\theta} \le 0,$$

which implies

$$\int_{\mathbb{R}^N} (u^-(t))^2\, dx \le \int_{\mathbb{R}^N} (u_0^-)^2\, dx, \quad \forall t \in [0, T].$$

Since $u_0(x) \ge 0$ a.e. in $\mathbb{R}^N$ we get $u^-(t) = 0$ a.e. in $\mathbb{R}^N$ and for any $t \in [0, T]$. Hence, $u(x, t) \ge 0$ a.e. in $\mathbb{R}^N$ and for any $t \in [0, T]$. □

## 5 Global Existence and Asymptotic Behavior

This section is concerned with the proof of the existence of global weak solutions to initial value problem (1.1). Moreover, we show the asymptotic behavior of these solutions as $t \to +\infty$. In doing so, we introduce the potential well set

$$\mathcal{W} = \{u \in W\ : E(u) < d,\ I(u) > 0\} \cup \{0\}\,. \tag{5.1}$$

Similarly to the previous section, we will employ the Galerkin method.

***Proof*** (Theorem 1.2) In what follows we only need to consider the case where $u_0 \in \mathcal{W}\backslash\{0\}$. Otherwise if we assume $u_0 = 0$, then initial value problem (1.1) admits a global solution $u = 0$ and there is nothing to prove. From (4.12) remembering that

$$\int_0^t \|u_{ms}(s)\|_2^2\, ds + E(u_m(t)) = E(u_{0m}), \quad t \in [0, T_m), \tag{5.2}$$

where $T_m$ is the maximal existence time of solution $u_m$ to initial value problem (4.1)–(4.2). Since we are assuming that $u_0 \in \mathcal{W}\backslash\{0\}$, that is

$$E(u_0) < d \quad \text{and} \quad I(u_0) > 0.$$

It is follows from (4.3) that

$$E(u_{0m}) < d \quad \text{and} \quad I(u_{0m}) > 0, \tag{5.3}$$

for $m$ large enough. This combined with (5.2), yields

$$\int_0^t \|u_{ms}(s)\|_2^2\,ds + E(u_m(t)) < d, \quad t \in [0, T_m). \tag{5.4}$$

for $m$ large enough. Next we prove $u_m \in \mathcal{W}$ for sufficiently large $m$ and $T_m = +\infty$. Assume that $u_m(t) \notin \mathcal{W}$ and let $t_0$ be the smallest time for which $u_m(t_0) \notin \mathcal{W}$. By the continuity of $u_m(t)$ we obtain that $u_m(x, t_0) \in \partial\mathcal{W}$, i.e.

$$E(u_m(t_0)) = d. \tag{5.5}$$

or

$$u_m(t_0) \neq 0 \quad \text{and} \quad I(u_m(t_0)) = 0. \tag{5.6}$$

Obviously, the case (5.5) could not occur due to (5.4), while if (5.6) holds, then by definition of $d$ in (1.7) we infer that $E(u_m(t_0)) \geq d$. This is also impossible, since it contradicts (5.4).

In conclusion $u_m(t) \in \mathcal{W}$ for all $t \in [0, T_m)$ and sufficiently large $m$, so that

$$E(u_m(t)) < d \quad \text{and} \quad I(u_m(t)) > 0. \quad \forall t \in [0, T_m). \tag{5.7}$$

This combined with (1.6), for $m$ large enough we get

$$d > E(u_m(t)) = \frac{2q - \theta p}{2q\theta p}\|u_m(t)\|^{\theta p} + \frac{1}{2q} I(u_m(t)) > \frac{2q - \theta p}{2q\theta p}\|u_m(t)\|^{\theta p}, \tag{5.8}$$

which gives

$$\|u_m(t)\| < \left(\frac{2q\theta p}{2q - \theta p}\right)^{\frac{1}{\theta p}}, \quad \forall t \in [0, T_m). \tag{5.9}$$

Furthermore, from (5.4) we observe that

$$\int_0^t \|u_{ms}(s)\|_2^2\,ds < d, \quad \forall t \in [0, T_m). \tag{5.10}$$

In view of (5.9), we infer that $T_m = +\infty$. Consequently, by (5.9)–(5.10), there exist $u$ and a subsequence of $\{u_m\}_m$, still denoted by $\{u_m\}_m$ such that, as $m \to \infty$

$$\begin{cases} u_m \rightharpoonup^* u \ \text{ in } L^\infty(0, +\infty; W)), \\ u_{mt} \rightharpoonup u_t \ \text{ in } L^2(0, +\infty; L^2(\mathbb{R}^N)), \end{cases} \tag{5.11}$$

Hence, in a similar manner to the previous section, applying (5.11) we conclude that initial value problem (1.1) admits a global positive solution, that is

$$\begin{aligned}\int_{\mathbb{R}^N} u_t(t)v\,dx + \|u(t)\|^{p(\theta-1)}\langle u(t), v\rangle \\ = \int_{\mathbb{R}^N} (\mathcal{K}_\mu * u^q(t)u^{q-1}(t)v\,dx, \quad \text{a.e. } t \in (0,+\infty),\ \forall v \in W. \end{aligned} \tag{5.12}$$

Thus, this completes the proof of Theorem (1.2). □

We give now the proof of Corollary 1.3.

