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Continuous Dependence and Optimal Control for a Class of Variational–Hemivariational Inequalities

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Abstract
The paper investigates control problems for a class of nonlinear elliptic variational–hemivariational inequalities with constraint sets. Based on the well posedness of a variational–hemivariational inequality, we prove some results on continuous dependence and existence of optimal pairs to optimal control problems. We obtain some continuous dependence results in which the strong dependence and weak dependence are considered, respectively. A frictional contact problem is given to illustrate our main results.

Keywords Continuous dependence · Optimal control · Variational–hemivariational inequality · Mosco convergence

Mathematics Subject Classification 47J20 · 49J40 · 74M10 · 74M15 · 90C26

1 Introduction

In this paper we present systematic approaches to continuous dependence and optimal control for a class of nonlinear elliptic variational–hemivariational inequalities. Theoretical results are delivered in the general framework of abstract inequalities in a reflexive Banach space. The main feature of such inequalities lies in the fact that they

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are governed by a nonlinear operator, a convex set of constraints and two nondifferentiable functionals, among which at least one is convex.

Variational–hemivariational inequalities appear in a variety of mechanical problems, for example, the unilateral contact problems in nonlinear elasticity, the problems describing the adhesive and friction effects, the nonconvex semipermeability problems, the masonry structures, and the delamination problems in multilayered composites (see e.g. [13,17]). Hemivariational inequalities have been introduced by Panagiotopoulos in 1981 as the variational formulation of important classes of unilateral and inequality problems in mechanics (see [18]). The notion of hemivariational inequality is a generalization of variational inequality for a case where the function involved is nonconvex and nonsmooth. The hemivariational inequalities are based on a concept of the generalized gradient of Clarke (see [3,4,13]). They cover boundary value problems for partial differential equations with nonmonotone, possibly multivalued and nonconvex nonlinearities. In the last few years many kinds of variational and hemivariational inequalities have been investigated (see [6,21]) and the study of variational–hemivariational inequalities has emerged today as a new and interesting branch of applied mathematics.

Various models in applied sciences can conveniently be formulated as variational–hemivariational inequality problems involving certain parameters. These parameters are known and they often characterize some physical properties of the underlying model. In recent years, the field of inverse and identification problems emerged as one of the most vibrant and developing branches of applied and industrial mathematics because of their wide applications, see [1,2,5,7,8] and the references therein. Stability of inverse problems with respect to perturbations of the original problem and of cost functional can be found in [10–12] and some related optimal control problem in [9,19].

The variational–hemivariational inequality studied in the present paper can be formulated as follows. Let X, $\mathcal{P}$ and $\mathcal{Q}$ be reflexive Banach spaces. For $p \in \mathcal{P}$, let $K(p)$ be a nonempty, closed and convex set of constraints in X. Given a nonlinear pseudomonotone operator $A\colon X \to X^*$, a linear continuous control operator $B : \mathcal{Q} \to X^*$, a convex functional $\varphi\colon X \times X \to \mathbb{R}$, a locally Lipschitz (in general nonconvex) functional $j\colon X \to \mathbb{R}$ and an element $f \in X^*$ with some properties to be specified later, the abstract variational–hemivariational inequality has the form:

Problem 1 For given $p \in \mathcal{P}$, $q \in \mathcal{Q}$, find $u \in K(p)$ such that

$$\langle Au, v-u\rangle_X + \varphi(u, v) - \varphi(u, u) + j^0(u; v-u) \ge \langle f + Bq, v-u\rangle_X, \quad \forall v \in K(p).$$

Here $j^0(u; v)$ stands for the generalized (Clarke) directional derivative of j at a point $u \in X$ in the direction $v \in X$.

In the first part of the paper we first provide results on the well posedness of the above abstract problem. The existence and uniqueness of solution to Problem 1 has been recently obtained in [14,19]. Then, we prove some continuous dependence results in which the strong dependence and weak dependence are considered, respectively, and show a new result saying that the mapping $\mathcal{P} \times \mathcal{Q} \ni (p, q) \mapsto u(p, q) \in X$ is continuous. This continuous dependence result is fundamental to obtain the existence of solution to an optimal control problem in which we minimize an appropriate cost functional F defined on the space of admissible parameters $\mathcal{P}_{ad} \times \mathcal{Q}_{ad} \subset \mathcal{P} \times \mathcal{Q}$. The novelty of this paper is that the constraint set K depends on p. The linear and nonlinear cases are considered.

The second part of the paper is devoted to an optimal control problem for a frictional unilateral contact problem in nonlinear elasticity in which u represents the displacement field. Moreover, since the abstract problem is a variational–hemivariational inequality, we are able to incorporate in this setting various complicated physical phenomena modeled by nonmonotone and nondifferentiable potentials which are met in industrial processes. On the other hand, the contact problem under consideration offers some nontrivial mathematical interest. Finally, we note that the techniques and results discussed in this paper could be also used in many other optimal control problems in mechanics.

The rest of this paper is organized as follows. In the next section, we will briefly introduce some necessary preliminary material. In Sect. 3, we give the result on the existence and uniqueness of solution to Problem 1. In Sect. 4, we provide some continuous dependence results for Problem 1. In Sect. 5, we establish some existence and convergence results for an optimal control problem. Finally, we elaborate on a frictional contact problem to illustrate our main results.

