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Global Existence and Blow-Up for a Parabolic Problem of Kirchhoff Type with Logarithmic Nonlinearity

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Abstract

In this paper, we study the following parabolic problem of Kirchhoff type with logarithmic nonlinearity:

$$\begin{cases} u_t + M([u]_s^2)\mathcal{L}_K u = |u|^{p-2}u \log |u|, & \text{in } \Omega \times (0, +\infty), \\ u(x,t) = 0, & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, +\infty), \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases}$$

where $[u]_s$ is the Gagliardo seminorm of u, $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $0 < s < 1$, $\mathcal{L}_K$ is a nonlocal integro-differential operator defined in (1.2), which generalizes the fractional Laplace operator $(-\Delta)^s$, u_0 is the initial function, and $M : [0, +\infty) \to [0, +\infty)$ is continuous. Let $J(u_0)$ be the initial energy (see (2.1) for the definition of J), $d > 0$ be the mountain-pass level given in (2.4), and $\widetilde{M} \in (0, d]$ be the constant defined in (2.6). Firstly, we get the conditions on global existence and finite time blow-up for $J(u_0) \leq d$. Then we study the lower and upper bounds of blow-up time to blow-up solutions under some appropriate conditions. Secondly, for $J(u_0) \leq \widetilde{M}$, the growth rate of the solution is got. Moreover, we give some blow-up conditions independent of d and study the upper bound of the blow-up time. Thirdly, the behavior of the energy functional as $t \to T$ is also discussed, where T is the blow-up time. In addition, for $J(u_0) \leq d$, we give some equivalent conditions for the solutions blowing up in finite time or existing globally. Finally, we consider the existence of ground state solutions and the asymptotical behavior of the general global solution.

Keywords Parabolic problem of Kirchhoff type · Logarithmic nonlinearity · Global existence · Blow-up · Ground-state solution

Mathematics Subject Classification Primary: 35K55; Secondary: 35R11 · 47G20 · 35B44 · 35Q91

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1 Introduction

In this paper, we study the global existence and blow-up phenomena for the following fractional Kirchhoff-type parabolic problem with logarithmic nonlinearity:

$$\begin{cases} u_t + M([u]_s^2)\mathcal{L}_K u = |u|^{p-2}u \log |u|, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in (\mathbb{R}^N \setminus \Omega),\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where

$$[u]_s^2 := \iint\limits_{\mathbb{R}^{2N}} |u(x,t) - u(y,t)|^2 K(x-y)dxdy,$$

$\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $\mathcal{L}_K$ is a nonlocal integro-differential operator, which is defined by

$$\mathcal{L}_K \varphi(x) := \frac{1}{2} \int\limits_{\mathbb{R}^N} (2\varphi(x) - \varphi(x+y) - \varphi(x-y))K(y)dy, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N). \tag{1.2}$$

Here, $K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^+$ is a function with the following properties:

(k_1) $\gamma K \in L^1(\mathbb{R}^N)$, with $\gamma(x) := \min\{|x|^2, 1\}$;
(k_2) there exists $K_0 > 0$ such that $K(x) \geq K_0|x|^{-N-2s}$ for all $x \in \mathbb{R}^N \setminus \{0\}$.

Furthermore, we make the following assumptions:

(M_1) $0 < s < 1$, $M(\tau) := a + b\tau^{\theta-1}$ for $\tau \in \mathbb{R}_0^+ := [0, +\infty)$ ($a \geq 0, b > 0$ are two constants), $\theta \in \left[1, 2_s^*/2\right)$, $p \in (2\theta, 2_s^*)$. Here,

$$2_s^* := \begin{cases} \dfrac{2N}{N-2s}, & \text{if } 2s < N; \\ \infty, & \text{if } 2s \geq N. \end{cases}$$

In the past few decades, more and more attention has been devoted to the study of Kirchhoff type problems. More specifically, Kirchhoff in 1883 proposed the following Kirchhoff model

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int\limits_0^L \left| \frac{\partial u(x)}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

which was as a generalization of the well-known D'Alembert wave equation for free vibrations of elastic strings, where the above constants have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension.

It is worth mentioning that the above equation received much attention after the work of Lions [24], where a functional analysis framework was proposed for the following higher dimension problem in presence of an external force term f:

$$\frac{\partial^2 u}{\partial t^2} - \left(a + b \int_\Omega |\nabla u(x)|^2 dx \right) \Delta u = f(x, u),$$

where Δ denotes the Euclidean Laplace operator.

Recently, in [14], Han and Li studied the following initial boundary value problem for a class of Kirchhoff type parabolic equation with a nonlinear term

$$\begin{cases} u_t - M\left(\int_\Omega |\nabla u|^2 dx \right) \Delta u = |u|^{q-1} u, & (x,t) \in \Omega \times (0,T), \\ u = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.3}$$

Here the diffusion coefficient $M(\tau) = a + b\tau$ with the parameters a, b being positive, $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bound domain with smooth boundary $\partial\Omega$, $3 < q \leq 2^* - 1$, where 2^* is the Sobolev conjugate of 2. By using the potential well theory and variational methods, the authors obtained the global existence and finite time blow-up of solutions when the initial energy was subcritical, critical and supercritical. After this work, in [15], the authors investigated the upper and lower bounds of blow-up time to the blow-up solutions of problem (1.3).

It is well known that many mathematical models involving fractional and nonlocal operators are actively studied in recent years. More precisely, this type of operators arises in a quite natural way in many applications, such as finance, physics, fluid dynamics, population dynamics, image processing, minimal surfaces and game theory. As for the research motivation, we would like to point out that Applebaum in [1] viewed the fractional Laplacian operators of the form $(-\Delta)^s$ as the infinitesimal generators of stable radially symmetric Lévy processes. Laskin in [19] deduced the fractional Schrödinger equation as a result of expanding the Feynman path integral, from the Brownian-like to the Lévy-like quantum mechanical paths. In particular, we would like to point out that $(-\Delta)^s$ can be reduced to the classical Laplace operator $-\Delta$ as $s \to 1^-$, see [9] for more details. For more recent results involving the fractional Laplacian, interested readers may refer to, for example, [2,3,9,11,18,26,36] and the references therein.

In particular, in [32], the authors studied the following parabolic equations of Kirchhoff type involving the fractional Laplacian:

$$\begin{cases} \partial_t u + M([u]_s^2)\mathcal{L}_K u = |u|^{p-2} u, & \text{in } \Omega \times (0, +\infty), \\ u(x,t) = 0, & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, +\infty), \\ u(x,0) = u_0(x), & \text{in } \Omega. \end{cases}$$

By using the Galerkin method and differential inequality technique, the local existence of weak solutions and the conditions on blow-up were studied.

In recent years, logarithmic nonlinearity appears frequently in partial differential equations which describes important physical phenomena (see [5,6,10,12,16,17,25, 29,42]) and the references therein). Especially, in the classical case, Chen and Tian [5] studied the following semilinear pseudo-parabolic equation with logarithmic nonlinearity:

$$u_t - \Delta u_t - \Delta u = u \log |u| \tag{1.4}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ with zero Dirichlet boundary condition. By using the logarithmic Sobolev inequality (see [7,8,21]), they studied the existence of global solution, blow-up at ∞ and behavior of vacuum isolation of the solutions, and they also compared the difference between logarithmic nonlinearity and polynomial nonlinearity.

Inspired by the above works, in the present article we consider model (1.1). To our best knowledge, this is the first attempt to study the properties of the solutions for Kirchhoff-type equation with logarithmic nonlinearity. In this paper, we mainly discuss the properties of global existence and finite time blow-up for the solutions of problem (1.1) when the initial energy is subcritical and critical by potential well method which was established by Payne and Sattinger [27] and the concavity method which was established by Levine [22,23], see also [12,13,33–35,40,41,43] and references therein for more applications of these two methods. Furthermore, we also obtain the growth estimates of blow-up solutions. Moreover, the blow-up conditions independent of mountain-pass level are also investigated. In particular, under some appropriate conditions, we obtain the upper and lower bounds of blow-up time to blow-up solutions of problem (1.1). Finally, we consider the ground state solutions for the stationary problem. Here we say the initial energy is subcritical and critical if $J(u_0) < d$ and $J(u_0) = d$ are satisfied respectively, where $J(u_0)$ denotes the initial energy defined in (2.1) and $d > 0$ is the mountain-pass level defined in (2.4). We remark that to handle the logarithmic nonlinear term of problem (1.1), we use some other methods instead of logarithmic Sobolev inequality, which is a key inequality to get the results in [4–6,20,21,38].

Throughout this paper, we denote by $(\cdot, \cdot)$ the $L^2(\Omega)$-inner product, i.e.

$$(\phi, \varphi) = \int_\Omega \phi(x)\varphi(x)dx, \quad \forall \phi, \varphi \in L^2(\Omega).$$

We also denote the norm of $L^\gamma(\Omega)$ for $1 \leq \gamma \leq \infty$ by $\| \cdot \|_\gamma$. That is, for any $u \in L^\gamma(\Omega)$,

$$\|u\|_\gamma = \begin{cases} \left(\int_\Omega |u(x)|^\gamma dx \right)^{\frac{1}{\gamma}}, & \text{if } 1 \leq \gamma < \infty; \\ \operatorname{ess}\sup_{x \in \Omega} |u(x)|, & \text{if } \gamma = \infty. \end{cases}$$

Now we recall some necessary properties of fractional Sobolev spaces which will be used later. Let X be the linear space of Lebesgue measurable function $u : \mathbb{R}^N \to \mathbb{R}$ whose restrictions to Ω belong to $L^2(\Omega)$ and such that

$$\text{the map } (x, y) \mapsto |u(x) - u(y)|^2 K(x - y) \text{ is in } L^1(\mathcal{Q}, dxdy),$$

where $\mathcal{Q} = \mathbb{R}^{2N} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ and $\mathcal{C}\Omega = \mathbb{R}^N \setminus \Omega$. The space X is endowed with the norm

$$\|\varphi\|_X = \left(\|\varphi\|_2^2 + \iint_{\mathcal{Q}} |u(x) - u(y)|^2 K(x - y) dxdy \right)^{\frac{1}{2}}, \tag{1.5}$$

for all $\varphi \in X$. We observe that bounded and Lipschitz functions belong to X, thus X is not reduced to $\{0\}$.

The functional space Z denotes the closure of $C_0^\infty(\Omega)$ in X. The scalar product defined for any $\varphi, \psi \in Z$ as

$$\langle \varphi, \psi \rangle_Z = \iint_{\mathcal{Q}} (\varphi(x) - \varphi(y))(\psi(x) - \psi(y)) K(x - y) dxdy, \tag{1.6}$$

makes Z a Hilbert space. The norm

$$\|\varphi\|_Z = \left(\iint_{\mathcal{Q}} |\varphi(x) - \varphi(y)|^2 K(x - y) dxdy \right)^{\frac{1}{2}} \tag{1.7}$$

is equivalent to the usual norm defined in (1.5). Note that in (1.5)–(1.7) the integrals can be extended to $\mathbb{R}^{2N}$, since $u = 0$ a.e. in $\mathcal{C}\Omega$. By Lemma 6 of [28] and (k_1), the Hilbert space $Z = (Z, \|\cdot\|_Z)$ is continuously embedded in $L^r(\Omega)$ for any $r \in [1, 2_s^*]$. Hence there exists $C_r > 0$ such that

$$\|u\|_r \le C_r \|u\|_Z \text{ for all } u \in Z \text{ and } r \in [1, 2_s^*]. \tag{1.8}$$

Next, we consider the eigenvalue of the operator $\mathcal{L}_K$ with homogeneous Dirichlet boundary data, namely the eigenvalue of the the problem (see [32])

$$\begin{cases} -\mathcal{L}_K u = \lambda u, & \text{in } \Omega; \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.9}$$

we denote by λ_1 the first eigenvalue of problem (1.9), i.e.

$$\lambda_1 = \inf_{u \in Z \setminus \{0\}} \frac{\|u\|_Z^2}{\|u\|_2^2} \in (0, \infty). \tag{1.10}$$

The rest of this paper is organized as follows. In Sect. 2, we state the main results of this paper. In Sect. 3, we give some important lemmas, which will be used in the proof of the main results. In Sect. 4, we give the proof of the main results.

2 Main Results

In this section, we will give the main results of this paper and we always assume (M_1) holds. The energy functional J and the Nehari functional I are as follows:

$$J(u) := \frac{a}{2}\|u\|_Z^2 + \frac{b}{2\theta}\|u\|_Z^{2\theta} - \frac{1}{p}\int_\Omega |u|^p \log|u| dx + \frac{1}{p^2}\|u\|_p^p, \tag{2.1}$$

and

$$I(u) := \langle J'(u), u\rangle = a\|u\|_Z^2 + b\|u\|_Z^{2\theta} - \int_\Omega |u|^p \log|u| dx, \tag{2.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the dual pairing between Z and Z'.

By (M_1), we know that $2 < p < 2_s^*$. Let $\varrho := 2_s^* - p > 0$. Since $\log(|u|^\varrho) \leq |u|^\varrho$, it follows from (1.8) that

$$\int_\Omega |u|^p \log|u| dx = \frac{1}{\varrho}\int_\Omega |u|^p \log\left(|u|^\varrho\right) dx \leq \frac{1}{\varrho}\int_\Omega |u|^{p+\varrho} dx \leq \frac{1}{\varrho}\left(C_{2_s^*}\|u\|_Z\right)^{2_s^*}$$

and $\|u\|_p \leq C_p\|u\|_Z$. So J and I are well-defined for $u \in Z$.

Obviously, from (2.1) and (2.2), we have

$$J(u) = \frac{1}{p}I(u) + \frac{(p-2)a}{2p}\|u\|_Z^2 + \frac{(p-2\theta)b}{2\theta p}\|u\|_Z^{2\theta} + \frac{1}{p^2}\|u\|_p^p. \tag{2.3}$$

Let

$$d := \inf_{u\in N} J(u) \tag{2.4}$$

denote the mountain-pass level, where N is the Nehari manifold, which is defined by

$$N := \{u \in Z \setminus \{0\} \mid I(u) = 0\}. \tag{2.5}$$

By Lemma 5, we know that d satisfies

$$d \geq \widetilde{M} := \frac{a\theta r_*^2(p-2) + br_*^{2\theta}(p-2\theta)}{2\theta p}, \tag{2.6}$$

where r_* is a positive constant defined in (3.6) of Lemma 4.

Moreover, we define

$$N_+ := \{u \in Z \mid I(u) > 0\}\,, \tag{2.7}$$
$$N_- := \{u \in Z \mid I(u) < 0\}\,. \tag{2.8}$$

Finally, the potential well W and its corresponding set V are defined by

$$W := \{u \in Z \mid I(u) > 0,\, J(u) < d\} \cup \{0\}, \tag{2.9}$$
$$V := \{u \in Z \mid I(u) < 0,\, J(u) < d\}. \tag{2.10}$$

To state the main results succinctly, we need the following two definitions.

Definition 1 (*Weak solution*) A function $u = u(t) \in L^\infty(0, T; Z)$ is called a weak solution of problem (1.1), if $u_t \in L^2(0, T; L^2(\Omega))$ and the following equality holds

$$\int_\Omega u_t\phi dx + \left(a + b\|u\|_Z^{2\theta-2}\right)\iint_Q (u(x,t) - u(y,t))(\phi(x) - \phi(y))K(x-y)dxdy = \int_\Omega |u|^{p-2}u\log|u|\phi dx \tag{2.11}$$

for all $\phi \in Z$. Moreover, the following inequality

$$\int_0^t \|u_\tau\|_2^2 d\tau + J(u(t)) \leq J(u_0) \tag{2.12}$$

holds for a.e. $t \in (0, T)$.

Definition 2 (*Maximal existence time*) Let $u = u(t)$ be a weak solution of problem (1.1). We define the maximal existence time T of u as follows:

(1) If u exists for all $0 \leq t < \infty$, then $T = \infty$;
(2) If there exists a $t_0 \in (0, \infty)$ such that u exists for $0 \leq t < t_0$, but doesn't exist at $t = t_0$, then $T = t_0$.

Based on the above preparations, the main results of this paper are as follows. The first result is about global existence.

Theorem 1 *Let (M_1) hold, $u_0 \in Z$. Assume that $J(u_0) < d$ and $I(u_0) > 0$. Then problem (1.1) admits a global weak solution $u(t) \in L^\infty(0, \infty; Z)$ with $u_t \in L^2(0, \infty; L^2(\Omega))$ and $u(t) \in W$ for $0 \leq t < \infty$. Furthermore, the weak solution is unique if it is bounded. Moreover, for any $\varepsilon \in (0, 2_s^* - p]$, if $J(u_0) < d(\varepsilon)$, then*

$$\|u\|_2^2 \leq F(\varepsilon) := \begin{cases} \|u_0\|_2^2 e^{-C_\varepsilon t}, & \text{if } \theta = 1, \\ \left(C_\varepsilon(\theta-1)t + \|u_0\|_2^{2-2\theta}\right)^{-\frac{1}{\theta-1}}, & \text{if } \theta \in \left(1, \frac{2_s^*}{2}\right), \end{cases}$$

where

$$d(\varepsilon) := \frac{(p-2\theta)br^{2\theta}(\varepsilon)}{2\theta p} \leq d,$$

$$C_\varepsilon := 2\lambda_1^\theta \left[b - \frac{C_*^{p+\varepsilon}}{\varepsilon} \left(\frac{2\theta p J(u_0)}{(p-2\theta)b} \right)^{\frac{p+\varepsilon-2\theta}{2\theta}} \right] > 0.$$

Here λ_1, $r(\varepsilon)$ *and* C_* *are defined in* (1.10), (3.2) *and* (3.3) *respectively.*

Remark 1 We show that $d(\varepsilon) \leq d$. In fact, for any $u \in N$. By (2) of Lemma 3, we get $\|u\|_Z > r(\varepsilon)$. Then it follows from from (2.3) that

$$\begin{aligned} J(u) &= \frac{1}{p} I(u) + \frac{(p-2)a}{2p}\|u\|_Z^2 + \frac{(p-2\theta)b}{2\theta p}\|u\|_Z^{2\theta} + \frac{1}{p^2}\|u\|_p^p \\ &\geq \frac{(p-2\theta)b}{2\theta p}\|u\|_Z^{2\theta} \geq \frac{(p-2\theta)br^{2\theta}(\varepsilon)}{2\theta p} = d(\varepsilon). \end{aligned}$$

Then by the definition of d in (2.4), we get $d(\varepsilon) \leq d$.