***Proof*** (Corollary 1.3) We divided the proof into two cases :

Case 1. $E(u_0) < d$ and $I(u_0) = 0$ In view of Remark 2 this case can not occurs.
Case 2. $E(u_0) = d$ and $I(u_0) \geq 0$ For $n = 2, 3, \ldots$ we let $\epsilon_n = 1 - \frac{1}{n}$, $u_{0n} = \epsilon_n u_0$ and consider the following problem

$$\begin{cases} u_t + \|u\|^{p(\theta-1)}[(-\Delta)^s_p u + V(x)|u|^{p-2}u] = (\mathcal{K}_\mu * |u|^q)|u|^{q-2}u & \text{in } \mathbb{R}^N \times (0,+\infty), \\ u(x,0) = u_{0n}(x), & \text{in } \quad \mathbb{R}^N, \end{cases} \tag{5.13}$$

According to the definition of $I$ we have

$$\begin{aligned} I(u_{0n}) &= \epsilon_n^{\theta p}\|u_0\|^{\theta p} - \epsilon_n^{2q}\int_{\mathbb{R}^N}(\mathcal{K}_\mu * u_0^q)u_0^q\,dx \\ &= \epsilon_n^{\theta p} I(u_0) + \epsilon_n^{2q}(\epsilon_n^{\theta p - 2q} - 1)\int_{\mathbb{R}^N}(\mathcal{K}_\mu * u_0^q)u_0^q\,dx \end{aligned}$$

In view of $I(u_0) \geq 0$ and $0 < \epsilon_n < 1$ it follows that

$$I(u_{0n}) > 0 \tag{5.14}$$

Moreover according to the definition of $E$ we know

$$\frac{d}{d\epsilon_n} E(\epsilon_n u_0) = \frac{1}{\epsilon_n} I(u_{0n}) > 0$$

Hence, this shows that $\epsilon_n \mapsto E(\epsilon_n u_0)$ is strictly increasing on $(0, 1)$, from where it follows that

$$E(u_{0n}) = E(\epsilon_n u_0) < E(u_0) = d. \tag{5.15}$$

On the other hand, it is easy to see that

$$u_{0n} \to u_0 \quad \text{in} \quad W, \quad \text{as } n \to \infty. \tag{5.16}$$

Combining (5.14)–(5.16) and by a similar way to the proof of Theorem 1.2, we obtain the existence of a global solution $u_n$ for initial value problem (5.13) satisfies that

$$\|u_n(t)\| < \left(\frac{2q\theta p}{2q-\theta p}\right)^{\frac{1}{\theta p}}, \quad \forall t \in [0, T_n), \tag{5.17}$$

and

$$\int_0^t \|u_{ns}(s)\|_2^2 \, ds < d, \quad \forall t \in [0, T_n), \tag{5.18}$$

for every $n \geq 2$. Therefore the reminder of the proof is quite similar to the proof of Theorem 1.1.

□

Now we turn our attention to the asymptotic behavior of global solutions as $t \to +\infty$. To this aim, we need to recall the following Lemma which due to Martinez [42].

**Lemma 5.1** *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function and $\sigma$ is a nonnegative constant such that*

$$\int_t^{+\infty} f^{1+\sigma}(s)\, ds \leq \frac{1}{\omega} f^\sigma(0) f(t), \quad \forall t \geq 0.$$

*then we have*

(1) $f(t) \leq f(0)e^{1-\omega t}$, *for all* $t \geq 0$, *whenever* $\sigma = 0$.
(2) $f(t) \leq f(0)\left(\frac{1+\sigma}{1+\omega\sigma t}\right)^{1/\sigma}$, *for all* $t \geq 0$, *whenever* $\sigma > 0$.

