Applied Mathematics & Optimization (2021) 84:1185–1208
https://doi.org/10.1007/s00245-020-09674-3

Travelling waves in the Fisher–KPP equation with nonlinear degenerate or singular diffusion

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Published online: 29 March 2020

Abstract
We consider a one-dimensional reaction–diffusion equation of Fisher–Kolmogoroff–Petrovsky–Piscounoff type. We investigate the effect of the interaction between the nonlinear diffusion coefficient and the reaction term on the existence and non-existence of travelling waves. Our diffusion coefficient is allowed to be degenerate or singular at both equilibrium points, 0 and 1, while the reaction term need not be differentiable. These facts influence the existence and qualitative properties of travelling waves in a substantial way.

Keywords Fisher–Kolmogoroff–Petrovsky–Piscounoff equation · Travelling wave · Degenerate and/or singular diffusion · Non-smooth reaction term · Existence and non-existence of travelling waves · An overdetermined first-order boundary value problem

Mathematics Subject Classification Primary 35Q92 · 92D25 · (34B08); Secondary 35K57 · 35K65 · (34B18)

1 Introduction

We are concerned with the travelling waves (particularly with their *speed* and *profile*) for the *Fisher–Kolmogoroff–Petrovsky–Piscounoff* population model with *nonlinear*

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diffusion, $d(u)$ (of porous medium type), and a *non-Lipschitzian* reaction term, $g(u)$:

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(d(u)\,\frac{\partial u}{\partial x}\right) = g(u) \quad \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}_+ . \tag{1.1}$$

We employ certain specific forms of the possibly degenerate or singular diffusion coefficient $d(u)$ and the nonlinear reaction function $g(u)$ that are motivated by classical population models by Fisher [15] and Kolmogoroff et al. [20], both from the same year of 1937. We allow both, $d(u)$ and $g(u)$, to depend continuously on the population density u. The reaction–diffusion equation (1.1) is briefly referred to as the *Fisher–KPP equation* (or *FKPP equation*).

In contrast with similar models that have been considered in the literature so far, particularly in Audrito and Vázquez [4,5], Corli and Malaguti [8], Corli et al. [9], King and McCabe [19], Malaguti and Marcelli [21], Murray [23,24], and Sánchez-Garduño and Maini [25], typically with a power-type diffusion coefficient $d(u)$ and a continuously differentiable (C^1-) reaction function $g(u)$, our diffusion term $d = d(u)$ and the reaction term $g = g(u)$ are much more general functions. Only in our simple examples (in Sect. 5) do we take functions $d(u)$ and $g(u)$ with *power-type* asymptotic behavior near the equilibrium points $u = 0$ and $u = 1$. In fact, the diffusion term $d = d(u)$ may degenerate or blow up as $u \to 0+$ and/or $u \to 1-$. In particular, to the authors' best knowledge, models with a discontinuous diffusion term d on $[0, 1]$ have not been considered in the literature so far. Our only restriction on d is that $d : \mathbb{R}\setminus\{0, 1\} \to \mathbb{R}$ be continuous and locally Lebesgue integrable near the (possibly) singular points $\{0, 1\}$. We expect some of our results, in particular, Proposition 4.2 (existence of a travelling wave in Sect. 4.1) and Proposition 4.3 (nonexistence of a travelling wave in Sect. 4.2), to stay valid even if d is only a locally Lebesgue integrable function. At the same time, the reaction term $g = g(u)$ need not be a Lipschitz continuous function in its domain of definition. While the role of the nonlinear reaction term $g = g(u)$ has been justified already in the original works [15,20] (which consider only constant diffusivity $d > 0$), the importance of the density-dependent diffusion term $d = d(u)$ in insect despersal models is emphasized in the monograph [24, Sect. 13.4, p. 449].

In a general biological *Fisher–KPP model* one naturally expects travelling waves $u(x,t) = U(x - ct)$ with a continuous wave profile U. However, requiring a smoother profile U does not seem to be biologically justified, see [24, Sect. 11.3] for a sketch of non-smooth profiles in Fig. 11.2 on p. 403. Non-smooth profiles for *doubly non-linear diffusion* (like ours in Eq. (1.1) above) have been suggested as "generalizations" in [19] (termed profiles with "sharp front") and treated in details much later in [4, Fig. 2, p. 7651] (with "slow" diffusion) and [5, Fig. 3, p. 217] (with "fast" diffusion). Taking into account this fact, we define a travelling wave for problem (1.1) in a rather general fashion that does not require differentiability of the profile; cf. Definition 2.1 below. In a higher space dimension (in $\mathbb{R}^N$), an appropriate definition in the sense of distributions is used; cf. [4,5,16,19]. However, in one space dimension (in $\mathbb{R}^1$) our Definition 2.1 is simpler and more natural. It yields useful qualitative properties of expected travelling waves (see Sect. 3) which permit to transform the original second-order Fisher–KPP equation (1.1) (for a travelling wave $u(x,t) = U(x - ct)$) into an equivalent first-order boundary value problem for the (first) derivative of the inverse function of U (see Sect. 4) under rather general hypotheses on d and g.

Density-dependent dispersal (modelled by density-dependent diffusion) has been observed in many insect populations, such as the antlion *Glenuroides japonicus*. Several authors propose to analyse the flux of ants throughout a compartmentally divided habitat which leads to the spatial segregation of a species. For greater details and numerous references to biological modelling, we refer the reader to [25, Sect. 2, pp. 164–166].

This article is organized as follows. Our new definition of a travelling wave is given in the next section (Sect. 2). Basic properties of a wave profile U, such as monotonicity, are studied in Sect. 3. A standard phase plane transformation applied to the equation for the wave profile U in Sect. 4 yields an overdetermined first-order, two-point boundary value problem, with a free parameter $c \in \mathbb{R}$, the wave speed. This is our basic tool for obtaining existence and nonexistence of a travelling wave. The last section (Sect. 5) is dedicated to studies with simple terms $d(u)$ and $g(u)$ that are nonlinear of *power-type* near the equilibrium points. As a conclusion, from the interaction between $d(u)$ and $g(u)$ we determine the asymptotic shape of travelling waves near the equilibrium points.

2 A Quasilinear Fisher–KPP Equation with Discontinuous Diffusion and Non-smooth Positive Reaction

As usual, we denote $\mathbb{R} \stackrel{\text{def}}{=} (-\infty, \infty)$, $\mathbb{R}_+ \stackrel{\text{def}}{=} [0, \infty)$, and assume that the diffusion coefficient d and the reaction term g satisfy the following basic hypotheses, respectively:

(H1) $d : \mathbb{R} \setminus \{0, 1\} \to \mathbb{R}$ is a continuous, but *not necessarily smooth* function, such that $d(s) > 0$ for every $s \in \mathbb{R} \setminus \{0, 1\}$, and the (Lebesgue) integral $\int_a^b d(s)\,\mathrm{d}s < \infty$ whenever $-\infty < a < b < +\infty$.

(H2) $g : \mathbb{R} \to \mathbb{R}$ is a continuous, but *not necessarily smooth* function, such that $g(0) = g(1) = 0$ together with $g(s) > 0$ for every $s \in (0, 1)$, and $g(s) < 0$ for every $s \in (-\infty, 0) \cup (1, \infty)$.

The reaction function g satisfying **(H2)** comprises also the so-called *generalized logistic growth* in the population model studied in Tsoularis and Wallace [26].

We reformulate Eq. (1.1) for $u(x,t)$ as an equivalent initial value problem for the unknown function $v(z,t) = u(z + ct, t) \equiv u(x,t)$ with the moving coordinate $z = x - ct$:

$$\frac{\partial v}{\partial t} - \frac{\partial}{\partial z}\left(d(v)\,\frac{\partial v}{\partial z}\right) - c\,\frac{\partial v}{\partial z} = g(v)\,, \quad (z,t) \in \mathbb{R} \times \mathbb{R}_+\,. \tag{2.1}$$

We will show below that every travelling wave $u(x,t) = U(x - ct)$ for (1.1) must have a monotone decreasing profile $U : \mathbb{R} \to \mathbb{R}$ satisfying

$$\lim_{z\to-\infty} U(z) = 1 \quad \text{and} \quad \lim_{z\to+\infty} U(z) = 0\,. \tag{2.2}$$

More precisely, $U : \mathbb{R} \to \mathbb{R}$ must be monotone decreasing with $U' < 0$ on a suitable open interval $(z_0, z_1) \subset \mathbb{R}$, such that

$$\lim_{z \to z_0+} U(z) = 1 \quad \text{and} \quad \lim_{z \to z_1-} U(z) = 0 , \tag{2.3}$$

by Proposition 3.4. We would like to remark that the cases of $z_0 > -\infty$ and/or $z_1 < +\infty$ render qualitatively different travelling waves than the classical case $(z_0, z_1) = \mathbb{R}$ which has been studied in the original works [15,20] and in the literature [2,3,14,17, 21,23–25].

In order to be able to give a workable definition of a travelling wave, we introduce the (Lebesgue) integral

$$D(s) \stackrel{\text{def}}{=} \int_0^s d(s')\,\mathrm{d}s' \quad \text{for every } s \in \mathbb{R} .$$

This is an absolutely continuous function on $\mathbb{R}$ which is continuously differentiable on $\mathbb{R} \setminus \{0,\ 1\}$ with the derivative $D'(s) = d(s)$ for every $s \in \mathbb{R} \setminus \{0,\ 1\}$. Using this setting, in Sect. 4 we are able to find a *first integral* for the second-order equation for U restricted to the open interval $(z_0, z_1) \subset \mathbb{R}$:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(d(U)\,\frac{\mathrm{d}U}{\mathrm{d}z} \right) + c\,\frac{\mathrm{d}U}{\mathrm{d}z} + g(U(z)) = 0 , \quad z \in (z_0, z_1) . \tag{2.4}$$

It is easy to observe that this equation is valid for every $z \in \mathbb{R} \setminus \{z_0,\ z_1\}$ (in the sense of Definition 2.1 below) provided U is extended by $U(z) = 1$ if $-\infty < z \le z_0$ and $U(z) = 0$ if $z_1 \le z < +\infty$.

