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Stability of a Timoshenko System with Localized Kelvin–Voigt Dissipation

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Abstract

We consider the Timoshenko beam with localized Kelvin–Voigt dissipation distributed over two components: one of them with constitutive law of the type C^1, and the other with discontinuous law. The third component is simply elastic, where the viscosity is not effective. Our main result is that the decay depends on the position of the components. We will show that the system is exponentially stable if and only if the component with discontinuous constitutive law is not in the center of the beam. When the discontinuous component is in the middle, the solution decays polynomially.

Keywords Timoshenko beam · Localized viscoelastic dissipative mechanism · Transmission problem · Exponential stability · Polynomial decay

Mathematics Subject Classification 35B40 · 35P05 · 35Q74

1 Introduction

We consider a Timoshenko beam configured in the interval $]0, \ell[$, and divided into three components: an elastic part configured over the interval I_E, without any dissipative mechanism, and two viscous components, one of them configured over I_C has a C^1 constitutive law, the other viscous component over I_D with discontinuous constitutive law. These components can be distributed over any of the intervals $I_1 =]0, \ell_0[$, $I_2 =]\ell_0, \ell_1[$, $I_3 =]\ell_1, \ell[$. Denoting by $\widetilde{I} = I_1 \cup I_2 \cup I_3$, we consider

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$$\rho_1\,\varphi_{tt} - S_x = 0 \text{ in } \widetilde{I} \times (0,+\infty), \tag{1.1}$$

$$\rho_2\,\psi_{tt} - M_x + S = 0 \text{ in } \widetilde{I} \times (0,+\infty), \tag{1.2}$$

with initial conditions

$$\varphi(x,0)=\varphi_0(x),\ \ \varphi_t(x,0)=\varphi_1(x),\ \ \psi(x,0)=\psi_0(x),\ \ \psi_t(x,0)=\psi_1(x)\quad \text{in } (0,\ell), \tag{1.3}$$

and Dirichlet boundary conditions:

$$\varphi(0,t)=\varphi(\ell,t)=\psi(0,t)=\psi(\ell,t)=0 \qquad \text{in } (0,\ +\infty) \tag{1.4}$$

Here, S and M are given, respectively, by:

$$S=\kappa(\varphi_x+\psi)+\widetilde{\kappa}(\varphi_{xt}+\psi_t), \qquad M=b\psi_x+\tilde{b}\psi_{xt} \tag{1.5}$$

where ρ_1, ρ_2, κ, and b positive constants for simplicity. To see more details of the model, we refer to [15]. The functions $\tilde{\kappa}$ and $\tilde{b}$ are non negative, where $\tilde{\kappa}=\kappa_0(x)+\kappa_1(x)$, $\tilde{b}=b_0(x)+b_1(x)$. Here $\kappa_0, b_0 \in C^1(I_D)$ are discontinuous functions of the first kind over $]0,\ \ell[$, vanishing outside of I_D and positive inside I_D. Instead, $\kappa_1(x)$ and $b_1(x)$, are C^1 functions vanishing outside of I_C and positive inside I_C.

Finally, we consider the transmission conditions,

$$\varphi(\ell_i^-)=\varphi(\ell_i^+), \quad \psi(\ell_i^-)=\psi(\ell_i^+), \quad S(\ell_i^-)=S(\ell_i^+), \quad M(\ell_i^-)=M(\ell_i^+). \tag{1.6}$$

for $i=0,1$. Note that condition (1.6) implies $S, M \in H^1(0,\ell)$. If we have more points of discontinuity, the set $\widetilde{I}$ have to be modified.

To get the uniform rate of decay, we consider the following hypotheses (to be used in Lemma 3.3)

$$|b_1'(x)|^2 \le c|b_1(x)|, \qquad |\kappa_1'(x)|^2 \le c|\kappa_1(x)|, \quad \forall x \in \overline{I_C} \tag{1.7}$$

Additionally, we assume that there exists positive constants C_1 and C_2 such that

$$C_1\kappa_1(x) \le b_1(x) \le C_2\kappa_1(x) \tag{1.8}$$

As a typical example of a function $\widetilde{\kappa}(x)$, ($\widetilde{b}(x)$ is similar) is given in the following graphics
In the case of Fig. 1 we have not exponential stability, and in case of Figs. 2 and 3 the system is exponentially stable.

In [11], the authors consider the transmission problem of Timoshenko beam composed by N components, each of them being either purely elastic (**E**), or a Kelvin–Voigt viscoelastic material (discontinuous constitutive law **V**), or an elastic material inserted with a frictional damping mechanism (**F**). The authors prove that the Timoshenko model is exponentially stable if and only if all the elastic components are connected with one component with frictional damping. Otherwise, there is no exponential stability, but a polynomial decay of the energy as $1/t^2$. On the other hand, Liu and Liu