Moreover, we recall the fractional Gagliardo-Nirenberg interpolation inequality that can be found in [43] : if $\Omega$ is a standard domain (i.e. $\mathbb{R}^N$, a half-space or it is bounded with Lipschitz boundary) then

$$\begin{aligned} &\|v\|_{W^{\tau,p}(\Omega)} \leq C\|v\|^{\eta}_{W^{s_1,p_1}(\Omega)} \|v\|^{1-\eta}_{W^{s_2,p_2}(\Omega)}, \quad \eta \in (0,1), \\ &\tau = \eta s_1 + (1-\eta)s_2, \quad \frac{1}{p} = \frac{\eta}{p_1} + \frac{1-\eta}{p_2}. \end{aligned} \tag{5.19}$$

as long as it fails that : $s_2$ is an integer $\geq 1$ and $p_2 = 1$ and $s_1 - s_2 \leq 1 - \frac{1}{p_1}$.

***Proof*** (Theorem 1.4) Multiplying (4.1) by $g_{jm}$ and summing over $j = 1, 2, \ldots, m$ gives

$$-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} u_m^2(t)\, dx = I(u_m(t)). \tag{5.20}$$

In view of the proof of Lemma 3.2, we know that

$$I(u_m(t)) \geq \|u_m(t)\|^{\theta p}\left(1 - C(N,\mu,r)\Lambda_{qr}^{-2q}\|u_m(t)\|^{2q-\theta p}\right), \quad \forall t \geq 0.$$

On the other hand, from (1.17) for $m$ large enough

$$E(u_{0m}) < \left(\frac{2q-\theta p}{2q\theta p}\right)\left(\frac{\Lambda_{rq}^{2q}}{C(N,\mu,r)}\right)^{\frac{\theta p}{2q-\theta p}}. \tag{5.21}$$

Combining (5.2) with (5.8) yields

$$\left(\frac{2q\theta p}{2q-\theta p}E(u_{0m})\right)^{\frac{1}{\theta p}} > \|u_m(t)\|, \quad \forall t \geq 0,$$

which implies that

$$I(u_m(t)) \geq \|u_m(t)\|^{\theta p}\left(1 - C(N,\mu,r)\Lambda_{rq}^{-2q}\left(\frac{2q\theta p}{2q-\theta p}E(u_{0m})\right)^{\frac{2q-\theta p}{\theta p}}\right), \quad \forall t \geq 0. \tag{5.22}$$

Integrating (5.20) over $(t, T)$, we infer from the continuous embedding $W \hookrightarrow L^2(\mathbb{R}^N)$ and (5.22) that

$$\int_t^T \|u_m(t)\|_2^{\theta p}\,dt \leq \frac{1}{\chi_m}\|u_m(t)\|_2^2 \tag{5.23}$$

where $\chi_m = \left(1 - C(N,\mu,r)\Lambda_{rq}^{-2q}\left(\frac{2q\theta p}{2q-\theta p}E(u_{0m})\right)^{\frac{2q-\theta p}{\theta p}}\right)$. Using (5.9) and by a similar argument to that used in proving (4.17) we have

$$u_m \to u \quad \text{in} \quad C([0,T], L^\nu(\mathbb{R}^N)), \quad \forall \nu \in [2, p_s^*). \tag{5.24}$$

On the other hand, we infer from $E(u_{0m}) \to E(u_0)$ as $m \to \infty$ that

$$\chi_m \to \left(1 - C(N,\mu,r)\Lambda_{rq}^{-2q}\left(\frac{2q\theta p}{2q-\theta p}E(u_0)\right)^{\frac{2q-\theta p}{\theta p}}\right) \quad \text{as} \quad m \to \infty. \tag{5.25}$$

Combining (5.24) with (5.25) and passing to the limit in (5.23) as $m$ goes to infinity, we conclude

$$\int_t^T \|u(t)\|_2^{\theta p}\,dt \leq \frac{1}{\chi}\|u(t)\|_2^2 \quad \forall t \geq 0, \tag{5.26}$$

where $\chi = \left(1 - C(N,\mu,r)\Lambda_{qr}^{-2q}\left(\frac{2q\theta p}{2q-\theta p}E(u_{0m})\right)^{\frac{2q-\theta p}{\theta p}}\right)$. Letting $T \to +\infty$ in (5.26) and using Lemma 5.1, we infer that

$$\|u(t)\|_2 \le \left(\frac{\theta p}{1+\chi(\theta p-1)t}\right)^{\frac{1}{\theta p-1}}, \quad \forall t \ge 0. \tag{5.27}$$