Definition 2.1 A function $u(x, t) = U(x - ct)$ of $(x, t) \in \mathbb{R} \times \mathbb{R}_+$ is called a *travelling wave* (or *TW*, for short) for problem (1.1) where $c \in \mathbb{R}$ is a constant called *wave speed* (or simply *speed*) and $U : \mathbb{R} \to \mathbb{R}$ is a continuous function called *wave profile* (or simply *profile*) with the following properties:

(a) $U(z) \ge 0$ holds for every $z \in \mathbb{R}$ and the limits in (2.2) are valid.
(b) The composition $z \mapsto (D \circ U)(z) = D(U(z)) : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function with the derivative $\frac{\mathrm{d}}{\mathrm{d}z} D(U(z))$ vanishing at every point $\xi \in \mathbb{R}$ such that $U(\xi) \in \{0,\ 1\}$.
(c) The following integral form of Eq. (2.4) is valid for all pairs $z, z^* \in \mathbb{R}$:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}z} D(U(z)) - \frac{\mathrm{d}}{\mathrm{d}z} D(U(z))\Big|_{z=z^*} \\ &+ c\,(U(z) - U(z^*)) + \int_{z^*}^{z} g(U(z'))\,\mathrm{d}z' = 0 . \end{aligned} \tag{2.5}$$

Remark 2.2 An important feature of our definition of a travelling wave for problem (1.1) above is the fact that we do *not* assume that its profile, $U : \mathbb{R} \to \mathbb{R}$, is a sufficiently smooth function that obeys the differential equation (2.4) in a classical sense. In fact, we will see in the next remark (Remark 2.3, Part (ii)) that the "weaker" integral form of Eq. (2.4), given in Eq. (2.5) above, easily yields also the "stronger" classical form (2.4) at every point $z \in \mathbb{R}$ such that $U(z) \notin \{0,\ 1\}$. In other words, in case the wave profile U is only continuous, but not differentiable, one has to take advantage of the integral form (2.5) only for $z \in \mathbb{R}$ near those points $\xi \in \mathbb{R}$ at which $U(\xi) \in \{0,\ 1\}$.

The integral form (2.5) enables us to use rather general, nonsmooth diffusion and reaction terms, d and g, respectively. Last but not least, our definition of a travelling wave covers both alternatives for travelling waves introduced in Sánchez-Garduño and Maini [25, Sect. 3, p. 167]: *front-type* and *sharp-type* travelling waves (see also [21, Sect. 2, pp. 473–474]). Such types of travelling waves (TW) are called "positive TW" and "finite TW", respectively, in Audrito and Vázquez [4,5]. It has been shown in Malaguti and Marcelli [21, Sect. 2, pp. 476–481] that the cases of $z_0 > -\infty$ and/or $z_1 < +\infty$ may occur if the nonlinear reaction function $g(s)$ and the diffusion term $d(s)$ are not differentiable at the points $s \in \{0,\ 1\}$. In accordance with [21, Remark 1, p. 478], we now persue the case of g and/or d being "nonsmooth" at the points $s \in \{0,\ 1\}$ in greater details.

Definition 2.1 has the following simple, but important technical consequences to be used in the sequel:

Remark 2.3 (i) Equation (2.4) being translation invariant ($z \mapsto z + \zeta : \mathbb{R} \to \mathbb{R}$, for $\zeta \in \mathbb{R}$ fixed), we are allowed to choose the profile U in such a way that $U(0) = 1/2$. This choice will determine the profile, U, uniquely if needed, thanks to the strict monotonicity of the profile throughout the open interval $(z_0, z_1) \subset \mathbb{R}$, by $U' < 0$; cf. Proposition 3.4 below. However, we do not assume $U(0) = 1/2$, in general, unless we need the uniqueness of U for a fixed speed $c \in \mathbb{R}$.

(ii) Hypothesis **(H1)** combined with Definition 2.1, Part (b), imply that, at every point $\xi \in \mathbb{R}$ with $U(\xi) \notin \{0,\ 1\}$, we have $d(U(\xi)) > 0$ and the derivative $U'(\xi)$ exists and satisfies $\frac{\mathrm{d}}{\mathrm{d}z}\, D(U(z))\Big|_{z=\xi} = d(U(\xi))\, U'(\xi)$.

(iii) There exist two sequences $\xi_n \in (n, n+1)$ and $\xi_n^* \in (-n-1, -n)$; $n = 1, 2, 3, \ldots$, such that

$$\frac{\mathrm{d}}{\mathrm{d}z}\, D(U(z))\Big|_{z=\xi_n} \longrightarrow 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}z}\, D(U(z))\Big|_{z=\xi_n^*} \longrightarrow 0 \quad \text{as } n \to \infty\,. \tag{2.6}$$

Indeed, we can apply the mean value theorem to the (continuously differentiable) function $z \mapsto D(U(z)) : \mathbb{R} \to \mathbb{R}$ in each of the intervals $(n, n+1)$ and $(-n-1, -n)$; $n = 1, 2, 3, \ldots$, to conclude that there are $\xi_n \in (n, n+1)$ and $\xi_n^* \in (-n-1, -n)$, such that

$$D(U(n+1)) - D(U(n)) = \frac{\mathrm{d}}{\mathrm{d}z}\, D(U(z))\Big|_{z=\xi_n} \quad \text{and}$$
$$D(U(-n)) - D(U(-n-1)) = \frac{\mathrm{d}}{\mathrm{d}z}\, D(U(z))\Big|_{z=\xi_n^*}\,.$$

The limits in Eq. (2.6) follow from Definition 2.1 combined with the limits in (2.2). Similarly, given any interval length $\lambda \in (0, \infty)$, analogous sequences $\xi_n \in (n\lambda,\ (n+1)\lambda)$ and $\xi_n^* \in (-(n+1)\lambda,\ -n\lambda)$ can be obtained for $n = 1, 2, 3, \dots$.

3 Basic Properties of a Wave Profile

Throughout this section we assume that $d, g : \mathbb{R} \to \mathbb{R}$ satisfy Hypotheses **(H1)** and **(H2)**. In this section we prove that every travelling wave profile obeying Definition 2.1 has some specific properties that permit us to take advantage of a phase plane transformation, thus reducing the second-order differential equation (2.4) for $U = U(z)$ to a first-order ordinary differential equation for the derivative $\mathrm{d}z/\mathrm{d}U$ of its inverse function $U \mapsto z = z(U)$ as a function of $U \in (0, 1)$; see Sect. 4. Next, we show that any wave profile $U : \mathbb{R} \to \mathbb{R}$ takes only values between 0 and 1.

Lemma 3.1 (Wave profile values.) *Let $(x, t) \mapsto u(x, t) = U(x - ct) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ be a TW with speed $c \in \mathbb{R}$ and profile $U : \mathbb{R} \to \mathbb{R}$. Then we have $0 \le U(z) \le 1$ for every $z \in \mathbb{R}$.*

Proof We have $U(z) \ge 0$ for every $z \in \mathbb{R}$, by Definition 2.1. By contradiction to $U(z) \le 1$ for every $z \in \mathbb{R}$, suppose there is a number $\xi \in \mathbb{R}$ such that $U(\xi) > 1$. We make use of the limits in (2.2) to conclude that there are numbers $\xi_1, \xi_2 \in \mathbb{R}$ such that $\xi_1 < \xi < \xi_2$ and $U(\xi) > \min\{U(\xi_1), U(\xi_2)\} > 1$. We may choose ξ_1 and ξ_2, close enough to ξ, in such a way that also $U(z) > 1$ holds for every $z \in [\xi_1, \xi_2]$. Denoting by $\xi_0 \in [\xi_1, \xi_2]$ a (global) maximizer for the function U over the compact interval $[\xi_1, \xi_2]$, we arrive at $\xi_0 \in (\xi_1, \xi_2)$, $U(\xi_0) \ge U(\xi) > 1$, $U'(\xi_0) = 0$, and

$$d(U(z))\, U'(z) - d(U(\xi_0))\, U'(\xi_0) = \, - c\, (U(z) - U(\xi_0)) - \int_{\xi_0}^{z} g(U(z'))\, \mathrm{d}z'$$

for all $z \in [\xi_1, \xi_2]$, by Eq. (2.5) and Remark 2.3, Part (i). Since $U'(\xi_0) = 0$, the last equation entails

$$d(U(z))\, \frac{U'(z) - U'(\xi_0)}{z - \xi_0} = \, - c\, \frac{U(z) - U(\xi_0)}{z - \xi_0} - \frac{1}{z - \xi_0} \int_{\xi_0}^{z} g(U(z'))\, \mathrm{d}z' \tag{3.1}$$

for all $z \in [\xi_1, \xi_2] \setminus \{\xi_0\}$. We apply the mean value theorem to the right-hand side of Eq. (3.1) to conclude that, for every $z \in [\xi_1, \xi_2]$, $z \ne \xi_0$, there is a number $\hat{z} \in [\xi_1, \xi_2]$ between ξ_0 and z, such that

$$d(U(z))\, \frac{U'(z) - U'(\xi_0)}{z - \xi_0} = \, - c\, U'(\hat{z}) - g(U(\hat{z}))\, .$$

Letting $z \to \xi_0$ we get also $\hat{z} \to \xi_0$ and, consequently, the second derivative $U''(\xi_0)$ of U at ξ_0 exists and satisfies

$$d(U(\xi_0))\, U''(\xi_0) = \,-\, c\, U'(\xi_0) - g(U(\xi_0)) = \,-\, g(U(\xi_0)) > 0\,,$$

where $d(U(\xi_0)) > 0$. Hence, $U'(\xi_0) = 0$ and $U''(\xi_0) > 0$ show that $\xi_0 \in (\xi_1, \xi_2)$ is also a strict local minimizer for the function U in the open interval (ξ_1, ξ_2). But this contradicts our construction of ξ_0 as a (global) maximizer for U over $[\xi_1, \xi_2]$.

This proves $U(z) \le 1$ for all $z \in \mathbb{R}$. □

Now we are ready to calculate the wave speed c explicitly from the wave profile U.

Lemma 3.2 (Wave speed.) *Let* $(x, t) \mapsto u(x, t) = U(x - ct) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ *be a TW with speed* $c \in \mathbb{R}$ *and profile* $U : \mathbb{R} \to \mathbb{R}$*. Then we have* $0 \le U(z) \le 1$ *and* $g(U(z)) \ge 0$ *for every* $z \in \mathbb{R}$*, together with*

$$0 < c = \int_{-\infty}^{+\infty} g(U(z'))\, \mathrm{d}z' < \infty\,. \tag{3.2}$$

Moreover, Eq. (2.5) *is equivalent with*

$$\frac{\mathrm{d}}{\mathrm{d}z}\, D(U(z)) + c\, U(z) - \int_z^{+\infty} g(U(z'))\, \mathrm{d}z' = 0 \quad \textit{for all } z \in \mathbb{R}\,. \tag{3.3}$$

Proof We have $0 \le U(z) \le 1$ for every $z \in \mathbb{R}$, by Lemma 3.1, which yields $g(U(z)) \ge 0$, by Hypothesis **(H2)**.

For every fixed $n = 1, 2, 3, \ldots$ we take the pair $(z^*, z) = (\xi_n^*, \xi_n)$ in Eq. (2.5), where the latter pair has been specified in Remark 2.3, Part (iii). Applying (2.2) and (2.6) to Eq. (2.5) and letting $n \to \infty$, we arrive at

$$-\, c + \int_{-\infty}^{+\infty} g(U(z'))\, \mathrm{d}z' = 0\,,$$

by the Lebesgue monotone convergence theorem. This proves Eq. (3.2) with $c \ge 0$. However, the integrand $g(U(z')) \ge 0$ cannot vanish identically for all $z' \in \mathbb{R}$, by the continuity of the wave profile $U : \mathbb{R} \to \mathbb{R}$ and the limits (2.2) which guarantee $U(\hat{z}) = \frac{1}{2} \in (0, 1)$ for some $\hat{z} \in \mathbb{R}$; hence, $g(U(\hat{z})) > 0$. Since also $g : \mathbb{R} \to \mathbb{R}$ is continuous, by Hypothesis **(H2)**, we must have $c > 0$, by Eq. (3.2).