2 Preliminaries

Let $(X, \|\cdot\|_X)$ be a Banach space. We denote by X^* its dual space and by $\langle \cdot, \cdot \rangle_X$ the duality pairing between X^* and X. We denote by "$\to$" the strong convergence and by "$\rightharpoonup$" the weak convergence.

We recall the following definitions, see [3,4,13].

Definition 2 A function $f : X \to \mathbb{R}$ is said to be

(i) (weakly) upper semicontinuous (u.s.c.) at x_0, if for any sequence $\{x_n\}_{n\geq 1} \subset X$ with $(x_n \rightharpoonup x_0)$ $x_n \to x_0$, we have $\limsup f(x_n) \leq f(x_0)$.
(ii) (weakly) lower semicontinuous (l.s.c.) at x_0, if for any sequence $\{x_n\}_{n\geq 1} \subset X$ with $(x_n \rightharpoonup x_0)$ $x_n \to x_0$, we have $f(x_0) \leq \liminf f(x_n)$.
(iii) f is said to be (weakly) u.s.c. (l.s.c.) on X, if for all $x \in X$ f is (weakly) u.s.c. (l.s.c.) at x.

Definition 3 Let $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and l.s.c. function. The mapping $\partial\varphi : X \to 2^{X^*}$ defined by

$$\partial\varphi(u) = \{u^* \in X^* \mid \langle u^*, v - u\rangle_X \leq \varphi(v) - \varphi(u) \text{ for all } v \in X\}$$

for $u \in X$, is called the subdifferential of φ. An element $x^* \in \partial\varphi(x)$ is called a subgradient of φ in u.

Definition 4 Given a locally Lipschitz function $\varphi : X \to \mathbb{R}$, we denote by $\varphi^0(u; v)$ the Clarke generalized directional derivative of φ at the point $u \in X$ in the direction $v \in X$ defined by

$$\varphi^0(u; v) = \limsup_{\lambda \to 0^+,\, \zeta \to u} \frac{\varphi(\zeta + \lambda v) - \varphi(\zeta)}{\lambda}.$$

The Clarke subdifferential or the generalized gradient of φ at $u \in X$, denoted by $\partial\varphi(u)$, is a subset of X^* given by

$$\partial\varphi(u) = \{\, u^* \in X^* \mid \varphi^0(u; v) \geq \langle u^*, v\rangle_X \text{ for all } v \in X \,\}.$$

Definition 5 A single-valued operator $F\colon X \to X^*$ is said to be pseudomonotone, if it is bounded and satisfies the inequality

$$\langle Fu, u - v\rangle \leq \liminf \langle Fu_n, u_n - v\rangle_X \text{ for all } v \in X,$$

where $u_n \rightharpoonup u$ in X with $\limsup \langle Fu_n, u_n - u\rangle_X \leq 0$.

Lemma 6 [13, Proposition 1.3.66] *Let $F\colon X \to X^*$ be a single-valued operator defined on a reflexive Banach space X. The operator F is pseudomonotone if and only if F is bounded and satisfies the following condition: if $u_n \rightharpoonup u$ in X and* $\limsup \langle Fu_n, u_n - u\rangle_X \leq 0$, *then $Fu_n \rightharpoonup Fu$ in X^* and* $\lim \langle Fu_n, u_n - u\rangle_X = 0$.

The following notion of the Mosco convergence of sets will be useful in the next sections. For the definitions, properties and other modes of set convergence, we refer to [4, Chap. 4.7] and [15].

Definition 7 Let $(X, \|\cdot\|)$ be a normed space and $\{K_n\}_{n\in\mathbb{N}} \subset 2^X \setminus \{\emptyset\}$. We say that K_n converge to K in the Mosco sense, as $n \to \infty$, denoted by $K_n \xrightarrow{M} K$ if and only if the two conditions hold
(m1) for each $x \in K$, there exists $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \in K_n$ and $x_n \to x$ in X as $n \to \infty$,
(m2) for each subsequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \in K_n$ and $x_n \rightharpoonup x$ in X, we have $x \in K$.

3 Existence and Uniqueness Result

In this section we provide a result on existence and uniqueness of solution to the variational–hemivariational inequality of the following form.

Problem 8 Find $u \in K$ such that

$$\langle Au, v - u\rangle_X + \varphi(u, v) - \varphi(u, u) + j^0(u; v - u) \geq \langle f, v - u\rangle_X, \quad \forall v \in K.$$

We need the following hypotheses on the data of Problem 8.