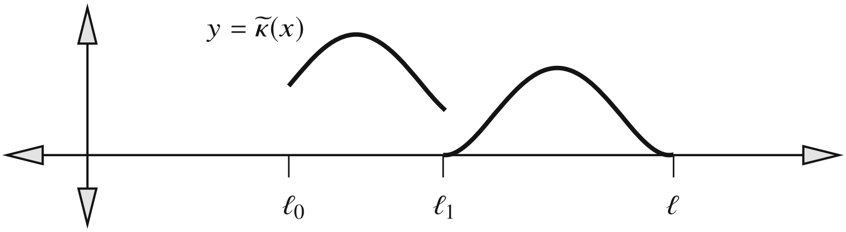

Fig. 1 The discontinuous component I_D is in the center of the beam

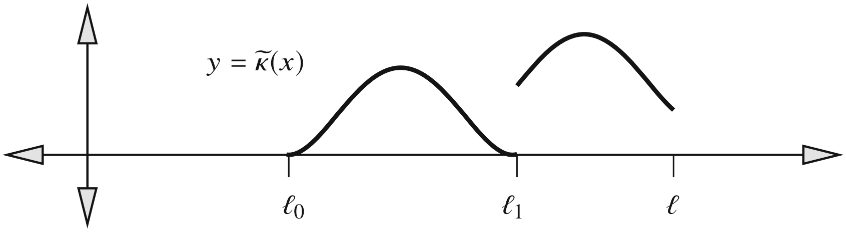

Fig. 2 Here the continuous component I_C is in the center of the beam

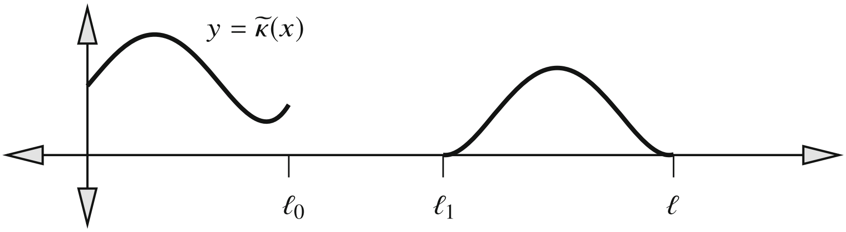

Fig. 3 The elastic component I_C is in the center of the beam

in [8] and Cheng et al. [3], proved that the wave equation with localized Kelvin–Voigt viscoelastic damping (with discontinuous constitutive law) is not exponentially stable. In [1] was proved that the corresponding semigroup decays polynomially to zero. On the other hand, Liu and Rao in [9] proved that when the localized viscoelastic damping has a C^1-constitutive law, then the corresponding semigroup is exponentially stable. Therefore, for localized viscoelastic damping, the regularity of the constitutive law is important and completely changes the asymptotic properties.

In this work we consider the two types of localized viscoelastic damping (continuous and discontinuous constitutive law) and we prove that the exponential stability depends on the order of the viscoelastic components of the beam. That is, we will show that the semigroup is exponentially stable if and only if the discontinuous component is not in the center of the beam. Furthermore, in case of lack of exponential stability, we show that the semigroup decays polynomially to zero.

The remainder part of this paper is organized as follows. In Sect. 2 we show the well-posedness of the model. In Sect. 3 we show the the exponential stability provided the discontinuous component is not in the center of the beam, and the polynomial stability, in case of the discontinuous component is in the center. Finally, in Sect. 4 we show the lack of exponential stability.

2 The Semigroup Approach

The energy of the system is given by:

$$E(t) = \frac{1}{2}\int_0^\ell \left(\rho_1|\varphi_t|^2 + \rho_2|\psi_t|^2 + b|\psi_x|^2 + \kappa|\varphi_x + \psi|^2\right) dx \tag{2.1}$$

Multiplying Eq. (1.1) by φ_t and Eq. (1.2) by ψ_t, summing up the product result we arrive to

$$\frac{d}{dt}E(t) = -\int_0^\ell \tilde{b}|\psi_{xt}|^2\,dx - \int_0^\ell \tilde{\kappa}|\varphi_{xt} + \psi_t|^2\,dx \le 0. \tag{2.2}$$

We denote by $\mathcal{H}$ the phase space given by:

$$\mathcal{H} = H_0^1(0,\ell)\times L^2(0,\ \ell)\times H_0^1(0,\ \ell)\times L^2(0,\ \ell)$$

For $U = (\varphi,\ \Phi,\ \psi,\ \Psi)^t$ we define

$$\|U\|_{\mathcal{H}}^2 = \int_0^\ell \rho_1|\Phi(s)|^2 + \rho_2|\Psi(s)|^2 + b|\psi_x(s)|^2 + \kappa|\varphi_x(s) + \psi(s)|^2\,ds$$

Taking $U = (\varphi, \psi, \varphi_t, \psi_t)^\top$, system (1.1)–(1.2) can be written as

$$U_t = AU$$

with $\mathcal{A}\colon D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is the linear operator defined by:

$$\mathcal{A}U = \left(\Phi\ ,\ \frac{1}{\rho_1}S_x\ ,\ \Psi\ ,\ \frac{1}{\rho_2}(M_x - S)\right)^\top$$

where M and S are given in (1.5). The domain is given by:

$$D(\mathcal{A}) = \{U \in \mathcal{H}\colon \Phi,\ \Psi \in H_0^1(0,\ \ell),\ \varphi,\ \psi \in H^2(I_E),\ S,\ M \in H^1(0,\ell)\}$$

Note the operator $\mathcal{A}$ is dissipative,

$$\mathrm{Re}\,\langle \mathcal{A}U,\ U\rangle = -\int_0^\ell \tilde{b}|\Psi_x|^2\,dx - \int_0^\ell \tilde{\kappa}|\Phi_x + \Psi|^2\,dx \le 0$$

So we have the following result.

Theorem 2.1 *The operator $\mathcal{A}$ defined by (2) is the infinitesimal generator of a contractions semigroup $S(t)$ over the space $\mathcal{H}$.*

Proof It is no difficult to show that $0 \in \rho(\mathcal{A})$. Hence, to use Theorem 1.2.4 in [10] to show the the desired result, we only need to prove that the domain $D(A)$ is dense. But this follows by using Theorem 4.6 Chapter 1 of [13], this because $\mathcal{H}$ is reflexive and $\mathcal{A}$ is a dissipative operator; thus, it is deduced that $D(\mathcal{A})$ is dense. We conclude that $\mathcal{A}$ is the infinitesimal generator of a contractions C_0-semigroup (see [4]). □

3 The Asymptotic Behavior

The main tool we use is the characterizations of the exponential and polynomial stabilization due to Prüss [14]–Huang [6]–Gearhart [5] and Borichev and Tomilov [2], respectively.