By using Theorem 1, we get the following corollary:

Corollary 1 *Let* (M_1) *hold,* $u_0 \in Z$. *Assume that* $J(u_0) \leq d$ *and* $I(u_0) \geq 0$. *Then problem* (1.1) *admits a global weak solution* $u(t) \in L^\infty(0, \infty; Z)$ *with* $u_t \in L^2(0, \infty; L^2(\Omega))$ *and* $u(t) \in \overline{W}$ *for* $0 \leq t < \infty$.

As the other side of the above theorem, we have the following blow-up result.

Theorem 2 *Let* (M_1) *hold,* $u_0 \in Z$. *If* $J(u_0) \leq d$, $I(u_0) < 0$, *and* $u = u(t)$ *is a corresponding solution of problem* (1.1), *then* $u(t)$ *blows up at some finite time* T *in the sense of*

$$\lim_{t \to T^-} \int_0^t \|u\|_2^2 d\tau = \infty.$$

Moreover,

1. if $J(u_0) < d$, then

$$T \leq \frac{4(p-1)\|u_0\|_2^2}{p(d-J(u_0))(p-2)^2};$$

2. if $2\theta < p < 2\theta + 2 - 4\theta/2_s^*$, then for any $\varepsilon \in \big(0, 2\theta + 2 - 4\theta/2_s^* - p\big)$, it holds

$$T > \frac{1}{2\widehat{C}(\zeta-1)\|u_0\|_2^{2(\zeta-1)}}$$

and

$$\|u\|_2 > \left(2\widehat{C}(\zeta-1)(T-t)\right)^{-\frac{1}{2(\zeta-1)}},$$

where

$$\zeta = \frac{\beta\theta(p+\varepsilon)}{2\theta-(1-\beta)(p+\varepsilon)} > 1 \quad \text{(see Remark 2)},$$
$$\widehat{C} = \left(\frac{\widetilde{C}^{p+\varepsilon}}{\varepsilon b^{\frac{(1-\beta)(p+\varepsilon)}{2\theta}}}\right)^{\frac{2\theta}{2\theta-(1-\beta)(p+\varepsilon)}}.$$

Here,

$$\tilde{C} = \sup_{u\in Z\setminus\{0\}} \frac{\|u\|_{p+\varepsilon}}{\|u\|_Z^{1-\beta}\|u\|_2^{\beta}} \in (0,\infty) \quad \text{(see Remak 2)} \tag{2.13}$$

and

$$\beta = \frac{2(2_s^*-p-\varepsilon)}{(p+\varepsilon)(2_s^*-2)} \in (0,1) \quad \text{(see Remak 2)}. \tag{2.14}$$

Remark 2 In this remark, we show that $\beta \in (0,1)$, $\tilde{C}$ is well-defined and $\zeta > 1$.

1. Since $\theta \in \left[1, 2_s^*/2\right)$, $\varepsilon \in \left(0, 2\theta+2-4\theta/2_s^*-p\right)$ and $2_s^* > 2$, we get

$$p+\varepsilon < 2\theta+2-\frac{4\theta}{2_s^*} = 2\left(1-\frac{2}{2_s^*}\right)\theta+2 < 2\left(1-\frac{2}{2_s^*}\right)\frac{2_s^*}{2}+2 = 2_s^*, \tag{2.15}$$

   which implies $\beta > 0$. On the other hand, by $p+\varepsilon > 2$, we obtain $2\cdot 2_s^* < 2_s^*(p+\varepsilon)$, i.e., $2\cdot 2_s^* - 2p - 2\varepsilon < 2_s^*(p+\varepsilon) - 2p - 2\varepsilon = (p+\varepsilon)(2_s^*-2)$, thus, we get $\beta \in (0,1)$.
2. Since $2 < p+\varepsilon < 2_s^*$, we get there exists a positive constant such that

$$\|u\|_{p+\varepsilon} \le C\|u\|_{2_s^*}^{1-\beta}\|u\|_2^{\beta},$$

   which, together with (1.8), implies

$$\|u\|_{p+\varepsilon} \le CC_{2_s^*}^{1-\beta}\|u\|_Z^{1-\beta}\|u\|_2^{\beta}.$$

   Then $\tilde{C}$ is well-defined. Here β satisfies

$$\frac{1}{p+\varepsilon} = \frac{1-\beta}{2_s^*} + \frac{\beta}{2},$$

   i.e., (2.14) holds.

3. Now, we prove $\zeta > 1$. In fact, from the definitions of ζ and β, by a direct computation, we have

$$\zeta = \frac{2\theta(2_s^* - p - \varepsilon)}{2\theta(2_s^* - 2) - 2_s^*(p + \varepsilon - 2)}. \tag{2.16}$$

Since $\varepsilon < 2\theta + 2 - 4\theta/2_s^* - p$, we get

$$2_s^*(p + \varepsilon - 2) < 2_s^* \left(2\theta - \frac{4\theta}{2_s^*}\right) = 2\theta(2_s^* - 2),$$

which, together with (2.15) and (2.16), implies

$$\begin{aligned} \zeta > 1 &\Leftrightarrow 2\theta(2_s^* - p - \varepsilon) > 2\theta(2_s^* - 2) - 2_s^*(p + \varepsilon - 2) \\ &\Leftrightarrow 2_s^*(p + \varepsilon - 2) > 2\theta(2_s^* - 2) - 2\theta(2_s^* - p - \varepsilon) = 2\theta(p + \varepsilon - 2). \end{aligned}$$

Then by $p + \varepsilon > 2$ and $\theta < 2_s^*/2$, we get $\zeta > 1$.

The next theorem shows lower bound of the growth rate for the solution got in above theorem under more specific assumptions on $J(u_0)$ and $I(u_0)$ (note that, by (2.6), $\widetilde{M} \le d$).

Theorem 3 *Let* (M_1) *hold,* $u_0 \in Z$ *satisfy* $I(u_0) < 0$ *and* $J(u_0) \le \widetilde{M}$*. Then for any* $\gamma \in \left[0, 2/2_s^*\right]$*, there exists a* $t_\gamma \in (0, T)$ *such that the weak solution* $u = u(t)$ *of problem* (1.1) *satisfies*

$$\|u\|_2^2 \ge C_\gamma (t^{\frac{p\gamma}{2}} - t^{\frac{p\gamma}{2}-1} t_\gamma)^{\frac{2}{2-p\gamma}}$$

for all $t \in [t_\gamma, T)$*, where*

$$C_\gamma := \left[\left(1 - \frac{p\gamma}{2}\right) G^{-\frac{p\gamma}{2}}(t_\gamma) G'(t_\gamma)\right]^{\frac{2}{2-p\gamma}},$$

and

$$G(t) := \int_0^t \|u\|_2^2 d\tau.$$

Remark 3 Since $p < 2_s^*$ and $\gamma \in \left[0, 2/2_s^*\right]$, we have $p\gamma < 2$. Then the constant C_γ is well-defined and $1 \le 2/(2 - p\gamma) < \infty$.

In view of the above results, one can see they all depend on the mountain-pass level d. Next, we give some blow-up results independent of d but related to λ_1, where $\lambda_1 > 0$ is the first eigenvalue of problem (1.9).

Theorem 4 *Let* (M_1) *hold and* $u = u(t)$ *be a weak solution to problem* (1.1)*. If*

$$J(u_0) < \frac{(p-2\theta)b\lambda_1^\theta}{2\theta p}\|u_0\|_2^{2\theta}, \tag{2.17}$$

then $u(t)$ *blows up at some finite time* T *in the sense of*

$$\lim_{t\to T^-}\int_0^t \|u\|_2^2 d\tau = \infty.$$

Moreover, we have

$$T \le \frac{8(p-1)\theta\|u_0\|_2^2}{(p-2)^2\left[(p-2\theta)b\lambda_1^\theta\|u_0\|_2^{2\theta} - 2p\theta J(u_0)\right]}.$$

Furthermore, $u(t)$ *grows exponentially with* L^2*-norm for all* $t \in [0, T)$*, that is,*

$$\|u\|_2^2 \ge \left(\|u_0\|_2^2 - \frac{2p}{A}J(u_0)\right)e^{At} + \frac{2p}{A}J(u_0),$$

where $A = \frac{(p-2\theta)b\lambda_1^\theta\|u_0\|_2^{2\theta-2}}{\theta}$.

The next theorem is about the asymptotic behavior of $J(u(t))$ as $t \to T$, where $u(t)$ is the blow-up solution got from the above theorems.

Theorem 5 *Let* $u(t)$ *be the blow-up solution of problem* (1.1) *with* $I(u_0) < 0$*,* $J(u_0) \le d$ *or* (2.17) *holds, and assume* T *is the maximum existence time of* $u(t)$*, then*

$$\lim_{t\to T} J(u(t)) = -\infty. \tag{2.18}$$

Next, we derive some sufficient and necessary conditions for the solutions blowing up in finite time.

Theorem 6 *Let* $u(t)$ *be a solution of problem* (1.1) *and* $T \in (0, +\infty]$ *be the maximal existence time of* $u(t)$*. Then*

1. *if* $u_0 \in Z \setminus \{0\}$ *and* $J(u_0) < d$*, we have following conclusions:*
   (1) $I(u_0) < 0 \Leftrightarrow T < +\infty \Leftrightarrow$ *there exists a* $t_0 \in [0, T)$ *such that* $J(u(t_0)) < 0$*;*
   (2) $I(u_0) > 0 \Leftrightarrow T = +\infty \Leftrightarrow J(u(t)) > 0$ *for all* $t \in [0, T)$*;*
2. *if* $u_0 \in Z \setminus \{N \cup \{0\}\}$ *and* $J(u_0) = d$*, we have following conclusions:*
   (3) $I(u_0) < 0 \Leftrightarrow T < +\infty \Leftrightarrow$ *there exists a* $t_0 \in [0, T)$ *such that* $J(u(t_0)) < 0$*;*
   (4) $I(u_0) > 0 \Leftrightarrow T = +\infty \Leftrightarrow J(u(t)) > 0$ *for all* $t \in [0, T)$*,*

where N *is defined in* (2.5)*.*

The next problem we will consider is can the mountain-pass level d defined in (2.4) be achieved by some $u \in N$? To this end, we consider the steady-state corresponding to problem (1.1), i.e., the following boundary value problem:

$$\begin{cases} M([u]_s^2)\mathcal{L}_K u = |u|^{p-2}u \log |u|, & x \in \Omega, \\ u = 0, & x \in (\mathbb{R}^N \setminus \Omega). \end{cases} \tag{2.19}$$

A function $u \in Z$ is called a weak solution of problem (2.19), if the following equality

$$\left(a + b\|u\|_Z^{2\theta-2}\right) \iint_{\mathcal{Q}} (u(x) - u(y))(\phi(x) - \phi(y))K(x - y)dxdy$$
$$= \int_{\Omega} |u|^{p-2}u \log |u|\phi dx$$

holds for all $\phi \in Z$. Then we introduce the set

$$\begin{aligned} \Gamma &= \{\text{weak solutions of problem (2.19)}\} \\ &= \{u \in Z : J'(u) = 0 \text{ in } Z'\} \\ &= \{u \in Z : \langle J'(u), \phi\rangle = 0, \ \forall \phi \in Z\}, \end{aligned} \tag{2.20}$$

where J is defined in (2.1), Z' is the dual space of Z, and $\langle \cdot, \cdot\rangle$ is the dual product between Z' and Z. We have the following two theorems:

Theorem 7 *Assume* (M_1) *hold. Let* N *be the set defined in* (2.5)*, then there exists a function* $v_0 \in N$ *such that*

(1) $J(v_0) = \inf_{u \in N} J(u) = d$;
(2) v_0 *is a ground-state solution of problem* (2.19)*, i.e.,* $v_0 \in \Gamma \setminus \{0\}$ *and* $J(v_0) = \inf_{u \in \Gamma \setminus \{0\}} J(u)$.

By Theorem 1, we know that the global solution converges to 0 as $t \to \infty$ when u_0 satisfies some special conditions, how about the general global solutions? For this question, we have the following results:

Theorem 8 *Assume* (M_1) *hold. Let* $u = u(t)$ *be a global solution to problem* (1.1)*. Then there exists a* $u^* \in \Gamma$ *and an increasing sequence* $\{t_k\}_{k=1}^{\infty}$ *with* $t_k \to \infty$ *as* $k \to \infty$ *such that*

$$\lim_{k \to \infty} \|u(t_k) - u^*\|_Z = 0.$$

3 Preliminaries

In this section, we give some lemmas, which will be needed in our proofs. Throughout this section, we denote by $u = u(t)$ the solution to problem (1.1) with initial value u_0, whose maximal existence time is T.

Let (M_1) hold. For any ε satisfying

$$0 < \varepsilon \leq 2_s^* - p, \tag{3.1}$$

we define

$$r(\varepsilon) := \left(\frac{b\varepsilon}{C_*^{p+\varepsilon}} \right)^{\frac{1}{p+\varepsilon-2\theta}} > 0, \tag{3.2}$$

where C_* is the optimal embedding constant of $Z \hookrightarrow L^{p+\varepsilon}(\Omega)$ (see (1.8)), i.e.

$$\frac{1}{C_*} = \inf_{u \in Z \setminus \{0\}} \frac{\|u\|_Z}{\|u\|_{p+\varepsilon}}. \tag{3.3}$$

The following lemma is used to derive the upper bound of the blow-up time.

Lemma 1 [22,23] *Suppose that* $0 < T \leq \infty$ *and suppose a nonnegative function* $F(t) \in C^2[0, T)$ *satisfies*

$$F''(t)F(t) - (1+\gamma)(F'(t))^2 \geq 0$$

for some constant $\gamma > 0$*. If* $F(0) > 0$*,* $F'(0) > 0$*, then*

$$T \leq \frac{F(0)}{\gamma F'(0)} < \infty$$

and $F(t) \to \infty$ *as* $t \to T$*.*

Lemma 2 [28] *For any bounded sequence* $\{v_j\}_{j=1}^{\infty}$ *in* Z *and any* $m \in [1, 2_s^*)$*, there exists a* $v \in L^m(\mathbb{R}^N)$*, with* $v = 0$ *a.e. in* $\mathbb{R}^N \setminus \Omega$*, such that up to a subsequence, still denoted by* $\{v_j\}_{j=1}^{\infty}$*,*

$$v_j \to v \quad \textit{strongly in } L^m(\Omega) \textit{ as } j \to \infty.$$

Lemma 3 *Assume* (M_1) *hold. Let* $u \in Z \setminus \{0\}$*. Then for any* ε *satisfying* (3.1) *we have*

(1) *if* $0 < \|u\|_Z \leq r(\varepsilon)$*, then* $I(u) > 0$*;*
(2) *if* $I(u) \leq 0$*, then* $\|u\|_Z > r(\varepsilon)$*,*

where $r(\varepsilon)$ *is defined in* (3.2)*.*

Proof Since $u \in Z \setminus \{0\}$, we get $|u(x)| > 0$ for a.e. $x \in \Omega$. By a simple computation, we know (for any $\varepsilon > 0$)

$$\log |u(x)| < \frac{|u(x)|^{\varepsilon}}{\varepsilon} \text{ for a.e. } x \in \Omega.$$

Then by the above inequality and the definition of $I(u)$, we have

$$
\begin{aligned}
I(u) &= a\|u\|_Z^2 + b\|u\|_Z^{2\theta} - \int_\Omega |u|^p \log|u| dx \\
&> b\|u\|_Z^{2\theta} - \frac{\|u\|_{p+\varepsilon}^{p+\varepsilon}}{\varepsilon}.
\end{aligned} \tag{3.4}
$$

For any ε satisfying (3.1), it follows from (3.3) that

$$
\|u\|_{p+\varepsilon}^{p+\varepsilon} \leq C_*^{p+\varepsilon} \|u\|_Z^{p+\varepsilon},
$$

then by (3.4) we get

$$
\begin{aligned}
I(u) &> b\|u\|_Z^{2\theta} - \frac{C_*^{p+\varepsilon}}{\varepsilon} \|u\|_Z^{p+\varepsilon} \\
&= \|u\|_Z^{2\theta} \left(b - \frac{C_*^{p+\varepsilon}}{\varepsilon} \|u\|_Z^{p+\varepsilon-2\theta} \right).
\end{aligned} \tag{3.5}
$$

(1) If $0 < \|u\|_Z \leq r(\varepsilon)$, then it follows from (3.2) that

$$
b - \frac{C_*^{p+\varepsilon}}{\varepsilon} \|u\|_Z^{p+\varepsilon-2\theta} \geq 0,
$$

so by (3.5) we obtain $I(u) > 0$.

(2) If $I(u) \leq 0$, according to (3.5), we get

$$
b - \frac{C_*^{p+\varepsilon}}{\varepsilon} \|u\|_Z^{p+\varepsilon-2\theta} < 0,
$$

which implies

$$
\|u\|_Z > r(\varepsilon).
$$

□

Lemma 4 *Assume (M_1) hold. With the notations in Lemma 3,*

$$
r_* := \sup_{\varepsilon \in (0, 2_s^* - p]} r(\varepsilon) \tag{3.6}
$$

exists and

$$
0 < r_* \leq r^* < \infty, \tag{3.7}
$$

where

$$r^* := \sup_{\varepsilon \in (0, 2_s^* - p]} \sigma(\varepsilon) \tag{3.8}$$

and

$$\sigma(\varepsilon) := \left(\frac{b\varepsilon}{\kappa^{p+\varepsilon}} \right)^{\frac{1}{p+\varepsilon-2\theta}} |\Omega|^{\frac{\varepsilon}{p(p+\varepsilon-2\theta)}}. \tag{3.9}$$

Here, $|\Omega|$ is the measure of Ω, κ is the optimal embedding constant of $Z \hookrightarrow L^p(\Omega)$, i.e.,

$$\frac{1}{\kappa} = \inf_{u \in Z \setminus \{0\}} \frac{\|u\|_Z}{\|u\|_p}. \tag{3.10}$$

Proof Obviously r_*, if it exists, is positive. So in order to prove the lemma. We only need to prove $r(\varepsilon) \leq \sigma(\varepsilon)$, r^* exists and $r^* < \infty$.