To verify also Eq. (3.3), we now take the pair $(z^*, z) = (\xi_n^*, z)$, where $\xi_n^* \in (-n-1, -n)$ is as above and $z \in \mathbb{R}$ is arbitrary. Applying (2.2) and (2.6) to Eq. (2.5) again and letting $n \to \infty$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}z}\, D(U(z)) + c\,(U(z) - 1) + \int_{-\infty}^{z} g(U(z'))\, \mathrm{d}z' = 0 \quad \text{for all } z \in \mathbb{R}\,.$$

Finally, we apply (3.2) to the last equation to derive (3.3). □

We continue with the constant sections of the travelling wave.

Lemma 3.3 (Constant sections.) *Let* $(x, t) \mapsto u(x, t) = U(x - ct) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ *be a TW with speed* $c \in \mathbb{R}$ *and profile* $U : \mathbb{R} \to \mathbb{R}$. *Assume that* $\xi \in \mathbb{R}$ *is such that* $U(\xi) \in \{0, 1\}$. *Then the following two alternatives are valid:*

(i) *If* $U(\xi) = 0$ *then* $U(z) \equiv 0$ *for every* $z \ge \xi$.
(ii) *If* $U(\xi) = 1$ *then* $U(z) \equiv 1$ *for every* $z \le \xi$.

Proof We recall that $0 \le U(z) \le 1$ for every $z \in \mathbb{R}$, by Lemma 3.1.

Alt. (i): Assume that $U(\xi) = 0$. Suppose there is some $\xi^* \in (\xi, +\infty)$ such that $U(\xi^*) > 0$. We can guarantee even $0 < U(\xi^*) < 1$, by taking $\xi^* \in (\xi, +\infty)$ closer to ξ. This implies $g(U(\xi^*)) > 0$ and, consequently, we have $\int_\xi^{+\infty} g(U(z'))\,\mathrm{d}z' > 0$. Furthermore, our definition of a travelling wave, Definition 2.1, Part (b), guarantees that also $\frac{\mathrm{d}}{\mathrm{d}z} D(U(z))\Big|_{z=\xi} = 0$, thanks to $U(\xi) = 0$. We insert these facts into Eq. (3.3) with $z = \xi$, which yields $\int_\xi^{+\infty} g(U(z'))\,\mathrm{d}z' = 0$, a contradiction with the inequality (> 0) above.

Alt. (ii): Now assume $U(\xi) = 1$ and suppose there is some $\xi^* \in (-\infty, \xi)$ such that $U(\xi^*) < 1$. Again, we can guarantee $0 < U(\xi^*) < 1$, by taking $\xi^* \in (-\infty, \xi)$ closer to ξ. This implies $g(U(\xi^*)) > 0$ and, as above, we have $\int_{-\infty}^{\xi} g(U(z'))\,\mathrm{d}z' > 0$. Definition 2.1, Part (b), guarantees also $\frac{\mathrm{d}}{\mathrm{d}z} D(U(z))\Big|_{z=\xi} = 0$, thanks to $U(\xi) = 1$. We insert these facts into Eq. (3.3) with $z = \xi$, which yields $\int_\xi^{+\infty} g(U(z'))\,\mathrm{d}z' = c$. A comparison of this equality with Eq. (3.2) forces $\int_{-\infty}^{\xi} g(U(z'))\,\mathrm{d}z' = 0$, a contradiction with the inequality (> 0) above.

The lemma is proved. $\square$

Finally, we establish the monotonicity of the travelling wave (see Definition 2.1).

Proposition 3.4 (Monotonicity.) *Let* $(x, t) \mapsto u(x, t) = U(x - ct) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ *be a TW with speed* $c \in \mathbb{R}$ *and profile* $U : \mathbb{R} \to \mathbb{R}$. *Then* $0 \le U(z) \le 1$ *and* $U'(z) \le 0$ *for all* $z \in \mathbb{R}$. *Moreover, there is an open interval* $(z_0, z_1) \subset \mathbb{R}$, $-\infty \le z_0 < z_1 \le +\infty$, *such that* $U' < 0$ *on* (z_0, z_1) *together with*

$$\begin{cases} \lim\limits_{z \to z_0+} U(z) = 1 \quad \text{and} \qquad U(z) = 1 \;\; \text{if} \;\; -\infty < z \le z_0\,, \\ \lim\limits_{z \to z_1-} U(z) = 0 \quad \text{and} \qquad U(z) = 0 \;\; \text{if} \;\; z_1 \le z < +\infty\,. \end{cases}$$

Proof Recalling Lemmas 3.1, 3.2, and 3.3 , we conclude that it remains to prove $U'(z) < 0$ for every $z \in \mathbb{R}$ satisfying $0 < U(z) < 1$. Suppose not; hence, there is some $\xi \in \mathbb{R}$ such that $U'(\xi) = 0$ and $0 < U(\xi) < 1$. Eq. (2.5) and Remark 2.3, Part (i), yield

$$d(U(z))\,U'(z) - d(U(\xi))\,U'(\xi) = \; - c\,(U(z) - U(\xi)) - \int_\xi^z g(U(z'))\,\mathrm{d}z' \tag{3.4}$$

for all $z \in \mathbb{R}$, in analogy with our proof of Lemma 3.1, Eq. (3.1).

Next, we show that every such point ξ must be a strict (i.e., isolated) local maximum satisfying $U''(\xi) < 0$. Let us choose $\xi_1, \xi_2 \in \mathbb{R}$ such that $\xi_1 < \xi < \xi_2$ and $0 < U(z) < 1$ holds for all $z \in [\xi_1, \xi_2]$. We apply the mean value theorem to the right-hand side of Eq. (3.4) to conclude that, for every $z \in [\xi_1, \xi_2]$, $z \neq \xi$, there is a number $\hat{z} \in [\xi_1, \xi_2]$ between ξ and z, such that

$$d(U(z))\, \frac{U'(z) - U'(\xi)}{z - \xi} = -\, c\, U'(\hat{z}) - g(U(\hat{z}))\,.$$

Letting $z \to \xi$ we conclude that $\hat{z} \to \xi$, $d(U(z)) \to d(U(\xi)) > 0$, and

$$d(U(\xi))\, U''(\xi) = -\, c\, U'(\xi) - g(U(\xi)) = -\, g(U(\xi)) < 0\,.$$

This yields $U''(\xi) < 0$.

Since $U(z) \to 1$ as $z \to -\infty$, and $U(\xi) < 1$, there is some $\xi_1' \in (-\infty, \xi)$ such that $U(\xi) < U(\xi_1') < 1$. Now let $\xi_0 \in [\xi_1', \xi]$ be a (global) minimizer for the function U over the compact interval $[\xi_1', \xi]$. With a help from $U'(\xi) = 0$ and $U''(\xi) < 0$, we arrive at $\xi_0 \in (\xi_1', \xi)$, $U(\xi_0) < U(\xi) < 1$, $U'(\xi_0) = 0$, and Eq. (3.4) with ξ_0 in place of ξ. But then, by what we have proved above, if also $U(\xi_0) > 0$ then we must have $U''(\xi_0) < 0$ as above. This contradicts our choice of ξ_0 to be a (global) minimizer for the function U over the open interval (ξ_1', ξ).

The case $U(\xi_0) = 0$ would lead to a contradiction, by Lemma 3.3. It would force $U(z) = 0$ for every $z \geq \xi_0$ and, in particular, also $U(\xi) = 0$, thus contradicting our choice of $\xi \in \mathbb{R}$.

We conclude that $U'(z) < 0$ holds for every $z \in (z_0, z_1)$. □

4 A Phase Plane Transformation

We use a phase plane transformation (cf. Murray [24], Sect. 13.2, pp. 440–441, Malaguti and Marcelli [21], Enguiça et al. [13, Sect. 1], Corli and Malaguti [7], and Drábek and Takáč [11]) in order to describe all monotone decreasing travelling waves $u(x,t) \equiv U(x - ct - \zeta)$ where $U : \mathbb{R} \to \mathbb{R}$ is the profile of a travelling wave normalized by $U(0) = 1/2$ as specified in Remark 2.3, Part (i), and $\zeta \in \mathbb{R}$ is a suitable translation constant; see also Proposition 3.4. We reduce the second-order differential equation for $U = U(z)$ to a first-order ordinary differential equation for the derivative $\mathrm{d}z/\mathrm{d}U$ of its inverse function $U \mapsto z = z(U)$ as a function of $U \in (0,1)$. In fact, below we find a nonlinear differential equation for the derivative

$$U'(z) = \left(\frac{\mathrm{d}z}{\mathrm{d}U}\right)^{-1} \equiv \frac{1}{z'(U)} < 0 \quad \text{as a function of } U \in (0,1)\,.$$

To this end, we make the substitution

$$V \stackrel{\text{def}}{=} -\, d(U)\, \frac{\mathrm{d}U}{\mathrm{d}z} > 0 \quad \text{for } z \in (z_0, z_1) \tag{4.1}$$

and consequently look for $V = V(U)$ as a function of $U \in (0, 1)$ that satisfies the following differential equation obtained from Eq. (2.4):

$$-\frac{\mathrm{d}V}{\mathrm{d}U} \cdot \frac{\mathrm{d}U}{\mathrm{d}z} + c\,\frac{\mathrm{d}U}{\mathrm{d}z} + g(U) = 0\,, \quad z \in (z_0, z_1),$$

that is,

$$\frac{\mathrm{d}V}{\mathrm{d}U} \cdot \frac{V}{d(U)} - c\,\frac{V}{d(U)} + g(U) = 0\,, \quad U \in (0, 1)\,. \tag{4.2}$$

Hence, we are looking for the inverse function $U \mapsto z(U)$ with the derivative

$$\frac{\mathrm{d}z}{\mathrm{d}U} = -\,\frac{d(U)}{V(U)} < 0 \quad \text{for } U \in (0, 1)\,, \quad \text{such that } z(1/2) = 0\,.$$

Finally, we multiply Eq. (4.2) by $d(U)$, make the substitution

$$y = V^2 = d(U)^2 \left|\frac{\mathrm{d}U}{\mathrm{d}z}\right|^2 = \left|\frac{\mathrm{d}}{\mathrm{d}z}\,D(U(z))\right|^2 > 0\,, \tag{4.3}$$

and write r in place of U, thus arriving at

$$\frac{1}{2} \cdot \frac{\mathrm{d}y}{\mathrm{d}r} - c\sqrt{y} + f(r) = 0\,, \quad r \in (0, 1)\,.$$

Here, the function $f : \mathbb{R} \setminus \{0,\ 1\} \to \mathbb{R}$ is defined by $f(r) \stackrel{\text{def}}{=} d(r)\,g(r)$ for every $r \in \mathbb{R} \setminus \{0,\ 1\}$. Observe that f is continuous on $\mathbb{R} \setminus \{0,\ 1\}$ with $f(r) > 0$ for every $r \in (0, 1)$, and $f(r) < 0$ for every $r \in (-\infty, 0) \cup (1, \infty)$. In our existence results in Sect. 4.1 we will assume also $\lim_{r \to 0+} f(r) = 0$ and $\lim_{r \to 1-} f(r) = 0$, that is, the restriction $f|_{(0,1)}$ of f to the open interval $(0, 1)$ can be extended to a continuous function $f|_{[0,1]}$ on $[0, 1]$ by setting $f(0) = f(1) = 0$.

