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Vectorial Variational Principles in L^∞ and Their Characterisation Through PDE Systems

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Abstract
We discuss two distinct minimality principles for general supremal first order functionals for maps and characterise them through solvability of associated second order PDE systems. Specifically, we consider Aronsson's standard notion of absolute minimisers and the concept of ∞-minimal maps introduced more recently by the second author. We prove that C^1 absolute minimisers characterise a divergence system with parameters probability measures and that C^2 ∞-minimal maps characterise Aronsson's PDE system. Since in the scalar case these different variational concepts coincide, it follows that the non-divergence Aronsson's equation has an equivalent divergence counterpart.

Keywords Calculus of variations in L^∞ · L^∞ variational principle · Aronsson system · ∞-Laplacian · Absolute minimisers · ∞-minimal maps

Mathematics Subject Classification Primary 35J47 · 35J62 · 53C24; Secondary 49J99

1 Introduction

Let $n, N \in \mathbb{N}$ and $\mathrm{H} \in C^2\big(\Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}\big)$ with $\Omega \subseteq \mathbb{R}^n$ an open set. In this paper we consider the supremal functional

$$\mathrm{E}_\infty(u, \mathcal{O}) := \operatorname*{ess\,sup}_{\mathcal{O}} \mathrm{H}(\cdot, u, \mathrm{D}u), \quad u \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{R}^N),\ \mathcal{O} \Subset \Omega, \tag{1.1}$$

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defined on maps $u : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^N$. In (1.1) and subsequently, we see the gradient as a matrix map $\mathrm{D}u = (\mathrm{D}_i u_\alpha)_{i=1...n}^{\alpha=1...N} : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^{N\times n}$. Variational problems for (1.1) have been pioneered by Aronsson in the 1960s in the scalar case $N = 1$ ([1–5]). Nowadays the study of such functionals (and of their associated PDEs describing critical points) form a fairly well-developed area of vivid interest, called Calculus of Variations in L^∞. For pedagogical general introductions to the theme we refer to [7,19,31].

One of the main difficulties in the study of (1.1) which prevents us from utilising the standard machinery of Calculus of Variations for conventional (integral) functionals as e.g. in [22] is that it is non-local, in the sense that a global minimisers u of $\mathrm{E}_\infty(\cdot,\Omega)$ in $W^{1,\infty}_g(\Omega;\mathbb{R}^N)$ for some fixed boundary data g may not minimise $\mathrm{E}_\infty(\cdot,\mathcal{O})$ in $W^{1,\infty}_u(\mathcal{O};\mathbb{R}^N)$. Namely, global minimisers are not generally local minimisers, a property which is automatic for integral functionals. The remedy proposed by Aronsson (adapted) to the vector case is to build locality into the minimality notion:

Definition 1 Let $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$. We say that u is an *absolute minimiser* of (1.1) on Ω if

$$\left.\begin{array}{l}\forall\, \mathcal{O} \Subset \Omega,\\ \forall\, \phi \in W^{1,\infty}_0(\mathcal{O};\mathbb{R}^N)\end{array}\right\} \implies \mathrm{E}_\infty(u,\mathcal{O}) \le \mathrm{E}_\infty(u+\phi,\mathcal{O}). \tag{1.2}$$

In the scalar case of $N = 1$, Aronsson's concept of absolute minimisers turns out to be the appropriate substitute of mere minimisers. Indeed, absolute minimisers possess the desired uniqueness properties subject to boundary conditions and, most importantly, the possibility to characterise them through a necessary (and sufficient) condition of satisfaction of a certain nonlinear nondivergence second order PDE, known as the Aronsson equation ([7,8,10–13,16–18,20,26,37,42]). The latter can be written for functions $u \in C^2(\Omega)$ as

$$\mathrm{H}_P(\cdot,u,\mathrm{D}u)\cdot \mathrm{D}\big(\mathrm{H}(\cdot,u,\mathrm{D}u)\big) = 0. \tag{1.3}$$

The Aronsson equation, being degenerate elliptic and non-divergence when formally expanded, is typically studied in the framework of viscosity solutions. In the above, $\mathrm{H}_P, \mathrm{H}_\eta, \mathrm{H}_x$ denotes the derivatives of $\mathrm{H}(x,\eta,P)$ with respect to the respective arguments and "$\cdot$" is the Euclidean inner product.

In this paper we are interested in characterising appropriately defined minimisers of (1.1) in the general vectorial case of $N \ge 2$ through solvability of associated PDE systems which generalise the Aronsson equation (1.3). As the wording suggests and we explain below, when $N \ge 2$ Aronsson's notion of Definition 1 is no longer the unique possible L^∞ variational concept. In any case, the extension of Aronsson's equation to the vectorial case reads

$$\begin{aligned}&\mathrm{H}_P(\cdot,u,\mathrm{D}u)\,\mathrm{D}\big(\mathrm{H}(\cdot,u,\mathrm{D}u)\big)\\ &\quad + \mathrm{H}(\cdot,u,\mathrm{D}u)\,[\mathrm{H}_P(\cdot,u,\mathrm{D}u)]^\perp\Big(\mathrm{Div}\big(\mathrm{H}_P(\cdot,u,\mathrm{D}u)\big) - \mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\Big) = 0.\end{aligned} \tag{1.4}$$

In the above, for any linear map $A : \mathbb{R}^n \longrightarrow \mathbb{R}^N$, $[A]^\perp$ symbolises the orthogonal projection $\mathrm{Proj}_{\mathrm{R}(A)^\perp}$ on the orthogonal complement of its range $\mathrm{R}(A) \subseteq \mathbb{R}^N$. We will refer to the PDE system (1.4) as the "Aronsson system", in spite of the fact it was actually derived by the second author in [27], wherein the connections between general vectorial variational problems and their associated PDEs were first studied, namely those playing the role of Euler-Lagrange equations in L^∞. The Aronsson system was derived through the well-known method of L^p-approximations and is being studied quite systematically since its discovery, see e.g. [27–30,32,36]. The additional normal term which is not present in the scalar case imposes an extra layer of complexity, as it might be discontinuous even for smooth solutions (see [28,30]).