$$K \text{ is a nonempty, closed, convex subset of } X. \tag{1}$$

$$\begin{cases} A\colon X \to X^* \text{ is such that} \\ \text{(a) there exists } \alpha_A > 0 \text{ such that} \\ \qquad \langle Au_1 - Au_2, u_1 - u_2\rangle_X \geq \alpha_A \, \|u_1 - u_2\|_X^2 \\ \quad \text{for all } u_1, u_2 \in X. \\ \text{(b) there exists } L_A > 0 \text{ such that} \\ \qquad \|Au_1 - Au_2\| \leq L_A \|u_1 - u_2\|_X \\ \text{for all } u_1, u_2 \in X. \end{cases} \tag{2}$$

$$\begin{cases} \varphi\colon X \times X \to \mathbb{R} \text{ is such that} \\ \text{(a) } \varphi(u, \cdot)\colon X \to \mathbb{R} \text{ is convex and l.s.c. on } X, \text{ for all } u \in X. \\ \text{(b) there exists } \alpha_\varphi > 0 \text{ such that} \\ \quad \varphi(u_1, v_2) - \varphi(u_1, v_1) + \varphi(u_2, v_1) - \varphi(u_2, v_2) \\ \qquad \leq \alpha_\varphi \, \|u_1 - u_2\|_X \|v_1 - v_2\|_X \\ \quad \text{for all } u_1, u_2, v_1, v_2 \in X. \\ \text{(c) } \varphi(u, \lambda v) = \lambda\varphi(u, v), \ \varphi(v, v) \geq 0 \text{ for all } u, v \in X, \lambda > 0. \end{cases} \tag{3}$$

$$\begin{cases} j\colon X \to \mathbb{R} \text{ is such that} \\ \text{(a) } j \text{ is locally Lipschitz.} \\ \text{(b) there exist } c_0, c_1 \geq 0 \text{ such that} \\ \qquad \|\partial j(u)\|_{X^*} \leq c_0 + c_1 \|u\|_X \\ \quad \text{for all } u \in X. \\ \text{(c) there exists } \alpha_j > 0 \text{ such that} \\ \qquad j^0(u_1; u_2 - u_1) + j^0(u_2; u_1 - u_2) \leq \alpha_j \, \|u_1 - u_2\|_X^2 \\ \quad \text{for all } u_1, u_2 \in X. \end{cases} \tag{4}$$

We have the following existence and uniqueness result.

Theorem 9 *Assume that (1)–(4) hold and the following smallness condition is satisfied*

$$\alpha_\varphi + \alpha_j < \alpha_A. \tag{5}$$

Then for any $f \in X^*$, Problem 8 has a unique solution $u \in X$. Moreover, u satisfies the following estimate

$$\|u\|_X \leq \frac{1}{\alpha_A - \alpha_j} (\|A0_X\|_{X^*} + \|f\|_{X^*} + c_0). \tag{6}$$

Note that a result on existence and uniqueness of solution to the variational–hemivariational inequality in Problem 8 has been recently provided in [14, Theorem 18]. The results on existence and uniqueness of solution and the estimate (6) follow from [19, Theorem 8] and [19, Lemma 10], respectively.

4 Continuous Dependence

In this section, we consider some continuous dependence results which play a crucial role in the study of the optimal control problem. At first, we have the following result which is a corollary of Theorem 9.

Theorem 10 *Assume that (1)–(5) hold. Then for every $f \in X^*, p \in \mathcal{P}, q \in \mathcal{Q}$, Problem 1 has a unique solution $u(p, q) \in K(p)$. Moreover, u satisfies the following estimate*

$$\|u(p, q)\|_X \le \frac{1}{\alpha_A - \alpha_j}(\|A0_X\|_{X^*} + \|f\|_{X^*} + \|B\|\|q\|_{\mathcal{Q}} + c_0). \tag{7}$$

We start with two continuous dependence results which play a crucial role in the study of the optimal control problem. We need the following hypotheses on the data.

$$K(p_n) \xrightarrow{M} K(p) \text{ as } p_n \to p. \tag{8}$$

$$\begin{cases} \text{For any } \{u_n\} \subset X \text{ with } u_n \rightharpoonup u \text{ in } X, \text{ and all } v \in X, \text{ we have} \\ \quad \limsup(\varphi(u_n, v) - \varphi(u_n, u_n)) \le \varphi(u, v) - \varphi(u, u). \end{cases} \tag{9}$$

$$\begin{cases} \text{For any } \{u_n\} \subset X \text{ with } u_n \rightharpoonup u \text{ in } X, \text{ and all } v \in X, \text{ we have} \\ \quad \limsup j^0(u_n; v - u_n) \le j^0(u; v - u). \end{cases} \tag{10}$$

$$\text{For any } \{q_n\} \subset \mathcal{Q} \text{ with } q_n \to q \text{ in } \mathcal{Q}, \text{ we have } Bq_n \to Bq \text{ in } X^*. \tag{11}$$

The first continuous dependence result reads as follows.