This means that the unknown function $y :\ (0, 1) \to (0, \infty)$ of r verifies also

$$\frac{\mathrm{d}y}{\mathrm{d}r} = 2\left(c\sqrt{y^+} - f(r)\right)\,, \quad r \in (0, 1)\,, \tag{4.4}$$

where $y^+ = \max\{y,\ 0\}$. Since we require that the function $z \mapsto D(U(z)) : \mathbb{R} \to \mathbb{R}$ be continuously differentiable with the derivative $\frac{\mathrm{d}}{\mathrm{d}z}\,D(U(z))$ vanishing at every point $\xi \in \mathbb{R}$ such that $U(\xi) \in \{0,\ 1\}$, that is, $\frac{\mathrm{d}}{\mathrm{d}z}\,D(U(z) \to 0$ as $z \to z_0+$ and $z \to z_1-$, the function $y = y(r) = |\mathrm{d}D(U(z))/\mathrm{d}z|^2$ must satisfy the boundary conditions

$$y(0) = y(1) = 0\,. \tag{4.5}$$

The results of our phase plane transformation are collected in the following lemma. Recall that, by Lemma 3.2, Eq. (3.2), any TW with speed $c \in \mathbb{R}$, if it exists, must have speed $c > 0$.

Lemma 4.1 (Existence of the wave profile.) *Assume that* d *and* g *satisfy* Hypotheses **(H1)** *and* **(H2)**, *respectively. Let* $c \in (0, \infty)$. *Then problem* (4.4), (4.5) *has a classical solution* $y \equiv y_c : (0, 1) \to (0, \infty)$ *if and only if problem* (2.4), (2.3) *has a solution* $U : (z_0, z_1) \to (0, \infty)$.

In Sects. 4.1 and 4.2 below we are concerned with the solvability of the overdetermined first-order boundary value problem (4.4), (4.5) with a free parameter $c \in \mathbb{R}$. We address the natural questions, such as existence and nonexistence, and uniqueness and nonuniqueness of a classical solution $y : (0, 1) \to (0, \infty)$. But first, we explain the method how to arrive at the existence and nonexistence results in Sects. 4.1 and 4.2, respectively, by *monotone iterations* (Hartman [18, Chapter III, Sect. 4]).

We begin by the observation that any classical solution $y : (0, 1) \to \mathbb{R}$ to problem (4.4), (4.5) must satisfy $y(r) > 0$ for every $r \in (0, 1)$. On the contrary, suppose that $y(r_0) \le 0$ for some $r_0 \in (0, 1)$. Owing to the zero boundary conditions (4.5), we may assume that y attains its global minimum at r_0, i.e., $y(r_0) = \min_{r\in(0,1)} y(r)$. Hence, we get $y'(r_0) = 0$. But then Eq. (4.4) at $r = r_0$ forces

$$0 = y'(r_0) - 2c\sqrt{y^+(r_0)} = \; -2\, f(r_0) < 0\,,$$

a contradiction. We conclude that Eq. (4.4) is equivalent with

$$\frac{\mathrm{d}}{\mathrm{d}r}\sqrt{y(r)} = c - \frac{f(r)}{\sqrt{y(r)}} \quad \text{where } y(r) > 0 \text{ for every } r \in (0,1)\,. \tag{4.6}$$

In this equation we substitute $Y(r) = c^{-1}\sqrt{y(r)}$ which transforms it into the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}r}\,Y(r) = 1 - \frac{f(r)}{c^2 Y(r)} \quad \text{where } Y(r) > 0 \text{ for every } r \in (0,1)\,. \tag{4.7}$$

Owing to $f(r) > 0$ for every fixed $r \in (0, 1)$, the right-hand side of this equation,

$$F(r,\,\cdot\,) : Y \longmapsto 1 - \frac{f(r)}{c^2 Y} : (0,\infty) \to \mathbb{R}\,,$$

is a strictly monotone increasing function of the variable $Y \in (0, \infty)$. To Eq. (4.7) we attach the initial condition $Y(0) = 0$ and consider the corresponding initial value problem (i.v.p., for short) for $Y(r)$ on an open interval $(0, \delta)$ where $\delta \in (0, 1]$. We use $\delta = 1$ for the existence result in Proposition 4.2 (Sect. 4.1), whereas $\delta \in (0, 1]$ will have to be taken small enough for the nonexistence result in Proposition 4.3 below (Sect. 4.2).

Our method of monotone iterations takes advantage of a standard comparison result from Hartman [18, Chapter III, Sect. 4], Theorem 1.1 (p. 26) and Corollary 4.4 (p. 29), proved by monotone iterations, as well.

4.1 An Existence Result

The following existence result for problem (4.4), (4.5) is essentially a special case of a result due to Enguiça et al. [13, Proposition 2, p. 176].

Proposition 4.2 (Existence of TW) *Assume that* $f = dg : \mathbb{R} \setminus \{0, 1\} \to \mathbb{R}$ *satisfies* $f(1) = 0$ *and*

$$0 < \mu \overset{\text{def}}{=} \sup_{r \in (0,1)} \frac{f(r)}{r} < +\infty\,. \tag{4.8}$$

Then there exists a number $c^* \in \left(0,\, 2\sqrt{\mu}\right]$ *such that problem* (4.4), (4.5) *with speed* $c \in \mathbb{R}$ *admits a unique positive solution if and only if* $c \geq c^*$. *Consequently, also problem* (1.1) *has a TW solution in the sense of* Definition 2.1.

In [13, Sect. 2], this proposition is derived from [13, Proposition 1, p. 176]. Below, we give a more detailed proof which hinges on the monotone iteration procedure (4.13) starting with $X_0(r) = \frac{1}{2} r$ for all $r \in (0, \delta)$.

Proof of Proposition 4.2 It follows from Eq. (4.7) that $Y_0(r) = r$ is a supersolution to our i.v.p. for $r \in (0, 1)$, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}r} Y_0(r) \geq 1 - \frac{f(r)}{c^2 Y_0(r)}\,, \quad r \in (0, 1)\,, \quad \text{and} \quad Y_0(0) = 0\,. \tag{4.9}$$

Recursively for $k = 1, 2, 3, \ldots$, let us define $Y_k(r)$ (for $0 \leq r < \delta$) by its derivative

$$\frac{\mathrm{d}}{\mathrm{d}r} Y_k(r) = 1 - \frac{f(r)}{c^2 Y_{k-1}(r)}\,, \quad r \in (0, \delta)\,, \quad \text{and} \quad Y_k(0) = 0\,. \tag{4.10}$$

If a classical solution $y : (0, 1) \to \mathbb{R}$ to problem (4.4), (4.5) exists, with $y(r) > 0$ for every $r \in (0, 1)$, then we must have

$$\begin{aligned} 0 < c^{-1}\sqrt{y(r)} \leq \cdots \leq Y_k(r) \leq Y_{k-1}(r) \leq \cdots \leq Y_1(r) < Y_0(r) \\ = r \quad \text{for } r \in (0, \delta)\,. \end{aligned} \tag{4.11}$$

On the other hand, given any number $\delta \in (0, 1]$, if $X_0(r) = \alpha r$ should be a subsolution to our i.v.p. for $r \in (0, \delta)$, where $\alpha \in (0, 1]$ is some constant, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}r} X_0(r) \leq 1 - \frac{f(r)}{c^2 X_0(r)}\,, \quad r \in (0, \delta)\,, \quad \text{and} \quad X_0(0) = 0\,, \tag{4.12}$$

then this property is equivalent with the inequality $f(r)/(c^2 r) \leq \alpha(1 - \alpha)$ for all $r \in (0, \delta)$. The least restrictive condition on the ratio $f(r)/r$ is thus obtained for $\alpha = \frac{1}{2}$, namely, $f(r)/r \leq c^2/4$ for all $r \in (0, \delta)$. We now use the subsolution $X_0(r) = \frac{1}{2} r$ (i.e., $\alpha = \frac{1}{2}$) to establish the desired existence result for problem (4.4), (4.5) by a method of monotone iterations analogous to Eq. (4.10) above: Recursively for $k = 1, 2, 3, \ldots$, we define $X_k(r)$ (for $0 \le r < \delta$) by its derivative

$$\frac{\mathrm{d}}{\mathrm{d}r} X_k(r) = 1 - \frac{f(r)}{c^2 X_{k-1}(r)}, \quad r \in (0, \delta), \quad \text{and} \quad X_k(0) = 0. \tag{4.13}$$

Starting with $X_0(r) = \frac{1}{2}r < Y_0(r) = r$ for all $r \in (0, \delta)$, we verify the induction step

$$\begin{aligned} & X_{k-1}(r) < Y_{k-1}(r) \quad (\text{for all } r \in (0, \delta)) \\ \Longrightarrow\ & X_{k-1}(r) \le X_k(r) < Y_k(r) \le Y_{k-1}(r) \quad (\text{for all } r \in (0, \delta)) \end{aligned}$$

for every $k = 1, 2, 3, \ldots$. Either of the monotone limits, $X_\infty(r) = \lim_{k\to\infty} X_k(r)$ and $Y_\infty(r) = \lim_{k\to\infty} Y_k(r)$, for $r \in (0, \delta)$, renders a classical solution $Y : (0, \delta) \to \mathbb{R}$ to the differential equation (4.7), with $Y(r) > 0$ for every $r \in (0, \delta)$ and $Y(0) = 0$.

Setting $c_0 = 2\sqrt{\mu}$ we observe that $f(r)/r \le c_0^2/4$ holds for all $r \in (0, 1)$, by Eq. (4.8). We treat Eq. (4.7) with $c = c_0$. Next, taking $\delta = 1$ and $X_0(r) = \frac{1}{2}r$ (for $r \in [0, 1]$) in Eqs. (4.12) and (4.13), we obtain a monotone increasing sequence of continuous functions $X_0(r) \le X_1(r) \le \cdots \le X_{k-1}(r) \le X_k(r) \le \ldots$ (for $r \in [0, 1]$) which satisfies $X_k(0) = 0$ and $X_k(r) \le Y_0(r) = r$ (for $r \in [0, 1]$). It follows from the integral form of Eq. (4.13) that $X_k(r) \nearrow Y(r)$ as $k \to \infty$ holds pointwise for every $r \in [0, 1]$ and the monotone limit function $Y : [0, 1] \to \mathbb{R}_+$ is continuous and satisfies Eq. (4.7) with $Y(0) = 0$.