For simplicity and in order to illustrate the main ideas in a manner which minimises technical complications, *in this paper we restrict our attention exclusively to regular minimisers and solutions*. In general, solutions to (1.4) are nonsmooth and the lack of divergence structure combined with its vectorial nature renders its study beyond the reach of viscosity solutions. To this end, the theory of $\mathcal{D}$-solutions introduced in [32] and subsequently utilised in several works (see e.g. [9,21,32,33]) offers a viable alternative for the study of general locally Lipschitz solutions to (1.4), and in fact it works far beyond the realm of Calculus of Variations in L^∞. We therefore leave the generalisation of the results herein to a lower regularity setting for future work.

Additionally to absolute minimisers, for reasons to be explained later, in the paper [29] a special case of the next L^∞ variational concept was introduced (therein for $\mathrm{H}(x, \eta, P) = |P|^2$):

Definition 2 Let $u \in C^1(\Omega; \mathbb{R}^N)$. We say that u is an ∞-*minimal map* for (1.1) on Ω if (i) and (ii) below hold true:
(i) u is a *rank-one absolute minimiser*, namely it minimises with respect to essentially scalar variations vanishing on the boundary along fixed unit directions:

$$\left.\begin{array}{l}\forall\, \mathcal{O} \Subset \Omega,\ \forall\, \xi \in \mathbb{R}^N \\ \forall\, \phi \in C^1_0(\overline{\mathcal{O}}; \mathrm{span}[\xi])\end{array}\right\} \implies \mathrm{E}_\infty(u, \mathcal{O}) \le \mathrm{E}_\infty(u + \phi, \mathcal{O}). \tag{1.5}$$

(ii) u has ∞-*minimal area*, namely it minimises with respect to variations which are normal to the range of the matrix field $\mathrm{H}_P(\cdot, u, \mathrm{D}u)$ and free on the boundary:

$$\left.\begin{array}{l}\forall\, \mathcal{O} \Subset \Omega,\ \ \forall\, \phi \in C^1(\mathbb{R}^n; \mathbb{R}^N) \\ \text{with } \phi^\top \mathrm{H}_P(\cdot, u, \mathrm{D}u) = 0 \text{ on } \mathcal{O}\end{array}\right\} \implies \mathrm{E}_\infty(u, \mathcal{O}) \le \mathrm{E}_\infty(u + \phi, \mathcal{O}). \tag{1.6}$$

In the above,

$$C^1_0(\overline{\mathcal{O}}; \mathbb{R}^N) := \big\{\psi \in C^1(\mathbb{R}^n; \mathbb{R}^N)\ :\ \psi = 0 \text{ on } \partial\mathcal{O}\big\}.$$

Note also that *when $N = 1$ absolute minimisers and ∞-minimal maps coincide*, at least when $\{\mathrm{H}_P = 0\} \subseteq \{\mathrm{H} = 0\}$. Further, in the event that $\mathrm{H}_P(\cdot, u, \mathrm{D}u)$ has discontinuous rank on $\mathcal{O}$, the only continuous normal vector fields ϕ may be only those vanishing on the set of discontinuities.

In [29] it was proved that C^2 ∞-minimal maps of full rank (namely immersions or submersions) are ∞-Harmonic, that is solutions to the so-called ∞-Laplace system. The latter is a special case of (1.4), corresponding to the choice $\mathrm{H}(x,\eta,P)=|P|^2$:

$$\mathrm{D}u\,\mathrm{D}\big(|\mathrm{D}u|^2\big)\;+\;|\mathrm{D}u|^2\,[\mathrm{D}u]^{\perp}\Delta u\;=\;0. \tag{1.7}$$

The fullness of rank was assumed because of the possible discontinuity of the coefficient $[\mathrm{D}u]^{\perp}$, which may well happen even for smooth solutions (for explicit examples see [28]). In this paper we bypass this difficulty by replacing the orthogonal projection $[\,\cdot\,]^{\perp}$ by the projection on the subspace of those normal vectors which have local normal C^1 extensions in a open neighbourhood:

Definition 3 Let $V:\mathbb{R}^n\supseteq\Omega\longrightarrow\mathbb{R}^{N\times n}$ be a matrix field and note that

$$\mathrm{R}(V(x))^{\perp}\,=\,\mathrm{N}(V(x)^{\top}),$$

where for any $x\in\Omega$, $\mathrm{N}(V(x)^{\top})$ is the nullspace of the transpose $V(x)^{\top}\in\mathbb{R}^{n\times N}$. We define the orthogonal projection

$$[\![V(x)]\!]^{\perp}:=\,\mathrm{Proj}_{\tilde{\mathrm{N}}(V(x)^{\top})},\quad [\![V(\cdot)]\!]^{\perp}:\;\mathbb{R}^n\supseteq\Omega\longrightarrow\mathbb{R}^{N\times N},$$

where $\tilde{\mathrm{N}}(V(x)^{\top})$ is the reduced nullspace, given by

$$\tilde{\mathrm{N}}(V(x)^{\top})\;:=\;\Big\{\xi\in\mathrm{N}(V(x)^{\top})\;\Big|\;\exists\,\varepsilon>0\;\;\&\;\exists\,\bar{\xi}\in C^1(\mathbb{R}^n;\mathbb{R}^N):\;\bar{\xi}(x)=\xi\;\;\&\;\bar{\xi}(y)\in\mathrm{N}(V(y)^{\top}),\;\forall\,y\in\mathbb{B}_{\varepsilon}(x)\Big\}.$$

It is a triviality to check that $\tilde{\mathrm{N}}(V(x)^{\top})$ is indeed a vector space and that

$$[\![V(x)]\!]^{\perp}[V(x)]^{\perp}\,=\,[\![V(x)]\!]^{\perp},$$

where $[V(x)]^{\perp}\,=\,\mathrm{Proj}_{\mathrm{N}(V(x)^{\top})}$. Note that the definition could be written in a more concise manner by using the algebraic language of *sheaves and germs*, but we refrained from doing so as there is no real benefit in this simple case.

The first main result in this paper is the next variational characterisation of the Aronsson system (1.4).

Theorem 4 (Variational Structure of Aronsson's system) *Let $u:\mathbb{R}^n\supseteq\Omega\longrightarrow\mathbb{R}^N$ be a map in $C^2(\Omega;\mathbb{R}^N)$. Then:*
(I) *If u is a rank-one absolute minimiser for* (1.1) *on Ω (Definition* 2*(i)), then it solves*

$$\mathrm{H}_P(\cdot,u,\mathrm{D}u)\,\mathrm{D}\big(\mathrm{H}(\cdot,u,\mathrm{D}u)\big)\;=\;0\;\;\text{on }\Omega. \tag{1.8}$$

The opposite is true if in addition H does not depend on $\eta\in\mathbb{R}^N$ and $\mathrm{H}_P(\cdot,\mathrm{D}u)$ has full rank on Ω.