Firstly, we prove $r(\varepsilon) \leq \sigma(\varepsilon)$. For any $u \in Z$, since (M_1) holds and $\varepsilon \in (0, 2_s^* - p]$, we have $u \in L^p(\Omega) \cap L^{p+\varepsilon}(\Omega)$. By Hölder's inequality we have

$$\int_\Omega |u|^p dx \leq |\Omega|^{\frac{\varepsilon}{p+\varepsilon}} \left(\int_\Omega |u|^{p+\varepsilon} dx \right)^{\frac{p}{p+\varepsilon}},$$

which, together with (3.3) and (3.10), implies

$$\begin{aligned} \frac{1}{C_*} &= \inf_{u \in Z \setminus \{0\}} \frac{\|u\|_Z}{\|u\|_{p+\varepsilon}} \\ &\leq |\Omega|^{\frac{\varepsilon}{p(p+\varepsilon)}} \inf_{u \in Z \setminus \{0\}} \frac{\|u\|_Z}{\|u\|_p} \\ &= \frac{1}{\kappa} |\Omega|^{\frac{\varepsilon}{p(p+\varepsilon)}}. \end{aligned} \tag{3.11}$$

Then it follows from (3.2) that

$$r(\varepsilon) = \left(\frac{b\varepsilon}{C_*^{p+\varepsilon}} \right)^{\frac{1}{p+\varepsilon-2\theta}} \leq \sigma(\varepsilon), \tag{3.12}$$

where $\sigma(\varepsilon)$ is defined in (3.9).

Secondly, we prove r^* exists and $r^* < \infty$. Since $\varepsilon \in (0, 2_s^* - p]$ and $\sigma(\varepsilon)$ is continuous on $[0, 2_s^* - p]$, we have r^* exists and

$$r^* = \sup_{\varepsilon \in (0, 2_s^* - p]} \sigma(\varepsilon) \leq \max_{\varepsilon \in [0, 2_s^* - p]} \sigma(\varepsilon) < \infty.$$

□

Based on the above two lemmas, we have the following corollary:

Corollary 2 *Assume* (M_1) *hold. Let* $u \in Z \setminus \{0\}$.

(1) *if* $0 < \|u\|_Z < r_*$, *then* $I(u) > 0$;
(2) *if* $I(u) \leq 0$, *then* $\|u\|_Z \geq r_*$,

where r_* *is defined in* (3.6) *of Lemma* 4.

Proof We only need to prove (1) since (2) is the direct result of (1). We fix $u \in Z \setminus \{0\}$ such that $0 < \|u\|_Z < r_*$. Then by the definition of r_* in (3.6), there exists a ε_0 satisfying (3.1) such that $\|u\|_Z \leq r(\varepsilon_0)$, where $r(\cdot)$ is defined in (3.2). Then by (1) of Lemma 3, $I(u) > 0$. □

Lemma 5 *Assume* (M_1) *hold. Then we have*

$$d \geq \frac{a\theta r_*^2(p-2) + br_*^{2\theta}(p-2\theta)}{2\theta p}, \tag{3.13}$$

where d *is defined in* (2.4) *and* r_* *is defined in* (3.6) *of Lemma* 4.

Proof For all $u \in N$, we have $u \in Z \setminus \{0\}$ and $I(u) = 0$. Thus by (2) of Corollary 2, we know $\|u\|_Z \geq r_*$, and then from (2.3) we get

$$\begin{aligned}
J(u) &= \frac{1}{p}I(u) + \frac{(p-2)a}{2p}\|u\|_Z^2 + \frac{(p-2\theta)b}{2\theta p}\|u\|_Z^{2\theta} + \frac{1}{p^2}\|u\|_p^p \\
&\geq \frac{(p-2)a}{2p}\|u\|_Z^2 + \frac{(p-2\theta)b}{2\theta p}\|u\|_Z^{2\theta} \\
&\geq \frac{(p-2)a}{2p}r_*^2 + \frac{(p-2\theta)b}{2\theta p}r_*^{2\theta} \\
&= \frac{a\theta r_*^2(p-2) + br_*^{2\theta}(p-2\theta)}{2\theta p},
\end{aligned}$$

which gives (3.13). □

Lemma 6 *Assume* (M_1) *hold. Let* $u \in Z$ *satisfy* $I(u) < 0$. *Then there exists a* $\lambda^* \in (0,1)$ *such that* $I(\lambda^* u) = 0$.

Proof We divide the proof into two cases.

Case 1: $a = 0$. Let

$$\phi(\lambda) := \lambda^{p-2\theta}\int_\Omega |u|^p \log|\lambda u|dx, \quad \lambda \in (0,\infty).$$

Then for any $\lambda > 0$, by the definition of $I(u)$, we have

$$\begin{aligned} I(\lambda u) &= b\lambda^{2\theta}\|u\|_Z^{2\theta} - \int_\Omega |\lambda u|^p \log|\lambda u| dx \\ &= \lambda^{2\theta}\left(b\|u\|_Z^{2\theta} - \lambda^{p-2\theta}\int_\Omega |u|^p \log|\lambda u| dx\right) \\ &= \lambda^{2\theta}\left(b\|u\|_Z^{2\theta} - \phi(\lambda)\right). \end{aligned} \tag{3.14}$$

Since $I(u) < 0$, by (3.14) and (2) of Corollary 2 we get

$$\phi(1) > b\|u\|_Z^{2\theta} \geq br_*^{2\theta} > 0. \tag{3.15}$$

On the other hand, by the definition of $\phi(\lambda)$, we have

$$\phi(\lambda) = \lambda^{p-2\theta}\int_\Omega |u|^p \log|u| dx + \lambda^{p-2\theta}\log\lambda\|u\|_p^p,$$

which, together with $p > 2\theta$, implies

$$\lim_{\lambda\to 0^+}\phi(\lambda) = 0.$$

So by (3.15), we get that there exists a $\lambda^* \in (0, 1)$ such that $\phi(\lambda^*) = b\|u\|_Z^{2\theta}$ and then $I(\lambda^* u) = 0$.

Case 2: $a > 0$. Let

$$\phi(\lambda) := \lambda^{p-2}\int_\Omega |u|^p \log|\lambda u| dx - b\lambda^{2\theta-2}\|u\|_Z^{2\theta}, \quad \lambda \in (0, \infty).$$

Then for any $\lambda > 0$, by the definition of $I(u)$, we have

$$\begin{aligned} I(\lambda u) &= a\lambda^2\|u\|_Z^2 + b\lambda^{2\theta}\|u\|_Z^{2\theta} - \int_\Omega |\lambda u|^p \log|\lambda u| dx \\ &= \lambda^2\left(a\|u\|_Z^2 + b\lambda^{2\theta-2}\|u\|_Z^{2\theta} - \lambda^{p-2}\int_\Omega |u|^p \log|\lambda u| dx\right) \\ &= \lambda^2\left(a\|u\|_Z^2 - \phi(\lambda)\right). \end{aligned} \tag{3.16}$$

Since $I(u) < 0$, by (3.16) and (2) of Corollary 2 we get

$$\phi(1) > a\|u\|_Z^2 \geq ar_*^2 > 0. \tag{3.17}$$

On the other hand, by the definition of $\phi(\lambda)$, we have

$$\phi(\lambda)=\lambda^{p-2}\int_{\Omega}|u|^p\log|u|dx+\lambda^{p-2}\log\lambda\|u\|_p^p-b\lambda^{2\theta-2}\|u\|_Z^{2\theta},$$

which, together with $p>2\theta\geq 2$, implies

$$\lim_{\lambda\to 0^+}\phi(\lambda)=0.$$

So by (3.17), we get that there exists a $\lambda^*\in(0,1)$ such that $\phi(\lambda^*)=a\|u\|_Z^2$ and then $I(\lambda^* u)=0$. □

Lemma 7 *Assume* (M_1) *hold. Let* $u\in Z$ *satisfy* $I(u)<0$. *Then*

$$I(u)<p\left(J(u)-d\right). \tag{3.18}$$

Proof First from Lemma 6 we know that there exists a $\lambda^*\in(0,1)$ such that $I(\lambda^* u)=0$. Set

$$g(\lambda):=pJ(\lambda u)-I(\lambda u),\quad \lambda>0.$$

By a direct computation, we obtain

$$g(\lambda)=\frac{a\lambda^2(p-2)}{2}\|u\|_Z^2+\frac{b\lambda^{2\theta}(p-2\theta)}{2\theta}\|u\|_Z^{2\theta}+\frac{\lambda^p}{p}\|u\|_p^p.$$

Then from (2) of Corollary 2, we get

$$\begin{aligned}g'(\lambda)&=a\lambda(p-2)\|u\|_Z^2+b\lambda^{2\theta-1}(p-2\theta)\|u\|_Z^{2\theta}+\lambda^{p-1}\|u\|_p^p\\&\geq b\lambda^{2\theta-1}(p-2\theta)\|u\|_Z^{2\theta}\\&>b\lambda^{2\theta-1}(p-2\theta)r_*^{2\theta}\\&>0,\end{aligned}$$

which implies that $g(\lambda)$ is strictly increasing for $\lambda>0$, hence according to $0<\lambda^*<1$ we get $g(1)>g(\lambda^*)$, namely

$$pJ(u)-I(u)>pJ(\lambda^* u)-I(\lambda^* u)=pJ(\lambda^* u)\geq pd,$$

where the last inequality we have used the $\lambda^* u\in N$ and $d=\inf_{\phi\in N}J(\phi)$, which gives (3.18) immediately. □

Lemma 8 *Let* (M_1) *hold and* $u=u(t)$ *be the corresponding solution to problem* (1.1). *Then for all* $t\in[0,T)$ *we have*

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2=-I(u). \tag{3.19}$$

Proof Let $\phi = u(t)$ in (2.11) of Definition 1, we get

$$\int_\Omega u_t u dx + \left(a + b\|u\|_Z^{2\theta-2}\right)\|u\|_Z^2 = \int_\Omega |u|^p \log|u| dx,$$

i.e.

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 = -a\|u\|_Z^2 - b\|u\|_Z^{2\theta} + \int_\Omega |u|^p \log|u| dx.$$

Thus, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 = -I(u).$$

□

Lemma 9 *If $J(u_0) \leq d$, then the sets N_- and N_+ are both invariant for $u(t)$, i.e., if $u_0 \in N_-$ (resp. $u_0 \in N_+$), then $u(t) \in N_-$ (resp. $u(t) \in N_+$) for all $t \in [0, T)$.*

Proof We only proof the invariance of N_- since the proof of the invariance of N_+ is similar.

Firstly, we consider the case $J(u_0) < d$. If the conclusion is not true, it follows $J(u(t)) \leq J(u_0) < d$ for $t \in [0, T)$ (see the energy inequality (2.12)) that there exists a $t_0 \in (0, T)$ such that

- $I(u(t_0)) = 0$ and $I(u(t)) < 0$ for $t \in [0, t_0)$.

From (2) of Corollary 2 we have $\|u\|_Z > r_* > 0$ for $t \in [0, t_0)$, then by the continuity of $\|u\|_Z$ with respect to t, we get $\|u(t_0)\|_Z \geq r_* > 0$, hence $u(t_0) \in N$. Then it follows from the definition of d in (2.4) that $J(u(t_0)) \geq d$, a contradiction.

Secondly, we consider the case $J(u_0) = d$. If the conclusion is not true, then by $I(u_0) < 0$, there must be a $t_1 \in (0, T)$ such that $I(u(t_1)) = 0$ and $I(u(t)) < 0$ for $t \in [0, t_1)$. On the one hand, we get from (2) of Corollary 2 that $\|u\|_Z > r_* > 0$ for $t \in [0, t_1)$, which implies that $u(t_1) \neq 0$. Then we have $u(t_1) \in N$ and then it follows from the definition of d in (2.4) that

$$J(u(t_1)) \geq d. \tag{3.20}$$

On the other hand, from $(u_t, u) = -I(u(t)) > 0$ (see Lemma 8) for $t \in [0, t_1)$ and $u(t)|_{\partial\Omega} = 0$ we deduce $u_t \neq 0$ and then $\int_0^{t_1} \|u_\tau\|_2^2 d\tau > 0$. So by (2.12) we obtain

$$J(u(t_1)) \leq J(u_0) - \int_0^{t_1} \|u_\tau\|_2^2 d\tau < d,$$

which conflicts with (3.20). □

4 Proof of the Theorems

Proof of Theorem 1 We divide the proof into three steps.

Step 1: Existence of a global weak solution Let ω_j, $j = 1, 2, \ldots$ be the eigenfunctions of the operator $\mathcal{L}_k$ subject to the Dirichlet boundary condition (see [32]):

$$\begin{cases} -\mathcal{L}_K \omega_j = \lambda_j \omega_j, & x \in \Omega, \\ \omega_j = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

we also normalize ω_j such that $\|\omega_j\|_2 = 1$. Then $\{\omega_j\}_{j=1}^{\infty}$ is a basis of Z.

First we construct the following approximate solutions $u_m(t)$ of problem (1.1):

$$u_m = \sum_{j=1}^{m} g_{jm}(t)\omega_j(x), \ m = 1, 2 \ldots \tag{4.1}$$

which satisfy

$$\begin{cases} \int_\Omega u_{mt}\omega_j dx + \left(a + b\|u_m\|_Z^{2\theta-2}\right) \iint_{\mathcal{Q}} (u_m(x) \\ -u_m(y))(\omega_j(x) - \omega_j(y))K(x-y)dxdy \\ = \int_\Omega |u_m|^{p-2}u_m \log|u_m|\omega_j dx, \\ (u_m(0), \omega_j) = \xi_{jm}, \end{cases} \tag{4.2}$$

for $j = 1, 2, \ldots, m$, where $(\cdot, \cdot)$ means the inner product of $L^2(\Omega)$ and ξ_{jm} are given constants such that

$$u_m(0) = \sum_{j=1}^{m} \xi_{jm}\omega_j(x) \to u_0 \text{ in } Z \tag{4.3}$$

as $m \to \infty$. Existence of such ξ_{jm} follows from $u_0 \in Z$, and $\{\omega_j\}_{j=1}^{\infty}$ is a basis of Z. The standard theory of ODEs, e.g. Peano's theorem, yields that there exists a $T > 0$ depending only on ξ_{jm}, $j = 1, 2, \ldots, m$, such that in $g_{jm} \in C^1[0, T]$ and $g_{jm}(0) = \xi_{jm}$. Thus $u_m \in C^1([0, T]; Z)$.

We now try to get a priori estimates for the approximate solution $u_m(t)$. Multiplying the first equation of (4.2) by $g'_{jm}(t)$, summing for j from 1 to m and integrating with respect to time from 0 to t, we can obtain

$$\int_0^t \|u_{m\tau}\|_2^2 d\tau + J(u_m(t)) = J(u_m(0)), \ \ 0 \le t \le T.$$

Due to (4.3) and $g_{jm}(0) = \xi_{jm}$, one has (note that we have assumed that $J(u_0) < d$ and $I(u_0) > 0$)

$$\lim_{m\to\infty} J(u_m(0)) = J(u_0) < d$$

and

$$\lim_{m\to\infty} I(u_m(0)) = I(u_0) > 0.$$

Therefore, for sufficiently large m, we have

$$\int_0^t \|u_{m\tau}\|_2^2 d\tau + J(u_m(t)) = J(u_m(0)) < d, \quad 0 \le t \le T, \tag{4.4}$$

and

$$I(u_m(0)) > 0,$$

which implies that $u_m(0) \in W$ for sufficiently large m [see the definition of W in (2.9)].

Next, we prove $u_m(t) \in W$ for sufficiently large m and any $t \in [0, T]$. Indeed, if it is false, there exists a sufficiently large m and a $t_0 \in (0, T]$ such that $u_m(t_0) \in \partial W$, which implies that $u_m(t_0) \in Z \setminus \{0\}$ and $J(u_m(t_0)) = d$ or $I(u_m(t_0)) = 0$. From (4.4), $J(u_m(t_0)) = d$ is not true. So $u_m(t_0) \in N$, then by the definition of d in (2.4), we have $J(u_m(t_0)) \ge d$, which also contradicts (4.4). Hence, $u_m(t) \in W$ for sufficiently large m and any $t \in [0, T]$.

By (4.4), $I(u_m(t)) > 0$ for sufficiently large m (since $u_m(t) \in W$ for sufficiently large m) and the fact that (see the definition of J and I in (2.1) and (2.2), respectively)

$$J(u_m(t)) = \frac{1}{p} I(u_m(t)) + \frac{(p-2)a}{2p}\|u_m\|_Z^2 + \frac{(p-2\theta)b}{2\theta p}\|u_m\|_Z^{2\theta} + \frac{1}{p^2}\|u_m\|_p^p,$$

we obtain

$$\int_0^t \|u_{m\tau}\|_2^2 d\tau + \frac{(p-2)a}{2p}\|u_m\|_Z^2 + \frac{(p-2\theta)b}{2\theta p}\|u_m\|_Z^{2\theta} + \frac{1}{p^2}\|u_m\|_p^p < d,$$

holds for sufficiently large m and any $t \in [0, T]$, which yields

$$\int_0^t \|u_{m\tau}\|_2^2 d\tau < d, \quad \forall t \in [0, T], \tag{4.5}$$

$$\|u_m\|_Z^{2\theta} < \frac{2\theta p d}{(p-2\theta)b}, \quad \forall t \in [0, T], \tag{4.6}$$

and

$$\|u_m\|_p^p < p^2 d, \quad \forall t \in [0, T]. \tag{4.7}$$

So $T = \infty$. Then $u_m(t) \in W$ for all $t \in [0, \infty)$ and all the above inequalities hold for $t \in [0, \infty)$.