Theorem 3.1 *Let $S(t)$ be a contraction C_0-semigroup, generated by $\mathcal{A}$ over a Hilbert space $\mathcal{H}$. Then, in Prüss [14] is established that there exists $C, \gamma > 0$ verifying*

$$\|S(t)\| \le Ce^{-\gamma t} \quad \Leftrightarrow \quad i\,\mathbb{R} \subset \varrho(\mathcal{A}) \text{ and } \|(i\,\lambda\,I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \le M, \quad \forall\,\lambda \in \mathbb{R}. \tag{3.1}$$

To polynomial stability, Borichev and Tomilov [2] *result establish that there exists $C > 0$ such that*

$$\|\mathcal{S}(t)\mathcal{A}^{-1}\| \le \frac{C}{t^{1/\alpha}} \quad \Leftrightarrow \quad i\mathbb{R} \subset \varrho(\mathcal{A}) \text{ and } \|(i\lambda\,I - \mathcal{A})^{-1}\| \le M|\lambda|^{\alpha}, \quad \forall\lambda \in \mathbb{R} \tag{3.2}$$

Hence, to show the uniform rate of decay we use the resolvent equation, given by:

$$i\lambda U - \mathcal{A}U = F \tag{3.3}$$

Taking $U = (\varphi,\ \Phi,\ \psi,\ \Psi)^t$ and $F = (f_1,\ f_2,\ f_3,\ f_4)^t$ we can rewrite (3.3) as

$$i\lambda\varphi - \Phi = f_1 \tag{3.4}$$

$$i\rho_1\lambda\Phi - S_x = \rho_1 f_2 \tag{3.5}$$

$$i\lambda\psi - \Psi = f_3 \tag{3.6}$$

$$i\rho_2\lambda\Psi - M_x + S = \rho_2 f_4 \tag{3.7}$$

Lemma 3.1 $i\mathbb{R} \subseteq \rho(\mathcal{A})$

Proof Since $0 \in \rho(A)$, the set

$$\mathcal{N} = \{s \in \mathbb{R}^+ : \,]-is,\ is[\subset \rho(\mathcal{A})\}$$

is not empty. Let us denote by $\sigma = \sup\mathcal{N}$. If $\sigma = \infty$ we have that $i\mathbb{R} \subset \rho(\mathcal{A})$, hence there is nothing to prove. So, let us suppose that $\sigma < \infty$, we will arrive to a contradiction. This implies that $i\mathbb{R} \nsubseteq \rho(\mathcal{A})$. Then, exists a sequence $\{\lambda_n\} \subseteq \mathbb{R}$ such that $\lambda_n \to \sigma < +\infty$ and

$$\|(i\lambda_n I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \to +\infty$$

Hence, exists a sequence $\{f_n\} \subseteq \mathcal{H}$ with $\|f_n\|_{\mathcal{H}} = 1$ and $\|(i\lambda_n I - \mathcal{A})^{-1} f_n\|_{\mathcal{H}} \to \infty$. Denoting by:

$$\tilde{U}_n = (i\lambda_n I - \mathcal{A})^{-1} f_n, \quad U_n = \frac{\tilde{U}_n}{\|\tilde{U}_n\|} \quad F_n = \frac{f_n}{\|\tilde{U}_n\|}$$

we get:

$$i\lambda_n U_n - \mathcal{A}U_n = F_n \to 0 \tag{3.8}$$

Note that $\|\mathcal{A}U_n\| \leq C$. Therefore U_n is bounded in $D(\mathcal{A})$. This implies in particular that Ψ_n and Φ_n are bounded in $H^1(0,\ \ell)$ and also ψ and φ are bounded in $H^2(I_E)$; therefore, there exists a subsequence (we still denote in the same way) such that:

$$\Phi_n \to \Phi, \qquad \Psi_n \to \Psi, \qquad \text{strong in } L^2(0,\ \ell) \tag{3.9}$$

$$\varphi_{n,x} + \psi_n \to \varphi_x + \psi, \qquad \psi_{n,x} \to \psi, \qquad \text{strong in } L^2(I_E) \tag{3.10}$$

Taking inner product to (3.8)

$$i\lambda_n \|U_n\|^2 - \langle \mathcal{A}U_n, U_n\rangle = \langle F_n,\ U_n\rangle \to 0$$

and taking real part:

$$-\operatorname{Re}\langle \mathcal{A}U_n, U_n\rangle = \int_0^\ell (\tilde{b}|\Psi_x^n|^2 + \tilde{\kappa}|\Phi_x^n + \Psi^n|^2)\,dx \to 0 \tag{3.11}$$

That implies:

$$\Phi_{n,x} + \Psi_n \to 0, \qquad \Psi_{n,x} \to 0 \qquad \text{strong in } \in L^2(I_C \cup I_D)$$

Therefore we have

$$\varphi_{n,x} + \psi_n \to 0, \qquad \psi_{n,x} \to 0 \qquad \text{strong in } \in L^2(I_C \cup I_D) \tag{3.12}$$

From (3.9), (3.10) and (3.12), we get that $U_n \to U$ strongly in $\mathcal{H}$. Since $\mathcal{A}$ is closed, we conclude that U satisfies:

$$i\sigma U - \mathcal{A}U = 0$$

Moreover, using (3.12) into (3.5)–(3.7) we conclude that $\Phi = \Psi = 0$, so relations (3.4)–(3.6) implies that $\varphi = \psi = 0$, hence $U \equiv 0$ over $I_C \cup I_D$. Since $I_E = [\alpha,\ \beta]$ is linked to I_C or I_D on α or β, we get that $U(\alpha) = 0$ or $U(\beta) = 0$. So we have that over $]\alpha,\ \beta[$ it verifies that:

$$-\sigma^2\varphi + \kappa(\varphi_x + \psi) = 0, \qquad -\sigma^2\psi + b\psi_{xx} + \kappa(\varphi_x + \psi) = 0$$

with

$$\varphi(\alpha) = \psi(\alpha) = \varphi_x(\alpha) = \psi_x(\alpha) = 0$$

It is a second order initial value problem verifying $\varphi = \psi = 0$ over $]\alpha, \beta[$. From where it follows that $U \equiv 0$ on $\mathcal{H}$, which is a contradiction. This finish the proof.□