(II) *If u has ∞-minimal area for (1.1) on Ω (Definition 2(ii)), then it solves*

$$\mathrm{H}(\cdot,u,\mathrm{D}u)\,[\![\mathrm{H}_P(\cdot,u,\mathrm{D}u)]\!]^\perp\Big(\mathrm{Div}\big(\mathrm{H}_P(\cdot,u,\mathrm{D}u)\big)-\mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\Big)=0 \;\; on\ \Omega. \tag{1.9}$$

The opposite is true if in addition for any $x\in\Omega$, $\mathrm{H}(x,\cdot,\cdot)$ is convex on $\mathbb{R}^n\times\mathbb{R}^{N\times n}$.
(III) *If u is ∞-minimal map for (1.1) on Ω, then it solves the (reduced) Aronsson system*

$$\begin{aligned}\mathrm{A}_\infty u \,:=\, &\mathrm{H}_P(\cdot,u,\mathrm{D}u)\,\mathrm{D}\big(\mathrm{H}(\cdot,u,\mathrm{D}u)\big)\\ &+\,\mathrm{H}(\cdot,u,\mathrm{D}u)\,[\![\mathrm{H}_P(\cdot,u,\mathrm{D}u)]\!]^\perp\Big(\mathrm{Div}\big(\mathrm{H}_P(\cdot,u,\mathrm{D}u)\big)-\mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\Big)\,=\,0.\end{aligned}$$

The opposite is true if in addition H does not depend on $\eta\in\mathbb{R}^N$, $\mathrm{H}_P(\cdot,\mathrm{D}u)$ has full rank on Ω and for any $x\in\Omega$ $H(x,\cdot)$ is convex in $\mathbb{R}^{N\times n}$.

The emergence of two distinct sets of variations and a pair of separate PDE systems comprising (1.4) might seem at first glance mysterious. However, it is a manifestation of the fact that the (reduced) Aronsson system in fact consists of two linearly independent differential operators because of the perpendicularity between $[\![\mathrm{H}_P]\!]^\perp$ and H_P; in fact, one may split $\mathrm{A}_\infty u=0$ to

$$\left\{\begin{aligned}\mathrm{H}_P(\cdot,u,\mathrm{D}u)\,\mathrm{D}\big(\mathrm{H}(\cdot,u,\mathrm{D}u)\big)\,&=\,0,\\ \mathrm{H}(\cdot,u,\mathrm{D}u)\,[\![\mathrm{H}_P(\cdot,u,\mathrm{D}u)]\!]^\perp\Big(\mathrm{Div}\big(\mathrm{H}_P(\cdot,u,\mathrm{D}u)\big)-\mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\Big)\,&=\,0.\end{aligned}\right.$$

Theorem 4 makes clear that Aronsson's absolute minimisers do **not** characterise the Aronsson system when $N\geq 2$, at least when the additional natural assumptions hold true. This owes to the fact that, unlike the scalar case, *the Aronsson system admits arbitrarily smooth non-minimising solutions*, even in the model case of the ∞-Laplacian. For details we refer to [36].

Since Aronsson's absolute minimisers do not characterise the Aronsson system, the natural question arises as to what is their PDE counterpart. The next theorem which is our second main result answers this question:

Theorem 5 (Divergence PDE characterisation of Absolute minimisers) *Let $u:\mathbb{R}^n\supseteq\Omega\longrightarrow\mathbb{R}^N$ be a map in $C^1(\Omega;\mathbb{R}^N)$. Fix also $\mathcal{O}\Subset\Omega$ and consider the following statements:*
(I) *u is a vectorial minimiser of $\mathrm{E}_\infty(\cdot,\mathcal{O})$ in $C^1_u(\overline{\mathcal{O}};\mathbb{R}^N)$.*
(II) *We have*

$$\max_{\mathrm{Argmax}\{\mathrm{H}(\cdot,u,\mathrm{D}u)\,:\,\overline{\mathcal{O}}\}}\Big[\mathrm{H}_P(\cdot,u,\mathrm{D}u):\mathrm{D}\psi\,+\,\mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\cdot\psi\Big]\,\geq\,0,$$

for any $\psi\in C^1_0(\overline{\mathcal{O}};\mathbb{R}^N)$.
(III) *For any $\psi\in C^1_0(\overline{\mathcal{O}};\mathbb{R}^N)$, there exists a non-empty compact set*

$$\mathrm{K}_\psi\equiv\mathrm{K}\,\subseteq\,\mathrm{Argmax}\big\{\mathrm{H}(\cdot,u,\mathrm{D}u)\,:\,\overline{\mathcal{O}}\big\} \tag{1.10}$$

such that,

$$\Big(\mathrm{H}_P(\cdot,u,\mathrm{D}u):\mathrm{D}\psi\;+\;\mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\cdot\psi\Big)\Big|_{\mathrm{K}}=0. \tag{1.11}$$

Then, (I) $\Longrightarrow$ (II) $\Longrightarrow$ (III). *If additionally* $\mathrm{H}(x,\cdot,\cdot)$ *is convex on* $\mathbb{R}^N\times\mathbb{R}^{N\times n}$ *for any fixed* $x\in\Omega$*, then* (III) $\Longrightarrow$ (I) *and all three statements are equivalent. Further, any of the statements above are deducible from the statement:*
(IV) *For any Radon probability measure* $\sigma\in\mathscr{P}(\overline{\mathcal{O}})$ *satisfying*

$$\mathrm{supp}(\sigma)\;\subseteq\;\mathrm{Argmax}\big\{\mathrm{H}(\cdot,u,\mathrm{D}u)\;:\;\overline{\mathcal{O}}\big\}, \tag{1.12}$$

we have

$$-\,\mathrm{div}\big(\mathrm{H}_P(\cdot,u,\mathrm{D}u)\sigma\big)\;+\;\mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\sigma\;=\;0, \tag{1.13}$$

in the dual space $(C^1_0(\overline{\mathcal{O}};\mathbb{R}^N))^*$.
Finally, all statement are equivalent if $\mathrm{K}=\mathrm{Argmax}\big\{\mathrm{H}(\cdot,u,\mathrm{D}u)\;:\;\overline{\mathcal{O}}\big\}$ *in* (III) *(this happens for instance when the argmax is a singleton set).*

The result above provides an interesting characterisation of Aronsson's concept of Absolute minimisers in terms of divergence PDE systems with measures as parameters. The exact distributional meaning of (1.13) is

$$\int_{\overline{\mathcal{O}}}\Big(\mathrm{H}_P(\cdot,u,\mathrm{D}u):\mathrm{D}\psi\;+\;\mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\cdot\psi\Big)\mathrm{d}\sigma\;=\;0$$

for all $\psi\in C^1_0(\overline{\mathcal{O}};\mathbb{R}^N)$, where the ":" notation in the PDE symbolises the Euclidean (Frobenius) inner product in $\mathbb{R}^{N\times n}$.