On the other hand, by a direct calculation, we know

$$\begin{aligned}
&\int_\Omega \left| |u_m(t)|^{p-2} u_m(t) \log |u_m(t)| \right|^{\frac{p}{p-1}} dx \\
&= \int_{\{x\in\Omega:|u_m(t)|\le 1\}} \left| |u_m(t)|^{p-1} \log |u_m(t)| \right|^{\frac{p}{p-1}} dx \\
&\quad + \int_{\{x\in\Omega:|u_m(t)|>1\}} \left| |u_m(t)|^{p-1} \log |u_m(t)| \right|^{\frac{p}{p-1}} dx.
\end{aligned} \tag{4.8}$$

Since

$$\inf_{\tau\in(0,1)} \tau^{p-1} \log \tau = \tau^{p-1} \log \tau \Big|_{\tau=e^{-1/(p-1)}} = -\frac{1}{(p-1)e},$$

we have

$$\int_{\{x\in\Omega;|u_m(t)|\le 1\}} \left| |u_m(t)|^{p-1} \log |u_m(t)| \right|^{\frac{p}{p-1}} dx \le \left(\frac{1}{(p-1)e} \right)^{\frac{p}{p-1}} |\Omega|, \quad \forall t \in [0, \infty). \tag{4.9}$$

Moreover, since $\log \tau \le \frac{1}{\mu} \tau^\mu$ for all $\mu,\ \tau \in (0, \infty)$, we can choose a positive constant μ such that $\frac{p(p+\mu-1)}{p-1} \in [1, 2_s^*]$, then we get from (4.6) that for m sufficiently large,

$$\begin{aligned}
&\int_{\{x\in\Omega;|u_m(t)|>1\}} \left| |u_m(t)|^{p-1} \log |u_m(t)| \right|^{\frac{p}{p-1}} dx \\
&\le \mu^{\frac{p}{1-p}} \int_{\{x\in\Omega:|u_m(t)|>1\}} \left| |u_m(t)|^{p+\mu-1} \right|^{\frac{p}{p-1}} dx \\
&= \mu^{\frac{p}{1-p}} \int_{\{x\in\Omega;|u_m(t)|>1\}} |u_m(t)|^{\frac{p(p+\mu-1)}{p-1}} dx \\
&\le \mu^{\frac{p}{1-p}} \|u_m\|_{\frac{p(p+\mu-1)}{p-1}}^{\frac{p(p+\mu-1)}{p-1}} \le \mu^{\frac{p}{1-p}} C_{**}^{\frac{p(p+\mu-1)}{p-1}} \|u_m\|_Z^{\frac{p(p+\mu-1)}{p-1}}
\end{aligned}$$

$$< \mu^{\frac{p}{1-p}} C_{**}^{\frac{p(p+\mu-1)}{p-1}} \left(\frac{2\theta pd}{(p-2\theta)b} \right)^{\frac{p(p+\mu-1)}{2\theta(p-1)}}, \quad \forall t \in [0, \infty), \tag{4.10}$$

where C_{**} is the optimal embedding constant of $Z \hookrightarrow L^{\frac{p(p+\mu-1)}{p-1}}(\Omega)$.

Then it follows from (4.8), (4.9) and (4.10) that, for m large enough and $t \in [0, \infty)$,

$$\begin{aligned} &\int_\Omega \left| |u_m(t)|^{p-2} u_m(t) \log |u_m(t)| \right|^{\frac{p}{p-1}} dx \\ &\le C_d := \left(\frac{1}{(p-1)e} \right)^{\frac{p}{p-1}} |\Omega| + \mu^{\frac{p}{1-p}} C_{**}^{\frac{p(p+\mu-1)}{p-1}} \left(\frac{2\theta pd}{(p-2\theta)b} \right)^{\frac{p(p+\mu-1)}{2\theta(p-1)}}. \end{aligned} \tag{4.11}$$

Therefore, by (4.5), (4.6) and (4.11), there is a function $u = u(t) \in L^\infty(0, \infty; Z)$ with $u_t \in L^2(0, \infty; L^2(\Omega))$, $\chi = \chi(t) \in L^2\left(0, \infty; L^{\frac{p}{p-1}}(\Omega)\right)$ and a subsequence of $\{u_m\}_{m=1}^\infty$ (still denoted by $\{u_m\}_{m=1}^\infty$) such that for each $\widetilde{T} > 0$, as $m \to \infty$,

$$u_{mt} \rightharpoonup u_t \text{ weakly in } L^2(0, \widetilde{T}; L^2(\Omega)), \tag{4.12}$$

$$u_m \rightharpoonup u \text{ weakly star in } L^\infty(0, \widetilde{T}; Z), \tag{4.13}$$

$$u_m \rightharpoonup u \text{ weakly in } L^2(0, \widetilde{T}; Z), \tag{4.14}$$

$$|u_m|^{p-2} u_m \log |u_m| \rightharpoonup \chi(t) \text{ weakly star in } L^\infty\left(0, \widetilde{T}; L^{\frac{p}{p-1}}(\Omega)\right), \tag{4.15}$$

$$|u_m|^{p-2} u_m \log |u_m| \rightharpoonup \chi(t) \text{ weakly in } L^2\left(0, \widetilde{T}; L^{\frac{p}{p-1}}(\Omega)\right). \tag{4.16}$$

Since $Z \hookrightarrow L^p(\Omega)$ compactly, by [39] we know that

$$\{u : u \in L^2(0, \widetilde{T}; Z), u_t \in L^2(0, \widetilde{T}; L^2(\Omega))\} \hookrightarrow L^2(0, \widetilde{T}; L^p(\Omega))$$

compactly. So, in view of (4.12) and (4.14), we can assume

$$u_m \to u \text{ strongly in } L^2(0, \widetilde{T}; L^p(\Omega)), \tag{4.17}$$

which implies $u_m \to u$ a.e. in $\Omega \times (0, \widetilde{T})$, and then $|u_m|^{p-2} u_m \log |u_m| \to |u|^{p-2} u \log |u|$ a.e. in $\Omega \times (0, \widetilde{T})$. Therefore, it follows from [39] that

$$\chi(t) = |u|^{p-2} u \log |u|. \tag{4.18}$$

To show that the limit function $u(t)$ obtained above is a weak solution to problem (1.1), we fix a positive integer k and choose a function $v \in C^1([0, \widetilde{T}]; Z)$ of the following form

$$v = \sum_{j=1}^k l_j(t) \omega_j(x), \tag{4.19}$$

where $\{l_j(t)\}_{j=1}^k$ are arbitrary given C^1 functions. Taking $m \geq k$ in the first equation of (4.2), multiplying the first equation of (4.2) by $l_j(t)$, summing for j from 1 to k, and integrating with respect to t from 0 to $\widetilde{T}$, we obtain

$$\int_0^{\widetilde{T}}\int_\Omega u_{mt} v dxdt + \int_0^{\widetilde{T}} (a+b\|u_m\|_Z^{2\theta-2}) \iint_Q (u_m(x)-u_m(y))(v(x) - v(y))K(x-y)dxdydt = \int_0^{\widetilde{T}}\int_\Omega |u_m|^{p-2}u_m \log|u_m| v dxdt. \tag{4.20}$$

Letting $m \to \infty$ in (4.20) and recalling (4.12), (4.14), (4.16) and (4.18) yield

$$\int_0^{\widetilde{T}}\int_\Omega u_t v dxdt + \int_0^{\widetilde{T}} (a+b\|u\|_Z^{2\theta-2}) \iint_Q (u(x) - u(y))(v(x)-v(y))K(x-y)dxdydt = \int_0^{\widetilde{T}}\int_\Omega |u|^{p-2}u \log|u| v dxdt. \tag{4.21}$$

Since the functions of the form in (4.19) are dense in $L^2(0, \widetilde{T}; Z)$, (4.21) also holds for all $v \in L^2(0, \widetilde{T}; Z)$. By arbitrariness of $\widetilde{T} > 0$, we know that

$$\int_\Omega u_t \phi dx + (a+b\|u\|_Z^{2\theta-2}) \iint_Q (u(x)-u(y))(\phi(x)-\phi(y))K(x-y)dxdy = \int_\Omega |u|^{p-2}u \log|u|\phi dx,$$

holds for a.e. $t \in (0, \infty)$ and any $\phi \in Z$.

In view of (4.12) and (4.14), we get $u_m(0) \rightharpoonup u(0)$ weakly in $L^2(\Omega)$. Then by (4.1), (4.3) and $g_{jm}(0) = \xi_{jm}$, we get $u(0) = u_0 \in Z$.

In view of Definition 1 and the above discussions, to show the limit function $u(t)$ got above is indeed a global weak solution to problem (1.1), we only need to prove (2.12) holds for a.e. $0 < t < \infty$. In fact, for a.e. $0 < t < \infty$, we choose $\widetilde{T} > t$. Then it follows from (4.17) that $u_m(t) \to u(t)$ strongly in $L^p(\Omega)$. So by (4.11) and (4.16), we have

$$\left| \int_\Omega |u_m|^p \log|u_m| dx - \int_\Omega |u|^p \log|u| dx \right|$$

$$
\begin{aligned}
&\leq \left| \int_{\Omega} (u_m - u) u_m |u_m|^{p-2} \log |u_m| dx \right| \\
&\quad + \left| \int_{\Omega} u \left(|u_m|^{p-2} u_m \log |u_m| - |u|^{p-2} u \log |u| \right) dx \right| \\
&\leq C_d^{\frac{p-1}{p}} \|u_m - u\|_p + \left| \int_{\Omega} u \left(|u_m|^{p-2} u_m \log |u_m| - |u|^{p-2} u \log |u| \right) dx \right| \\
&\to 0,
\end{aligned} \tag{4.22}
$$

as $m \to \infty$.

From the convergence of (4.12), (4.14), (4.17), the definition of J in (2.1), (4.1), (4.3), (4.4), (4.22) and $g_{jm}(0) = \xi_{jm}$, we obtain

$$
\begin{aligned}
&\frac{a}{2}\|u\|_Z^2 + \frac{b}{2\theta}\|u\|_Z^{2\theta} + \frac{1}{p^2}\|u\|_p^p + \int_0^t \|u_\tau\|_2^2 d\tau \\
&\quad \leq \frac{a}{2} \liminf_{m\to\infty} \|u_m\|_Z^2 + \frac{b}{2\theta} \liminf_{m\to\infty} \|u_m\|_Z^{2\theta} + \frac{1}{p^2} \liminf_{m\to\infty} \|u_m\|_p^p \\
&\qquad + \liminf_{m\to\infty} \int_0^t \|u_{m\tau}\|_2^2 d\tau \\
&\quad \leq \liminf_{m\to\infty} \left(\frac{a}{2}\|u_m\|_Z^2 + \frac{b}{2\theta}\|u_m\|_Z^{2\theta} + \frac{1}{p^2}\|u_m\|_p^p + \int_0^t \|u_{m\tau}\|_2^2 d\tau \right) \\
&\quad = \liminf_{m\to\infty} \left(J(u_m) + \frac{1}{p}\int_{\Omega} |u_m|^p \log |u_m| dx + \int_0^t \|u_{m\tau}\|_2^2 d\tau \right) \\
&\quad = \lim_{m\to\infty} \left(J(u_m(0)) + \frac{1}{p}\int_{\Omega} |u_m|^p \log |u_m| dx \right) \\
&\quad = J(u_0) + \frac{1}{p}\int_{\Omega} |u|^p \log |u| dx,
\end{aligned}
$$

which implies (2.12) holds for a.e. $t \in (0, \infty)$. So the limit function $u(t)$ got above is a global weak solution to problem (1.1). Furthermore, by using $u_0 \in W$ and (2.12), one can get $u(t) \in W$ for $0 \leq t < \infty$ and the proof is same as the proof of $u_m(t) \in W$.

Step 2: Uniqueness of bounded global weak solution To show the uniqueness of bounded global weak solution, we assume that $u, v \in L^\infty(0, \infty; L^\infty(\Omega))$ are two global weak solutions to problem (1.1). Then for any $\phi \in Z$, we have

$$(u_t, \phi) + a\langle u, \phi\rangle_Z + b\|u\|_Z^{2\theta-2}\langle u, \phi\rangle_Z = (|u|^{p-2}u\log|u|, \phi),$$

and

$$(v_t, \phi) + a\langle v, \phi\rangle_Z + b\|v\|_Z^{2\theta-2}\langle v, \phi\rangle_Z = (|v|^{p-2}v\log|v|, \phi).$$

Subtracting the above two inequalities, taking $\phi = u - v \in Z$, we obtain

$$\begin{aligned}\int_\Omega \phi_t\phi dx + a\|\phi\|_Z^2 + b\|u\|_Z^{2\theta-2}\langle u, \phi\rangle_Z - b\|v\|_Z^{2\theta-2}\langle v, \phi\rangle_Z \\ = \int_\Omega \Big(|u|^{p-2}u\log|u| - |v|^{p-2}v\log|v|\Big)\phi dx.\end{aligned} \tag{4.23}$$

Moreover, by using $\langle u, v\rangle_Z \le \frac{\|u\|_Z^2+\|v\|_Z^2}{2}$, we have

$$\begin{aligned}&a\|\phi\|_Z^2 + b\|u\|_Z^{2\theta-2}\langle u, \phi\rangle_Z - b\|v\|_Z^{2\theta-2}\langle v, \phi\rangle_Z \\ &\ge b\|u\|_Z^{2\theta-2}\langle u, u-v\rangle_Z - b\|v\|_Z^{2\theta-2}\langle v, u-v\rangle_Z \\ &= b\|u\|_Z^{2\theta-2}\langle u, u\rangle_Z - b\|u\|_Z^{2\theta-2}\langle u, v\rangle_Z - b\|v\|_Z^{2\theta-2}\langle u, v\rangle_Z + b\|v\|_Z^{2\theta-2}\langle v, v\rangle_Z \\ &= b\|u\|_Z^{2\theta} - b\|u\|_Z^{2\theta-2}\langle u, v\rangle_Z - b\|v\|_Z^{2\theta-2}\langle u, v\rangle_Z + b\|v\|_Z^{2\theta} \\ &\ge b\|u\|_Z^{2\theta} - b\|u\|_Z^{2\theta-2}\cdot\frac{\|u\|_Z^2+\|v\|_Z^2}{2} - b\|v\|_Z^{2\theta-2}\cdot\frac{\|u\|_Z^2+\|v\|_Z^2}{2} + b\|v\|_Z^{2\theta} \\ &= \frac{b\|u\|_Z^{2\theta-2}}{2}(\|u\|_Z^2 - \|v\|_Z^2) + \frac{b\|v\|_Z^{2\theta-2}}{2}(\|v\|_Z^2 - \|u\|_Z^2) \\ &= \frac{b}{2}(\|u\|_Z^2 - \|v\|_Z^2)(\|u\|_Z^{2\theta-2} - \|v\|_Z^{2\theta-2}) \ge 0.\end{aligned} \tag{4.24}$$

Then combining (4.23) and (4.24) we have

$$\begin{aligned}\int_\Omega \phi_t\phi dx &\le \int_\Omega \Big(|u|^{p-2}u\log|u| - |v|^{p-2}v\log|v|\Big)\phi dx \\ &= \int_\Omega \left[\int_0^1 \frac{d}{d\omega}\Big(|\omega|^{p-2}\omega\log|\omega|\Big)\Big|_{\omega=\vartheta u+(1-\vartheta)v} d\vartheta\right]\phi^2 dx \\ &\le D^{p-2}\big[(p-1)\log D + 1\big]\|\phi\|_2^2,\end{aligned}$$

where

$$D := \max\left\{\|u\|_{L^\infty(0,\infty;L^\infty(\Omega))}, \|v\|_{L^\infty(0,\infty;L^\infty(\Omega))}\right\}.$$

Then we have

$$\begin{cases} \frac{d}{dt}\|\phi\|_2^2 \le 2D^{p-2}\left[(p-1)\log D + 1\right]\|\phi\|_2^2, \quad t > 0, \\ \|\phi(0)\|_2^2 = 0, \end{cases}$$

which implies $\|\phi\|_2^2 = 0$ for $t \ge 0$. Thus $\phi(t)(x) = 0$ a.e. in $\Omega \times (0, \infty)$ and the uniqueness of bounded global weak solution follows.

Step 3: Decay estimates Since $d(\varepsilon) \le d$ (see (2.4)), by step 1 we know problem (1.1) admits a global solution $u \in L^\infty(0, \infty; Z)$ with $u_t \in L^2(0, \infty; L^2(\Omega))$ and $u(t) \in W$ for $0 \le t < \infty$. So by the definition of W in (2.9), we have $I(u) \ge 0$. Then it follows from (2.3) and (2.12) that

$$\begin{aligned} J(u_0) \ge J(u) &= \frac{1}{p}I(u) + \frac{(p-2)a}{2p}\|u\|_Z^2 + \frac{(p-2\theta)b}{2\theta p}\|u\|_Z^{2\theta} + \frac{1}{p^2}\|u\|_p^p \\ &\ge \frac{(p-2\theta)b}{2\theta p}\|u\|_Z^{2\theta}, \end{aligned} \tag{4.25}$$

which, together with (3.3), implies

$$\|u\|_{p+\varepsilon} \le C_*\|u\|_Z \le C_*\left(\frac{2\theta p J(u_0)}{(p-2\theta)b}\right)^{\frac{1}{2\theta}}. \tag{4.26}$$

In view of (3.3) and (4.26), we obtain

$$\begin{aligned} \|u\|_{p+\varepsilon}^{p+\varepsilon} &= \|u\|_{p+\varepsilon}^{p+\varepsilon-2\theta}\|u\|_{p+\varepsilon}^{2\theta} \\ &\le C_*^{2\theta}\|u\|_{p+\varepsilon}^{p+\varepsilon-2\theta}\|u\|_Z^{2\theta} \\ &\le C_*^{p+\varepsilon}\left(\frac{2\theta p J(u_0)}{(p-2\theta)b}\right)^{\frac{p+\varepsilon-2\theta}{2\theta}}\|u\|_Z^{2\theta}. \end{aligned} \tag{4.27}$$

By Lemma 8, we have

$$\frac{d}{dt}\|u\|_2^2 = -2I(u) = -2\left(a\|u\|_Z^2 + b\|u\|_Z^{2\theta} - \int_\Omega |u|^p \log|u|dx\right). \tag{4.28}$$

Then for $J(u_0) < d(\varepsilon)$, it follows from (1.10), (4.27) and $\log|u| < \frac{1}{\varepsilon}|u|^\varepsilon$ (for any $\varepsilon > 0$) that

$$\begin{aligned} \frac{d}{dt}\|u\|_2^2 &\le -2b\|u\|_Z^{2\theta} + 2\int_\Omega |u|^p \log|u|dx \\ &\le -2b\|u\|_Z^{2\theta} + \frac{2}{\varepsilon}\|u\|_{p+\varepsilon}^{p+\varepsilon} \end{aligned}$$

$$
\begin{aligned}
&\leq -2b\|u\|_Z^{2\theta} + \frac{2C_*^{p+\varepsilon}}{\varepsilon}\left(\frac{2\theta pJ(u_0)}{(p-2\theta)b}\right)^{\frac{p+\varepsilon-2\theta}{2\theta}}\|u\|_Z^{2\theta} \\
&= -2\|u\|_Z^{2\theta}\left[b - \frac{C_*^{p+\varepsilon}}{\varepsilon}\left(\frac{2\theta pJ(u_0)}{(p-2\theta)b}\right)^{\frac{p+\varepsilon-2\theta}{2\theta}}\right] \\
&\leq -2\lambda_1^{\theta}\|u\|_2^{2\theta}\left[b - \frac{C_*^{p+\varepsilon}}{\varepsilon}\left(\frac{2\theta pJ(u_0)}{(p-2\theta)b}\right)^{\frac{p+\varepsilon-2\theta}{2\theta}}\right],
\end{aligned} \tag{4.29}
$$

which implies

$$
\|u\|_2^2 \leq F(\varepsilon) := \begin{cases} \|u_0\|_2^2 e^{-C_\varepsilon t}, & \text{if } \theta = 1, \\ \left(C_\varepsilon(\theta-1)t + \|u_0\|_2^{2-2\theta}\right)^{-\frac{1}{\theta-1}}, & \text{if } \theta \in \left(1, \frac{2_s^*}{2}\right), \end{cases} \tag{4.30}
$$

where

$$
C_\varepsilon = 2\lambda_1^{\theta}\left[b - \frac{C_*^{p+\varepsilon}}{\varepsilon}\left(\frac{2\theta pJ(u_0)}{(p-2\theta)b}\right)^{\frac{p+\varepsilon-2\theta}{2\theta}}\right] > 0.
$$

□

Proof of Corollary 1 If $u_0 = 0$, then problem (1.1) admits a global solution $u(t) \equiv 0$, and the proof is complete. So in the following, we assume $u_0 \in Z \setminus \{0\}$ and the proof is divided into three cases.