Theorem 11 *Assume that (1)–(5) hold. Suppose also that (8)–(11) hold. Then,*

$$p_n \to \overline{p} \text{ in } \mathcal{P}, q_n \to \overline{q} \text{ in } \mathcal{Q} \Rightarrow u(p_n, q_n) \to u(\overline{p}, \overline{q}) \text{ in } X \quad \text{as } n \to \infty.$$

Proof Let $p_n \in \mathcal{P}$, $q_n \in \mathcal{Q}$ and $u_n = u(p_n, q_n) \in K(p_n)$ be a unique solution to Problem 1, i.e.,

$$\begin{aligned} &\langle Au_n, v - u_n\rangle_X + \varphi(u_n, v) - \varphi(u_n, u_n) + j^0(u_n; v - u_n) \\ &\quad \ge \langle f + Bq_n, v - u_n\rangle_X, \quad \forall v \in K(p_n). \end{aligned} \tag{12}$$

From (7) we have

$$\|u_n\|_X \le \frac{1}{\alpha_A - \alpha_j}(\|A0_X\|_{X^*} + \|f\|_{X^*} + \|B\|\|q_n\|_{\mathcal{Q}} + c_0). \tag{13}$$

Now, let $\{p_n\} \subset \mathcal{P}$ with $p_n \to p$ in $\mathcal{P}$ for some $p \in \mathcal{P}$ and $\{q_n\} \subset \mathcal{Q}$ with $q_n \to q$ in $\mathcal{Q}$ for some $q \in \mathcal{Q}$. It follows from (13) that $\{u_n\}$ is a bounded sequence in X. Therefore, by the reflexivity of X, we may suppose, passing to a subsequence if necessary, that $u_n \rightharpoonup \overline{u}$ in X as $n \to \infty$ with $\overline{u} \in X$. Since $u_n \in K(p_n)$ and $K(p_n) \xrightarrow{M} K(p)$, by the condition $(m2)$ of Definition 7, we have $\overline{u} \in K(p)$. From $(m1)$ in Definition 7, we find a sequence $\{u'_n\}$ such that $u'_n \in K(p_n)$ and $u_n \to \overline{u}$ in X, as $n \to \infty$. We set $v = u'_n$ in (1), and obtain

$$\begin{aligned}
&\limsup \langle Au_n, u_n - \overline{u}\rangle_X \\
&\quad = \limsup \langle Au_n, u_n - u'_n + u'_n - \overline{u}\rangle_X \\
&\quad \le \limsup \langle Au_n, u_n - u'_n\rangle_X + \limsup \langle Au_n, u'_n - \overline{u}\rangle_X \\
&\quad \le \limsup \langle Au_n, u_n - u'_n\rangle_X \\
&\quad \le \langle f + Bq_n, u_n - u'_n\rangle_X + \varphi(u_n, u'_n) - \varphi(u_n, u_n) + j^0(u_n; u'_n - u_n).
\end{aligned}$$

Using hypotheses (9)–(11) we have

$$\limsup \langle Au_n, u_n - \overline{u}\rangle_X \le 0.$$

It is well known that a monotone Lipschitz continuous operator is pseudomonotone and hence, (2) implies that A is pseudomonotone. Therefore, we infer

$$\liminf \langle Au_n, u_n - v\rangle_X \ge \langle A\overline{u}, \overline{u} - v\rangle_X, \quad \forall v \in X.$$

Subsequently, we are in a position to pass to the limit in (12). Let $w \in K(p)$. From $(m1)$ in Definition 7, we find a sequence $\{w_n\}$ such that $w_n \in K(p_n)$ and $w_n \to w$ in X, as $n \to \infty$. We set $v = w_n$ in (12), and obtain

$$\begin{aligned}
&\langle Au_n, w_n - u_n\rangle_X + \varphi(u_n, w_n) - \varphi(u_n, u_n) + j^0(u_n; w_n - u_n) \\
&\quad \ge \langle f + Bq_n, w_n - u_n\rangle_X.
\end{aligned}$$

Then from (9)–(11), we have

$$\begin{aligned}
&\langle A\overline{u}, \overline{u} - w\rangle_X \\
&\quad \le \limsup \langle Au_n, u_n - w\rangle_X \\
&\quad \le \limsup \langle Au_n, u_n - w_n\rangle_X + \limsup \langle Au_n, w_n - w\rangle_X \\
&\quad \le \limsup \Big(\langle f + Bq_n, w_n - u_n\rangle_X + \varphi(u_n, w_n) - \varphi(u_n, u_n) + j^0(u_n; w_n - u_n)\Big) \\
&\quad \le \langle f + Bq, w - \overline{u}\rangle_X + \varphi(w, \overline{u}) - \varphi(\overline{u}, \overline{u}) + j^0(\overline{u}; w - \overline{u}).
\end{aligned}$$

Since $w \in K(p)$ is arbitrary, we have shown that

$$\begin{aligned}
&\langle A\overline{u}, w - \overline{u}\rangle_X + \varphi(w, \overline{u}) - \varphi(\overline{u}, \overline{u}) + j^0(\overline{u}; w - \overline{u}) \\
&\quad \ge \langle f + Bq, w - \overline{u}\rangle_X, \quad \forall w \in K(p),
\end{aligned}$$

which implies that $\overline{u} \in K(p)$ solves Problem 1. Since every subsequence $\{u_n\}$ converges weakly to the same limit, the whole original sequence $\{u_n\}$ converges weakly to $\overline{u} \in K(p)$.