The idea of Theorem 5 is inspired by the paper [24] of Evans and Yu, wherein a particular case of the divergence system is derived (in the special scalar case $N=1$ for the ∞-Laplacian and only for $\Omega=\mathcal{O}$), as well as by new developments on higher order Calculus of variations in L^∞ in [34,35,38].

Note that, it does not suffice to consider only $\Omega=\mathcal{O}$ as in [24] in order to describe absolute minimisers. For a subdomain $\mathcal{O}\subseteq\Omega$, it may well happen that the only measure σ "charging" the points of $\overline{\mathcal{O}}$ where the energy density $\mathrm{H}(\cdot,u,\mathrm{D}u)$ is maximised is the Dirac measure at a single point $x\in\partial\mathcal{O}$. This is for instance the case for the standard "Aronsson solution" of the ∞-Laplacian on $\mathbb{R}^2$, given by $u(x,y)=|x|^{4/3}-|y|^{4/3}$, as well as for any other ∞-Harmonic function which is nowhere Eikonal (i.e. $|\mathrm{D}u|$ is non-constant on all open subsets).

We conclude this introduction by noting that the two vectorial variational concepts we are considering herein (Definitions 1–2) do not exhaust the plethora variational concepts in L^∞. In particular, in the paper [41] the concept of *tight maps* was introduced in the case of $\mathrm{H}(x,\eta,P)=\|P\|$ where $\|\cdot\|$ is the operator norm on $\mathbb{R}^{N\times n}$. Additionally, in the papers [9,33] a concept of special affine variations was considered which also characterises the Aronsson system, in fact in the generality of merely locally

Lipschitz $\mathcal{D}$-solutions. Finally, in the paper [6] new concepts of absolute minimisers for constrained minimisation problems have been proposed, whilst results relevant to variational principles in L^∞ and applications appear in [14,15,17,25,39,40].

2 Proofs and a Maximum–Minimum Principle for H(·, *u*, D*u*)

In this section we prove our main results Theorems 4–5. Before delving into that, we establish a result of independent interest, which generalises a corresponding result from [29].

Proposition 6 (Maximum–Minimum Principles) *Suppose Let* $u \in C^2(\Omega;\mathbb{R}^N)$ *be a solution to* (1.8)*, such that* H *satisfies*
(a) $\mathrm{H}_P(\cdot,u,\mathrm{D}u)$ *has full rank on* Ω*,*
(b) *there exists* $c>0$ *such that*

$$\big(\xi^\top \mathrm{H}_P(x,\eta,P)\big)\cdot\big(\xi^\top P\big)\,\geq c\big|\xi^\top\mathrm{H}_P(x,\eta,P)\big|^2,$$

for all $\xi\in\mathbb{R}^N$ *and all* $(x,\eta,P)\in\Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n}$.
Then, for any $\mathcal{O}\Subset\Omega$ *we have:*

$$\sup_{\mathcal{O}}\mathrm{H}(\cdot,u,\mathrm{D}u)\;=\;\max_{\partial\mathcal{O}}\mathrm{H}(\cdot,u,\mathrm{D}u), \tag{2.1}$$

$$\inf_{\mathcal{O}}\mathrm{H}(\cdot,u,\mathrm{D}u)\;=\;\min_{\partial\mathcal{O}}\mathrm{H}(\cdot,u,\mathrm{D}u). \tag{2.2}$$

The proof is based on the usage of the following flow with parameters:

Lemma 7 *Let* $u\in C^2(\Omega;\mathbb{R}^N)$*. Consider the parametric ODE system*

$$\begin{cases}\dot\gamma(t)\;=\;\xi^\top\mathrm{H}_P(\cdot,u,\mathrm{D}u)\big|_{\gamma(t)}, & t\neq 0,\\ \gamma(0)\;=\;x,\end{cases} \tag{2.3}$$

for given $x\in\Omega$ *and* $\xi\in\mathbb{R}^N$*. Then, we have*

$$\frac{d}{dt}\Big(\mathrm{H}(\cdot,u,\mathrm{D}u)\big|_{\gamma(t)}\Big)\;=\;\xi^\top\mathrm{H}_P(\cdot,u,\mathrm{D}u)\,\mathrm{D}\big(\mathrm{H}(\cdot,u,\mathrm{D}u)\big)\big|_{\gamma(t)}, \tag{2.4}$$

$$\frac{d}{dt}\xi^\top u\big(\gamma(t)\big)\;\geq\;c\Big|\xi^\top\mathrm{H}_P(\cdot,u,\mathrm{D}u)\big|_{\gamma(t)}\Big|^2. \tag{2.5}$$

Proof of Lemma 7. The identity (2.4) follows by a direct computation and (2.3). For the inequality (2.5), we have

$$\begin{aligned}\frac{d}{dt}\xi^\top u\big(\gamma(t)\big)\;&=\;\Big(\xi^\top\mathrm{D}u\big(\gamma(t)\big)\Big)\cdot\dot\gamma(t)\\ &=\;\Big(\xi^\top\mathrm{D}u\big(\gamma(t)\big)\Big)\cdot\Big(\xi^\top\mathrm{H}_P(\cdot,u,\mathrm{D}u)\big|_{\gamma(t)}\Big)\\ &\geq\;c\Big|\xi^\top\mathrm{H}_P(\cdot,u,\mathrm{D}u)\big|_{\gamma(t)}\Big|^2.\end{aligned}$$