A key result that we are going to use in this work, is given by the following:

Lemma 3.2 *For $g \in H^1(a, b)$:*

$$\int_a^b |g|^2 \, dx \leq C \left| \int_a^b g \, dx \right|^2 + \int_a^b |g_x|^2 \, dx$$

Proof In fact, for any $a \leq x < y \leq b$ we have

$$g(y) - g(x) = \int_x^y g_x \, ds \quad \Rightarrow \quad (b-a)g(y) - \int_a^b g(x) \, dx = \int_a^b \int_x^y g_x \, ds \, dx,$$

therefore, taking absolute value

$$(b-a)|g(y)| \leq \left| \int_a^b g(x) \, dx \right| + (b-a) \int_a^b |g_x| \, dx,$$

Since $(b-a) \int_a^b |g_x| \, dx \leq (b-a)^{3/2} \left(\int_a^b |g_x|^2 \, dx \right)^{1/2}$, squaring and integrating once more over $[a, b]$ our conclusion follows. .□

The dissipativity of the operator $\mathcal{A}$ implies that

$$\int_{I_C} \kappa_1 |\Phi_x + \Psi|^2 + b_1 |\Psi_x|^2 \, dx + \int_{I_D} \kappa_0 |\Phi_x + \Psi|^2 + b_0 |\Psi_x|^2 \, dx \\ = \mathrm{Re}\,(U, F)_{\mathcal{H}} \leq \|U\| \|F\| \tag{3.13}$$

Lemma 3.3 *Let us suppose that condition* (1.7)–(1.8) *holds, then any solution of* (3.4)–(3.7) *satisfies*

$$\int_{I_C} \kappa_1 |\lambda \Phi|^2 + b_1 |\lambda \Psi|^2 \, dx \leq C_\varepsilon \|U\| \|F\| + C_\varepsilon \|F\|^2 + \varepsilon \|U\|^2$$

Proof The resolvent system over I_C is written as:

$$i\lambda\rho_1\Phi - [\kappa(\varphi_x + \psi)]_x - [\kappa_1(\Phi_x + \Psi)]_x = \rho_1 f_2, \quad \text{in} \quad I_C \tag{3.14}$$

$$i\lambda\rho_2\Psi - (b\psi_x)_x - (b_1\Psi_x)_x + \kappa(\varphi_x + \psi) + \kappa_1(\Phi_x + \Psi) = \rho_2 f_4, \quad \text{in} \quad I_C \tag{3.15}$$

Multiplying (3.14) by $\overline{i\lambda\kappa_1\Phi}$ and integrating over $I_C = [a, b]$

$$\int_a^b \rho_1\kappa_1|\lambda\Phi|^2\,dx = \int_a^b [\kappa(\varphi_x+\psi)+\kappa_1(\Phi_x+\Psi)]_x\overline{i\lambda\kappa_1\Phi}\,dx + \int_a^b \rho_1 f_2\overline{i\lambda\kappa_1\Phi}\,dx = \mathfrak{G}+\mathfrak{G}_0+\int_a^b \rho_1 f_2\overline{i\lambda\kappa_1\Phi}\,dx \tag{3.16}$$

where $\mathfrak{G} = \int_a^b[\kappa_1(\Phi_x+\Psi)]i\lambda\overline{(\kappa_1'\Phi+\kappa_1\Phi_x)}\,dx$ and $\mathfrak{G}_0 = \int_a^b[\kappa(\varphi_x+\psi)]i\lambda\overline{(\kappa_1'\Phi+\kappa_1\Phi_x)}\,dx$. Estimating $\mathfrak{G}$ (the estimation of $\mathfrak{G}_0$ is similar after using Eqs. (3.4) and (3.6))

$$\mathfrak{G} = \int_a^b[\kappa_1(\Phi_x+\Psi)]i\lambda\overline{(\kappa_1'\Phi+\kappa_1(\Phi_x+\Psi))}\,dx - \int_a^b[\kappa_1(\Phi_x+\Psi)]i\lambda\overline{(\kappa_1\Psi)}\,dx$$

Taking the real part of the above relation and using (3.13), we get:

$$\operatorname{Re}\mathfrak{G} = \operatorname{Re}\int_a^b[\kappa_1(\Phi_x+\Psi)]i\lambda\overline{(\kappa_1'\Phi)}\,dx - \operatorname{Re}\int_a^b[\kappa_1(\Phi_x+\Psi)]i\lambda\overline{(\kappa_1\Psi)}\,dx \le \epsilon\|\lambda\Phi\|^2+\epsilon\|\lambda\Psi\|^2+C_\epsilon\|U\|\|F\| \tag{3.17}$$

Similarly, using (3.4), (3.6) and (3.13), we get:

$$\operatorname{Re}\int_a^b[\kappa(\varphi_x+\psi)]i\lambda\overline{(\kappa_1'\Phi+\kappa_1\Phi_x)}\,dx = \operatorname{Re}\int_a^b[\kappa(\Phi_x+\Psi)+\kappa(f_{1,x}+f_3)]\overline{(\kappa_1'\Phi+\kappa_1\Phi_x)}\,dx \le \epsilon\int_a^b \|\Phi\|^2+\|\Psi\|^2\,dx + C_\epsilon\|U\|\|F\|+C_\epsilon\|F\|^2 \tag{3.18}$$

Thus, substitution of (3.17) and (3.18) into (3.16) yields

$$\int_a^b \kappa_1|\lambda\Phi|^2\,dx \le \epsilon\|\lambda\Phi\|^2+\epsilon\|\lambda\Psi\|^2+C_\epsilon\|U\|\|F\|+C_\epsilon\|F\|^2 \tag{3.19}$$

for $|\lambda| > 1$. Multiplying (3.15) by $\overline{i\lambda b_1\Psi}$ and using the same above procedure, we get

$$\int_a^b \rho_2 b_1|\lambda\Psi|^2\,dx \le \epsilon\|\lambda\Psi\|^2+C_\epsilon\|U\|\|F\|+C_\epsilon\|F\|^2$$