Now, we can apply (5.19) with $s_1 = s$, $s_2 = 0$, $\eta = \frac{s-\epsilon}{s}$, $p_2 = p_1 = p$, to find

$$\|u(t)\|_{W^{s-\epsilon,p}(\mathbb{R}^N)} \le C\|u(t)\|^{\eta}_{W^{s,p}(\mathbb{R}^N)}\|u(t)\|_p^{1-\eta} \le \|u(t)\|^{\eta}\|u(t)\|_p^{1-\eta}. \tag{5.28}$$

From (5.9) and (4.16), it follows that

$$\|u(t)\| \le \liminf_{m\to\infty}\|u_m(t)\| \le \left(\frac{2q\theta p}{2q-\theta p}\right)^{\frac{1}{\theta p}}, \quad \forall t \ge 0. \tag{5.29}$$

Since $2 < p < p_s^*$, then by using the classical interpolation inequality and the continuous embedding $W \hookrightarrow L^{p_s}(\mathbb{R}^N)$ we obtain

$$\|u(t)\|_p \le \|u(t)\|_2^{\beta}\|u(t)\|_{p_s^*}^{1-\beta} \le \Lambda_{p_s^*}^{\beta-1}\|u(t)\|_2^{\beta}\|u(t)\|^{1-\beta}. \tag{5.30}$$

where $\beta \in (0,1)$ satisfies that

$$\frac{1}{p} = \frac{\beta}{2} + \frac{1-\beta}{p_s^*}.$$

By (5.27), (5.28), (5.29) and (5.30), we deduce

$$\begin{aligned}
\|u(t)\|_{W^{s-\epsilon,p}(\mathbb{R}^N)} &\le C\|u(t)\|^{\eta}_{W^{s,p}(\mathbb{R}^N)}\|u(t)\|_p^{1-\eta} \\
&\le \|u(t)\|^{\eta+(1-\eta)(1-\beta)}\left(\frac{\theta p}{1+\chi(\theta p-1)t}\right)^{\frac{\beta(1-\eta)}{\theta p-1}} \\
&\le \Lambda_{p_s^*}^{(\beta-1)(1-\eta)}\left(\frac{2q\theta p}{2q-\theta p}\right)^{\frac{\eta+(1-\eta)(1-\beta)}{\theta p}}\left(\frac{\theta p}{1+\chi(\theta p-1)t}\right)^{\frac{\beta(1-\eta)}{\theta p-1}},
\end{aligned}$$

for all $t \ge 0$. This completes the proof. □

## 6 Blow-up Phenomena

In this section we prove that the local weak solutions of initial value problem (1.1) blow up in finite time. Moreover, we give an estimate for the lower and upper bounds of the blow-up time.

In order to find an upper bound estimate for the blow-up time we need the following lemma, which is found in [44, Lemma 2.2].

**Lemma 6.1** *Suppose that a positive, twice differentiable function* $\psi$ *satisfies the inequality*

$$\psi''(t)\psi(t) - (1+\varsigma)(\psi'(t))^2 \geq 0$$

*where* $\varsigma > 0$. *If* $\psi(0) > 0$, $\psi'(0) > 0$, *then* $\psi(t) \to +\infty$ *as* $t \to t_*$ *and* $t_* \leq \frac{\psi(0)}{\varsigma\psi'(0)}$.

***Proof*** (Theorem 1.5) Hereafter, $T = T_{\max}$ is the maximal existence time of solutions to (1.1). Note that due to (1.20) we have $u_0 \neq 0$ and

$$\|u_0\| > \left(\frac{2q\Lambda_{rq}^{2q}}{\theta p C(N,\mu,r)}\right)^{\frac{1}{2q-\theta p}}. \tag{5.1}$$

Indeed, by the Hardy–Littlewood–Sobolev inequality

$$\frac{1}{\theta p}\|u_0\|^{\theta p} < \frac{1}{2q}C(N,\mu,r)\|u_0\|_{qr}^{2q} \leq \frac{1}{2q\Lambda_{rq}^{2q}}C(N,\mu,r)\|u_0\|^{2q},$$

which implies (5.2) holds. Moreover, from (1.12) and the condition that $E(u_0) < 0$ we have $E(u(t)) < 0$ for all $t \in [0,T)$. Thus in a similar way to (5.1) we infer that

$$\|u(t)\| > \left(\frac{2q\Lambda_{rq}^{2q}}{\theta p C(N,\mu,r)}\right)^{\frac{1}{2q-\theta p}}, \quad \forall t \in [0,T). \tag{5.2}$$