Finally, we show that $u_n \to \overline{u}$, as $n \to \infty$. Since $K(p_n) \xrightarrow{M} K(p)$ as $n \to \infty$, by the condition $(m1)$ of Definition 7, we can find a sequence $\{\tilde{u}_n\}$, $\tilde{u}_n \in K(p_n)$ such that $\tilde{u}_n \to \overline{u}$, as $n \to \infty$. Choosing $v = \tilde{u}_n$ in (1), we have

$$\begin{aligned} m_A \|u_n - \tilde{u}_n\|_X^2 &\le \langle Au_n - A\tilde{u}_n, u_n - \tilde{u}_n\rangle_X \\ &= \langle Au_n, u_n - \tilde{u}_n\rangle_X - \langle A\tilde{u}_n, u_n - \tilde{u}_n\rangle_X \\ &\le \varphi(u_n, \tilde{u}_n) - \varphi(u_n, u_n) + j^0(u_n; \tilde{u}_n - u_n) \\ &\quad + \langle f + Bq_n - A\tilde{u}_n, u_n - \tilde{u}_n\rangle_X. \end{aligned}$$

Passing to the upper limit in the last inequality, as $n \to \infty$, and exploiting (9)–(11) and Lemma 6, we deduce $\limsup \|u_n - \tilde{u}_n\|_X^2 \le 0$. Hence, we obtain $\|u_n - \tilde{u}_n\|_X \to 0$. Finally, we have

$$0 \le \lim \|u_n - \overline{u}\|_X \le \lim \|u_n - \tilde{u}_n\|_X + \lim \|\tilde{u}_n - \overline{u}\|_X = 0,$$

which implies that $u_n \to \overline{u}$ in X, as $n \to \infty$. This completes the proof. □

We need the following hypotheses on the data.

$$K(p_n) \xrightarrow{M} K(p) \text{ as } p_n \rightharpoonup p. \tag{14}$$

$$\text{For any } \{q_n\} \subset \mathcal{Q} \text{ with } q_n \rightharpoonup q \text{ in } \mathcal{Q}, \text{ we have } Bq_n \to Bq \text{ in } X^*. \tag{15}$$

Similar to Theorem 11, we have the following result.

Theorem 12 *Assume that* (1)–(5) *hold. Suppose also that* (9), (10), (14), (15) *hold. Then,*

$$p_n \rightharpoonup \overline{p} \text{ in } \mathcal{P},\, q_n \rightharpoonup \overline{q} \text{ in } \mathcal{Q} \Rightarrow u(p_n, q_n) \to u(\overline{p}, \overline{q}) \text{ in } X \quad \text{as } n \to \infty.$$

Consider the constraint sets $K(p)$ satisfy the following hypothesis

$$\begin{cases} K(p) = c(p)K + d(p)\theta \text{ is such that} \\ \quad \text{(a)}\ \ K \text{ is a nonempty, closed and convex subset of } X. \\ \quad \text{(b)}\ \ 0_X \in K(p) \text{ and } \theta \text{ is a given element of } X. \\ \quad \text{(c)}\ \ c, d \colon \mathcal{P} \to \mathbb{R} \text{ are continuous and } c(p) > 0 \text{ for all } p \in \mathcal{P}. \end{cases} \tag{16}$$

Remark 13 We observe that if $K(p)$, for $p \in \mathcal{P}$, is defined by (16), then $K(p_n) \xrightarrow{M} K(p)$ as $p_n \to p$. In fact, if $p_n \to p$, then $c(p_n) \to c(p)$ and $d(p_n) \to d(p)$, and hence $K(p_n) \xrightarrow{M} K(p)$, see [15].

We also need the additional hypothesis on function φ to obtain the second continuous dependence result.

$$\begin{cases} \varphi\colon X \times X \to \mathbb{R} \text{ is such that there exists function } c_\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+ \\ \quad \text{and for each } k \in \mathbb{N} \text{ there exists a constant } N_k > 0 \text{ such that} \\ \quad c_\varphi(r) \le N_k \text{ for all } r \in [0, k] \text{ and} \\ \quad \varphi(u, v_1) - \varphi(u, v_2) \le c_\varphi(\|u\|_X)\|v_1 - v_2\|_X, \\ \text{for all } u, v_1, v_2 \in X. \end{cases} \tag{17}$$

Theorem 14 *Assume that* (2), (3), (4) *and* (5) *hold. Suppose also that* (11), (16) *and* (17) *hold. Then,*

$$p_n \to \overline{p} \text{ in } \mathcal{P}, q_n \to \overline{q} \text{ in } \mathcal{Q} \Rightarrow u(p_n, q_n) \to u(\overline{p}, \overline{q}) \text{ in } X \quad \text{as } n \to \infty.$$

Proof The existence and uniqueness of solution follow from Remark 13 and Theorem 9. Let $p_n, p \in \mathcal{P}, q_n, q \in \mathcal{Q}$ with $p_n \to p$ and $q_n \to q$ as $n \to +\infty$ and $u_n = u(p_n, q_n) \in K(p_n)$, $u = u(p, q) \in K(p)$ be the corresponding solutions, that is,

$$\begin{aligned} &\langle Au, v - u\rangle_X + \varphi(u, v) - \varphi(u, u) + j^0(u; v - u) \\ &\quad \ge \langle f + Bq, v - u\rangle_X, \quad \forall v \in K(p), \end{aligned} \tag{18}$$

$$\begin{aligned} &\langle Au_n, v_n - u_n\rangle_X + \varphi(u_n, v_n) - \varphi(u_n, u_n) + j^0(u_n; v_n - u_n) \\ &\quad \ge \langle f + Bq_n, v_n - u_n\rangle_X, \quad \forall v_n \in K(p_n). \end{aligned} \tag{19}$$