The lemma ensues. □

Proof of Proposition 6 Fix $\mathcal{O} \Subset \Omega$. Without loss of generality, we may suppose $\mathcal{O}$ is connected. Consider first the case where $\mathrm{rk}\big(\mathrm{H}_P(\cdot, u, \mathrm{D}u)\big) \equiv n \leq N$. Then, the matrix-valued map $\mathrm{H}_P(\cdot, u, \mathrm{D}u)$ is pointwise left invertible. Therefore, by (1.8),

$$\big(\mathrm{H}_P(\cdot, u, \mathrm{D}u)\big)^{-1}\mathrm{H}_P(\cdot, u, \mathrm{D}u)\,\mathrm{D}\big(\mathrm{H}(\cdot, u, \mathrm{D}u)\big) = 0$$

which, by the connectivity of $\mathcal{O}$, gives $\mathrm{H}(\cdot, u, \mathrm{D}u) \equiv \mathrm{const}$ on $\mathcal{O}$. The latter equality readily implies the desired conclusion. Consider now the case where $\mathrm{rk}\big(\mathrm{H}_P(\cdot, u, \mathrm{D}u)\big) \equiv N \leq n$. Fix $x \in \mathcal{O}$ and a unit vector $\xi \in \mathbb{R}^n$ and consider the parametric ODE system (2.3) of Lemma 7. By the fullness of the rank of $\mathrm{H}_P(\cdot, u, \mathrm{D}u)\big)$, we have that

$$\big|\xi^\top \mathrm{H}_P(\cdot, u, \mathrm{D}u)\big)\big| \geq c_1 > 0 \quad \text{on } \mathcal{O}.$$

We will now show that the trajectory $\gamma(t)$ reaches $\partial\mathcal{O}$ in finite time. To this end, we estimate

$$\|\mathrm{D}u\|_{L^\infty(\mathcal{O})}\mathrm{diam}(\mathcal{O}) \geq \|\mathrm{D}u\|_{L^\infty(\mathcal{O})}\big|\gamma(t) - \gamma(0)\big| \geq \left|\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{\hat{t}}\xi^\top u(\gamma(t))\right| t,$$

for some $\hat{t} \in (0, t)$, by the mean value theorem. Hence,

$$\begin{aligned}
\|\mathrm{D}u\|_{L^\infty(\mathcal{O})}\mathrm{diam}(\mathcal{O}) &\geq \left|\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{\hat{t}}\xi^\top u(\gamma(t))\right| t \\
&= \left|\xi^\top \mathrm{D}u(\gamma(\hat{t}))\cdot\dot{\gamma}(\hat{t})\right| t \\
&= \left|\xi^\top \mathrm{D}u(\gamma(\hat{t}))\cdot\Big(\xi^\top \mathrm{H}_P(\cdot, u, \mathrm{D}u)\big|_{\gamma(\hat{t})}\Big)\right| t \\
&\geq c_0\left|\xi^\top \mathrm{H}_P(\cdot, u, \mathrm{D}u)\big|_{\gamma(\hat{t})}\right|^2 t \\
&\geq (c_0 c_1^2)\, t.
\end{aligned}$$

This proves the desired claim. Further, since u solves (1.8), by (2.4) of Lemma 7 it follows that $\mathrm{H}(\cdot, u, \mathrm{D}u)$ is constant along the trajectory. Thus, if $x \in \mathcal{O}$ is chosen as a point realising either the maximum or the minimum in $\overline{\mathcal{O}}$, then by moving along the trajectory, we reach a point $y \in \partial\mathcal{O}$ such that $\mathrm{H}(\cdot, u, \mathrm{D}u)\big|_x = \mathrm{H}(\cdot, u, \mathrm{D}u)\big|_y$. This establishes both the maximum and minimum principle. The proposition ensues. □

Remark 8 *(Danskin's theorem)* The central ingredient in the proofs of Theorems 4–5 is the next consequence of Danskin's theorem: for any $\mathcal{O} \Subset \Omega$ and any $u, \phi \in C^1(\Omega; \mathbb{R}^N)$, we have the identities

$$\begin{cases} \left.\dfrac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0^+} \mathrm{E}_\infty(u+t\phi,\mathcal{O}) \,=\, \max\limits_{\mathcal{O}(u)} \Big(\mathrm{H}_P(\cdot,u,\mathrm{D}u):\mathrm{D}\phi \,+\, \mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\cdot\phi\Big), \\ \left.\dfrac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0^-} \mathrm{E}_\infty(u+t\phi,\mathcal{O}) \,=\, \min\limits_{\mathcal{O}(u)} \Big(\mathrm{H}_P(\cdot,u,\mathrm{D}u):\mathrm{D}\phi \,+\, \mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\cdot\phi\Big), \end{cases} \tag{2.6}$$

where

$$\mathcal{O}(u) \,:=\, \mathrm{Argmax}\big\{\mathrm{H}(\cdot,u,\mathrm{D}u) \,:\, \overline{\mathcal{O}}\big\}.$$

Indeed, by [23, Theorem 1, page 643] and the chain rule we have

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0^+} \mathrm{E}_\infty(u+t\phi,\mathcal{O}) \,&=\, \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0^+} \Big(\max_{\overline{\mathcal{O}}} \mathrm{H}\big(\cdot,u+t\phi,\mathrm{D}u+t\mathrm{D}\phi\big)\Big) \\ &=\, \max_{\mathcal{O}(u)} \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0^+} \mathrm{H}\big(\cdot,u+t\phi,\mathrm{D}u+t\mathrm{D}\phi\big)\right) \\ &=\, \max_{\mathcal{O}(u)} \Big(\mathrm{H}_P(\cdot,u,\mathrm{D}u):\mathrm{D}\phi \,+\, \mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\cdot\phi\Big). \end{aligned}$$

This establishes the first identity of (2.6). The second one follows through the substitutions $\phi \rightsquigarrow -\phi$, $t \rightsquigarrow -t$.

Now we may establish Theorem 4.