Upper bound Define

$$I(t) = \frac{1}{2}\int_0^t \|u(s)\|_2^2\,ds + \frac{T-t}{2}\|u_0\|_2^2 + \beta(\sigma+t)^2, \quad 0 < t < T.$$

where $\beta, \sigma > 0$ are to be determined. In view of the regularity of the weak local solutions stated in Definition 1.1 and [33, Proposition 1.2] we have

$$I''(t) = \int_{\mathbb{R}^N} u_t(t)u(t)\,dx + 2\beta.$$

Taking $u = v$ as a test function in Definition 1.1 yields

$$I''(t) = -\|u(t)\|^{\theta p} + \int_{\mathbb{R}^N}(\mathcal{K}_\mu * u^q(t))u^q(t)\,dx + 2\beta.$$

According to the definition of the energy functional

$$E(u(t)) = \frac{1}{\theta p}\|u(t)\|^{\theta p} - \frac{1}{2q}\int_{\mathbb{R}^N}(\mathcal{K}_\mu * u^q(t))u^q(t)\,dx,$$

we have

$$I''(t) = -\theta p E(u(t)) + \left(1 - \frac{\theta p}{2q}\right) \int_{\mathbb{R}^N} (\mathcal{K}_\mu * u^q(t))u^q(t)\,dx + 2\beta.$$

Since $2q > \theta p$, then the last equality implies

$$I''(t) \geq -(\lambda + 1)E(u(t)) + 2\beta,$$

where $\lambda = \theta p - 1 > 0$. In view of (1.12), we have

$$I''(t) \geq (\lambda + 1) \int_0^t \|u_s(t)\|^2\,ds - (\lambda + 1)E(u_0) + 2\beta. \tag{5.3}$$

Since

$$\begin{aligned} I'(t) &= \frac{1}{2} \int_{\mathbb{R}^N} u^2(s)\,dxds - \frac{1}{2} \int_{\mathbb{R}^N} u_0^2\,dx + 2\beta(\sigma + t) \\ &= \int_0^t \int_{\mathbb{R}^N} u_s(s)u(s)\,dxds + 2\beta(\sigma + t). \end{aligned}$$

It is follows that,

$$\begin{aligned} [I'(t)]^2 &\leq 2 \int_0^t \|u_s(s)\|_2^2\,ds \int_0^t \|u(s)\|_2^2\,ds + 8\beta^2(\sigma + t)^2 \\ &\leq \left(4 \int_0^t \|u_s(s)\|_2^2\,ds + 8\beta\right) I(t). \end{aligned} \tag{5.4}$$

Combining the above estimates, we find that for $\nu > 0$

$$\begin{aligned} &I''(t)I(t) - \frac{(\lambda + 1)}{4}[I'(t)]^2 \\ &\geq I(t)\left[\left((\lambda + 1) - (\lambda + 1)\right) \int_0^t \|u_s(s)\|_2^2\,ds - (\lambda + 1)E(u_0) - 2\beta\lambda\right] \\ &= I(t)\left[-(\lambda + 1)E(u_0) - 2\beta\lambda\right] \geq 0, \end{aligned}$$

provided $\beta \in \left(0, \frac{-\theta p E(u_0)}{2(\theta p - 1)}\right]$. Since $\theta \in (2, N/(N - sp))$ and $p > 2$ it follows that $\theta p > 4$. Invoking Lemma 6.1, yields

$$\lim_{t \to T^-} I(t) = +\infty, \tag{5.5}$$

and

$$T \leq \frac{4I(0)}{(\theta p - 4)I'(0)} = \frac{2(T\|u_0\|_2^2 + 2\beta\sigma^2)}{2\beta\sigma(\theta p - 4)} = \frac{\|u_0\|_2^2}{\beta\sigma(\theta p - 4)}T + \frac{\sigma}{(\theta p - 4)}.$$

This inequality can be rewritten as

$$T\left(1-\frac{\|u_0\|_2^2}{\beta\sigma(\theta p-4)}\right)\leq\frac{\sigma}{(\theta p-4)}. \tag{5.6}$$

for any $\beta \in \left(0, \frac{-\theta pE(u_0)}{2(\theta p-1)}\right]$ and $\sigma > 0$. Now if we choose $\sigma \in \left(\frac{\|u_0\|_2^2}{\beta(\theta p-4)}, +\infty\right)$ we infer that

$$0<\frac{\|u_0\|_2^2}{\sigma\beta(\theta p-4)}<1.$$

This combined with (5.6) yields

$$T\leq\frac{\sigma}{(\theta p-4)}\left(1-\frac{\|u_0\|_2^2}{\beta\sigma(\theta p-4)}\right)^{-1}=\frac{\beta\sigma^2}{\beta\sigma(\theta p-4)-\|u_0\|_2^2}. \tag{5.7}$$