From the last two inequalities, our conclusion follows. □

Let us introduce the following notations

$$\mathcal{E}_\varphi = \frac{(\kappa q\rho_1)'}{2}|\Phi|^2 + \frac{q'}{2}|S|^2, \qquad \mathcal{I}_\varphi = \frac{\kappa q\rho_1}{2}|\Phi|^2 + \frac{q}{2}|S|^2 \tag{3.20}$$

$$\mathcal{E}_\psi = \frac{(bq\rho_2)'}{2}|\Psi|^2 + \frac{q'}{2}|M|^2, \qquad \mathcal{I}_\psi = \frac{bq\rho_2}{2}|\Phi|^2 + \frac{q}{2}|M|^2 \tag{3.21}$$

$$\mathcal{E} = \mathcal{E}_\varphi + \mathcal{E}_\psi, \qquad \mathcal{I} = \mathcal{I}_\varphi + \mathcal{I}_\psi \tag{3.22}$$

and

$$\mathcal{L} = \int_a^b \mathcal{E}(s)\,ds - \int_a^b \rho_1 q\Phi\overline{\psi}\,dx + \int_a^b \kappa q S\bar{M}\,dx \tag{3.23}$$

Taking $q(x) = \dfrac{\mathrm{e}^{nx} - \mathrm{e}^{na}}{n}$ we have $q'(x) = e^{nx} \gg q(x)$, for n large. Note that

$$\left|\int_a^b \rho_1 q\Phi\overline{\psi}\,dx\right| \le \frac{1}{n}\int_a^b \rho_1 q'\left|\Phi\overline{\psi}\right|\,dx \le \frac{c}{n}\int_a^b \mathcal{E}(s)\,ds$$

similarly we have

$$\left|\int_a^b \kappa q S\bar{M}\,dx\right| \le \frac{c}{n}\int_a^b \mathcal{E}(s)\,ds$$

Hence, for n large enough we have

$$C_0\int_a^b \mathcal{E}\,dx \le \mathcal{L} \le C_1\int_a^b \mathcal{E}\,dx \tag{3.24}$$

Remark 3.1 Recalling the definition of S and M we get

$$\int_a^b |S|^2\,dx \le c\int_a^b \kappa|\varphi_x + \psi|^2\,dx + \int_a^b |\widetilde{\kappa}(\Phi_x + \Psi)|^2\,dx$$

Using the dissipative properties

$$\int_a^b |S|^2\,dx \le c\int_a^b |\varphi_x + \psi|^2\,dx + c\|U\|\|F\|$$

Similarly

$$\int_a^b |M|^2\,dx \le c\int_a^b |\psi_x|^2\,dx + c\|U\|\|F\|$$

from where it follows that

$$\int_a^b |\Phi|^2 + |\varphi_x + \psi|^2 + |\Psi|^2 + |\psi_x|^2\,dx \le \int_a^b \mathcal{E}\,dx + c\|U\|\|F\|$$
$$\int_a^b \mathcal{E}\,dx \le c\int_a^b |\Phi|^2 + |\varphi_x + \psi|^2 + |\Psi|^2 + |\psi_x|^2\,dx + c\|U\|\|F\|$$

Lemma 3.4 *Over* $[a,b] \subset I_C \cup I_E$ *we have*

$$\left|\mathcal{L} - \mathcal{I}(s)\Big|_a^b\right| \le C_\varepsilon\|U\|\|F\| + C_\varepsilon\|F\|^2 + \varepsilon\|U\|^2$$

Instead over $I_D = [a,b]$

$$\left|\mathcal{L} - \mathcal{I}(s)\Big|_a^b\right| \le \varepsilon\|U\|^2 + C_\varepsilon|\lambda|^2\|U\|\|F\|^2 + \|F\|^2$$

Proof Multiplying (3.5) by $q\bar{S}$ and (3.7) by $q\bar{M}$ we have

$$i\lambda\rho_1\Phi q\bar{S} - S_x q\bar{S} = \rho_1 f_2 q\bar{S},$$
$$i\lambda\rho_2\Psi q\bar{M} - M_x q\bar{M} + qS\bar{M} = \rho_2 f_4 q\bar{M}.$$

The above equations implies

$$-\frac{\rho_1\kappa q}{2}\frac{d}{dx}|\Phi|^2 - \frac{q}{2}\frac{d}{dx}|S|^2 = \rho_1 f_2 q\bar{S} + \rho_1 q\Phi\kappa(\overline{f_{1,x} + f_3}) - i\lambda\rho_1 q\Phi(\kappa\overline{\psi}$$
$$+\tilde{\kappa}\overline{(\Phi_x + \Psi)})$$
$$-\frac{\rho_2 bq}{2}\frac{d}{dx}|\Psi|^2 - \frac{1}{2}q\frac{d}{dx}|M|^2 + qS\bar{M} = \rho_2 f_4 q\bar{M}$$
$$+\rho_2 q\Psi b\overline{f_{3,x}} - i\lambda\rho_2\Psi q[\overline{\tilde{b}\Psi_x}]$$

Summing up the two equations we get

$$-\frac{d}{dx}\mathcal{I}(x) + \mathcal{E}(x) = R_3 + \rho_1\kappa q\Phi\overline{\Psi} - qS\bar{M} \underbrace{-i\lambda\rho_1\tilde{\kappa}q\Phi\overline{(\Phi_x + \Psi)} - i\lambda\rho_2\tilde{b}q\Psi\overline{\Psi_x}}_{:=J(x)},$$

where $R_3 = \rho_1 f_2 q\bar{S} + \rho_1 q\Phi\kappa(\overline{f_{1,x} + f_3}) + \rho_2 f_4 q\bar{M} + \rho_2 q\Psi b\overline{f_{3,x}}$. Note that when $[a,b] \subset I_C \cup I_E$, from Lemma 3.3 we get

$$\left|\int_a^b J(x)\,dx\right| \le C_\varepsilon\|U\|\|F\| + C_\varepsilon\|F\|^2 + \varepsilon\|U\|^2 \tag{3.25}$$

Over I_D we get

$$\left|\int_{I_D} J(x)\,dx\right| \leq \varepsilon\|U\|^2 + C_\varepsilon|\lambda|^2\|U\|\|F\|^2 + \|F\|^2 \tag{3.26}$$

for λ large enough. After an integration using the above inequalities our conclusion follows. □

Now, we are in condition to establish our main result.