Proof of Theorem 4 (I) Suppose first that u is a rank-one absolute minimiser on Ω. The aim is to show that (1.8) is satisfied on Ω. This conclusion in fact follows by the results in [27], but below we provide a new shorter proof. To this end, fix $x \in \Omega$ and $\rho \in (0, \mathrm{dist}(x, \partial\Omega))$ and let $\mathcal{O} := \mathbb{B}_\rho(x)$. We fix also $\xi \in \mathbb{R}^N$ and choose

$$\phi(y) \,:=\, \xi\big(|y-x|^2-\rho^2\big).$$

Then, $\phi \in C^1_0\big(\bar{\mathbb{B}}_\rho(x);\mathrm{span}[\xi]\big)$. By Remark 8 and our minimality assumption, the definition of one-sided derivatives yields

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0^-} \mathrm{E}_\infty(u+t\phi,\mathcal{O}) \,\le\, 0 \,\le\, \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0^+} \mathrm{E}_\infty(u+t\phi,\mathcal{O}). \tag{2.7}$$

Hence, by (2.7), (2.6) and continuity there exists a point x_ρ with $|x_\rho - x| \le \rho$ which lies in the argmax set

$$(\mathbb{B}_\rho(x))(u) \,=\, \mathrm{Argmax}\big\{\mathrm{H}(\cdot,u,\mathrm{D}u) \,:\, \bar{\mathbb{B}}_\rho(x)\big\}$$

such that

$$\Big(\mathrm{H}_P(\cdot,u,\mathrm{D}u):\mathrm{D}\phi \,+\, \mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\cdot\phi\Big)\Big|_{x_\rho} \,=\, 0. \tag{2.8}$$

Therefore,

$$\xi^\top\Big(2\mathrm{H}_P(\cdot,u,\mathrm{D}u)\big|_{x_\rho}(x_\rho - x) + \mathrm{H}_\eta(\cdot,u,\mathrm{D}u)\big|_{x_\rho}\big(|x_\rho - x|^2 - \rho^2\big)\Big) = 0. \quad (2.9)$$

If x_ρ lies in the interior of $\mathbb{B}_\rho(x)$, then it is an interior maximum and therefore

$$\mathrm{D}\big(\mathrm{H}(\cdot,u,\mathrm{D}u)\big)\big|_{x_\rho} = 0.$$

This means that (1.8) is satisfied at x_ρ. If x_ρ lies on the boundary of $\mathbb{B}_\rho(x)$, then this means that

$$\forall\ y \in \bar{\mathbb{B}}_\rho(x),\ \text{ we have } \mathrm{H}(\cdot,u,\mathrm{D}u)\big|_y \leq \mathrm{H}(\cdot,u,\mathrm{D}u)\big|_{x_\rho}.$$

The above can be rewritten as

$$\bar{\mathbb{B}}_\rho(x) \subseteq \mathcal{H}(x_\rho) := \Big\{\mathrm{H}(\cdot,u,\mathrm{D}u) \leq \mathrm{H}(\cdot,u,\mathrm{D}u)\big|_{x_\rho}\Big\},$$

and note also that $x_\rho \in \partial\mathbb{B}_\rho(x) \cap \partial\mathcal{H}(x_\rho)$. Hence, the sublevel set $\mathcal{H}(x_\rho)$ satisfied an interior sphere condition at x_ρ. If $\mathrm{D}\big(\mathrm{H}(\cdot,u,\mathrm{D}u)\big)\big|_{x_\rho} = 0$ then (1.8) is again satisfied at x_ρ. If on the other hand

$$\mathrm{D}\big(\mathrm{H}(\cdot,u,\mathrm{D}u)\big)\big|_{x_\rho} \neq 0$$

then $\partial\mathcal{H}(x_\rho)$ is a C^1 manifold near x_ρ and the gradient above is the normal vector at the point x_ρ. Due to the interior sphere condition, this implies that this is also the normal vector to the sphere $\partial\mathbb{B}_\rho(x)$ at x_ρ. Thus, there exists $\lambda \neq 0$ such that

$$x_\rho - x = \lambda\mathrm{D}\big(\mathrm{H}(\cdot,u,\mathrm{D}u)\big)\big|_{x_\rho}. \quad (2.10)$$

By inserting (2.10) into (2.9) and noting that $|x_\rho - x| = \rho$, we infer that

$$2\lambda\,\xi^\top\Big(\mathrm{H}_P(\cdot,u,\mathrm{D}u)\mathrm{D}\big(\mathrm{H}(\cdot,u,\mathrm{D}u)\big)\Big)\big|_{x_\rho} = 0.$$

By dividing by 2λ and letting $\rho \to 0$, we deduce that (1.8) is satisfied at the arbitrary $x \in \Omega$.

Conversely, suppose that u satisfies (1.8) on Ω, together with the additional assumptions of the statement. Fix $\mathcal{O} \Subset \Omega$ and $\phi \in C^1_0(\overline{\mathcal{O}};\mathrm{span}[\xi])$. Without loss of generality, we may suppose $\mathcal{O}$ is connected. Since $\phi = (\xi^\top\phi)\xi$, for convenience we set $g := \xi^\top\phi$ and then we may write $\phi = g\xi$ with $g \in C^1_0(\overline{\mathcal{O}})$. Then, the matrix-valued map $\mathrm{H}_P(\cdot,\mathrm{D}u)$ is pointwise left invertible. Therefore, by (1.8)

$$\big(\mathrm{H}_P(\cdot,\mathrm{D}u)\big)^{-1}\mathrm{H}_P(\cdot,\mathrm{D}u)\,\mathrm{D}\big(\mathrm{H}(\cdot,\mathrm{D}u)\big) = 0 \ \text{ on } \mathcal{O},$$

which, by the connectivity of $\mathcal{O}$, gives

$$\mathrm{H}(\cdot, \mathrm{D}u) \equiv \text{ const } \text{ on } \mathcal{O}.$$

Since $g \in C^1(\mathbb{R}^n)$ with $g = 0$ on $\partial\mathcal{O}$, there exists at least one interior critical point $\bar{x} \in \mathcal{O}$ such that $\mathrm{D}g(\bar{x}) = 0$. By the previous, we have

$$\begin{aligned} \mathrm{E}_\infty(u, \mathcal{O}) &= \mathrm{H}\big(\bar{x}, \mathrm{D}u(\bar{x})\big) \\ &= \mathrm{H}\Big(\bar{x}, \mathrm{D}u(\bar{x}) + \xi \otimes \mathrm{D}g(\bar{x})\Big) \\ &= \mathrm{H}\Big(\bar{x}, \mathrm{D}u(\bar{x}) + \mathrm{D}\phi(\bar{x})\Big) \\ &\le \sup_{x \in \mathcal{O}} \mathrm{H}\Big(x, \mathrm{D}u(x) + \mathrm{D}\phi(x)\Big) \\ &= \mathrm{E}_\infty(u + \phi, \mathcal{O}). \end{aligned}$$