In what follows for the sake of simplicity we put $\kappa=\frac{-\theta pE(u_0)}{2(\theta p-1)}$. Let us consider the following set

$$\begin{aligned}\Sigma&=\left\{(\beta,\sigma):\ \beta\in(0,\kappa],\ \sigma\in\left(\frac{\|u_0\|_2^2}{\beta(\theta p-4)},+\infty\right)\right\}\\&=\left\{(\beta,\sigma):\ \sigma\in\left(\frac{\|u_0\|_2^2}{\kappa(\theta p-4)},+\infty\right),\ \beta\in\left(\frac{\|u_0\|_2^2}{\sigma(\theta p-4)},\kappa\right]\right\}\end{aligned}$$

In view of (5.7), we have

$$T\leq\inf_{(\beta,\sigma)\in\Sigma}\frac{\beta\sigma^2}{\beta\sigma(\theta p-4)-\|u_0\|_2^2}.$$

Set

$$g(\sigma,\tau)=\frac{\tau\sigma}{\tau(\theta p-4)-\|u_0\|_2^2}.$$

where $\tau=\beta\sigma$. Through straightforward computation one can show that $g(\sigma,\tau)$ is decreasing in $\tau$. Thus, it follows that

$$T\leq\inf_{\sigma\in\left(\frac{\|u_0\|_2^2}{\kappa(\theta p-4)},+\infty\right)}\frac{\kappa\sigma^2}{\sigma\kappa(\theta p-4)-\|u_0\|_2^2}. \tag{5.8}$$

Minimizing the right hand side of (5.8) one has

$$T \leq \frac{4\|u_0\|_2^2}{\kappa(\theta p - 4)^2}. \tag{5.9}$$

Now by substituting the value of $\kappa$ in (5.9) we get

$$T \leq \frac{8\|u_0\|_2^2(\theta p - 1)}{-\theta p E(u_0)(\theta p - 4)^2}. \tag{5.10}$$

According to the definition of $I$ and (5.5), we have

$$\lim_{t\to T^-} \|u(t)\|_2^2 = +\infty. \tag{5.11}$$

Lower bound Now we turn to prove the second part. Taking $u = v$ as a test function in Definition 1.1 we get

$$\frac{d}{dt}\int_{\mathbb{R}^N} u^2(t)\,dx = -2\|u(t)\|^{\theta p} + 2\int_{\mathbb{R}^N} (\mathcal{K}_\mu * u^q(t))u^q(t)\,dx.$$

Using the Hardy–Littlewood–Sobolev inequality we deduce

$$\frac{d}{dt}\int_{\mathbb{R}^N} u^2(t)\,dx \leq -2\|u(t)\|^{\theta p} + 2C(N,\mu,r)\|u(t)\|_{rq}^{2q}.$$

Since $2 < rq < p_s^*$, it follows from the interpolation inequality that

$$\frac{d}{dt}\int_{\mathbb{R}^N} u^2(t)\,dx \leq -2\|u(t)\|^{\theta p} + 2C(N,\mu,r)\|u(t)\|_2^{2q\gamma}\,\|u(t)\|_{p_s^*}^{2q(1-\gamma)},$$

where $\gamma \in (0,1)$ satisfies

$$\frac{1}{qr} = \frac{\gamma}{2} + \frac{1-\gamma}{p_s^*}.$$

Now, by using a similar argument that in the proof of Theorem 1.1, for any $\varepsilon > 0$ we have

$$\frac{d}{dt}\int_{\mathbb{R}^N} u^2(t)\,dx \leq -2\|u(t)\|^{\theta p} + C(\varepsilon)\left(\|u(t)\|_2^2\right)^\alpha + \varepsilon\|u(t)\|^{\theta p} \tag{5.12}$$

where

$$\alpha = \frac{\theta p q\gamma}{\theta p - 2q(1-\gamma)} > 1,$$

and

$$C(\varepsilon)=\varepsilon^{-\frac{2\theta^2p^2}{\theta p-2q(1-\gamma)}}\Lambda_{p_s^*}^{-\frac{2q(1-\gamma)\theta p}{\theta p-2q(1-\gamma)}}.$$

Put

$$g(t)=\int_{\mathbb{R}^N}u^2(t)\,dx=\|u(t)\|_2^2.$$

Then (5.12) can be rewritten as

$$\frac{dg(t)}{dt}\leq -2\|u(t)\|^{\theta p}+C(\varepsilon)g^{\alpha}(t)+\varepsilon\|u(t)\|^{\theta p}.$$