Theorem 3.2 *The system is exponentially stable if the viscous discontinuous part I_D is not in the center of the beam.*

Proof Since I_D is not in the middle then $0 \in I_D$ or $\ell \in I_D$; hence, because of the boundary conditions, Poincaré inequality is valid for Φ and Ψ. So we have

$$\int_{I_D} |\Psi|^2\,dx \leq C_p \int_{I_D} |\Psi_x|^2\,dx \leq C\|U\|\|F\| \tag{3.27}$$

Using that $i\lambda\psi = \Psi + f_3$ we get

$$\int_{\ell_1}^{\ell} |\psi_x|^2 + |\Psi|^2\,dx \leq C\|U\|\|F\| + C\|F\|^2$$

Using Poincare's and the triangular inequality, we get

$$\int_{I_D} |\Phi|^2\,dx \leq c\int_{I_D} |\Phi_x|^2\,dx \leq C\int_{I_D} \kappa|\Phi_x+\Psi|^2+|\Psi|^2\,dx \leq C\|U\|\|F\|+C\|F\|^2$$

So we have

$$\int_{I_D} |\Phi|^2 + |\psi_x|^2 + |\varphi_x + \psi|^2 + |\Psi|^2\,dx \leq C\|U\|\|F\| + C\|F\|^2$$

Integrating (3.5) and (3.7) over $[a, b] \subset I_C$, we get

$$i\lambda\rho_1 \int_a^b \Phi\,dx - S(b^-) + S(a^+) = \int_a^b \rho_1 f_2\,dx \tag{3.28}$$

$$i\lambda\rho_2 \int_a^b \Psi\,dx - M(b^-) + M(a^+) = \int_a^b \rho_2 f_4\,dx \tag{3.29}$$

From Lemma 3.4 we get

$$\left|\int_a^b \Phi\,dx\right| + \left|\int_a^b \Psi\,dx\right| \leq \frac{C}{|\lambda|}\|U\|^{1/2}\|F\|^{1/2} + \frac{C}{|\lambda|}\|U\| + \frac{C}{|\lambda|}\|F\| \tag{3.30}$$

$$\int_a^b |\Psi|^2\,dx \le c\left|\int_a^b \Psi\,dx\right|^2 + C\int_a^b b_1|\Psi_x|^2\,dx \le C\|U\|\|F\| + \frac{C}{|\lambda|^2}\|U\|^2 + \frac{C}{|\lambda|^2}\|F\|^2$$

Using (3.4), (3.6) and (3.13), we get

$$\int_a^b |\Phi|^2 + |\psi_x|^2 + |\varphi_x+\psi|^2 + |\Psi|^2\,dx \le C\|U\|\|F\| + \frac{C}{|\lambda|^2}\|U\|^2 + C\|F\|^2$$

From Lemma 3.4

$$\mathcal{I}(a) \le C\|U\|\|F\| + \varepsilon\|U\|^2 + C\|F\|^2 \tag{3.31}$$

Since I_D is not in the center, then $\overline{I_C} \cup \overline{I_E} = [0, \ell_2]$ or $I_C \cup I_E = [\ell_0, \ell]$. Let us assume the later case. Using the observability Lemma 3.4 once more

$$\begin{aligned}\int_{\ell_0}^a |\Phi|^2 + |\psi_x|^2 + |\varphi_x+\psi|^2 + |\Psi|^2\,dx &\le c\mathcal{I}(a) + C\|U\|\|F\| \\ &\quad + \frac{C}{|\lambda|^2}\|U\|^2 + C\|F\|^2 \\ &\le C\|U\|\|F\| + \varepsilon\|U\|^2 + C\|F\|^2\end{aligned} \tag{3.32}$$

for λ large. Using the observability over the interval $[a, \ell]$, we get

$$\int_a^\ell |\Phi|^2 + |\psi_x|^2 + |\varphi_x+\psi|^2 + |\Psi|^2\,dx \le C\|U\|\|F\| + \varepsilon\|U\|^2 + C\|F\|^2 \tag{3.33}$$

From (3.27), (3.32) and (3.33) we get

$$\begin{aligned}\|U\|^2 &= \int_0^\ell |\Phi|^2 + |\psi_x|^2 + |\varphi_x+\psi|^2 + |\Psi|^2\,dx \\ &\le C\|U\|\|F\| + \varepsilon\|U\|^2 + C\|F\|^2\end{aligned} \tag{3.34}$$

from where we get that $\|U\| \le C\|F\|$. So our conclusion follows. □

Finally, we finish this section showing the polynomial decay when the discontinuous viscous part is in the center of the beam. We use the result given in [2].