Case 1: $I(u_0) > 0$ and $J(u_0) < d$. The conclusion follows from Theorem 1.

Case 2: $I(u_0) = 0$ and $J(u_0) < d$. This case does not happen because in this case $u_0 \in N$, then it follows from the definition of d in (2.4) that $J(u_0) \geq d$.

Case 3: $I(u_0) \geq 0$ and $J(u_0) = d$. Let $\lambda_m = 1 - \frac{1}{m}$ and $m = 2, 3, \ldots$. Consider the following approximate problem:

$$
\begin{cases} u_t + M([u]_s^2)\mathcal{L}_K u = |u|^{p-2}u\log|u|, & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,\infty), \\ u(x,0) = u_{0m}(x) := \lambda_m u_0, & \text{in } \Omega. \end{cases} \tag{4.31}
$$

Since $u_0 \in Z \setminus \{0\}$, $\lambda_m \in (0,1)$ and $I(u_0) \geq 0$ (i.e. $a\|u_0\|_Z^2 + b\|u_0\|_Z^{2\theta} \geq \int_\Omega |u_0|^p \log|u_0|dx$), then we have

$$
\begin{aligned}
I(u_{0m}) = a\lambda_m^2\|u_0\|_Z^2 + b\lambda_m^{2\theta}\|u_0\|_Z^{2\theta} - \lambda_m^p \int_\Omega |u_0|^p \log|u_0|dx \\
-\lambda_m^p \log\lambda_m \int_\Omega |u_0|^p dx
\end{aligned}
$$

$$> a\lambda_m^2\|u_0\|_Z^2 + b\lambda_m^{2\theta}\|u_0\|_Z^{2\theta} - \lambda_m^p\int_\Omega |u_0|^p\log|u_0|dx$$

$$= \lambda_m^2\left(a\|u_0\|_Z^2 + b\lambda_m^{2\theta-2}\|u_0\|_Z^{2\theta} - \lambda_m^{p-2}\int_\Omega |u_0|^p\log|u_0|dx\right). \quad (4.32)$$

Next, we will discuss the sign of $I(u_{0m})$ on two aspects: $\int_\Omega |u_0|^p\log|u_0|dx \leq 0$ and $\int_\Omega |u_0|^p\log|u_0|dx > 0$.

(1) When $\int_\Omega |u_0|^p\log|u_0|dx \leq 0$, from (4.32) we get

$$I(u_{0m}) > \lambda_m^2\left(a\|u_0\|_Z^2 + b\lambda_m^{2\theta-2}\|u_0\|_Z^{2\theta}\right) > 0. \quad (4.33)$$

(2) When $\int_\Omega |u_0|^p\log|u_0|dx > 0$, from (4.32) we get

$$\begin{aligned} I(u_{0m}) &> \lambda_m^2\left(a\lambda_m^{2\theta-2}\|u_0\|_Z^2 + b\lambda_m^{2\theta-2}\|u_0\|_Z^{2\theta} - \lambda_m^{p-2}\int_\Omega |u_0|^p\log|u_0|dx\right)\\ &= \lambda_m^{2\theta}\left(a\|u_0\|_Z^2 + b\|u_0\|_Z^{2\theta} - \lambda_m^{p-2\theta}\int_\Omega |u_0|^p\log|u_0|dx\right)\\ &> 0. \end{aligned} \quad (4.34)$$

On the other hand, by a simply computation, we obtain

$$\begin{aligned} &\frac{d}{d\lambda_m}J(\lambda_m u)\\ &= a\lambda_m\|u\|_Z^2 + b\lambda_m^{2\theta-1}\|u\|_Z^{2\theta} - \lambda_m^{p-1}\int_\Omega |u|^p\log|u|dx - \lambda_m^{p-1}\log\lambda_m\int_\Omega |u|^p dx\\ &= \frac{1}{\lambda_m}\left(a\lambda_m^2\|u\|_Z^2 + b\lambda_m^{2\theta}\|u\|_Z^{2\theta} - \lambda_m^p\int_\Omega |u|^p\log|u|dx - \lambda_m^p\log\lambda_m\int_\Omega |u|^p dx\right)\\ &= \frac{1}{\lambda_m}I(\lambda_m u). \end{aligned} \quad (4.35)$$

Then by (4.33), (4.34) and (4.35), we have

$$\frac{d}{d\lambda_m}J(\lambda_m u_0) = \frac{1}{\lambda_m}I(\lambda_m u_0) = \frac{1}{\lambda_m}I(u_{0m}) > 0,$$

which implies that $J(\lambda_m u_0)$ is strictly increasing with respect to λ_m. So we can get

$$J(u_{0m}) = J(\lambda_m u_0) < J(u_0) = d.$$

From Theorem 1, it follows that for each $m = 2, 3, \ldots$, problem (4.31) admits a global weak solution $u_m(t) \in L^\infty(0, \infty; Z)$ with $u_{mt} \in L^2(0, \infty; L^2(\Omega))$, which satisfies $u_m(t) \in W$ for $0 \le t < \infty$ and

$$\begin{aligned} &\int_\Omega u_{mt}\phi dx + \left(a + b\|u_m\|_Z^{2\theta-2}\right) \iint_Q (u_m(x) - u_m(y))(\phi(x) \\ &\quad - \phi(y))K(x-y)dxdy \\ &= \int_\Omega |u_m|^{p-2}u_m \log|u_m|\phi dx, \end{aligned}$$

holds for any $\phi \in Z$ and a.e. $t > 0$. Moreover,

$$\int_0^t \|u_{m\tau}\|_2^2 d\tau + J(u_m(t)) = J(u_{0m}) < d. \tag{4.36}$$

From (4.36) and the fact that

$$J(u_m(t)) = \frac{1}{p}I(u_m(t)) + \frac{(p-2)a}{2p}\|u_m\|_Z^2 + \frac{(p-2\theta)b}{2\theta p}\|u_m\|_Z^{2\theta} + \frac{1}{p^2}\|u_m\|_p^p,$$

we obtain

$$\int_0^t \|u_{m\tau}\|_2^2 d\tau + \frac{(p-2)a}{2p}\|u_m\|_Z^2 + \frac{(p-2\theta)b}{2\theta p}\|u_m\|_Z^{2\theta} + \frac{1}{p^2}\|u_m\|_p^p < d.$$

Then the remainder of the proof is similar to that in the proof of Theorem 1. □

Proof of Theorem 2 We divide the proof into three steps.

Step 1: Blow-up in finite time

We divide the proof into two cases.

Case 1: $J(u_0) < d$. Let $u = u(t)$, $t \in [0, T)$ be a weak solution of problem (1.1) with $J(u_0) < d$ and $I(u_0) < 0$, where T is the maximal existence time. Then from Lemma 9, we have $u(t) \in V$. Next let us prove that $u(t)$ blows up in finite time. Arguing by contradiction, we suppose that $T = +\infty$ and define

$$M(t) := \int_0^t \|u\|_2^2 d\tau, \quad t \in [0, T).$$

Then we have

$$M'(t) = \|u\|_2^2, \tag{4.37}$$

and

$$M''(t) = 2(u(t), u_t(t)) = -2I(u(t)). \tag{4.38}$$

By (2.3) and (2.12), one has

$$\int_0^t \|u_\tau\|_2^2 d\tau + \frac{1}{p} I(u) + \frac{(p-2)a}{2p}\|u\|_Z^2 + \frac{(p-2\theta)b}{2\theta p}\|u\|_Z^{2\theta} + \frac{1}{p^2}\|u\|_p^p \leq J(u_0),$$

hence

$$-2I(u(t)) \geq 2p \int_0^t \|u_\tau\|_2^2 d\tau + (p-2)a\|u\|_Z^2 + \frac{(p-2\theta)b}{\theta}\|u\|_Z^{2\theta} + \frac{2}{p}\|u\|_p^p - 2pJ(u_0),$$

so by (1.8) and the above inequality, we have

$$\begin{aligned} M''(t) = -2I(u) &\geq 2p \int_0^t \|u_\tau\|_2^2 d\tau + \frac{(p-2\theta)b}{\theta}\|u\|_Z^{2\theta} - 2pJ(u_0) \\ &\geq 2p \int_0^t \|u_\tau\|_2^2 d\tau + \frac{(p-2\theta)b}{\theta C_2^{2\theta}}\|u\|_2^{2\theta} - 2pJ(u_0). \end{aligned} \tag{4.39}$$

In addition, from

$$\int_0^t (u_\tau, u) d\tau = \frac{1}{2}\int_0^t \frac{d}{d\tau}\|u\|_2^2 d\tau = \frac{1}{2}\left(\|u\|_2^2 - \|u_0\|_2^2\right),$$

we obtain

$$\begin{aligned} \left(\int_0^t (u_\tau, u) d\tau\right)^2 &= \frac{1}{4}\left(\|u\|_2^4 - 2\|u_0\|_2^2\|u\|_2^2 + \|u_0\|_2^4\right) \\ &= \frac{1}{4}\left((M'(t))^2 - 2\|u_0\|_2^2 M'(t) + \|u_0\|_2^4\right). \end{aligned} \tag{4.40}$$

Hence, by (4.39), (4.40) and the Schwartz's inequality we deduce that

$$\begin{aligned} & M(t)M''(t) - \frac{p}{2}(M'(t))^2 \\ & \geq 2p\int_0^t \|u_\tau\|_2^2 d\tau \int_0^t \|u\|_2^2 d\tau - 2p\left(\int_0^t (u_\tau, u)d\tau\right)^2 + \frac{p}{2}\|u_0\|_2^4 \\ & \quad + \frac{(p-2\theta)b}{\theta C_2^{2\theta}}\|u\|_2^{2\theta} M(t) - p\|u_0\|_2^2 M'(t) - 2pJ(u_0)M(t) \\ & \geq \frac{(p-2\theta)b}{\theta C_2^{2\theta}}(M'(t))^\theta M(t) - p\|u_0\|_2^2 M'(t) - 2pJ(u_0)M(t). \end{aligned} \tag{4.41}$$

Moreover, since $M''(t) = -2I(u(t)) > 0$ (note that $u(t) \in V$ for $t \in [0, T)$), so we have $M'(t) > M'(0) = \|u_0\|_2^2 > 0$. Then by (4.41) we obtain

$$\begin{aligned} M(t)M''(t) - \frac{p}{2}(M'(t))^2 \geq & \frac{(p-2\theta)b\|u_0\|_2^{2\theta-2}}{\theta C_2^{2\theta}} M(t)M'(t) - p\|u_0\|_2^2 M'(t) \\ & -2pJ(u_0)M(t). \end{aligned} \tag{4.42}$$

From Lemma 7 one has

$$-2I(u(t)) > 2p(d - J(u(t))), \quad 0 \leq t < \infty.$$

By (2.12) and (4.38) we have

$$\begin{aligned} M''(t) &= -2I(u(t)) \\ &> 2p(d - J(u(t))) \\ &\geq 2p(d - J(u_0)) \\ &:= C_1 > 0, \quad 0 \leq t < \infty. \end{aligned} \tag{4.43}$$

Then we can obtain

$$\begin{aligned} & M'(t) \geq C_1 t + M'(0) = C_1 t + \|u_0\|_2^2 > C_1 t, \quad 0 \leq t < \infty, \\ & M(t) > \frac{C_1}{2}t^2 + M(0) = \frac{C_1}{2}t^2, \quad 0 \leq t < \infty. \end{aligned}$$

Therefore,

$$\lim_{t\to\infty} M(t) = \infty, \qquad \lim_{t\to\infty} M'(t) = \infty.$$

Hence there exists a $t_0 \geq 0$ such that

$$\frac{(p-2\theta)b\|u_0\|_2^{2\theta-2}}{2\theta C_2^{2\theta}}M(t) > p\|u_0\|_2^2, \quad t_0 \leq t < \infty,$$
$$\frac{(p-2\theta)b\|u_0\|_2^{2\theta-2}}{2\theta C_2^{2\theta}}M'(t) > 2pJ(u_0), \quad t_0 \leq t < \infty,$$

which combined with (4.42) give the inequality

$$\begin{aligned} M(t)M''(t) &- \frac{p}{2}(M'(t))^2 \\ &\geq \left(\frac{(p-2\theta)b\|u_0\|_2^{2\theta-2}}{2\theta C_2^{2\theta}}M(t) - p\|u_0\|_2^2\right)M'(t) \\ &\quad + \left(\frac{(p-2\theta)b\|u_0\|_2^{2\theta-2}}{2\theta C_2^{2\theta}}M'(t) - 2pJ(u_0)\right)M(t) > 0, \quad t_0 \leq t < \infty. \end{aligned}$$

Then we get from Lemma 1 that the maximal existence time T_1 of $M(t)$ satisfying $T_1 < \infty$ and

$$\lim_{t \to T_1} M(t) = \infty,$$

which contradicts $T = \infty$.

Case 2: $J(u_0) = d$ Since $I(u(t)) < 0$ for $t \geq 0$ (see Lemma 9), it follows that

$$(u_t, u) = -I(u(t)) > 0, \quad t \geq 0.$$

Hence we can get $\|u_t\|_2^2 > 0$ for $t \geq 0$. Thus by (2.12), there exists a $t_1 > 0$ such that

$$J(u(t_1)) \leq J(u_0) - \int_0^{t_1} \|u_\tau\|_2^2 d\tau < d.$$

If we take t_1 as the initial time, then similar to the Case 1 in the proof of this section, we can obtain the finite time blow up result. The proof of Step 1 is complete.

Step 2: Upper bound estimate of the blow-up time

Let $u = u(t)$ be a solution of problem (1.1) with initial value u_0 satisfying $I(u_0) < 0$ and $J(u_0) < d$. By Step 1, the maximal existence time $T < +\infty$. Let

$$\mu(t) := \left(\int_0^t \|u\|_2^2 d\tau\right)^{\frac{1}{2}}, \quad \nu(t) := \left(\int_0^t \|u_\tau\|_2^2 d\tau\right)^{\frac{1}{2}}, \quad \forall t \in [0, T).$$

By (2.12), Lemmas 8 and 9, we have

$(R1)$ $J(u(t)) + \nu^2(t) \le J(u_0), \forall t \in [0, T)$;
$(R2)$ $\frac{d}{dt}\|u\|_2^2 = -2I(u(t)), \forall t \in [0, T)$;
$(R3)$ $u(t) \in N_-$, i.e., $I(u(t)) < 0, \forall t \in [0, T)$.

Consider the following functional:

$$F(t) := \mu^2(t) + (T - t)\|u_0\|_2^2 + \beta(t + \alpha)^2, \quad \forall t \in [0, T), \tag{4.44}$$

where α and β are two positive constants to be determined later. Then by $(R2)$ and $(R3)$, we have

$$\begin{aligned} F'(t) &= \|u\|_2^2 - \|u_0\|_2^2 + 2\beta(t + \alpha) \\ &\ge 2\beta(t + \alpha) > 0, \quad t \in [0, T), \end{aligned} \tag{4.45}$$

which implies

$$F(t) \ge F(0) = T\|u_0\|_2^2 + \beta\alpha^2 > 0, \quad t \in [0, T) \tag{4.46}$$

and (by (R1), (R2) and Lemma 7)

$$\begin{aligned} F''(t) &= -2I(u(t)) + 2\beta > 2p(d - J(u(t))) + 2\beta \\ &\ge 2p(d - J(u_0)) + 2p\nu^2(t) + 2\beta, \quad t \in [0, T). \end{aligned} \tag{4.47}$$

By Schwartz's inequality, we have

$$\begin{aligned} \frac{1}{2}\int_0^t \frac{d}{d\tau}\|u\|_2^2 d\tau &= \int_0^t (u, u_\tau) d\tau \\ &\le \int_0^t \|u\|_2 \|u_\tau\|_2 d\tau \le \mu(t)\nu(t), \quad t \in [0, T), \end{aligned}$$

which, together with the definition of $F(t)$, implies

$$\begin{aligned} &\left(F(t) - (T - t)\|u_0\|_2^2\right)\left(\nu^2(t) + \beta\right) \\ &= \left(\mu^2(t) + \beta(t + \alpha)^2\right)\left(\nu^2(t) + \beta\right) \\ &= \mu^2(t)\nu^2(t) + \beta\mu^2(t) + \beta(t + \alpha)^2\nu^2(t) + \beta^2(t + \alpha)^2 \\ &\ge \mu^2(t)\nu^2(t) + 2\beta\mu(t)\nu(t)(t + \alpha) + \beta^2(t + \alpha)^2 \\ &= (\mu(t)\nu(t) + \beta(t + \alpha))^2 \\ &\ge \left[\frac{1}{2}\int_0^t \frac{d}{d\tau}\|u\|_2^2 d\tau + \beta(t + \alpha)\right]^2, \quad t \in [0, T). \end{aligned}$$

Then it follows from (4.45) and the above inequality that

$$\begin{aligned}\left(F'(t)\right)^2 &= 4\left(\frac{1}{2}\int_0^t \frac{d}{d\tau}\|u\|_2^2 ds + \beta(t+\alpha)\right)^2 \\ &\le 4F(t)\left(v^2(t)+\beta\right), \quad t\in[0,T).\end{aligned} \tag{4.48}$$

Combining (4.46), (4.47) and (4.48), we get

$$\begin{aligned}&F(t)F''(t) - \frac{p}{2}\left(F'(t)\right)^2 \\ &> F(t)\left[2p(d-J(u_0)) + 2pv^2(t) + 2\beta - 2pv^2(t) - 2p\beta\right] \\ &= F(t)\left[2p(d-J(u_0)) - 2(p-1)\beta\right], \quad t\in[0,T),\end{aligned}$$

which is nonnegative if we take β small enough such that

$$0<\beta\le\frac{p(d-J(u_0))}{p-1}. \tag{4.49}$$

Then it follows from Lemma 1 that

$$T\le\frac{F(0)}{\left(\frac{p}{2}-1\right)F'(0)} = \frac{1}{p-2}\left(\alpha+\frac{\|u_0\|_2^2}{\beta\alpha}T\right). \tag{4.50}$$