The conclusion ensues.
(II) Suppose that u has ∞-minimal area. Fix $x \in \Omega$ and $\rho \in (0, \mathrm{dist}(x, \partial\Omega))$. Fix

$$\xi \in \tilde{\mathrm{N}}\Big(\mathrm{H}_P(\cdot, u, \mathrm{D}u)^\top\big|_x\Big),$$

noting also that by Definition 3 the above set is the reduced nullspace of $\mathrm{H}_P(\cdot, u, \mathrm{D}u)^\top$ at x. This implies that there exists a C^1 extension $\bar{\xi} \in C^1(\mathbb{R}^n; \mathbb{R}^N)$ such that $\bar{\xi}(x) = \xi$ and $(\bar{\xi})^\top \mathrm{H}_P(\cdot, u, \mathrm{D}u) = 0$ on the closed ball $\bar{\mathbb{B}}_\varepsilon(x)$ for some $\varepsilon \in (0, \rho)$. By differentiating the relation $(\bar{\xi})^\top \mathrm{H}_P(\cdot, u, \mathrm{D}u) = 0$ and taking its trace, we obtain

$$\bar{\xi} \cdot \mathrm{div}\big(\mathrm{H}_P(\cdot, u, \mathrm{D}u)\big) + \mathrm{D}\bar{\xi} : \mathrm{H}_P(\cdot, u, \mathrm{D}u) = 0, \tag{2.11}$$

on $\bar{\mathbb{B}}_\varepsilon(x)$. Since u has ∞-minimal area and $\bar{\xi}$ is an admissible normal variation, by using Remark 8 and arguing as in the beginning of part (I), it follows that

$$\Big(\bar{\xi} \cdot \mathrm{H}_\eta(\cdot, u, \mathrm{D}u) + \mathrm{D}\bar{\xi} : \mathrm{H}_P(\cdot, u, \mathrm{D}u)\Big)\Big|_{x_\varepsilon} = 0 \tag{2.12}$$

for some $x_\varepsilon \in (\mathbb{B}_\varepsilon(x))(u)$, where

$$(\mathbb{B}_\varepsilon(x))(u) = \mathrm{Argmax}\big\{\mathrm{H}(\cdot, u, \mathrm{D}u) : \bar{\mathbb{B}}_\varepsilon(x)\big\}.$$

By (2.11)–(2.12), we infer that

$$\bar{\xi}(x_\varepsilon) \cdot \Big(\mathrm{div}\big(\mathrm{H}_P(\cdot, u, \mathrm{D}u)\big) - \mathrm{H}_\eta(\cdot, u, \mathrm{D}u)\Big)\Big|_{x_\varepsilon} = 0$$

and by letting $\varepsilon \to 0$, we deduce that

$$\xi \cdot \Big(\mathrm{div}\big(\mathrm{H}_P(\cdot, u, \mathrm{D}u)\big) \,-\, \mathrm{H}_\eta(\cdot, u, \mathrm{D}u)\Big)\Big|_x \,=\, 0,$$

for any $\xi \in \tilde{\mathrm{N}}\big(\mathrm{H}_P(\cdot, u, \mathrm{D}u)^\top\big|_x\big)$. Hence, u satisfies (1.9) at the arbitrary $x \in \Omega$.

Conversely, suppose that u solves (1.9) on Ω. Fix $\mathcal{O} \Subset \Omega$ and $\phi \in C^1(\mathbb{R}^n; \mathbb{R}^N)$ such that $\phi^\top \mathrm{H}_P(\cdot, u, \mathrm{D}u) = 0$ on $\mathcal{O}$. Note further that by the continuity up to the boundary of all functions involved, the latter identity in fact holds on $\overline{\mathcal{O}}$. By the satisfaction of (1.9) and Definition 3, it follows that

$$\phi \cdot \Big(\mathrm{div}\big(\mathrm{H}_P(\cdot, u, \mathrm{D}u)\big) \,-\, \mathrm{H}_\eta(\cdot, u, \mathrm{D}u)\Big) \,=\, 0,$$

on $\overline{\mathcal{O}} \subseteq \Omega$. By differentiating $\phi^\top \mathrm{H}_P(\cdot, u, \mathrm{D}u) = 0$, we obtain

$$\phi \cdot \mathrm{div}\big(\mathrm{H}_P(\cdot, u, \mathrm{D}u)\big) \,+\, \mathrm{D}\phi : \mathrm{H}_P(\cdot, u, \mathrm{D}u) \,=\, 0,$$

on $\overline{\mathcal{O}}$. By the above two identities, we deduce

$$\phi \cdot \mathrm{H}_\eta(\cdot, u, \mathrm{D}u) \,+\, \mathrm{D}\phi : \mathrm{H}_P(\cdot, u, \mathrm{D}u) \,=\, 0,$$

on $\overline{\mathcal{O}}$. Since $\mathcal{O}(u) \subseteq \overline{\mathcal{O}}$, Remark 8 yields that u is a critical point since the left and right derivative of $\mathrm{E}_\infty(u + t\phi, \mathcal{O})$ at $t = 0$ coincide and vanish. Since by assumption $\mathrm{H}(x, \cdot, \cdot)$ is convex on $\mathbb{R}^N \times \mathbb{R}^{N\times n}$, it follows that $\mathrm{E}_\infty(\cdot, \mathcal{O})$ is convex on $C^1(\overline{\mathcal{O}}; \mathbb{R}^N)$. Hence, the critical point u is in fact a minimum point for this class of variations. This establishes our claim.
(III) This is an immediate corollary of items (I) and (II). □

Now we conclude by establishing Theorem 5.