Theorem 3.3 *Suppose that the viscoelastic discontinuous part V_D is in the center of the beam. Then, the energy of the system decays polynomially, and:*

$$\|S(t)U_0\| \le Ct^{-1}\|U_0\|_{D(\mathcal{A})} \tag{3.35}$$

Proof Denoting $V_D = [\ell_0, \ell_1]$. Using (3.28) and (3.29) for $a = \ell_0$ and $b = \ell_1$ we have:

$$\int_{I_D} |\Phi|^2 + |\psi_x|^2 + |\varphi_x + \psi|^2 + |\Psi|^2\, dx \le C\|U\|\|F\| + \frac{C}{|\lambda|^2}\|U\|^2 + C\|F\|^2 \tag{3.36}$$

Using the same procedure as in Theorem 3.2, we get

$$\int_{I_C} |\Phi|^2 + |\psi_x|^2 + |\varphi_x + \psi|^2 + |\Psi|^2\, dx \le C\|U\|\|F\| + \frac{C}{|\lambda|^2}\|U\|^2 + C\|F\|^2 \tag{3.37}$$

Let us suppose that $\ell_1 \in \overline{I_E}$. Using Lemma 3.4 over $I_D =]\ell_0, \ell_1[$, we have

$$\mathcal{I}(\ell_1^+) \le \int_{I_D} |\Phi|^2 + |\psi_x|^2 + |\varphi_x + \psi|^2 + |\Psi|^2\, dx + \varepsilon\|U\|^2 + C_\varepsilon|\lambda|^2\|F\|^2 \tag{3.38}$$

Since $S(\ell_1^-) = S(\ell_1^+)$ and $M(\ell_1^-) = M(\ell_1^+)$ we have

$$\begin{aligned}\int_{I_E} |\Phi|^2 + |\psi_x|^2 + |\varphi_x + \psi|^2 + |\Psi|^2\, dx &\le \mathcal{I}(\ell_1^-) + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}\\ &\le \varepsilon\|U\|^2 + C_\varepsilon|\lambda|^2\|F\|^2 \end{aligned}\tag{3.39}$$

From the above inequality we get

$$\|U\|^2 \le C_\varepsilon|\lambda|^2\|F\|^2 + \varepsilon\|U\|^2$$

and the polynomial decay is a consequence of the Borichev-Tomilov theorem. □

4 Lack of Exponential Stability

Our starting point is the boundary estimate of the Timoshenko system.

$$\begin{aligned}\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x &= 0\\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) &= 0\end{aligned}\tag{4.1}$$

over some interval $[a, b]$, then we have

Theorem 4.1 *Let us suppose that the solution of system (4.1) $(\varphi, \varphi_t, \psi, \psi_t)$ is bounded in $C(0, T; [H^1(a, b) \times L^2(a, b)]^2)$. Then we have that*

$$\int_0^t \rho_1|\varphi_t(\alpha, \tau)|^2 + \rho_2|\psi_t(\alpha, \tau)|^2 + \kappa|\varphi_x(\alpha, \tau)|^2 + b|\psi_x(\alpha, \tau)|^2 d\tau \le C_E$$

for any $\alpha \in [a, b]$.

Proof The proof is well known now, we develop here only the main ideas for completeness. Multiplying $(4.1)_1$ by $q\varphi_x$ and $(4.1)_2$ by $q\psi_x$ to get

$$\frac{d}{dt}(\rho_1\varphi_t q\varphi_x) - q\rho_1\varphi_t\varphi_{tx} - q\kappa\varphi_{xx}\varphi_x = -\kappa\psi_x q\varphi_x$$
$$\frac{d}{dt}(\rho_2\psi_t q\psi_x) - q\rho_2\psi_t\psi_{tx} - qb\psi_{xx}\psi_x = -\kappa\varphi_x q\psi_x - \kappa\psi q\psi_x$$

Summing up the above inequalities we get

$$-\frac{q}{2}\frac{d}{dx}\left(\rho_1|\varphi_t|^2 + \kappa|\varphi_x|^2 + \rho_2|\psi_t|^2 + b\psi_x|^2\right) = -\kappa\psi q\psi_x$$
$$-q\frac{d}{dt}(\rho_1\varphi_t\varphi_x + \rho_2\psi_t\psi_x)$$

Integrating over $[\alpha, \beta] \times [0, t]$, with $\beta \in [a, b]$, and taking $q = x - \beta$, we get:

$$\left|\int_0^t\int_\alpha^\beta q\frac{d}{dt}(\rho_1\varphi_t\varphi_x + \rho_2\psi_t\psi_x)\ dxd\tau\right| = \left|\int_\alpha^\beta (\rho_1\varphi_t\varphi_x + \rho_2\psi_t\psi_x)|_{t=0}^{\tau=t}\ dx\right| \le C_E,$$

the last inequality is a consequence of the hypotheses, where C_E is a positive constant that depends on the initial data. So, our conclusion follows.

Here we consider that I_D is in the middle of the beam. Our main tool is the following theorem due to [12].

Theorem 4.2 *Let H be a Hilbert space and H_0 a closed subspace of H. Let $S(t)$ be a contractions semigroup over H and $S_0(t)$ an unitary group over H_0. If the difference $S(t) - S_0(t)$ is a compact operator from H_0 over H, then $S(t)$ is not exponentially stable.* □

Theorem 4.3 *The semigroup $S(t)$ is not exponentially stable when the viscous discontinuous part is in the center of the beam.*

Proof Let be the spaces:

$$\mathbb{L}_0 = \{f \in L^2(0,\ \ell)\colon f\Big|_{[\ell_0,\,\ell]} = 0\}, \quad V_0 = H_0^1(0,\ \ell) \cap \mathbb{L}_0,$$
$$H_0 = V_0 \times \mathbb{L}_0 \times V_0 \times \mathbb{L}_0$$

Let us consider the model over $[0,\ \ell_0]$:

$$\rho_1\tilde\varphi_{tt} - \kappa(\tilde\varphi_x + \tilde\psi)_x = 0$$
$$\rho_2\tilde\psi_{tt} - b\tilde\psi_{xx} + \kappa(\tilde\varphi_x + \tilde\psi) = 0$$
$$\tilde\varphi(0,t) = \tilde\varphi(\ell_0,t) = \tilde\psi(0,t) = \tilde\psi(\ell_0,t) = 0 \tag{4.2}$$