By taking α large enough such that

$$\alpha>\frac{\|u_0\|_2^2}{(p-2)\beta}, \tag{4.51}$$

we get from (4.50) that

$$T\le\frac{\beta\alpha^2}{(p-2)\beta\alpha-\|u_0\|_2^2}. \tag{4.52}$$

The above analysis shows that ($\rho:=\alpha\beta$)

$$T\le\inf_{(\rho,\alpha)\in\Phi} f(\rho,\alpha), \tag{4.53}$$

where

$$\begin{aligned}\Phi &:= \left\{(\rho,\alpha):\rho>\frac{\|u_0\|_2^2}{p-2}, \alpha\ge\frac{(p-1)\rho}{p(d-J(u_0))}\right\}, \\ f(\rho,\alpha) &:= \frac{\rho\alpha}{(p-2)\rho-\|u_0\|_2^2}.\end{aligned}$$

It is easily to find that $f(\rho, \alpha)$ is increasing with α. Then

$$\begin{aligned} T &\le \inf_{\rho > \frac{\|u_0\|_2^2}{p-2}} f\left(\rho, \frac{(p-1)\rho}{p(d-J(u_0))}\right) \\ &= \inf_{\rho > \frac{\|u_0\|_2^2}{p-2}} \frac{(p-1)\rho^2}{p(d-J(u_0))\left((p-2)\rho - \|u_0\|_2^2\right)} \\ &= \left.\frac{(p-1)\rho^2}{p(d-J(u_0))\left((p-2)\rho - \|u_0\|_2^2\right)}\right|_{\rho = \frac{2\|u_0\|_2^2}{p-2}} \\ &= \frac{4(p-1)\|u_0\|_2^2}{p(d-J(u_0))(p-2)^2}. \end{aligned}$$

Step 3: Lower bound estimate of the blow-up time

From Step 1 we know that the weak solution $u(t) = u(x, t)$ of problem (1.1) blows up at finite time T. Now, we estimate the lower bound of T and blow-up rate. To this end, we define a function

$$f(t) := \frac{1}{2}\|u\|_2^2,$$

then we have

$$f(T) = \infty. \tag{4.54}$$

According to Lemma 8, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 = -I(u) = -a\|u\|_Z^2 - b\|u\|_Z^{2\theta} + \int_\Omega |u|^p \log|u| dx. \tag{4.55}$$

Moreover, by Lemma 9, we know $I(u) < 0$. Thus, by (2.13) and the inequality $\log|u(x)| < \frac{|u(x)|^\varepsilon}{\varepsilon}$ (for any $\varepsilon > 0$), we deduce

$$\begin{aligned} \|u\|_{p+\varepsilon}^{p+\varepsilon} &\le \widetilde{C}^{p+\varepsilon}\left(\|u\|_Z^2\right)^{\frac{(1-\beta)(p+\varepsilon)}{2}} \cdot \left(\|u\|_2^2\right)^{\frac{\beta(p+\varepsilon)}{2}} \\ &= \frac{\widetilde{C}^{p+\varepsilon}}{b^{\frac{(1-\beta)(p+\varepsilon)}{2\theta}}}\left(b\|u\|_Z^{2\theta}\right)^{\frac{(1-\beta)(p+\varepsilon)}{2\theta}} \cdot \left(\|u\|_2^2\right)^{\frac{\beta(p+\varepsilon)}{2}} \\ &\le \frac{\widetilde{C}^{p+\varepsilon}}{b^{\frac{(1-\beta)(p+\varepsilon)}{2\theta}}}\left(a\|u\|_Z^2 + b\|u\|_Z^{2\theta}\right)^{\frac{(1-\beta)(p+\varepsilon)}{2\theta}} \cdot \left(\|u\|_2^2\right)^{\frac{\beta(p+\varepsilon)}{2}} \\ &< \frac{\widetilde{C}^{p+\varepsilon}}{b^{\frac{(1-\beta)(p+\varepsilon)}{2\theta}}}\left(\int_\Omega |u|^p \log|u| dx\right)^{\frac{(1-\beta)(p+\varepsilon)}{2\theta}} \cdot \left(\|u\|_2^2\right)^{\frac{\beta(p+\varepsilon)}{2}} \end{aligned}$$

$$< \frac{\widetilde{C}^{p+\varepsilon}}{(b\varepsilon)^{\frac{(1-\beta)(p+\varepsilon)}{2\theta}}} \left(\|u\|_{p+\varepsilon}^{p+\varepsilon}\right)^{\frac{(1-\beta)(p+\varepsilon)}{2\theta}} \cdot \left(\|u\|_2^2\right)^{\frac{\beta(p+\varepsilon)}{2}}. \tag{4.56}$$

Since $0 < \varepsilon < 2\theta+2-\frac{4\theta}{2_s^*}-p, 2\theta < p < 2\theta+2-\frac{4\theta}{2_s^*}$ and $\beta = \frac{2\cdot 2_s^*-2p-2\varepsilon}{(p+\varepsilon)(2_s^*-2)} \in (0,1)$, we have

$$\frac{(1-\beta)(p+\varepsilon)}{2\theta} < 1.$$

Thus, by (4.56), we can get

$$\|u\|_{p+\varepsilon}^{p+\varepsilon} < \left(\frac{\widetilde{C}^{p+\varepsilon}}{(b\varepsilon)^{\frac{(1-\beta)(p+\varepsilon)}{2\theta}}}\right)^{\frac{2\theta}{2\theta-(1-\beta)(p+\varepsilon)}} \left(\|u\|_2^2\right)^{\frac{\beta\theta(p+\varepsilon)}{2\theta-(1-\beta)(p+\varepsilon)}}. \tag{4.57}$$

Moreover, by remark 2, we know

$$\zeta := \frac{\beta\theta(p+\varepsilon)}{2\theta-(1-\beta)(p+\varepsilon)} > 1.$$

Then combining (4.55) and (4.57) we have

$$\begin{aligned} f'(t) &= -a\|u\|_Z^2 - b\|u\|_Z^{2\theta} + \int_\Omega |u|^p \log|u| dx \\ &\le \int_\Omega |u|^p \log|u| dx \le \frac{1}{\varepsilon}\|u\|_{p+\varepsilon}^{p+\varepsilon} \\ &< \widehat{C}\left(\|u\|_2^2\right)^\zeta = 2^\zeta \widehat{C}\,(f(t))^\zeta, \end{aligned} \tag{4.58}$$

where $\widehat{C} = \left(\frac{\widetilde{C}^{p+\varepsilon}}{\varepsilon b^{\frac{(1-\beta)(p+\varepsilon)}{2\theta}}}\right)^{\frac{2\theta}{2\theta-(1-\beta)(p+\varepsilon)}}$.

We can prove by contradiction that for any $t \in [0,T)$, $f(t) > 0$. If not, there exists a $t_1 \ge 0$ such that $\|u(t_1)\|_2^2 = 0$, which contradicts (4.57). Then by (4.58) we have

$$\frac{f'(t)}{(f(t))^\zeta} < 2^\zeta \widehat{C}. \tag{4.59}$$

Integrating the above inequality from 0 to t, we have

$$(f(0))^{1-\zeta} - (f(t))^{1-\zeta} < 2^\zeta \widehat{C}(\zeta-1)t, \tag{4.60}$$

letting $t \to T$ in (4.60) and using (4.54) we can conclude that

$$T > \frac{(f(0))^{1-\zeta}}{2^\zeta \widehat{C}(\zeta-1)} = \frac{\|u_0\|_2^{2-2\zeta}}{2\widehat{C}(\zeta-1)}.$$

Similarly, integrating the inequality (4.59) from t to T, by (4.54) we have

$$f(t) > \left(2^{\zeta}\widehat{C}(\zeta-1)(T-t)\right)^{\frac{1}{1-\zeta}},$$

then by the definition of $f(t)$ we can see that

$$\|u\|_2 > \left(2\widehat{C}(\zeta-1)(T-t)\right)^{\frac{1}{2(1-\zeta)}}.$$

□

Proof of Theorem 3 Let $u = u(t)$ be the weak solution of problem (1.1) with $I(u_0) < 0$ and $J(u_0) \le \widetilde{M}$. Then according to Lemma 9 we have $I(u) < 0$ for all $t \in [0, T)$.

Let

$$G(t) := \int_0^t \|u\|_2^2 d\tau, \quad t \in [0, T),$$

then

$$G'(t) = \|u\|_2^2$$

and

$$G''(t) = -2I(u) > 0. \tag{4.61}$$

Then by (2.3), (2.6), (2.12), (4.61) and Corollary 2, we get

$$\begin{aligned} G''(t) &= -2pJ(u) + (p-2)a\|u\|_Z^2 + \frac{(p-2\theta)b}{\theta}\|u\|_Z^{2\theta} + \frac{2}{p}\|u\|_p^p \\ &\ge -2pJ(u_0) + 2p\int_0^t \|u_\tau\|_2^2 d\tau + (p-2)a\|u\|_Z^2 + \frac{(p-2\theta)b}{\theta}\|u\|_Z^{2\theta} \\ &\ge -2pJ(u_0) + (p-2)ar_*^2 + \frac{(p-2\theta)b}{\theta}r_*^{2\theta} + 2p\int_0^t \|u_\tau\|_2^2 d\tau \\ &= 2p(\widetilde{M} - J(u_0)) + 2p\int_0^t \|u_\tau\|_2^2 d\tau. \end{aligned} \tag{4.62}$$

Since

$$\left(\int_0^t (u,u_\tau)d\tau\right)^2 = \frac{1}{4}\left(\int_0^t \frac{d}{d\tau}\|u\|_2^2 d\tau\right)^2$$
$$= \frac{1}{4}(G'(t)-G'(0))^2$$
$$= \frac{1}{4}[(G'(t))^2 - 2G'(t)G'(0) + (G'(0))^2],$$

so we have

$$(G'(t))^2 = 4\left(\int_0^t (u,u_\tau)d\tau\right)^2 + 2\|u_0\|_2^2 G'(t) - \|u_0\|_2^4. \tag{4.63}$$

Combining (4.62) and (4.63), and using the Schwartz inequality, we get

$$\begin{aligned}
&G(t)G''(t) - \frac{p}{2}(G'(t))^2 \\
&\geq 2p\int_0^t \|u_\tau\|_2^2 d\tau \int_0^t \|u\|_2^2 d\tau - 2p\left(\int_0^t (u,u_\tau)d\tau\right)^2 \\
&\quad + 2p(\widetilde{M} - J(u_0))G(t) - p\|u_0\|_2^2 G'(t) + \frac{p}{2}\|u_0\|_2^4 \\
&\geq 2p(\widetilde{M} - J(u_0))G(t) - p\|u_0\|_2^2 G'(t) \\
&\geq -p\|u_0\|_2^2 G'(t).
\end{aligned}$$

Then for any $\gamma \in \left[0, \frac{2}{2_s^*}\right]$, we have

$$G(t)G''(t) - \frac{p\gamma}{2}(G'(t))^2 \geq \frac{p(1-\gamma)}{2}(G'(t))^2 - p\|u_0\|_2^2 G'(t). \tag{4.64}$$

Moreover, by Theorem 2, we know that $u(t)$ blows up at some finite time, so we have

$$\lim_{t\to T^-} G'(t) = \lim_{t\to T^-} \|u\|_2^2 = +\infty.$$

Then it follows from (4.64) that there exists a $t_\gamma \in (0,T)$ such that for all $t \in [t_\gamma, T)$

$$G(t)G''(t) - \frac{p\gamma}{2}(G'(t))^2 > 0. \tag{4.65}$$

Since

$$\left(G^{1-\frac{p\gamma}{2}}(t)\right)' = \left(1-\frac{p\gamma}{2}\right) G^{-\frac{p\gamma}{2}}(t)G'(t),$$

it follows from (4.65) that for all $t \in [t_\gamma, T)$,

$$\left(G^{1-\frac{p\gamma}{2}}(t)\right)'' = \left(1-\frac{p\gamma}{2}\right) G^{-\frac{p\gamma}{2}-1}(t)\left[G(t)G''(t) - \frac{p\gamma}{2}(G'(t))^2\right] > 0.$$

Then by $2 - p\gamma \geq 2 - \frac{2p}{2_s^*} > 0$ and $G(t_\gamma) > 0$, we have

$$\begin{aligned}
G(t) &= (G^{1-\frac{p\gamma}{2}}(t))^{\frac{2}{2-p\gamma}} \\
&= \left[G^{1-\frac{p\gamma}{2}}(t_\gamma) + \int_{t_\gamma}^{t} \left(G^{1-\frac{p\gamma}{2}}(\tau)\right)' d\tau\right]^{\frac{2}{2-p\gamma}} \\
&\geq \left[G^{1-\frac{p\gamma}{2}}(t_\gamma) + (t-t_\gamma)\left(G^{1-\frac{p\gamma}{2}}(\tau)\right)'\Big|_{\tau=t_\gamma}\right]^{\frac{2}{2-p\gamma}} \\
&= \left[G^{1-\frac{p\gamma}{2}}(t_\gamma) + \left(1-\frac{p\gamma}{2}\right)(t-t_\gamma)G^{-\frac{p\gamma}{2}}(t_\gamma)G'(t_\gamma)\right]^{\frac{2}{2-p\gamma}} \\
&\geq \left[\left(1-\frac{p\gamma}{2}\right)(t-t_\gamma)G^{-\frac{p\gamma}{2}}(t_\gamma)G'(t_\gamma)\right]^{\frac{2}{2-p\gamma}} \\
&= C_\gamma (t-t_\gamma)^{\frac{2}{2-p\gamma}},
\end{aligned} \tag{4.66}$$

where

$$C_\gamma := \left[\left(1-\frac{p\gamma}{2}\right) G^{-\frac{p\gamma}{2}}(t_\gamma)G'(t_\gamma)\right]^{\frac{2}{2-p\gamma}}.$$

Since $G''(t) > 0$ for all $t \in [0, T)$, thus we have

$$\int_0^t G'(\tau)d\tau \leq tG'(t),$$

i.e. (for all $t \in [0, T)$),

$$t\|u\|_2^2 \geq G(t).$$

Combining with (4.66) and the above inequality, we can deduce that for any $0 \leq \gamma \leq \frac{2}{2_s^*}$ and $t \in [t_\gamma, T)$,

$$\|u\|_2^2 \geq \frac{C_\gamma (t - t_\gamma)^{\frac{2}{2-p\gamma}}}{t} = C_\gamma (t^{\frac{p\gamma}{2}} - t^{\frac{p\gamma}{2}-1} t_\gamma)^{\frac{2}{2-p\gamma}}.$$

□

Proof of Theorem 4 To complete this proof, we use some ideas from [30,31] and we divide the proof into three steps.

Step 1: Blow-up in finite time

First, it follows from the definition of (2.3) and the assumption that

$$\begin{aligned} I(u_0) &= pJ(u_0) - \frac{(p-2)a}{2}\|u_0\|_Z^2 - \frac{(p-2\theta)b}{2\theta}\|u_0\|_Z^{2\theta} - \frac{1}{p}\|u_0\|_p^p \\ &\leq pJ(u_0) - \frac{(p-2\theta)b\lambda_1^\theta}{2\theta}\|u_0\|_2^{2\theta} < 0. \end{aligned}$$

Actually, we may claim that $I(u(t)) < 0$ for all $t \in [0, T)$. Otherwise, there exists a $t_0 \in (0, T)$ such that $I(u(t_0)) = 0$ and $I(u(t)) < 0$ for all $t \in [0, t_0)$. By Lemma 8 we know that $\|u\|_2^2$ is strictly increasing with respect to t for $t \in [0, t_0)$, and therefore

$$J(u_0) < \frac{(p-2\theta)b\lambda_1^\theta}{2\theta p}\|u_0\|_2^{2\theta} < \frac{(p-2\theta)b\lambda_1^\theta}{2\theta p}\|u(t_0)\|_2^{2\theta}. \tag{4.67}$$

On the other hand, by (2.3) and (2.12), we can get

$$\frac{(p-2\theta)b\lambda_1^\theta}{2\theta p}\|u(t_0)\|_2^{2\theta} \leq \frac{(p-2\theta)b}{2\theta p}\|u(t_0)\|_Z^{2\theta} \leq J(u(t_0)) \leq J(u_0),$$

which contradicts (4.67), so we obtain $I(u(t)) < 0$ for all $t \in [0, T)$.

Next, we are going to prove the blow-up of the solution $u(t)$ by contradiction. Fix a $\widetilde{T} = \frac{(4(p-1)\|u_0\|_2^2+1)^2+1}{\varrho(p-2)^2}$, where $\varrho := \frac{(p-2\theta)b\lambda_1^\theta}{\theta}\|u_0\|_2^{2\theta} - 2pJ(u_0) > 0$, then we suppose that $u(t)$ exists globally on $[0, \widetilde{T}]$ and let

$$G(t) := \int_0^t \|u\|_2^2 d\tau + (\widetilde{T} - t)\|u_0\|_2^2 + \sigma(t+\epsilon)^2, \quad t \in [0, \widetilde{T}],$$

where σ, ϵ are two positive constants which will be specified later.

Then, for any $t \in [0, \widetilde{T}]$, by a simply computation, we obtain

$$\begin{cases} G(0) = \widetilde{T}\|u_0\|_2^2 + \sigma\epsilon^2 > 0, \\ G'(t) = \|u\|_2^2 - \|u_0\|_2^2 + 2\sigma(t+\epsilon) > 2\sigma(t+\epsilon) > 0, \\ G'(0) = 2\sigma\epsilon > 0, \end{cases}$$

and

$$
\begin{aligned}
G''(t) = -2I(u(t)) + 2\sigma &\geq (p-2)a\|u\|_Z^2 + \frac{(p-2\theta)b}{\theta}\|u\|_Z^{2\theta} + \frac{2}{p}\|u\|_p^p \\
&\quad - 2pJ(u(t)) + 2\sigma \\
&\geq \frac{(p-2\theta)b}{\theta}\|u\|_Z^{2\theta} - 2pJ(u(t)) + 2\sigma \\
&\geq \frac{(p-2\theta)b\lambda_1^\theta}{\theta}\|u\|_2^{2\theta} - 2pJ(u_0) + 2p\int_0^t \|u_\tau\|_2^2 d\tau + 2\sigma \\
&\geq \frac{(p-2\theta)b\lambda_1^\theta}{\theta}\|u_0\|_2^{2\theta} - 2pJ(u_0) + 2p\int_0^t \|u_\tau\|_2^2 d\tau \\
&\quad + 2\sigma > 0,
\end{aligned}
$$

thus, we can get $G(t) \geq G(0) > 0$ for all $t \in [0, \widetilde{T}]$.