By the definition of $K(p_n)$, we get $\frac{u_n - d(p_n)\theta}{c(p_n)} \in K$. Let $c_n = \frac{c(p)}{c(p_n)}$. Taking $v = c_n(u_n - d(p_n)\theta) + d(p)\theta \in K(p)$ in (18) we obtain

$$\begin{aligned} &\langle Au - f - Bq, c_n(u_n - d(p_n)\theta) + d(p)\theta - u\rangle_X \\ &\quad + \varphi(u, c_n(u_n - d(p_n)\theta) + d(p)\theta) - \varphi(u, u) \\ &\quad + j^0(u; c_n(u_n - d(p_n)\theta) + d(p)\theta - u) \ge 0. \end{aligned}$$

Taking $v_n = \frac{1}{c_n}(u - d(p)\theta) + d(p_n)\theta \in K(p_n)$ in (19) and multiplying by c_n we obtain

$$\begin{aligned} &\langle Au_n - f - Bq_n, u - d(p)\theta - c_n(u_n - d(p_n)\theta)\rangle_X \\ &\quad + \varphi(u_n, u - d(p)\theta + c_n d(p_n)\theta) - \varphi(u_n, c_n u_n) \\ &\quad + j^0(u_n; u - d(p)\theta - c_n(u_n - d(p_n)\theta)) \ge 0. \end{aligned}$$

Adding the above two inequalities we obtain

$$\begin{aligned} &\langle Au_n - Au, u_n - u\rangle_X \\ &\quad \le \langle Au_n - Au, -d(p)\theta - (c_n - 1)u_n + c_n d(p_n)\theta\rangle_X \\ &\qquad + \langle Bq - Bq_n, u - d(p)\theta - c_n(u_n - d(p_n)\theta)\rangle_X \\ &\qquad + \varphi(u, c_n(u_n - d(p_n)\theta) + d(p)\theta) - \varphi(u, u) \\ &\qquad + \varphi(u_n, u - d(p)\theta + c_n d(p_n)\theta) - \varphi(u_n, c_n u_n) \end{aligned}$$

$$
\begin{aligned}
&+ j^0(u; c_n(u_n - d(p_n)\theta) + d(p)\theta - u) \\
&+ j^0(u_n; u - d(p)\theta - c_n(u_n - d(p_n)\theta)).
\end{aligned}
$$

From (3)(c) and (17) we have

$$
\begin{aligned}
&\varphi(u, c_n(u_n - d(p_n)\theta) + d(p)\theta) - \varphi(u, u) \\
&\quad + \varphi(u_n, u - d(p)\theta + c_n d(p_n)\theta) - \varphi(u_n, c_n u_n) \\
&\quad = \varphi(u, c_n(u_n - d(p_n)\theta) + d(p)\theta) - \varphi(u, u - d(p)\theta + c_n d(p_n)\theta) \\
&\quad + \varphi(u_n, u - d(p)\theta + c_n d(p_n)\theta) - \varphi(u_n, c_n(u_n - d(p_n)\theta) + d(p)\theta) \\
&\quad + \varphi(u, u - d(p)\theta + c_n d(p_n)\theta) - \varphi(u, u) \\
&\quad + \varphi(u_n, c_n(u_n - d(p_n)\theta) + d(p)\theta) - \varphi(u_n, c_n u_n) \\
&\quad \le \alpha_\varphi \|u_n - u\|_X \|c_n u_n - u + 2d(p)\theta - 2c_n d(p_n)\theta\|_X \\
&\quad + \|c_n d(p_n)\theta - d(p)\theta\|(c_\varphi(\|u_n\|_X) + c_\varphi(\|u\|_X)).
\end{aligned}
$$

Next, using the identity

$$
c_n u_n - u = u_n - u + (c_n - 1)u_n,
$$

we obtain

$$
\begin{aligned}
&\varphi(u, c_n(u_n - d(p_n)\theta) + d(p)\theta) - \varphi(u, u) \\
&\quad + \varphi(u_n, u - d(p)\theta + c_n d(p_n)\theta) - \varphi(u_n, c_n u_n) \\
&\quad \le \alpha_\varphi \|u_n - u\|_X \|c_n u_n - u + 2d(p)\theta - 2c_n d(p_n)\theta\|_X \\
&\quad + \|c_n d(p_n)\theta - d(p)\theta\|(c_\varphi(\|u_n\|_X) + c_\varphi(\|u\|_X)) \\
&\quad \le \alpha_\varphi \|u_n - u\|_X^2 + \alpha_\varphi \|u_n - u\|_X \|(c_n - 1)u_n + 2d(p)\theta - 2c_n d(p_n)\theta\|_V \\
&\quad + \|c_n d(p_n)\theta - d(p)\theta\|(c_\varphi(\|u_n\|_X) + c_\varphi(\|u\|_X)).
\end{aligned}
$$