Let S_0 be the semigroup over H_0 (null extensions on $[\ell_0, \ell]$) associated to (4.2). So we have

$$\|S_0(t)U_0\|^2 = \|U_0\|^2, \quad \forall U_0 \in H_0 \tag{4.3}$$

Now, we are going to prove that $S(t) - S_0(t)\colon H_0 \to H$ is a compact operator, where

$$S(t)U_0^m = (\varphi^m, \varphi_t^m, \psi^m, \psi_t^m) \in H, \qquad S_0(t)U_0^m = (\tilde{\varphi}^m, \tilde{\varphi}_t^m, \tilde{\psi}^m, \tilde{\psi}_t^m) \in H_0$$

Let be: $v^m := \varphi^m - \tilde{\varphi}^m, \qquad w^m := \psi^m - \tilde{\psi}^m$. By definition we have

$$v^m(x,t) = \begin{cases} \varphi^m - \tilde{\varphi}^m, & \text{if } x \in [0, \ \ell_0] \\ \varphi^m &, \text{if } x \notin [0, \ \ell_0] \end{cases} \quad ; \quad w^m(x,t) = \begin{cases} \psi^m - \tilde{\psi}^m, & \text{if } x \in [0, \ \ell_0] \\ \psi^m &, \text{if } x \notin [0, \ \ell_0] \end{cases}$$

Moreover, v and w verifies

$$\rho_1 v_{tt} - \kappa(v_x + w)_x - \tilde{\kappa}(v_{xt} + w_t)_x = 0 \tag{4.4}$$

$$\rho_2 w_{tt} - b w_{xx} - \tilde{b} w_{xxt} + \kappa(v_x + w) + \tilde{\kappa}(v_{xt} + w_t) = 0 \tag{4.5}$$

Multiplying (4.4) by v_t, (4.5) by w_t, and integrating over $[0, \ \ell]$, we obtain:

$$\int_0^\ell \left(\rho_1|v_t|^2 + \rho_2|w_t|^2 + b|w_x|^2 + \kappa|v_x + w|^2\right) dx =$$
$$\kappa v_x v_t\Big|_0^{\ell_0} + b w_x w_t\Big|_0^{\ell_0} - \int_{\ell_0}^{\ell} \tilde{\kappa}|v_{xt} + w_t|^2 + \tilde{b}|w_{xt}|^2 \, dx \tag{4.6}$$

Using the boundary conditions we get

$$\kappa v_x v_t\Big|_0^{\ell_0} + b w_x w_t\Big|_0^{\ell_0} = -\kappa\tilde{\varphi}_x(\ell_0^-, t)\varphi_t(\ell_0^-, t) - b\tilde{\psi}_x(\ell_0^-, t)\psi_t(\ell_0^-, t)$$

Denoting by $\mathfrak{U}^m(t) = [S(t) - S_0(t)]U_0^m = (v^m, v_t^m, w^m, w_t^m)$, integrating (4.6) over $[0, \ t]$, recalling the definition of the norm of the phase space $\mathcal{H}$ we get

$$\int_0^t \|\mathfrak{U}^m(t)\|_{\mathcal{H}}^2 \, dt + \int_0^t \int_{\ell_0}^{\ell} \tilde{\kappa}|v_{xt}^m + w_t^m|^2 + \tilde{b}|w_{xt}^m|^2 \, dx \, dt$$
$$= -\int_0^t (\kappa\tilde{\varphi}_x^m(\ell_0^-, t)\varphi_t^m(\ell_0^-, t) + b\tilde{\psi}_x(\ell_0^-, t)\psi_t(\ell_0^-, t)) \, dt \tag{4.7}$$

using Theorem 4.1 we have that $\tilde{\varphi}_x^m(\ell_0^-, t)$ and $\tilde{\psi}_x^m(\ell_0^-, t)$ are bounded. So there exists a subsequence, we still denote in the same way, such that

$$\tilde{\varphi}_x^m(\ell_0^-, t) \to \tilde{\varphi}_x(\ell_0^-, t) \text{ weak in } L^2(0, \ T), \ \tilde{\psi}_x^m(\ell_0^-, t) \to \tilde{\psi}_x(\ell_0^-, t) \text{ weak in } L^2(0, \ T)$$

We only need to prove that

$$\big(\varphi_x^m(\ell_0^-,t)\ ,\ \psi_x^m(\ell_0^-,t)\big)\to\big(\varphi_x(\ell_0^-,t)\ ,\ \psi_x(\ell_0^-,t)\big)\ \text{strong in } L^2(0,\ T)\times L^2(0,\ T) \tag{4.8}$$

which implies the norm convergence in (4.7). To do that we use (3.13) and (1.1)–(1.2) to get

$$\varphi_t^m,\psi_t^m\in L^2(0,T;H^1(I_D)),\quad \varphi_{tt}^m,\psi_{tt}^m\in L^2(0,T;H^{-1}(I_D))$$

Since $H^1\subset H^{1-\delta}\subset H^{-1}$ where the first inclusion is a compact embedding, the compactness Theorem of Lions-Aubin (see [7]) implies that there exists a subsequence (we still denote in the same way) such that

$$(\varphi_t^m,\psi_t^m)\to(\varphi_t,\psi_t)\ \text{strong in } L^2(0,\ T;H^{1-\delta}(I_D)\times H^{1-\delta}(I_D)).$$

Using that the embedding $H^{1-\delta}(I_D)\subset C(\overline{I_D})$ is compact, we have:

$$(\varphi_t^m,\psi_t^m)\to(\varphi_t,\psi_t)\ \text{strong in } L^2(0,\ T;C(V_D)\times C(V_D))$$

This implies (4.8). Hence inequality (4.7) implies the convergence in norm of $\mathfrak{U}^m$. So, $S(t)-\tilde{S}_0(t)$ is a compact operator. Then our conclusion follows. □

In summary, we can state the following theorem:

Theorem 4.4 *The Timoshenko system is exponentially stable if and only if the viscoelastic discontinuous part is not in the middle of the beam. Otherwise, the system only has polynomial rate of decay.*

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