Let

$$
\mu(t) := \left(\int_0^t \|u\|_2^2 d\tau\right)^{\frac{1}{2}}, \quad \nu(t) := \left(\int_0^t \|u_\tau\|_2^2 d\tau\right)^{\frac{1}{2}}.
$$

By using Hölder's inequality, we have

$$
\begin{aligned}
&\left[\int_0^t \|u\|_2^2 d\tau + \sigma(t+\epsilon)^2\right]\left[\int_0^t \|u_\tau\|_2^2 d\tau + \sigma\right] - \left[\frac{1}{2}(\|u\|_2^2 - \|u_0\|_2^2) + \sigma(t+\epsilon)\right]^2 \\
&= \left[\mu^2(t) + \sigma(t+\epsilon)^2\right]\left[\nu^2(t) + \sigma\right] - \left[\frac{1}{2}\int_0^t \frac{d}{d\tau}\|u\|_2^2 d\tau + \sigma(t+\epsilon)\right]^2 \\
&\geq \left[\mu^2(t) + \sigma(t+\epsilon)^2\right]\left[\nu^2(t) + \sigma\right] - \left[\int_0^t \|u\|_2\|u_\tau\|_2 d\tau + \sigma(t+\epsilon)\right]^2 \\
&\geq \left[\mu^2(t) + \sigma(t+\epsilon)^2\right]\left[\nu^2(t) + \sigma\right] - [\mu(t)\nu(t) + \sigma(t+\epsilon)]^2 \\
&= \left[\sqrt{\sigma}\mu(t)\right]^2 - 2\sigma(t+\epsilon)\mu(t)\nu(t) + \left[\sqrt{\sigma}(t+\epsilon)\nu(t)\right]^2 \\
&= \left[\sqrt{\sigma}\mu(t) - \sqrt{\sigma}(t+\epsilon)\nu(t)\right]^2 \geq 0.
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
-(G'(t))^2 &= -4\left(\frac{1}{2}(\|u\|_2^2-\|u_0\|_2^2)+\sigma(t+\epsilon)\right)^2 \\
&= 4\left(\int_0^t \|u\|_2^2 d\tau+\sigma(t+\epsilon)^2\right)\left(\int_0^t \|u_\tau\|_2^2 d\tau+\sigma\right) \\
&\quad -4\left(\frac{1}{2}(\|u\|_2^2-\|u_0\|_2^2)+\sigma(t+\epsilon)\right)^2 \\
&\quad -4\left(G(t)-(\widetilde{T}-t)\|u_0\|_2^2\right)\left(\int_0^t \|u_\tau\|_2^2 d\tau+\sigma\right) \\
&\geq -4G(t)\left(\int_0^t \|u_\tau\|_2^2 d\tau+\sigma\right).
\end{aligned}
$$

The above calculations show that

$$
\begin{aligned}
&G(t)G''(t)-\frac{p}{2}\left(G'(t)\right)^2 \\
&\geq G(t)\left(G''(t)-2p\left(\int_0^t \|u_\tau\|_2^2 d\tau+\sigma\right)\right) \\
&\geq G(t)\left(\frac{(p-2\theta)b\lambda_1^\theta}{\theta}\|u_0\|_2^{2\theta}-2pJ(u_0)-2(p-1)\sigma\right).
\end{aligned}
$$

We choose $\sigma=\frac{\varrho}{4(p-1)}$, then it follows that $G(t)G''(t)-\frac{p}{2}\left(G'(t)\right)^2\geq 0$. Therefore, by Lemma 1 it is seen that

$$
T\leq \frac{2G(0)}{(p-2)G'(0)}=\frac{\|u_0\|_2^2}{\sigma\epsilon(p-2)}\widetilde{T}+\frac{\epsilon}{p-2} \text{ and } \lim_{t\to T} G(t)=+\infty.
$$

Next, we choose $\epsilon=\frac{4(p-1)\|u_0\|_2^2+1}{\varrho(p-2)}$, then we have $T<\widetilde{T}$, which is a contraction. Hence, $u(t)$ will blow-up at some finite time T.

Step 2: Upper bound estimate of the blow-up time
For any $T_1\in(0,T)$, let

$$
F(t):=\int_0^t \|u\|_2^2 d\tau+(T-t)\|u_0\|_2^2+\sigma(t+\epsilon)^2,\quad t\in[0,T_1],
$$

where σ, ϵ are two positive constants which will be specified later.

Then similar to Step 1 we can get

$$F(t)F''(t)-\frac{p}{2}\left(F'(t)\right)^2\geq F(t)\left(\frac{(p-2\theta)b\lambda_1^{\theta}}{\theta}\|u_0\|_2^{2\theta}-2pJ(u_0)-2(p-1)\sigma\right).$$

We choose σ small enough, such that

$$\sigma\in\left(0,\frac{\varrho}{2(p-1)}\right],\tag{4.68}$$

then it follows that $F(t)F''(t)-\frac{p}{2}\left(F'(t)\right)^2\geq 0$. Therefore, by Lemma 1 we obtain

$$T_1\leq\frac{2F(0)}{(p-2)F'(0)}=\frac{\|u_0\|_2^2}{\sigma\epsilon(p-2)}T+\frac{\epsilon}{p-2},\quad\forall T_1\in[0,T).$$

Hence, letting $T_1\to T$, we have

$$T\leq\frac{\|u_0\|_2^2}{\sigma\epsilon(p-2)}T+\frac{\epsilon}{p-2}.\tag{4.69}$$

Let ϵ be large enough such that

$$\epsilon\in\left(\frac{\|u_0\|_2^2}{(p-2)\sigma},+\infty\right),\tag{4.70}$$

then by (4.69), we can get

$$T\leq\frac{\sigma\epsilon^2}{\sigma\epsilon(p-2)-\|u_0\|_2^2}.$$

In view of (4.68) and (4.70), we define

$$\begin{aligned}\Lambda:&=\left\{(\sigma,\epsilon):\sigma\in\left(0,\frac{\varrho}{2(p-1)}\right],\epsilon\in\left(\frac{\|u_0\|_2^2}{(p-2)\sigma},+\infty\right)\right\}\\&=\left\{(\sigma,\epsilon):\sigma\in\left(\frac{\|u_0\|_2^2}{(p-2)\epsilon},\frac{\varrho}{2(p-1)}\right],\epsilon\in\left(\frac{2(p-1)\|u_0\|_2^2}{(p-2)\varrho},+\infty\right)\right\},\end{aligned}$$

then

$$T\leq\inf_{(\sigma,\epsilon)\in\Lambda}\frac{\sigma\epsilon^2}{\sigma\epsilon(p-2)-\|u_0\|_2^2}.$$

Let $\varsigma = \sigma\epsilon$ and

$$f(\epsilon, \varsigma) := \frac{\varsigma\epsilon}{\varsigma(p-2) - \|u_0\|_2^2}.$$

Since $f(\epsilon, \varsigma)$ is decreasing with ς and we obtain

$$\begin{aligned} T &\leq \inf_{\epsilon \in \left(\frac{2(p-1)\|u_0\|_2^2}{(p-2)\varrho}, +\infty\right)} f\left(\epsilon, \frac{\varrho\epsilon}{2(p-1)}\right) \\ &= \inf_{\epsilon \in \left(\frac{2(p-1)\|u_0\|_2^2}{(p-2)\varrho}, +\infty\right)} \frac{\varrho\epsilon^2}{\varrho\epsilon(p-2) - 2(p-1)\|u_0\|_2^2} \\ &= \frac{\varrho\epsilon^2}{\varrho\epsilon(p-2) - 2(p-1)\|u_0\|_2^2}\Bigg|_{\epsilon = \frac{4(p-1)\|u_0\|_2^2}{(p-2)\varrho}} \\ &= \frac{8(p-1)\|u_0\|_2^2}{(p-2)^2\varrho}. \end{aligned}$$

Hence, by the definition of ϱ and the above inequality, we have

$$T \leq \frac{8(p-1)\theta\|u_0\|_2^2}{(p-2)^2\left[(p-2\theta)b\lambda_1^\theta\|u_0\|_2^{2\theta} - 2p\theta J(u_0)\right]}.$$

Step 3: Growth estimates

First, similar to Step 1, we can get $I(u) < 0$ for all $t \in [0, T)$, so by Lemma 8, we know that $\|u\|_2^2$ is strictly increasing with respect to t. Then by Lemma 8 and (2.3) we know

$$\begin{aligned} &\frac{d}{dt}\left(\|u\|_2^2 - \frac{2p\theta}{(p-2\theta)b\lambda_1^\theta\|u_0\|_2^{2\theta-2}}J(u_0)\right) \\ &\quad = -2I(u) = -2pJ(u) + (p-2)a\|u\|_Z^2 + \frac{(p-2\theta)b}{\theta}\|u\|_Z^{2\theta} + \frac{2}{p}\|u\|_p^p. \end{aligned}$$

According to (2.12), we know $J(u(t)) \leq J(u_0)$ for all $t \in [0, T)$. So we can deduce from (1.10) and the above equality that

$$\begin{aligned} &\frac{d}{dt}\left(\|u\|_2^2 - \frac{2p\theta}{(p-2\theta)b\lambda_1^\theta\|u_0\|_2^{2\theta-2}}J(u_0)\right) \\ &\quad \geq -2pJ(u_0) + \frac{(p-2\theta)b\lambda_1^\theta}{\theta}\|u\|_2^{2\theta} \\ &\quad \geq -2pJ(u_0) + \frac{(p-2\theta)b\lambda_1^\theta\|u_0\|_2^{2\theta-2}}{\theta}\|u\|_2^2 \end{aligned}$$

$$= \frac{(p-2\theta)b\lambda_1^\theta\|u_0\|_2^{2\theta-2}}{\theta}\left(\|u\|_2^2 - \frac{2p\theta}{(p-2\theta)b\lambda_1^\theta\|u_0\|_2^{2\theta-2}}J(u_0)\right),$$

which implies

$$\|u\|_2^2 \geq \left(\|u_0\|_2^2 - \frac{2p}{A}J(u_0)\right)e^{At} + \frac{2p}{A}J(u_0),$$

where $A = \frac{(p-2\theta)b\lambda_1^\theta\|u_0\|_2^{2\theta-2}}{\theta}$. □

Proof of Theorem 5 Let $u(t)$ be the blow-up solution with $I(u_0) < 0$, $J(u_0) \leq d$ or (2.17) holds, and assume T is the maximal existence time, then by Theorem 2 and Theorem 4 we know

$$\lim_{t\to T}\|u\|_2 = +\infty. \tag{4.71}$$

Moreover, by Lemma 9 and Step 1 of Theorem 4, we can get $I(u) < 0$ for all $t \in [0, T)$, so by Lemma 8, we can infer that $\|u\|_2^2$ is strictly increasing for all $t \in [0, T)$.

By Hölder's inequality, we obtain

$$\int_0^t \|u_\tau\|_2^2 d\tau \geq \frac{1}{t}\left(\int_0^t \|u_\tau\|_2 d\tau\right)^2.$$

By [37, page 75, Proposition 3.3], we know

$$\int_0^t \|u_\tau\|_2 d\tau \geq \left\|\int_0^t u_\tau d\tau\right\|_2 = \|u(t) - u_0\|_2 \geq \|u\|_2 - \|u_0\|_2.$$

Since $\|u\|_2^2$ is strictly increasing on $[0, T)$, we know $\|u\|_2 - \|u_0\|_2 \geq 0$, then the above inequalities imply

$$\int_0^t \|u_\tau\|_2^2 d\tau \geq \frac{1}{t}(\|u\|_2 - \|u_0\|_2)^2.$$

So it follows from (2.12) that

$$\begin{aligned} J(u(t)) &\leq J(u_0) - \int_0^t \|u_\tau\|_2^2 d\tau \\ &\leq J(u_0) - \frac{1}{t}(\|u\|_2 - \|u_0\|_2)^2. \end{aligned}$$

Let $t \to T$ in the above inequality and using (4.71), we get

$$\lim_{t\to T} J(u(t)) = -\infty.$$

□

Proof of Theorem 6 We divide the proof into two cases.

Case 1: $u_0 \in Z \setminus \{0\}$ and $J(u_0) < d$.

First, we claim that $I(u_0) \neq 0$. Indeed, since $u_0 \neq 0$, if $I(u_0) = 0$, then by the definition of d, we get $J(u_0) \geq d$, which contradicts $J(u_0) < d$.

(1) If $I(u_0) < 0$, then together with $J(u_0) < d$ and Theorem 2, we know the solution blows up in finite time, so we get $T < +\infty$. Next we claim that if $T < +\infty$, then we have $I(u_0) < 0$. Indeed, if $I(u_0) > 0$, then together with $J(u_0) < d$ and Theorem 1, we get $T = +\infty$, which contradicts $T < +\infty$. Since $I(u_0) \neq 0$, so the claim is true. Moreover, if $I(u_0) < 0$, then according to $J(u_0) < d$ and Theorem 5, we can get there must exist a $t_0 \in [0, T)$ such that $J(u(t_0)) < 0$. Hence, in order to complete this proof, we now only need to show

$$\text{there exists a } t_0 \in [0, T) \text{ such that } J(u(t_0)) < 0 \Rightarrow I(u_0) < 0.$$

Since there exists a $t_0 \in [0, T)$ such that $J(u(t_0)) < 0$, so by (2.3), we have $I(u(t_0)) < 0$, then taking t_0 as the initial time, by Theorem 2, we see the solution blows up in finite time, so by Theorem 1 we know that $I(u_0) > 0$ cannot happen. Since $I(u_0) \neq 0$, so there must be $I(u_0) < 0$.

(2) If $I(u_0) > 0$, together with $J(u_0) < d$ and Theorem 1, we get $T = +\infty$. Next, we claim that if $T = +\infty$, then we have $I(u_0) > 0$. Indeed, if $I(u_0) < 0$, then by (1), we have $T < +\infty$, which is a contradiction. Since $I(u_0) \neq 0$, so the claim is true. Moreover, if $I(u_0) > 0$, then according to $J(u_0) < d$ and Lemma 9, we know $I(u(t)) > 0$ for all $t \in [0, +\infty)$. Thus, by (2.3) we deduce that $J(u(t)) > 0$ for all $t \in [0, +\infty)$. So we only need to prove

$$J(u(t)) > 0 \text{ for all } t \in [0, T) \Rightarrow I(u_0) > 0.$$

Since $I(u_0) \neq 0$, if $I(u_0) < 0$, then together with $J(u_0) < d$ and Theorem 5, we have $\lim_{t\to T} J(u(t)) = -\infty$. By the continuity of $J(u(t))$ with respect to t, we know there exists a t_0 such that $J(u(t_0)) < 0$, which contradicts $J(u(t)) > 0$ for all $t \in [0, T)$ and our proof is complete.

Case 2: $u_0 \in Z \setminus \{N \cup \{0\}\}$ and $J(u_0) = d$.

First, since $u_0 \in Z \setminus \{N \cup \{0\}\}$, so we have $I(u_0) \neq 0$.

(3) If $I(u_0) < 0$, then together with $J(u_0) = d$ and Theorem 2, we know the solution blows up in finite time, so we get $T < +\infty$. Next we claim that if $T < +\infty$, then we have $I(u_0) < 0$. Indeed, if $I(u_0) > 0$, then together with $J(u_0) = d$ and Corollary 1, we get $T = +\infty$, which contradicts $T < +\infty$. Since $I(u_0) \neq 0$, so the claim is true. Moreover, if $I(u_0) < 0$, then according to $J(u_0) = d$ and Theorem 5,

we can get there must exist a $t_0 \in [0, T)$ such that $J(u(t_0)) < 0$. Hence, in order to complete this proof, we now only need to show

$$\text{there exists a } t_0 \in [0, T) \text{ such that } J(u(t_0)) < 0 \Rightarrow I(u_0) < 0.$$

Since there exists a $t_0 \in [0, T)$ such that $J(u(t_0)) < 0$, so by (2.3), we have $I(u(t_0)) < 0$, then taking t_0 as the initial time, by Theorem 2, we see the solution blows up in finite time, so by Corollary 1 we know that $I(u_0) > 0$ cannot happen. Since $I(u_0) \neq 0$, so there must be $I(u_0) < 0$.