Our function $[cY(r)]^2$ just obtained may be used in [13, Proposition 1, p. 176] in place of the function $s(u)$ in order to obtain the desired existence result. If the existence of a classical solution $Y^* : [0, 1] \to \mathbb{R}$ to the differential equation (4.7), with $Y^*(r) > 0$ for every $r \in (0, 1)$ and $Y^*(0) = 0$, is known for some speed $c = c^*$ satisfying $0 < c^* \le c_0 = 2\sqrt{\mu}$, then we may take any $c \ge c^*$ in Eq. (4.7) and conclude that the function $X_0(r) = Y^*(r) > 0$ satisfies in Eq. (4.12) for $r \in (0, 1)$. We proceed as above, in Eq. (4.13), to construct a sequence of continuous functions $X_k : [0, 1] \to \mathbb{R}_+$; $k = 0, 1, 2, \ldots$, that converges to $Y : [0, 1] \to \mathbb{R}_+$ as $k \to \infty$. Again, the desired existence result follows from [13, Proposition 1, p. 176]. $\square$

Some more related existence results can be found in Audrito and Vázquez [4, Theorem 1.3, p. 7653] and [5, Theorem 2.1, p. 217], and Malaguti and Marcelli [21, Theorems 2 and 3, pp. 474–475] for travelling waves distinguished by the *front-* or *sharp-type*; see our Figs. 1, 2 or 3, respectively.

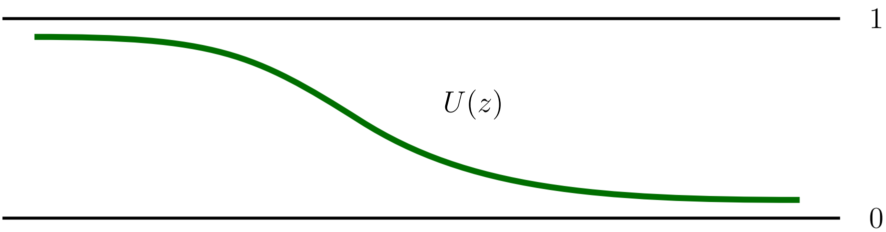

Fig. 1 Travelling wave of front-type with $z_0 = -\infty$, $z_1 = +\infty$

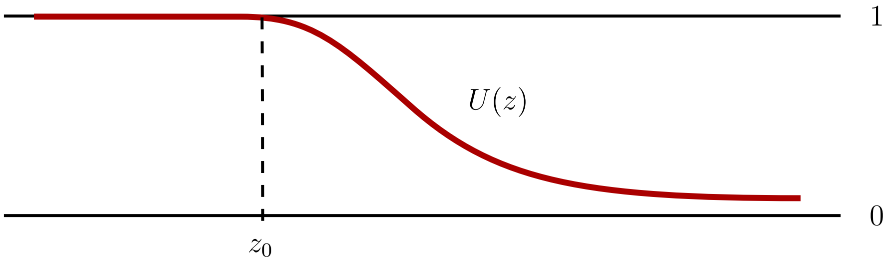

Fig. 2 Travelling wave of front-type with $z_0 > -\infty$, $z_1 = +\infty$

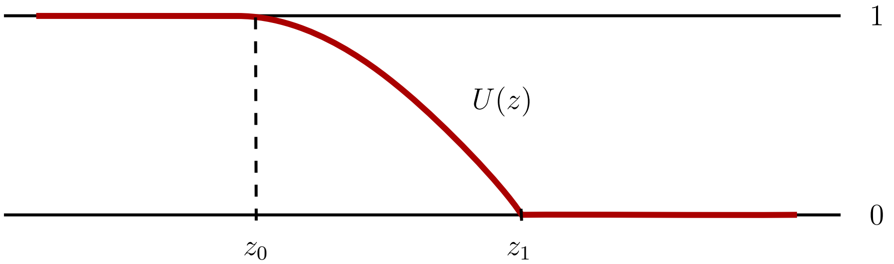

Fig. 3 Travelling wave of sharp-type with $z_0 > -\infty$, $z_1 < +\infty$

4.2 A Nonexistence Result

Now we prove a nonexistence result for a TW $u(x,t) \equiv U(x - ct - \zeta)$ whose profile $U : \mathbb{R} \to \mathbb{R}$ should satisfy the boundary value problem (2.4), (2.3).

Proposition 4.3 (Nonexistence of TW) *Let speed $c \in (0, \infty)$ be arbitrary and assume that there exist $\delta \in (0, 1]$ and $\mu_0 > \frac{1}{4}$ such that the function $f = dg : \mathbb{R} \setminus \{0, 1\} \to \mathbb{R}$ satisfies the following* growth rate *condition,*

$$f(r)/r \geq \mu_0 c^2 \quad \textit{for all } \; r \in (0, \delta)\,. \tag{4.14}$$

Then problem (4.4), (4.5) has no classical solution $y : (0, 1) \to \mathbb{R}$.

In particular, if f satisfies

$$\nu = \liminf_{r \to 0+} \frac{f(r)}{r} > \frac{1}{4}\, c^2\,,$$

or the stronger condition

$$\lim_{r \to 0+} \frac{f(r)}{r} = +\infty\,, \tag{4.15}$$

then problem (4.4), (4.5) has no classical solution $y : (0, 1) \to \mathbb{R}$ for any $c \in \mathbb{R}$. Consequently, also problem (1.1) has no TW solution in the sense of Definition 2.1.

Our nonexistence result generalizes Lemma 3.1 in Enguiça et al. [13, p. 177]. Indeed, we require neither the continuity of the function $f = dg$ on [0, 1], nor the hypothesis $\mu < \infty$ (assumed in [13, p. 177] and in our existence result in Proposition 4.2, Eq. (4.8), as well).

Proof of Proposition 4.3 On the contrary, assume that $y : (0, 1) \to \mathbb{R}$ is a classical solution to problem (4.4), (4.5). Then it must satisfy $y(r) > 0$ for every $r \in (0, 1)$. We recall from Eq. (4.9) that $Y_0(r) = r$ is a supersolution to our i.v.p. for $r \in (0, 1)$. For the number $\delta \in (0, 1]$ specified in Eq. (4.14), let us consider the sequence of functions $Y_k : [0, \delta) \to \mathbb{R}$ defined by Eq. (4.10) recursively for $k = 1, 2, 3, \dots$. We recall that, if a classical solution $y : (0, 1) \to \mathbb{R}$ to problem (4.4), (4.5) exists, with $y(r) > 0$ for every $r \in (0, 1)$, then this sequence satisfies the inequalities in (4.11) for every $r \in (0, \delta)$. Consequently, in order to derive the desired nonexistence result, it will suffice to guarantee that there is a number $r_0 \in (0, \delta)$ such that $\lim_{k\to\infty} Y_k(r_0) = 0$, thus contradicting $y(r_0) > 0$.

We take advantage of the *growth rate* condition (4.14), where $\delta \in (0, 1)$ is some number and $\mu_0 > \frac{1}{4}$ is to be determined below. Then problem (4.10) for $k = 1$ and $Y_1(0) = 0$ has the solution

$$\begin{aligned} Y_1(r) &= r - c^{-2} \int_0^r \frac{f(s)}{Y_0(s)}\,\mathrm{d}s = r - c^{-2} \int_0^r \frac{f(s)}{s}\,\mathrm{d}s \\ &\le r - \mu_0 r = (1 - \mu_0) r \quad \text{for all } r \in (0, \delta)\,. \end{aligned} \tag{4.16}$$

Repeating this step for $k = 2$ and $Y_2(0) = 0$ we arrive at

$$\begin{aligned} Y_2(r) &= r - c^{-2} \int_0^r \frac{f(s)}{Y_1(s)}\,\mathrm{d}s < r - c^{-2} \int_0^r \frac{f(s)}{(1-\mu_0)s}\,\mathrm{d}s \\ &\le r - \mu_0 (1-\mu_0)^{-1} r = \left[1 - \frac{\mu_0}{1-\mu_0}\right] r \quad \text{for all } r \in (0, \delta)\,. \end{aligned} \tag{4.17}$$

Performing this iterative process for all $k = 1, 2, 3, \dots$, as long as $Y_{k-1}(r) > 0$ for every $r \in (0, \delta)$, we finally obtain the estimate

$$\begin{aligned} & Y_k(r) \le a_k r \quad \text{for all } r \in (0, \delta)\,, \quad \text{where} \\ & a_0 = 1 \quad \text{and} \quad a_k = 1 - \mu_0 / a_{k-1}\,;\ k = 1, 2, 3, \dots\,. \end{aligned} \tag{4.18}$$

Recalling our contradictory hypothesis that assumes the existence of a positive classical solution $y : (0, 1) \to \mathbb{R}$ to problem (4.4), (4.5), we deduce from the inequalities in (4.11) that the inequalities

$$0 < c^{-1} \sqrt{y(r)} \le \dots \le a_k r \le a_{k-1} r \le \dots \le a_2 r \le a_1 r \le a_0 r = r$$

hold for all $r \in (0, \delta)$, together with

$$1 = a_0 \ge a_1 \ge a_2 \ge \dots \ge a_{k-1} \ge a_k \ge a_\infty = \lim_{k\to\infty} a_k > 0\,.$$

In particular, taking the limit $k \to \infty$ in Eq. (4.18), we get $a_\infty(1-a_\infty) = \mu_0$. Thanks to $0 < a_\infty \le 1$, the last equation forces $\mu_0 \le \frac{1}{4}$ which is a contradiction to our hypothesis $\mu_0 > \frac{1}{4}$.

We conclude that if $\mu_0 > 1/4$, then problem (4.4), (4.5) has no classical solution $y : [0, 1] \to \mathbb{R}$, such that $y(r) > 0$ for all $r \in (0, 1)$. □

Remark 4.4 In fact, in Proposition 4.2 (in Sect. 4.1), c^* is the minimal travelling wave speed and Eq. (4.8) provides an *upper* bound, $c^* \le 2\sqrt{\mu}$.

On the other hand, our nonexistence result in Proposition 4.3 above provides a *lower* bound for c^*. Indeed, we have shown that

$$c^* \ge 2\sqrt{\nu}\,, \quad \text{where} \quad \nu = \liminf_{r\to 0+} \frac{f(r)}{r}\,.$$

The inequality $\nu > \frac{1}{4}c^2$ is equivalent with $\mu_0 > \frac{1}{4}$ in condition (4.14), where $\delta \in (0, 1]$ is sufficiently small.

We have thus obtained the following estimates on the minimal travelling wave speed, $2\sqrt{\nu} \le c^* \le 2\sqrt{\mu}$.

Remark 4.5 Notice that conditions (4.8) and (4.14) impose a restriction on the mutual relation between the diffusion $d(r)$ and the reaction $g(r)$ as $r \to 0+$. In particular, given a reaction function $g : \mathbb{R} \to \mathbb{R}$ satisfying Hypothesis **(H2)**, diffusion $d(r)$ that degenerates to zero "suitably fast" as $r \to 0+$ may guarantee the *existence* of a solution to problem (4.4), (4.5). On the other hand, diffusion $d(r)$ that blows up to $+\infty$ "suitably fast" as $r \to 0+$ may prevent the *existence* of a solution to (4.4), (4.5).