Taking $\varepsilon=2$ we arrive at inequality

$$\frac{dg(t)}{dt}\leq Cg^{\alpha}(t).$$

Integrating this inequality from 0 to $t$ yields

$$\frac{1}{C}\int_{g(0)}^{g(t)}\frac{1}{\tau^{\alpha}}\,d\tau\leq t.$$

Thus, it turns out that

$$\frac{1}{C(1-\alpha)}g^{1-\alpha}(t)+\frac{1}{C(\alpha-1)}g^{1-\alpha}(0)\leq t,$$

which together with $\lim_{t\to T^-}g(t)=\lim_{t\to T^-}\|u(t)\|_2^2=+\infty$ and $\alpha>1$ give

$$\frac{1}{C(\alpha-1)}g^{1-\alpha}(0)\leq T, \tag{5.13}$$

where $g(0)=\|u_0\|_2^2>0$ and $C=2^{-\frac{2\theta^2p^2}{\theta p-2q(1-\gamma)}}\Lambda_{p_s^*}^{-\frac{2q(1-\gamma)\theta p}{\theta p-2q(1-\gamma)}}$. Hence from (5.10) and (5.13) we reach the conclusion.

□

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## References


1. Caffarelli, L.: Nonlocal equations, drifts and games. Nonlinear Partial Differ. Equ. Abel Symp. **7**, 37–52 (2012)
2. Gilboa, G., Osher, S.: Nonlocal operators with applications to image processing. Multiscale Model. Simul **7**, 1005–1028 (2008)
3. Applebaum, D.: Lévy processes from probability to finance and quantum groups. Notices Am. Math. Soc. **51**, 1336–1347 (2004)
4. Vázquez, J.L. : The evolution fractional p-Laplacian equation in $\mathbb{R}^N$. Fundamental solution and asymptotic behaviour. Nonlinear Anal. **199**, 112034. arXiv: 2004.05799
5. Mingqi, X., Rădulescu, D.V., Zhang, B.L.: Nonlocal Kirchhoff diffusion problems: local existence and blow-up of solutions. Nonlinearity **31**, 3228–3250 (2018)
6. Vázquez, J.L.: Recent progress in the theory of nonlinear diffusion with fractional Laplacian operators. Discret. Contin. Dyn. Syst. Ser. S **7**, 857–885 (2014)
7. Willem, M.: Minimax Theorems. Birkhäuser, Boston (1996)
8. Pekar, S.: Untersuchung über die Elektronentheorie der Kristalle. Akademie Verlag, Berlin (1954)
9. Penrose, R.: On gravity's role in quantum state reduction. Genet. Relativ. Gravit. **28**(5), 581–600 (1996)
10. Alves, C.O., Figueiredo, G.M., Yang, M.: Multiple semiclassical solutions for a nonlinear Choquard equation with magnetic field. Asymptot. Anal. **96**(2), 135–159 (2016)
11. Alves, C.O., Yang, M.: Existence of semiclassical ground state solutions for a generalized Choquard equation. J. Differ. Equ. **257**(11), 4133–4164 (2014)
12. Liang, S., Pucci, P., Zhang, B.: Multiple solutions for critical Choquard–Kirchhoff type equations. Adv. Nonlinear Anal. **10**(1), 400–419 (2021)
13. Pucci, P., Xiang, M., Zhang, B.: Existence results for Schrdinger–Choquard–Kirchhoff equations involving the fractional $p$-Laplacian. Adv. Calc. Var. **12**(3), 253–275 (2019)
14. Gobbino, M.: Quasilinear degenerate parabolic equations of Kirchhoff type. Math. Methods Appl. Sci. **22**(5), 375–388 (1999)
15. Ghisi, M., Gobbino, M.: Kirchhoff equations from quasi-analytic to spectral-gap data. Bull. Lond. Math. Soc. **43**(2), 374–385 (2011)
16. Ghisi, M., Gobbino, M.: Spectral gap global solutions for degenerate Kirchhoff equations. Nonlinear Anal. **71**(9), 4115–4124 (2009)
17. Ghisi, M., Gobbino, M.: Kirchhoff equations with strong damping. J. Evol. Equ. **16**, 441–482 (2016)
18. Bisci, G.M., Rădulescu, V., Servadei, R.: Variational Methods for Nonlocal Fractional Equations. Encyclopedia of Mathematics and its Applications , vol. 162. Cambridge University Press, Cambridge (2016)
19. Papadopulos, P.G., Stavrakakis, N.M.: Global existence and blow-up results for an equations of Kirchhoff type on $\mathbb{R}^N$. Top. Methods Nonlinear Anal. **17**, 91–109 (2001)
20. D'Ancona, P., Shibata, Y.: On global solvability for the degenerate Kirchhoff equation in the analytic category. Math. Methods Appl. Sci. **17**, 477–489 (1994)
21. D'Ancona, P., Spagnolo, S.: Nonlinear perturbations of the Kirchhoff equation. Commun. Pure Appl. Math. **47**, 1005–1029 (1994)
22. Gobbino, M.: Quasilinear degenerate parabolic equations of Kirchhoff type. Math. Meth. Appl. Sci. **22**, 375–388 (1999)
23. Han, Y., Gao, W., Sun, Z., Li, H.: Upper and lower bounds of blow-up time to a parabolic type Kirchhoff equation with arbitrary initial energy. Comput. Math. Appl. **76**(10), 2477–2483 (2018)
24. Han, Y., Li, Q.: Threshold results for the existence of global and blow-up solutions to Kirchhoff equations with arbitrary initial energy. Comput. Math. Appl. **75**(9), 3283–3297 (2018)
25. Alves, C.O., Boudjeriou, T.: Existence of solution for a class of nonvariational Kirchhoff type problem via dynamical methots. Nonlinear Anal. **197**, 1–17 (2020)
26. Pan, N., Pucci, P., Xu, R., Zhang, B.: Degenerate Kirchhoff-type wave problems involving the fractional Laplacian with nonlinear damping and source terms. J. Evol. Equ. **19**(3), 615–643 (2019)
27. Pan, N., Pucci, P., Zhang, B.: Degenerate Kirchhoff-type hyperbolic problems involving the fractional Laplacian. J. Evol. Equ. **18**(2), 385–409 (2018)
28. Fiscella, A., Valdinoci, E.: A critical Kirchhoff type problem involving a nonlocal operator. Nonliear Anal. **94**, 156–170 (2014)
29. Ding, H., Zhou, J.: Local existence, global existence and blow-up of solutions to a nonlocal Kirchhoff diffusion problem. Nonlinearity **33**(1046) (2020)