From (4)(b) we have

$$
\begin{aligned}
&j^0(u; c_n(u_n - d(p_n)\theta) + d(p)\theta - u) \\
&\quad + j^0(u_n; u - d(p)\theta - c_n(u_n - d(p_n)\theta)) \\
&\quad \le \alpha_j \|u_n - u\|_X \|c_n(u_n - d(p_n)\theta) + d(p)\theta - u\|_X \\
&\quad \le \alpha_j \|u_n - u\|_X \|u_n - u + (c_n - 1)u_n - c_n d(p_n)\theta + d(p)\theta\|_X \\
&\quad \le \alpha_j \|u_n - u\|_X^2 + \alpha_j \|u_n - u\|_X \|c_n - 1)u_n - c_n d(p_n)\theta + d(p)\theta\|_X.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&(\alpha_A - \alpha_\varphi - \alpha_j)\|u_n - u\|_X^2 \\
&\quad \le L_A \| - d(p)\theta - (c_n - 1)u_n + c_n d(p_n)\theta\|_X \|u_n - u\|_X \\
&\quad + \|Bq_n - Bq\|_{X^*} \|u - d(p)\theta - c_n(u_n - d(p_n)\theta)\|_X \\
&\quad + \alpha_\varphi \|u_n - u\|_X \|(c_n - 1)u_n + 2d(p)\theta - 2c_n d(p_n)\theta\|_X
\end{aligned}
$$

$$
\begin{aligned}
&+\|c_n d(p_n)\theta - d(p)\theta\|(c_\varphi(\|u_n\|_X) + c_\varphi(\|u\|_X)) \\
&+\alpha_j\|u_n - u\|_X\|\frac{c(p)}{(c(p_n)} - 1)u_n - c_n d(p_n)\theta + d(p)\theta\|_X \\
\le &\Big(L_A|(c_n - 1|\|u_n\|_X + |c_n d(p_n) - d(p)|\|\theta\|_X) \\
&+\alpha_\varphi(|c_n - 1|\|u_n\|_X + |2d(p) - 2c_n d(p_n)|\|\theta\|_X) \\
&+\alpha_j(|\frac{c(p)}{(c(p_n)} - 1|\|u_n\|_X + |c_n d(p_n) - d(p)|\|\theta\|_X\Big)\|u_n - u\|_X \\
&+\|Bq_n - Bq\|_{X^*}(\|u\|_X + |d(p)|\|\theta\|_X + |c_n|\|u_n\|_X + |d(p_n)|\|\theta\|_X) \\
&+|c_n d(p_n) - d(p)|\|\theta\|_X(c_\varphi(\|u_n\|_X) + c_\varphi(\|u\|_X)).
\end{aligned}
$$

Let

$$
k = \frac{1}{\alpha_A - \alpha_j}(\|A0_X\|_{X^*} + \|f\|_{X^*} + \|B\|(\|q\|_Q + 1) + c_0).
$$

Then, for sufficiently large n we have

$$
\begin{cases} c_\varphi(\|u_n\|_X) \le N_k, \\ c_\varphi(\|u\|_X) \le N_k. \end{cases} \tag{20}
$$

and hence

$$
\begin{aligned}
&(\alpha_A - \alpha_\varphi - \alpha_j)\|u_n - u\|_X^2 \\
&\quad\le \Big(L_A|(c_n - 1|k + |c_n d(p_n) - d(p)|\|\theta\|_X) \\
&\quad+\alpha_\varphi(|c_n - 1|k + |2d(p) - 2c_n d(p_n)|\|\theta\|_X) \\
&\quad+\alpha_j(|\frac{c(p)}{(c(p_n)} - 1|k + |c_n d(p_n) - d(p)|\|\theta\|_X\Big)\|u_n - u\|_X \\
&\quad+\|Bq_n - Bq\|_{X^*}(k + |d(p)|\|\theta\|_X + |c_n|k + |d(p_n)|\|\theta\|_X) \\
&\quad+2N_k|c_n d(p_n) - d(p)|\|\theta\|_X.
\end{aligned}
$$

From the following inequality

$$
x, a, b \ge 0 \text{ and } x^2 \le ax + b \quad \Rightarrow x \le a + \sqrt{b},
$$

it follows that

$$
\begin{aligned}
&\|u_n - u\|_X \\
&\quad\le \frac{1}{\alpha_A - \alpha_\varphi - \alpha_j}\Big(L_A|(c_n - 1|k + |c_n d(p_n) - d(p)|\|\theta\|_X) \\
&\quad+\alpha_\varphi(|c_n - 1|k + |2d(p) - 2c_n d(p_n)|\|\theta\|_X)
\end{aligned}
$$

$$
\begin{aligned}
&+\alpha_j(|\frac{c(p)}{(c(p_n)}-1|k+|c_n d(p_n)-d(p)|\|\theta\|_X\Big)\\
&+\sqrt{\frac{1}{\alpha_A-\alpha_\varphi-\alpha_j}}\Big(\|Bq_n-Bq\|_{X^*}(k+|d(p)|\|\theta\|_X+|c_n|k+|d(p_n)|\|\theta\|_X)\\
&\quad+2N_k|c_n d(p_n)-d(p)|\|\theta\|_X\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Since $c(p_n)\to c(p)$, $d(p_n)\to d(p)$ and $Bq_n\to Bq$ as $n\to+\infty$, we deduce that the right-hand side of above inequality tends to 0 as $n\to+\infty$, and hence $u_n\to u$ as $n\to+\infty$. The proof is complete. □

Remark 15 We observe that if $d(p)=0$ for all $p\in\mathcal{P}$ in (16), then from the proof of above theorem, we can omit the condition (17).