5 Interaction Between Diffusion and Reaction, Asymptotic Shape of Travelling Waves

In this section we prove a number of specialized results on the profile of a travelling wave for some simple forms of the nonlinearities $d(r)$ and $g(r)$ involved. Our main goal here is to illustrate the biological meaning of our mathematical results rather than to treat mathematically general cases. We restrict ourselves to diffusion and reaction terms $d(r)$ and $g(r)$ having the following *power-type asymptotic behavior* as $r \to 0+$ and $r \to 1-$, respectively, where γ_0, γ_1, δ_0, and δ_1 are some real constants:

$$\begin{cases} \lim\limits_{r\to 0+} \dfrac{g(r)}{r^{\gamma_0}} \overset{\text{def}}{=} g_0 \in (0, \infty)\,, \\ \lim\limits_{r\to 1-} \dfrac{g(r)}{(1-r)^{\gamma_1}} \overset{\text{def}}{=} g_1 \in (0, \infty)\,, \\ \lim\limits_{r\to 0+} \dfrac{d(r)}{r^{\delta_0}} \overset{\text{def}}{=} d_0 \in (0, \infty)\,, \\ \lim\limits_{r\to 1-} \dfrac{d(r)}{(1-r)^{\delta_1}} \overset{\text{def}}{=} d_1 \in (0, \infty)\,. \end{cases} \tag{5.1}$$

The following restrictions on the parameters γ_0, γ_1, δ_0, and δ_1 are imposed by Hypotheses **(H1)** and **(H2)**:

$$\text{Hypothesis } \mathbf{(H1)} \quad \Longrightarrow \quad \delta_0 > -1 \text{ and } \delta_1 > -1\,,$$
$$\text{Hypothesis } \mathbf{(H2)} \quad \Longrightarrow \quad \gamma_0 > 0 \text{ and } \gamma_1 > 0\,.$$

In addition, recalling $f(r) = d(r)\,g(r)$ for every $r \in \mathbb{R} \setminus \{0,\ 1\}$, and f continuous on $[0, 1]$ with $f(0) = f(1) = 0$, we get also the restrictions

$$\gamma_0 + \delta_0 > 0 \quad \text{and} \quad \gamma_1 + \delta_1 > 0\,.$$

In what follows we treat the profile of the travelling wave $r = U(z)$ for values near the equilibrium points $r = 0$ (in Sect. 5.1) and $r = 1$ (in Sect. 5.2).

5.1 Existence of TWs and Asymptotics (5.1) Near 0

Let us define the following parameter sets, see Fig. 4,

$$\mathscr{M}_0^1 \stackrel{\text{def}}{=} \{(\gamma_0, \delta_0) \in \mathbb{R}^2 : \gamma_0 > 0,\ \delta_0 > -1,\ 0 < \gamma_0 + \delta_0 < 1\}\,,$$
$$\mathscr{M}_0^2 \stackrel{\text{def}}{=} \{(\gamma_0, \delta_0) \in \mathbb{R}^2 : \gamma_0 > 0,\ \delta_0 > -1,\ \gamma_0 + \delta_0 \geq 1\}\,.$$

For the parameter pairs $(\gamma_0, \delta_0) \in \mathscr{M}_0^1 \cup \mathscr{M}_0^2$ we have the following conclusions on the existence of travelling waves; see Propositions 4.3 and 4.2 above for further details.

Theorem 5.1 (i) $(\gamma_0, \delta_0) \in \mathscr{M}_0^1$ *implies Eq.* (4.15) *and, hence, no travelling wave exists, by* Proposition 4.3.

(ii) $(\gamma_0, \delta_0) \in \mathscr{M}_0^2$ *implies Eq.* (4.8) *and, hence, a travelling wave exists, by* Proposition 4.2.

5.2 Profile Asymptotics Near 1

Here, we need the following parameter sets, see Fig. 5,

$$\mathscr{M}_1^1 \stackrel{\text{def}}{=} \{(\gamma_1, \delta_1) \in \mathbb{R}^2 : 0 < \gamma_1 < 1 + \delta_1,\ 0 < \gamma_1 + \delta_1 \leq 1\}\,,$$
$$\mathscr{M}_1^2 \stackrel{\text{def}}{=} \{(\gamma_1, \delta_1) \in \mathbb{R}^2 : 0 < 1 + \delta_1 \leq \gamma_1,\ 0 < \gamma_1 + \delta_1 \leq 1\}\,,$$
$$\mathscr{M}_1^3 \stackrel{\text{def}}{=} \{(\gamma_1, \delta_1) \in \mathbb{R}^2 : 0 < \gamma_1 < 1,\ \gamma_1 + \delta_1 > 1\}\,,$$
$$\mathscr{M}_1^4 \stackrel{\text{def}}{=} \{(\gamma_1, \delta_1) \in \mathbb{R}^2 : \gamma_1 \geq 1,\ \delta_1 > -1,\ \gamma_1 + \delta_1 > 1\}\,.$$

From Sect. 4 we recall that $r \mapsto y \equiv y_{\mathrm{c}}(r) : (0, 1) \to (0, \infty)$ is a classical solution of problem (4.4), (4.5).

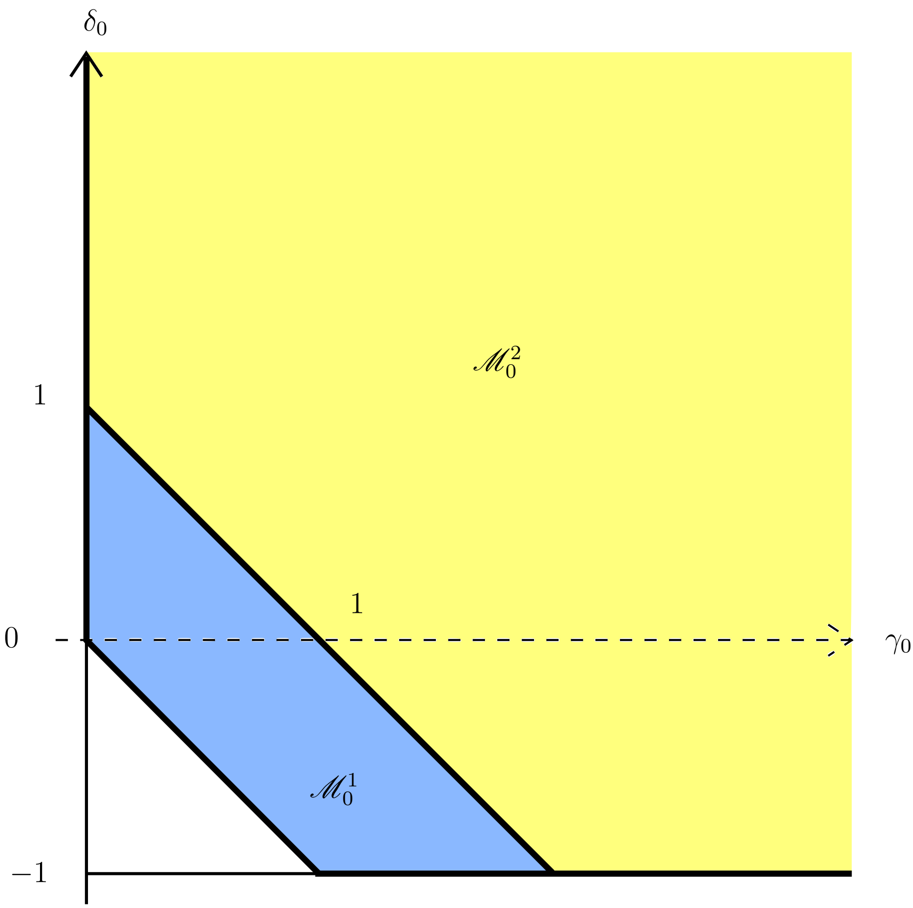

Fig. 4 The sets $\mathscr{M}_0^1$ and $\mathscr{M}_0^2$

In what follows we assume $(\gamma_0, \delta_0) \in \mathscr{M}_0^2$, i.e., $y \equiv y_c(r)$ exists as a solution to the nonlinear two-point boundary value problem (4.4), (4.5) for the unknown function $y : (0, 1) \to (0, \infty)$ with some speed $c > 0$, by Theorem 5.1(ii) and Proposition 4.2. Consequently, a travelling wave with the profile $U : z \mapsto U(z)$ is obtained by the phase plane transformation described in Sect. 4, Lemma 4.1. We classify the parameters γ_1 and δ_1 according to whether $z_0 > -\infty$ or $z_0 = -\infty$. For the parameter pairs $(\gamma_1, \delta_1) \in \cup_{i=1}^4 \mathscr{M}_1^i$ we will prove the following conclusions.

Theorem 5.2 *Assume* $(\gamma_0, \delta_0) \in \mathscr{M}_0^2$. *Then we have:*
(i) $z_0 > -\infty$ *provided* $(\gamma_1, \delta_1) \in \mathscr{M}_1^1 \cup \mathscr{M}_1^3$.
(ii) $z_0 = -\infty$ *provided* $(\gamma_1, \delta_1) \in \mathscr{M}_1^2 \cup \mathscr{M}_1^4$.

Proof We begin with

Case 1 $(\gamma_1, \delta_1) \in \mathscr{M}_1^1$. We will compare the classical solution $y \equiv y_c : (0, 1) \to (0, \infty)$ specified above with the function $w_\kappa(r) \stackrel{\text{def}}{=} \kappa\,(1 - r)^{\gamma_1+\delta_1+1}$ of $r \in [0, 1]$,

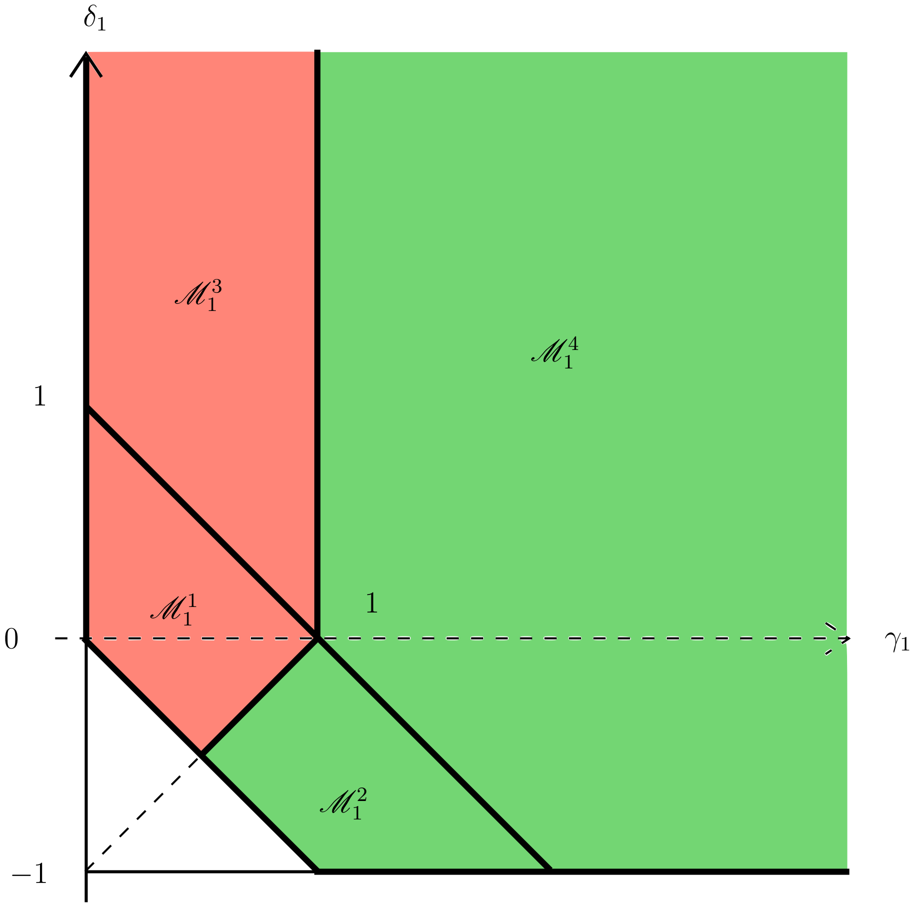

Fig. 5 The sets $\mathscr{M}_1^1$, $\mathscr{M}_1^2$, $\mathscr{M}_1^3$, and $\mathscr{M}_1^4$

where $\kappa > 0$ is a suitable number to be determined later. We set $f_1 = d_1 g_1$ (>0) and write $f(r) = (f_1 + \eta(r))\,(1-r)^{\gamma_1+\delta_1}$, where $\eta : [0,1] \to \mathbb{R}$ is a continuous function with $\eta(1) = 0$.