30. Pucci, P., Xiang, M., Zhang, B.: A diffusion problem of Kirchhoff type involving the nonlocal fractional p-Laplacian. Discret. Contin. Dyn. Syst. **37**, 4035–4051 (2017)
31. Fu, Y., Pucci, P.: On solutions of space-fractional diffusion equations by means of potential wells. Electron. J. Qual. Theory Differ. Equ. Paper No. 70 (2016)
32. Di Nezza, E., Palatucci, G., Valdinoci, E.: Hitchhiker's guide to the fractional Sobolev spaces. Bull. Sci. Math. **136**, 519–527 (2012)
33. Showalter, E.R.: Monotone operators in Banach spaces and nonlinear partial differential equations. In: Mathematical Surveys and Monographs, vol. 49. American Mathematical Society, Providence
34. Sattinger, D.H.: On global solution of nonlinear hyperbolic equations. Arch. Ration. Mech. Math. **30**, 148–172 (1968)
35. Boudjeriou, T.: Global existence and blow-up for the fractional p-Laplacian with logarithmic nonlinearity. Mediterr. J. Math. **17**, 162 (2020). https://doi.org/10.1007/s00009-020-01584-6
36. Alves, C.O., Boudjeriou, T.: Existence of solution for a class of heat equation in whole $\mathbb{R}^N$. Discret. Contin. Dyn. Syst. https://doi.org/10.3934/dcds.2021031
37. Lieb, E., Loss, M.: Analysis, 2nd ed., Grad. Stud. Math. 14, American Mathematical Society, Providence (2001)
38. Bogachev, V.I.: Measure Theory II, vol. I. Springer, Berlin (2007)
39. Simon, J.: Compact sets in the space $L^p(0, T, B)$. Ann. Math. Pura. Appl. (4) **146**, 65–96 (1987)
40. Pedregal, P.: Parametrized Measures and Variational Principles. Birkhäuser, Basel (1991)
41. Zheng, S.: Nonlinear Evolution Equations, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, vol. 133. Chapman & Hall/CRC, Boca Raton, FL (2004)
42. Martinez, P.: A new method to obtain decay rate estimates for dissipative systems. ESAIM Control Optim. Calc. Var. **4**, 419–444 (1999)
43. Brezis, H., Mironescu, P.: Gagliardo-Nirenberg inequalities and non-inequalities: the full story. Ann. l'Inst. Henri Poincare (C) Anal Non Lineaire **35**(5), 1355–1376 (2018)
44. Han, Y., Gao, W., Sun, Z., Li, H.: Upper and lower bounds of blow-up time to a parabolic type Kirchhoff equation with arbitrary initial energy. Comput. Math. Appl. **76**(10), 2477–2483 (2018)