(4) If $I(u_0) > 0$, together with $J(u_0) = d$ and Corollary 1, we get $T = +\infty$. Next, we claim that if $T = +\infty$, then we have $I(u_0) > 0$. Indeed, if $I(u_0) < 0$, then by (3), we have $T < +\infty$, which is a contradiction. Since $I(u_0) \neq 0$, so the claim is true. Moreover, if $I(u_0) > 0$, then according to $J(u_0) = d$ and Lemma 9, we know $I(u(t)) > 0$ for all $t \in [0, +\infty)$. Thus, by (2.3) we deduce that $J(u(t)) > 0$ for all $t \in [0, +\infty)$. So we only need to prove

$$J(u(t)) > 0 \text{ for all } t \in [0, T) \Rightarrow I(u_0) > 0.$$

Since $I(u_0) \neq 0$, if $I(u_0) < 0$, then together with $J(u_0) = d$ and Theorem 5, we have $\lim_{t \to T} J(u(t)) = -\infty$. By the continuity of $J(u(t))$ with respect to t, we know there exists a t_0 such that $J(u(t_0)) < 0$, which contradicts $J(u(t)) > 0$ for all $t \in [0, T)$ and our proof is complete. □

Proof of Theorem 7 (1) By the definition of d in (2.4), N in (2.5) and (2.3), we get

$$d = \inf_{u \in N} J(u) = \inf_{u \in N} \left[\frac{(p-2)a}{2p} \|u\|_Z^2 + \frac{(p-2\theta)b}{2\theta p} \|u\|_Z^{2\theta} + \frac{1}{p^2} \|u\|_p^p \right].$$

Then a minimizing sequence $\{u_k\}_{k=1}^{\infty} \subset N$ exists such that

$$\lim_{k \to \infty} J(u_k) = \lim_{k \to \infty} \left[\frac{(p-2)a}{2p} \|u_k\|_Z^2 + \frac{(p-2\theta)b}{2\theta p} \|u_k\|_Z^{2\theta} + \frac{1}{p^2} \|u_k\|_p^p \right] = d. \tag{4.72}$$

Since $p \in (2\theta, 2_s^*)$, (4.72) ensures that $\{u_k\}_{k=1}^{\infty}$ is bounded in Z, i.e., there exists a constant ϑ independent of k such that

$$\|u_k\|_Z \leq \vartheta, \quad k = 1, 2, \ldots, \tag{4.73}$$

which, together with Z is reflexive, implies there exists a subsequence of $\{u_k\}_{k=1}^{\infty}$, still denoted by $\{u_k\}_{k=1}^{\infty}$, and a v_0 such that

$$u_k \rightharpoonup v_0 \text{ weakly in } Z \text{ as } k \to \infty. \tag{4.74}$$

Moreover, by Lemma 2, we have

$$u_k \to v_0 \text{ strongly in } L^p(\Omega) \text{ as } k \to \infty. \tag{4.75}$$

Since (4.73) holds, similar to the proof (4.11), there exists a positive constant C_χ independent of k such that

$$\int_\Omega ||u_k|^{p-2}u_k \log |u_k||^{\frac{p}{p-1}} \leq C_\chi. \tag{4.76}$$

In view of (4.75) and (4.76), similar to get (4.18), there exists a subsequence of $\{u_k\}_{k=1}^\infty$, still denoted by $\{u_k\}_{k=1}^\infty$ such that

$$|u_k|^{p-2}u_k \log |u_k| \rightharpoonup |v_0|^{p-2}v_0 \log |v_0| \text{ weakly in } L^{\frac{p}{p-1}}(\Omega) \text{ as } k \to \infty. \tag{4.77}$$

Similar to the proof of (4.22), by (4.75) and (4.77), we have, as $k \to \infty$,

$$\begin{aligned} &\left| \int_\Omega |u_k|^p \log |u_k| dx - \int_\Omega |v_0|^p \log |v_0| dx \right| \\ &\leq \sqrt[\frac{p}{p-1}]{C_\chi} \|u_k - v_0\|_p \\ &\quad + \left| \int_\Omega v_0 \left(|u_k|^{p-2}u_k \log |u_k| - |v_0|^{p-2}v_0 \log |v_0| \right) dx \right| \\ &\to 0. \end{aligned} \tag{4.78}$$

Since $\{u_k\}_{k=1}^\infty \in N$, by the definition of N in (2.5), we get

$$a\|u_k\|_Z^2 + b\|u_k\|_Z^{2\theta} = \int_\Omega |u_k|^p \log |u_k| dx,$$

which, together with $\|\cdot\|_Z$ is weakly lower semi-continuous, (4.74) and (4.78), implies

$$\begin{aligned} a\|v_0\|_Z^2 + b\|v_0\|_Z^{2\theta} &\leq \liminf_{k\to\infty} (a\|u_k\|_Z^2 + b\|u_k\|_Z^{2\theta}) \\ &= \lim_{k\to\infty} \int_\Omega |u_k|^p \log |u_k| dx \\ &= \int_\Omega |v_0|^p \log |v_0| dx. \end{aligned} \tag{4.79}$$

Next, we claim that $I(v_0) = a\|v_0\|_Z^2 + b\|v_0\|_Z^{2\theta} - \int_\Omega |v_0|^p \log |v_0| dx = 0$. Indeed, if the claim is not true, then by (4.79), we get $a\|v_0\|_Z^2 + b\|v_0\|_Z^{2\theta} < \int_\Omega |v_0|^p \log |v_0| dx$. Obviously, we have $v_0 \neq 0$. Then by Lemma 6, there exists a $\lambda^* \in (0, 1)$ such that $\lambda^* v_0 \in N$.

By (4.72), the first inequality of (4.79) and (4.75), we get

$$
\begin{aligned}
d &= \lim_{k\to\infty}\left[\frac{(p-2)a}{2p}\|u_k\|_Z^2 + \frac{(p-2\theta)b}{2\theta p}\|u_k\|_Z^{2\theta} + \frac{1}{p^2}\|u_k\|_p^p\right] \\
&\ge \frac{p-2}{2p}\liminf_{k\to\infty} a\|u_k\|_Z^2 + \frac{p-2\theta}{2\theta p}\liminf_{k\to\infty} b\|u_k\|_Z^{2\theta} + \frac{1}{p^2}\liminf_{k\to\infty}\|u_k\|_p^p \\
&\ge \frac{(p-2)a}{2p}\|v_0\|_Z^2 + \frac{(p-2\theta)b}{2\theta p}\|v_0\|_Z^{2\theta} + \frac{1}{p^2}\|v_0\|_p^p.
\end{aligned}
$$

Then by $I(\lambda^* v_0) = 0$, $\lambda^* \in (0, 1)$, we obtain

$$
\begin{aligned}
J(\lambda^* v_0) &= \frac{(p-2)a}{2p}\lambda^{*2}\|v_0\|_Z^2 + \frac{(p-2\theta)b}{2\theta p}\lambda^{*2\theta}\|v_0\|_Z^{2\theta} + \frac{\lambda^{*p}}{p^2}\|v_0\|_p^p \\
&< \frac{(p-2)a}{2p}\|v_0\|_Z^2 + \frac{(p-2\theta)b}{2\theta p}\|v_0\|_Z^{2\theta} + \frac{1}{p^2}\|v_0\|_p^p \\
&\le d.
\end{aligned}
$$

However, since $\lambda^* v_0 \in N$, it follows from the definition of d in (2.4) that $J(\lambda^* v_0) \ge d$, a contradiction. So the claim holds and we get from (4.79) that

$$
\lim_{k\to\infty}\|u_k\|_Z = \|v_0\|_Z,
$$

which, together with Z is uniformly convex and (4.74), implies

$$
u_k \to v_0 \text{ strongly in } Z \text{ as } k \to \infty.
$$

Then by (2.3) and (4.72), we get

$$
\begin{aligned}
J(v_0) &= \frac{(p-2)a}{2p}\|v_0\|_Z^2 + \frac{(p-2\theta)b}{2\theta p}\|v_0\|_Z^{2\theta} + \frac{1}{p^2}\|v_0\|_p^p \\
&= \lim_{k\to\infty}\left[\frac{(p-2)a}{2p}\|u_k\|_Z^2 + \frac{(p-2\theta)b}{2\theta p}\|u_k\|_Z^{2\theta} + \frac{1}{p^2}\|u_k\|_p^p\right] \\
&= d,
\end{aligned}
$$

which implies $v_0 \ne 0$. Then by $I(v_0) = 0$, we get $v_0 \in N$ and $d = J(v_0) = \inf_{u\in N} J(u)$.

(2) Finally, we prove v_0 is a ground-state solution of problem (2.19). That is, $v_0 \in \Gamma \setminus \{0\}$ and

$$
J(v_0) = \inf_{u\in\Gamma\setminus\{0\}} J(u), \tag{4.80}
$$

where Γ is defined in (2.20).

In fact, by conclusion (1) and the definition of N in (2.5) we know that

$$v_0 \in N = \{u \in Z \setminus \{0\} : \langle J'(u), u\rangle = I(u) = 0\}$$

and

$$J(v_0) = d = \inf_{u \in N} J(u). \tag{4.81}$$

Therefore, $v_0 \neq 0$, and by the theory of Lagrange multipliers, there exists a constant $\mu \in \mathbb{R}$ such that

$$J'(v_0) - \mu I'(v_0) = 0. \tag{4.82}$$

Then

$$\mu\langle I'(v_0), v_0\rangle = \langle J'(v_0), v_0\rangle = I(v_0) = 0. \tag{4.83}$$

On the other hand, for any $u \in Z$, we can deduce

$$\begin{aligned}
\langle I'(v_0), u\rangle &= \frac{d}{d\tau} I(v_0 + \tau u)\Big|_{\tau=0} \\
&= \frac{d}{d\tau}\Big[a\|v_0 + \tau u\|_Z^2 + b\|v_0 + \tau u\|_Z^{2\theta} \\
&\quad - \int_\Omega |v_0 + \tau u|^p \log|v_0 + \tau u| dx\Big]\Big|_{\tau=0} \\
&= 2a\langle v_0, u\rangle_Z + 2\theta b\|v_0\|_Z^{2\theta-2}\langle v_0, u\rangle_Z \\
&\quad - \int_\Omega \Big[p|v_0|^{p-2}v_0 u \log|v_0| + |v_0|^{p-2}v_0 u\Big] dx,
\end{aligned}$$

which implies

$$\langle I'(v_0), v_0\rangle = 2a\|v_0\|_Z^2 + 2\theta b\|v_0\|_Z^{2\theta} - \int_\Omega \big[p|v_0|^p \log|v_0| + |v_0|^p\big] dx.$$

Since $I(v_0) = 0$, we get from (2.2) that $a\|v_0\|_Z^2 + b\|v_0\|_Z^{2\theta} = \int_\Omega |v_0|^p \log|v_0| dx$. Then it follows from the above equality, $v_0 \neq 0$, and $p \in (2\theta, 2_s^*)$ that

$$\langle I'(v_0), v_0\rangle = 2a\|v_0\|_Z^2 + 2\theta b\|v_0\|_Z^{2\theta} - ap\|v_0\|_Z^2 - bp\|v_0\|_Z^{2\theta} - \|v_0\|_p^p < 0,$$

which, together with (4.83), implies $\mu = 0$. Then we get from (4.82) that $J'(v_0) = 0$, so $v_0 \in \Gamma \setminus \{0\}$. Furthermore, by (4.81) and $\Gamma \setminus \{0\} \subset N$, we get (4.80). □

Proof of Theorem 8 Let $u = u(t)$ be a global solution of problem (1.1). Without loss of generality, we may assume that

$$0 \leq J(u(t)) \leq J(u_0), \quad t \in [0, \infty). \tag{4.84}$$

Indeed, the second inequality follows from (2.12). Now we prove the first inequality by contradiction argument. If there is a $t_0 \in [0, \infty)$ such that $J(u(t_0)) < 0$, then by (2.3) we have $I(u(t_0)) < 0$, so it follows from Theorem 2 that $u(t)$ blows up in finite time, which contradicts the assumption that $u(t)$ is global.

Since $J(u(t)) \in [0, J(u_0)]$, so there must exist a subsequence $\{t_m\}_{m=1}^{\infty}$ and a constant $c \in [0, J(u_0)]$ such that

$$\lim_{t_m \to \infty} J(u(t_m)) = c.$$

By (2.12), we have

$$\int_0^{t_m} \|u_\tau\|_2^2 d\tau + J(u(t_m)) \leq J(u_0).$$

Letting $t_m \to \infty$ in the above inequality, we get

$$\int_0^{\infty} \|u_\tau\|_2^2 d\tau \leq J(u_0) - c \leq J(u_0),$$

which implies there is an increasing sequence $\{t_k\}_{k=1}^{\infty}$ with $t_k \to \infty$ as $k \to \infty$ satisfying

$$\lim_{k \to \infty} \|u_t(t_k)\|_2 = 0. \tag{4.85}$$

By the definition of J in (2.1) and (2.11), for any $\psi \in Z$, we have

$$\begin{aligned}
\langle J'(u(t)), \psi \rangle &= \frac{d}{d\tau} J(u(t) + \tau\psi) \Big|_{\tau=0} \\
&= a\langle u, \psi \rangle_Z + b\|u\|_Z^{2\theta-2} \langle u, \psi \rangle_Z - \int_\Omega |u|^{p-2} u\psi \log |u| dx \\
&= -\int_\Omega u_t \psi dx,
\end{aligned}$$

which, together with (4.85), implies

$$\|J'(u(t_k))\|_{Z'} = \sup_{\|\psi\|_Z \leq 1} |\langle J'(u(t_k)), \psi \rangle|$$

$$
\begin{aligned}
&\leq \sup_{\|\psi\|_Z\leq 1} \|u_t(t_k)\|_2\|\psi\|_2 \\
&\leq \frac{1}{\sqrt{\lambda_1}} \sup_{\|\psi\|_Z\leq 1} \|u_t(t_k)\|_2\|\psi\|_Z \\
&\leq \frac{1}{\sqrt{\lambda_1}}\|u_t(t_k)\|_2 \\
&\to 0 \qquad (4.86)
\end{aligned}
$$

as $k \to \infty$, where λ_1 is the first eigenvalue of problem (1.9).

By (2.2) and (4.86), there exists a positive constant σ such that

$$
\begin{aligned}
\frac{1}{p}|I(u(t_k))| &= \frac{1}{p}|\langle J'(u(t_k)), u(t_k)\rangle| \\
&\leq \frac{1}{p}\|J'(u(t_k))\|_{Z'}\|u(t_k)\|_Z \\
&\leq \sigma\|u(t_k)\|_Z.
\end{aligned}
$$

Then it follows from (2.1), (2.2), (4.84) and $p \in (2\theta, 2_s^*)$ that

$$
\begin{aligned}
J(u_0) + \sigma\|u(t_k)\|_Z &\geq J(u(t_k)) - \frac{1}{p}I(u(t_k)) \\
&= \frac{(p-2)a}{2p}\|u(t_k)\|_Z^2 + \frac{(p-2\theta)b}{2\theta p}\|u(t_k)\|_Z^{2\theta} + \frac{1}{p^2}\|u(t_k)\|_p^p \\
&\geq \frac{(p-2\theta)b}{2\theta p}\|u(t_k)\|_Z^{2\theta},
\end{aligned}
$$

which implies there exists a positive constant L independent of k such that

$$
\|u(t_k)\|_Z \leq L, \quad k = 1, 2, \ldots \qquad (4.87)
$$

Indeed, if $\|u(t_k)\|_Z$ is unbounded, then there must be a $\tilde{t}_k$ such that $\|u(\tilde{t}_k)\|_Z \to \infty$, which, together with $\theta \in \left[1, \frac{2_s^*}{2}\right)$ and $p > 2\theta$, implies

$$
J(u_0) + \sigma\|u(\tilde{t}_k)\|_Z - \frac{(p-2\theta)b}{2\theta p}\|u(\tilde{t}_k)\|_Z^{2\theta} < 0,
$$

a contradiction.

Then by similar arguments as to get (4.74) and (4.75), there exists an increasing subsequence of the sequence $\{t_k\}_{k=1}^{\infty}$, still denoted by $\{t_k\}_{k=1}^{\infty}$, and a $u^* \in Z$ such that $u_k := u(t_k)$ satisfies

$$
u_k \rightharpoonup u^* \text{ weakly in } Z \text{ as } k \to \infty, \qquad (4.88)
$$

and

$$u_k \to u^* \text{ strongly in } L^p(\Omega) \text{ as } k \to \infty. \tag{4.89}$$

By the definition of J in (2.1) and (1.7) we obtain

$$\begin{aligned}
&\langle J'(u_k), u_k - u^* \rangle \\
&= \frac{d}{d\tau} J(u_k + \tau(u_k - u^*))\Big|_{\tau=0} \\
&= \frac{d}{d\tau}\left[\frac{a}{2}\|u_k + \tau(u_k - u^*)\|_Z^2 + \frac{b}{2\theta}\|u_k + \tau(u_k - u^*)\|_Z^{2\theta}\right]\Big|_{\tau=0} \\
&\quad - \frac{d}{d\tau}\left[\frac{1}{p}\int_\Omega |u_k + \tau(u_k - u^*)|^p \log|u_k + \tau(u_k - u^*)| dx\right. \\
&\quad \left. - \frac{1}{p^2}\|u_k + \tau(u_k - u^*)\|_p^p\right]\Big|_{\tau=0} \\
&= (a + b\|u_k\|_Z^{2\theta-2})\langle u_k, u_k - u^* \rangle_Z - \int_\Omega |u_k|^{p-2} u_k (u_k - u^*) \log|u_k| dx.
\end{aligned} \tag{4.90}$$

Similarly, we have

$$\begin{aligned}
\langle J'(u^*), u_k - u^* \rangle = (a + b\|u^*\|_Z^{2\theta-2})\langle u^*, u_k - u^* \rangle_Z \\
- \int_\Omega |u^*|^{p-2} u^* (u_k - u^*) \log|u^*| dx.
\end{aligned} \tag{4.91}$$

So, we can get

$$\begin{aligned}
\langle J'(u_k) - J'(u^*), u_k - u^* \rangle &= a\|u_k - u^*\|_Z^2 + b\langle \|u_k\|_Z^{2\theta-2} u_k \\
&\quad - \|u^*\|_Z^{2\theta-2} u^*, u_k - u^* \rangle_Z + \rho \\
&\geq C_{\theta,b}\|u_k - u^*\|_Z^{2\theta} + \rho,
\end{aligned} \tag{4.92}$$

where $C_{\theta,b}$ is a positive constant independent of k and

$$\rho := \int_\Omega |u^*|^{p-2} u^* (u_k - u^*) \log|u^*| - |u_k|^{p-2} u_k (u_k - u^*) \log|u_k| dx.$$

According to (4.87), then similar to the proof of (4.11), we have

$$\| |u_k|^{p-1} \log|u_k| \|_{\frac{p}{p-1}} \leq C_L, \tag{4.93}$$

where C_L is a positive constant independent of k.

Therefore, by Hölder's inequality, (4.89) and (4.93), we have

$$\begin{aligned}|\rho| &\le \left(\||u^*|^{p-1}\log|u^*|\|_{\frac{p}{p-1}} + \||u_k|^{p-1}\log|u_k|\|_{\frac{p}{p-1}}\right)\|u_k-u^*\|_p \\ &\le \left(\||u^*|^{p-1}\log|u^*|\|_{\frac{p}{p-1}} + C_L\right)\|u_k-u^*\|_p \\ &\to 0\end{aligned} \tag{4.94}$$

as $k\to\infty$. By (4.88), (4.89) and (4.91), we get

$$\langle J'(u^*), u_k-u^*\rangle \to 0 \tag{4.95}$$

as $k\to\infty$. By (4.86) and (4.87), we have

$$\begin{aligned}|\langle J'(u_k), u_k-u^*\rangle| &\le \|J'(u_k)\|_{Z'}(\|u_k\|_Z+\|u^*\|_Z) \\ &\le (L+\|u^*\|_Z)\|J'(u_k)\|_{Z'} \\ &\to 0\end{aligned} \tag{4.96}$$

as $k\to\infty$.

Then it follows from (4.92), (4.94), (4.95) and (4.96) that

$$\begin{aligned}C_{\theta,b}\|u_k-u^*\|_Z^{2\theta} &\le \langle J'(u_k)-J'(u^*), u_k-u^*\rangle - \rho \\ &\to 0\end{aligned}$$

as $k\to\infty$. Then we get

$$J'(u^*) = \lim_{k\to\infty} J'(u_k)$$

in Z', which, together with (4.86), implies $J'(u^*)=0$, i.e., $u^*\in\Gamma$. □

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