Then the differential operator in Eq. (4.4) takes the form

$$\mathscr{A}(y)(r) \overset{\text{def}}{=} \frac{\mathrm{d}y}{\mathrm{d}r} - 2c\sqrt{y^+} + 2f(r)\,, \quad r \in (0,1)\,. \tag{5.2}$$

In particular, for the function $w_{\underline{\kappa}}$ defined above, with $\underline{\kappa} > 0$ small enough, we calculate

$$\begin{aligned}\mathscr{A}(w_{\underline{\kappa}})(r) = &- \underline{\kappa}(\gamma_1 + \delta_1 + 1)\,(1-r)^{\gamma_1+\delta_1} - 2c\sqrt{\underline{\kappa}}\,(1-r)^{(\gamma_1+\delta_1+1)/2} \\ &+ 2\,(f_1 + \eta(r))\,(1-r)^{\gamma_1+\delta_1}\,, \quad r \in (0,1)\,.\end{aligned}$$

Since $(\gamma_1, \delta_1) \in \mathscr{M}_1^1$ implies $\gamma_1 + \delta_1 \le \frac{1}{2}(\gamma_1 + \delta_1 + 1)$, the first and third terms above dominate the second one in the following sense, for $r \in (0, 1)$ close enough to 1:

$$\mathscr{A}(w_{\underline{\kappa}})(r) = (1-r)^{\gamma_1+\delta_1} \\ \times \left[-\underline{\kappa}(\gamma_1+\delta_1+1) + 2\left(f_1 + \eta(r)\right) - 2c\sqrt{\underline{\kappa}}\,(1-r)^{(1-\gamma_1-\delta_1)/2} \right], \tag{5.3}$$

provided $\underline{\kappa} > 0$ is chosen small enough, relative to $f_1 > 0$. This way we are able to guarantee

$$\mathscr{A}(w_{\underline{\kappa}})(r) \ge f_1\,(1-r)^{\gamma_1+\delta_1} > 0 \quad \text{for all } r \in (0,1) \text{ close to } 1.$$

Hence, there is a sufficiently small number $\varrho \in (0, 1)$ such that $w_{\underline{\kappa}} : r \mapsto w_{\underline{\kappa}}(r)$ is a *subsolution* for the backward initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}r} = 2\left(c\sqrt{y^+} - f(r)\right), \quad r \in (1-\varrho, 1)\,; \qquad y(1) = 0\,. \tag{5.4}$$

Recall that $c > 0$. Observing that the nonlinearity $y \mapsto \sqrt{y^+}$ is a monotone, nondecreasing function, we conclude that the backward initial value problem (5.4) possesses a unique classical solution $y \equiv y_c(r)$ on the interval $(1-\varrho, 1)$. By a similar monotonicity argument, we arrive at $y_c(r) \ge w_{\underline{\kappa}}(r) = \underline{\kappa}\,(1-r)^{\gamma_1+\delta_1+1}$ for all $r \in (1-\varrho, 1)$. After returning to the original variables from Eqs. (4.1) and (4.3) we obtain

$$V(U) \ge \sqrt{\underline{\kappa}}\,(1-U)^{(\gamma_1+\delta_1+1)/2} \quad \text{for all } U \in (1-\varrho, 1)\,.$$

We combine this inequality with the last limit in (5.1) to conclude that there is a constant $c_1 > 0$ such that

$$-\frac{\mathrm{d}z}{\mathrm{d}U} = \frac{d(U)}{V(U)} \le \frac{c_1}{(1-U)^{(\gamma_1-\delta_1+1)/2}} \quad \text{for all } U \in (1-\varrho, 1)\,. \tag{5.5}$$

Notice that the relation $(\gamma_1, \delta_1) \in \mathscr{M}_1^1$ implies also $\frac{1}{2}(\gamma_1 - \delta_1 + 1) < 1$. We fix an arbitrary number $\tilde{U} \in (1-\varrho, 1)$, denote $\tilde{z} = z(\tilde{U}) \in (z_0, z_1)$ with $U \mapsto z(U) : (0,1) \to (z_0, z_1)$ being the inverse function of $U : (z_0, z_1) \to (0, 1)$, and integrate in Eq. (5.5) with respect to $U \in (\tilde{U}, 1)$, thus arriving at

$$\tilde{z} - z_0 = \int_{z_0}^{\tilde{z}} \mathrm{d}z = \int_1^{\tilde{U}} \frac{\mathrm{d}z}{\mathrm{d}U}\,\mathrm{d}U = -\int_{\tilde{U}}^1 \frac{\mathrm{d}z}{\mathrm{d}U}\,\mathrm{d}U \le c_1 \int_{\tilde{U}}^1 \frac{\mathrm{d}U}{(1-U)^{(\gamma_1-\delta_1+1)/2}} < \infty\,.$$

This estimate forces $z_0 > -\infty$.

Case 2 $(\gamma_1, \delta_1) \in \mathscr{M}_1^3$. Here we compare $y \equiv y_c : (0,1) \to (0,\infty)$ with the new function $w_\kappa(r) \stackrel{\text{def}}{=} \kappa\,(1-r)^{2(\gamma_1+\delta_1)}$ of $r \in [0, 1]$, where $\kappa > 0$ is a suitable number to be determined later again. Using Eq. (5.2), for $\underline{\kappa} > 0$ small enough, we calculate

$$\begin{aligned} \mathscr{A}(w_{\underline{\kappa}})(r) = &- 2\underline{\kappa}(\gamma_1 + \delta_1)\,(1-r)^{2(\gamma_1+\delta_1)-1} - 2c\,\sqrt{\underline{\kappa}}\,(1-r)^{\gamma_1+\delta_1} \\ &+ 2\,(f_1 + \eta(r))\,(1-r)^{\gamma_1+\delta_1}\,, \quad r \in (0,1)\,. \end{aligned} \tag{5.6}$$

Since $(\gamma_1, \delta_1) \in \mathscr{M}_1^3$ implies $2(\gamma_1 + \delta_1) - 1 > \gamma_1 + \delta_1$, the second and third terms above dominate the first one in the following sense, for $r \in (0, 1)$ close enough to 1:

$$\mathscr{A}(w_{\underline{\kappa}})(r) \geq f_1\,(1-r)^{\gamma_1+\delta_1} > 0\,,$$

provided $\underline{\kappa} > 0$ is chosen small enough, relative to $f_1 > 0$. Hence, there is a sufficiently small number $\varrho \in (0, 1)$ such that $w_{\underline{\kappa}} : r \mapsto w_{\underline{\kappa}}(r)$ is a *subsolution* for the backward initial value problem (5.4). It follows that the backward initial value problem (5.4) possesses a unique classical solution $y \equiv y_c(r)$ on the interval $(1-\varrho, 1)$ which satisfies $y_c(r) \geq w_{\underline{\kappa}}(r) = \underline{\kappa}\,(1-r)^{2(\gamma_1+\delta_1)}$ for all $r \in (1-\varrho, 1)$. After returning to the original variables from Eqs. (4.1) and (4.3) we obtain

$$V(U) \geq \sqrt{\underline{\kappa}}\,(1-U)^{\gamma_1+\delta_1} \quad \text{for all } U \in (1-\varrho, 1)\,.$$

We combine this inequality with the last limit in (5.1) to conclude that there is a constant $c_2 > 0$ such that

$$-\frac{\mathrm{d}z}{\mathrm{d}U} = \frac{d(U)}{V(U)} \leq \frac{c_2}{(1-U)^{\gamma_1}} \quad \text{for all } U \in (1-\varrho, 1)\,. \tag{5.7}$$

Notice that the relation $(\gamma_1, \delta_1) \in \mathscr{M}_1^3$ implies also $\gamma_1 < 1$. Consequently, fixing an arbitrary number $\tilde{U} \in (1-\varrho, 1)$, denoting $\tilde{z} = z(\tilde{U}) \in (z_0, z_1)$, and integrating in Eq. (5.7) with respect to $U \in (\tilde{U}, 1)$, we arrive at

$$\tilde{z} - z_0 = -\int_{\tilde{U}}^{1} \frac{\mathrm{d}z}{\mathrm{d}U}\,\mathrm{d}U \leq c_2 \int_{\tilde{U}}^{1} \frac{\mathrm{d}U}{(1-U)^{\gamma_1}} < \infty\,,$$

which forces $z_0 > -\infty$.

Case 3 $(\gamma_1, \delta_1) \in \mathscr{M}_1^2$. This time we compare $y \equiv y_c : (0,1) \to (0,\infty)$ with the function $w_\kappa(r) \stackrel{\text{def}}{=} \kappa\,(1-r)^{\gamma_1+\delta_1+1}$ of $r \in [0, 1]$, where $\kappa > 0$ is a suitable number to be determined later again. From Eq. (5.3) we deduce that there is a sufficiently large number $\bar{\kappa} > 0$ such that

$$\mathscr{A}(w_{\bar{\kappa}})(r) \leq -\bar{\kappa}(\gamma_1 + \delta_1)\,(1-r)^{\gamma_1+\delta_1} < 0 \quad \text{for all } r \in (0,1) \text{ close to } 1.$$

Hence, there is a sufficiently small number $\varrho \in (0, 1)$ such that $w_{\bar{\kappa}} : r \mapsto w_{\bar{\kappa}}(r)$ is a *supersolution* for the backward initial value problem (5.4).