Proof of Theorem 5 Fix $\mathcal{O} \Subset \Omega$ and $u, \phi \in C^1(\Omega; \mathbb{R}^N)$. We show that (I) $\Longrightarrow$ (II) $\Longrightarrow$ (III) and that (III) $\Longrightarrow$ (I) under the additional convexity assumption. By recalling Remark 8, note that if

$$\mathrm{E}_\infty(u + t\phi, \mathcal{O}) \,\geq\, \mathrm{E}_\infty(u, \mathcal{O}), \quad \text{for all } t \in \mathbb{R}, \tag{2.13}$$

then directly by (2.13) and the definition of one-sided derivatives, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0^-} \mathrm{E}_\infty(u + t\phi, \mathcal{O}) \,\leq\, 0 \,\leq\, \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0^+} \mathrm{E}_\infty(u + t\phi, \mathcal{O}). \tag{2.14}$$

This shows (I) $\Longrightarrow$ (II). If (II) holds, note that one also has that

$$\min_{\mathrm{Argmax}\{\mathrm{H}(\cdot,u,\mathrm{D}u)\,:\,\overline{\mathcal{O}}\}} \Big[\mathrm{H}_P(\cdot, u, \mathrm{D}u) : \mathrm{D}\phi \,+\, \mathrm{H}_\eta(\cdot, u, \mathrm{D}u) \cdot \phi\Big] \,\leq\, 0,$$

for any $\phi \in C_0^1(\overline{\mathcal{O}}; \mathbb{R}^N)$. By (2.6) we see that (2.14) is satisfied and by continuity we obtain the existence of a non-empty compact set $\mathrm{K} = \mathrm{K}_\phi \subseteq \mathcal{O}(u)$ such that

$$\Big(\mathrm{H}_P(\cdot, u, \mathrm{D}u) : \mathrm{D}\phi \,+\, \mathrm{H}_\eta(\cdot, u, \mathrm{D}u) \cdot \phi\Big)\Big|_{\mathrm{K}} \,=\, 0. \tag{2.15}$$

Hence, (III) ensues. If now (2.15) holds true for some non-empty compact set $\mathrm{K} \subseteq \mathcal{O}(u)$, then by (2.6) we have that (2.14) is true. If further $\mathrm{H}(x, \cdot, \cdot)$ is convex for all $x \in \Omega$, then by Lemma 9 given right after the proof, $t \mapsto \mathrm{E}_\infty(u + t\phi, \mathcal{O})$ is minimised at $t = 0$ and (2.13) holds true.
(IV) $\Longrightarrow$ (III): Let $\sigma \in \mathscr{P}(\overline{\mathcal{O}})$ be any Radon probability measure satisfying (1.12). Then, by assumption

$$\int_{\overline{\mathcal{O}}} \Big(\mathrm{H}_P(\cdot, u, \mathrm{D}u) : \mathrm{D}\phi \,+\, \mathrm{H}_\eta(\cdot, u, \mathrm{D}u) \cdot \phi\Big)\mathrm{d}\sigma \,=\, 0$$

for all $\phi \in C_0^1(\overline{\mathcal{O}}; \mathbb{R}^N)$. Fix any point $\bar{x} \in \mathcal{O}(u)$. By choosing the Dirac measure $\bar{\sigma} \in \mathscr{P}(\overline{\mathcal{O}})$ given by

$$\bar{\sigma} \,:=\, \delta_{\bar{x}}$$

which evidently satisfies $\mathrm{supp}(\bar{\sigma}) = \{\bar{x}\} \subseteq \mathcal{O}(u)$, we obtain

$$\begin{aligned} &\Big(\mathrm{H}_P(\cdot, u, \mathrm{D}u) : \mathrm{D}\phi \,+\, \mathrm{H}_\eta(\cdot, u, \mathrm{D}u) \cdot \phi\Big)\Big|_{\bar{x}} \\ &\quad= \int_{\overline{\mathcal{O}}} \Big(\mathrm{H}_P(\cdot, u, \mathrm{D}u) : \mathrm{D}\phi \,+\, \mathrm{H}_\eta(\cdot, u, \mathrm{D}u) \cdot \phi\Big)\mathrm{d}\bar{\sigma} \\ &\quad= 0, \end{aligned}$$

for any $\bar{x} \in \mathcal{O}(u)$. The conclusion ensues with $\mathrm{K} = \mathcal{O}(u)$.
(III) $\Longrightarrow$ (IV): If we have $\mathrm{K} = \mathcal{O}(u)$ and

$$\Big(\mathrm{H}_P(\cdot, u, \mathrm{D}u) : \mathrm{D}\phi \,+\, \mathrm{H}_\eta(\cdot, u, \mathrm{D}u) \cdot \phi\Big)\Big|_{\mathrm{K}} \,=\, 0,$$

then for any Radon probability measure $\sigma \in \mathscr{P}(\overline{\mathcal{O}})$ with $\mathrm{supp}(\sigma) \subseteq \mathrm{K}$, we have

$$\int_{\overline{\mathcal{O}}} \Big(\mathrm{H}_P(\cdot, u, \mathrm{D}u) : \mathrm{D}\phi \,+\, \mathrm{H}_\eta(\cdot, u, \mathrm{D}u) \cdot \phi\Big)\mathrm{d}\sigma \,=\, 0$$

for all $\phi \in C_0^1(\overline{\mathcal{O}}; \mathbb{R}^N)$. Hence, we have shown that

$$-\mathrm{div}\big(\mathrm{H}_P(\cdot, u, \mathrm{D}u)\sigma\big) \,+\, \mathrm{H}_\eta(\cdot, u, \mathrm{D}u)\sigma \,=\, 0,$$

in the dual space $(C_0^1(\overline{\mathcal{O}}; \mathbb{R}^N))^*$. ◻

The next result which was utilised in the proof of Theorem 5 completes our arguments.

Lemma 9 *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function. If the one-sided derivatives $f'(0^{\pm})$ exist and $f'(0^-) \leq 0 \leq f'(0^+)$, then $f(0)$ is the global minimum of f on $\mathbb{R}$.*

Proof of Lemma 9 By the convexity of f on $\mathbb{R}$, for any fixed $s \in \mathbb{R}$ there exists a sub-differential $p_s \in \mathbb{R}$ such that

$$f(t) - f(s) \;\geq\; p_s(t-s), \quad \text{for all } t \in \mathbb{R}. \tag{2.16}$$

For the choice $t = 0$ and $s > 0$, we have

$$\frac{f(s) - f(0)}{s} \;\leq\; p_s$$

and note also that since convex functions are locally Lipschitz, the set $(p_s)_{0<s<1}$ is bounded. Thus, since $f'(0^+)$ exists and is non-negative, the above inequality yields

$$0 \;\leq\; f'(0^+) \;\leq\; \liminf_{s \to 0^+} p_s \;<\; \infty.$$

Hence, by passing to the limit as $s \to 0^+$ in the inequality (2.16) for $t > 0$ fixed, we obtain $f(t) - f(0) \geq 0$. The case of $t < 0$ follows by arguing similarly. □

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Compliance with Ethical Standards

Conflict of interest The authors declare that they have no conflict of interest.

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