By similar arguments as above, we have $y_c(r) \leq w_{\bar{\kappa}}(r) = \bar{\kappa}\,(1-r)^{\gamma_1+\delta_1+1}$ for all $r \in (1-\varrho, 1)$. After returning to the original variables from Eqs. (4.1) and (4.3) we obtain, with a constant $c_3 > 0$,

$$-\frac{\mathrm{d}z}{\mathrm{d}U} = \frac{d(U)}{V(U)} \geq \frac{c_3}{(1-U)^{(\gamma_1-\delta_1+1)/2}} \quad \text{for all } U \in (1-\varrho, 1)\,. \tag{5.8}$$

Notice that the relation $(\gamma_1, \delta_1) \in \mathscr{M}_1^2$ implies also $\frac{1}{2}(\gamma_1 - \delta_1 + 1) \geq 1$. Again, we fix an arbitrary number $\tilde{U} \in (1-\varrho, 1)$, denote $\tilde{z} = z(\tilde{U}) \in (z_0, z_1)$, and integrate in Eq. (5.8) with respect to $U \in (\tilde{U}, 1)$, thus arriving at

$$\tilde{z} - z_0 = -\int_{\tilde{U}}^1 \frac{\mathrm{d}z}{\mathrm{d}U}\,\mathrm{d}U \geq c_3 \int_{\tilde{U}}^1 \frac{\mathrm{d}U}{(1-U)^{(\gamma_1-\delta_1+1)/2}} = +\infty\,.$$

This estimate forces $z_0 = -\infty$.

Case 4 $(\gamma_1, \delta_1) \in \mathscr{M}_1^4$. Finally, we compare $y \equiv y_c : (0,1) \to (0,\infty)$ with the function $w_\kappa(r) \stackrel{\text{def}}{=} \kappa\,(1-r)^{2(\gamma_1+\delta_1)}$ of $r \in [0,1]$, where $\kappa > 0$ is a suitable number to be determined. From Eq. (5.6) we deduce that there is a sufficiently large number $\bar{\kappa} > 0$ such that

$$\mathscr{A}(w_{\bar{\kappa}})(r) \leq -2\bar{\kappa}(\gamma_1+\delta_1)\,(1-r)^{2(\gamma_1+\delta_1)-1} < 0 \quad \text{for all } r \in (0,1) \text{ close to } 1.$$

Hence, there is a sufficiently small number $\varrho \in (0,1)$ such that $w_{\bar{\kappa}} : r \mapsto w_{\bar{\kappa}}(r)$ is a *supersolution* for the backward initial value problem (5.4).

Similarly as above, we have $y_c(r) \leq w_{\bar{\kappa}}(r) = \bar{\kappa}\,(1-r)^{2(\gamma_1+\delta_1)}$ for all $r \in (1-\varrho, 1)$. After returning to the original variables from Eqs. (4.1) and (4.3) we obtain, with a constant $c_4 > 0$,

$$-\frac{\mathrm{d}z}{\mathrm{d}U} = \frac{d(U)}{V(U)} \geq \frac{c_4}{(1-U)^{\gamma_1}} \quad \text{for all } U \in (1-\varrho, 1)\,. \tag{5.9}$$

Notice that the relation $(\gamma_1, \delta_1) \in \mathscr{M}_1^4$ implies also $\gamma_1 \geq 1$. Again, we fix an arbitrary number $\tilde{U} \in (1-\varrho, 1)$, denote $\tilde{z} = z(\tilde{U}) \in (z_0, z_1)$, and integrate in Eq. (5.9) with respect to $U \in (\tilde{U}, 1)$, thus arriving at

$$\tilde{z} - z_0 = -\int_{\tilde{U}}^1 \frac{\mathrm{d}z}{\mathrm{d}U}\,\mathrm{d}U \geq c_4 \int_{\tilde{U}}^1 \frac{\mathrm{d}U}{(1-U)^{\gamma_1}} = +\infty\,,$$

which forces $z_0 = -\infty$.

The theorem is proved. □

5.3 Comparisons with Previous Results

The first result on the existence of travelling waves of the so-called *sharp-type* for $c = c^*$ was obtained in Sánchez-Garduño and Maini [25, Theorem 2, p. 187]. The authors assume $d(0) = 0$, $d > 0$ in $(0,1]$, $g(0) = g(1) = 0$, $g > 0$ in $(0,1)$, and impose the following additional smoothness assumptions: $d \in C^2([0,1])$, $d'(s) > 0$

and $d''(s) \neq 0$ for all $s \in [0, 1]$, $g \in C^2([0, 1])$, $g'(0) > 0$ and $g'(1) < 0$. These assumptions are weakened to $d \in C([0, 1]) \cap C^1((0, 1])$ and $g \in C([0, 1])$ in Malaguti and Marcelli [21], in Theorems 2, 3, and 14 (pp. 474, 475, and 493). The authors in [21] allow even for $d'(0) = +\infty$ and $d(1) = 0$; of particular interest to us are the existence results for travelling waves of *sharp-type* [21, Theorems 2(b) and 14(b)].

Our results are related to the existence results in [21]. However, our results cover more general asymptotic behavior of both terms, d and g, near the equilibrium points 0 and 1. Indeed, their existence result [21, Theorem 2, p. 474] corresponds to the following parameter values in our case: $\gamma_0 > 0$, $\delta_0 = 1$, $\gamma_1 > 0$, and $\delta_1 = 0$. Another existence result in [21, Theorem 3, p. 475] corresponds to our parameter values $\gamma_0 + \delta_0 > 1$, $0 < \delta_0 < 1$, $\gamma_1 > 0$, and $\delta_1 = 0$. Furthermore, the existence result for doubly degenerate diffusion in [21, Theorem 14, p. 493] corresponds to $\gamma_0 > 0$, $\delta_0 = 1$, $\gamma_1 > 0$, and $\delta_1 = 1$. In each of these cases, for $0 < \gamma_1 < 1$, we obtain a wave profile U with $z_0 > -\infty$, while for $\gamma_1 \geq 1$ we have $z_0 = -\infty$.

Some more related results on qualitative properties of travelling waves can be found also in [1,6,10,12,22].

Acknowledgements The work of Pavel Drábek was supported in part by the Grant Agency of the Czech Republic (GAČR) under Grant No. #18-03253S. Both authors would like to thank Professor Juan Luis Vázquez (Universidad Autónoma de Madrid, Spain) for drawing their attention to several pertinent references and for further useful comments on an earlier version of this manuscript. Very constructive comments and suggestions by an anonymous referee are also highly appreciated with sincere gratitude.

Compliance with Ethical Standards

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Alfaro, M., Giletti, T.: Interplay of nonlinear diffusion, initial tails and Allee effect on the speed of invasions. arXiv:1711.10364v1 (2017)
2. Aronson, D.G., Weinberger, H.F.: Nonlinear diffusion in population genetics, combustion, and nerve propagation. In: Goldstein, J.A. (ed.) Partial Differential Equations and Related Topics, pp. 5–49. Springer, Berlin (1975)
3. Aronson, D.G., Weinberger, H.F.: Multidimensional nonlinear diffusion arising in population genetics. Adv. Math. **30**, 33–76 (1978)
4. Audrito, A., Vázquez, J.-L.: The Fisher–KPP problem with doubly nonlinear diffusion. J. Differ. Equ. **263**, 7647–7708 (2017). https://doi.org/10.1016/j.jde.2017.08.025
5. Audrito, A., Vázquez, J.-L.: The Fisher–KPP problem with doubly nonlinear "fast" diffusion. Nonlinear Anal. **157**, 212–248 (2017). https://doi.org/10.1016/j.na.2017.03.015
6. Calvo, J., Campos, J., Caselles, V., Sánchez, O., Soler, J.: Pattern formation in a flux limited reaction-diffusion equation of porous media type. Invent. Math. **206**, 57–108 (2016). https://doi.org/10.1007/s00222-016-0649-5
7. Corli, A., Malaguti, L.: Semiwavefront solutions in models of collective movements with density-dependent diffusivity. Dyn. PDEs **13**(4), 297–331 (2016)
8. Corli, A., Malaguti, L.: Viscous profiles in models of collective movement with negative diffusivity. Z. Angew. Math. Phys. **70**(2), 47 (2019). https://doi.org/10.1007/s00033-019-1094-2
9. Corli, A., di Ruvo, L., Malaguti, L.: Sharp profiles in models of collective movements. Nonlinear Differ. Equ. Appl. (NoDEA) **24**(4), 40 (2017). https://doi.org/10.1007/s00030-017-0460-z
10. de Pablo, A., Sánchez, A.: Travelling wave behaviour for a Porous–Fisher equation. Euro. J. Appl. Math. **9**, 285–304 (1998)

11. Drábek, P., Takáč, P.: New patterns of travelling waves in the generalized Fisher–Kolmogorov equation. Nonlinear Differ. Equ. Appl. (NoDEA) **23**(2), 7 (2016). https://doi.org/10.1007/s00030-016-0365-2
12. Engler, H.: Relations between travelling wave solutions of quasilinear parabolic equations. Proc. Am. Math. Soc. **93**, 297–302 (1985)
13. Enguiça, R., Gavioli, A., Sanchez, L.: A class of singular first order differential equations with applications in reactiondiffusion. Discrete Contin. Dyn. Syst. **33**(1), 173–191 (2013)
14. Fife, P.C., McLeod, J.B.: The approach of solutions of nonlinear diffusion equations to travelling front solutions. Arch. Ration. Mech. Anal. **65**(4), 335–361 (1977)
15. Fisher, R.A.: The wave of advance of advantageous genes. Ann. Eugenics **7**, 355–369 (1937)
16. Gilding, B.H., Kersner, R.: Travelling waves in nonlinear diffusion convection reaction. In: Brezis, H. (ed.) Progress in Nonlinear Differential Equations and their Applications. Birkhäuser, Basel (2004)
17. Hamel, F., Nadirashvili, N.: Travelling fronts and entire solutions of the Fisher–KPP equation in $\mathbb{R}^N$. Arch. Ration. Mech. Anal. **157**, 91–163 (2001)
18. Hartman, Ph: Ordinary Differential Equations, 2nd edn. Birkhäuser, Boston (1982)
19. King, J.R., McCabe, PhM: On the Fisher–KPP equation with fast nonlinear diffusion. Proc. R. Soc. Lond. A **459**(2038), 2529–2546 (2003)
20. Kolmogoroff, A.N., Petrovsky, I.G., Piscounoff, N.S.: Ètude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique, Bulletin Université d'État à Moscou (Bjul. Moskowskogo Gos. Univ.). Série internationale A **1**, 1–25 (1937)
21. Malaguti, L., Marcelli, C.: Sharp profiles in degenerate and doubly degenerate Fisher–KPP equations. J. Differ. Equ. **195**, 471–496 (2003). https://doi.org/10.1016/j.jde.2003.06.005
22. McCabe, PhM, Leach, J.A., Needham, D.J.: A note on the nonexistence of permanent form travelling wave solutions in a class of singular reactiondiffusion problems. Dyn. Syst. **17**(2), 131–135 (2002)
23. Murray, J.D.: Mathematical biology. In: Levin, S.A. (ed.) Biomathematics Texts. Springer, Berlin (1993)
24. Murray, J.D.: Mathematical biology I: an introduction. In: Murray, J.D. (ed.) Interdisciplinary Applied Mathematics, vol. 17. Springer, Berlin (2002)
25. SánchezGarduño, F., Maini, PhK: Existence and uniqueness of a sharp travelling wave in degenerate nonlinear diffusion Fisher–PKK equation. J. Math. Biol. **33**, 163–192 (1994)
26. Tsoularis, A., Wallace, J.: Analysis of logistic growth models. Math. Biosci. **179**(1), 21–55 (2002)