Consider the constraint sets $K(p)$ satisfy the following hypothesis

$$
\begin{cases}
K(p)=c(p)K+d(p)\theta \text{ is such that}\\
\text{(a) } K \text{ is a nonempty, closed and convex subset of } X.\\
\text{(b) } 0_X\in K(p) \text{ and } \theta \text{ is a given element of } X.\\
\text{(c) } c,d\colon\mathcal{P}\to\mathbb{R} \text{ are linear and continuous and } c(p)>0 \text{ for all } p\in\mathcal{P}.
\end{cases}
\tag{21}
$$

Remark 16 We observe that if $K(p)$, for $p\in\mathcal{P}$, is defined by (21), then $K(p_n)\xrightarrow{M} K(p)$ as $p_n\rightharpoonup p$. In fact, if $p_n\rightharpoonup p$, since $c,d\colon(0,+\infty)\to\mathbb{R}$ are linear and continuous, it is weakly continuous, then $c(p_n)\to c(p)$ and $d(p_n)\to d(p)$ in $\mathbb{R}$. By Remark 13, we obtain $K(p_n)\xrightarrow{M} K(p)$.

Similar to Theorem 14, we have the following result.

Theorem 17 *Assume that* (1)–(5) *hold. Suppose also that* (15)–(21) *hold. Then,*

$$
p_n\rightharpoonup\overline{p} \text{ in } \mathcal{P},\ q_n\rightharpoonup\overline{q} \text{ in } \mathcal{Q}\Rightarrow u(p_n,q_n)\to u(\overline{p},\overline{q}) \text{ in } X \quad \text{as } n\to\infty.
$$

5 Optimal Control Problems

In this section we provide some existence results for an optimal control problem which state is described by the variational–hemivariational inequality formulated in Problem 1.

Consider the following optimal control problem. Given admissible subsets of parameters $\mathcal{P}_{ad}\subset\mathcal{P}$, $\mathcal{Q}_{ad}\subset\mathcal{Q}$ and a cost functional $F\colon\mathcal{P}\times\mathcal{Q}\times X\to\mathbb{R}$, find a solution $(p^*,q^*)\in\mathcal{P}_{ad}\times\mathcal{Q}_{ad}$ to the following problem

$$
F(p^*,q^*,u(p^*,q^*))=\min\ F(p,q,u(p,q)),
\tag{22}
$$

where $u=u(p,q)\in K(p)$ denotes the unique solution of Problem 1 corresponding to parameters p,q.

We are now in a position to state the main result on the existence of solutions to problem (22). We admit the following hypotheses

$$\mathcal{P}_{ad} \text{ is a compact subset of } \mathcal{P}, \tag{23}$$

$$\mathcal{P}_{ad} \text{ is a weakly compact subset of } \mathcal{P}, \tag{24}$$

$$\mathcal{Q}_{ad} \text{ is a compact subset of } \mathcal{Q}, \tag{25}$$

$$\mathcal{Q}_{ad} \text{ is a weakly compact subset of } \mathcal{Q}, \tag{26}$$

$$F : \mathcal{P} \times \mathcal{Q} \times X \to \mathbb{R} \text{ is l.s.c. on } \mathcal{P}_{ad} \times \mathcal{Q}_{ad} \times X, \tag{27}$$

$$F : \mathcal{P} \times \mathcal{Q} \times X \to \mathbb{R} \text{ is weakly-weakly-strongly l.s.c. on } \mathcal{P}_{ad} \times \mathcal{Q}_{ad} \times X. \tag{28}$$

Theorem 18 *Assume hypotheses of Theorem 11, (23), (25) and (27) hold. Then the problem (22) has at least one solution.*

Proof Let $\{(p_n, q_n, u_n)\} \subset \mathcal{P}_{ad} \times \mathcal{Q}_{ad} \times X$ be a minimizing sequence of the functional F, i.e.,

$$\lim F(p_n, q_n, u_n) = \inf\{ F(p, q, u) \mid p \in \mathcal{P}_{ad}, q \in \mathcal{Q}_{ad}\},$$

where $p_n \in \mathcal{P}_{ad}, q_n \in \mathcal{Q}_{ad}$ and $u_n \in K(p_n)$ is the unique solution of Problem 1 that corresponds to p_n, q_n, i.e., $u_n = u(p_n, q_n)$. From (23) and (25), there are subsequence of $\{p_n\}$ and $\{q_n\}$, denoted in the same way, such that $p_n \to \overline{p}$ in $\mathcal{P}$ with some $\overline{p} \in \mathcal{P}_{ad}$ and $q_n \to \overline{q}$ in $\mathcal{Q}$ with some $\overline{q} \in \mathcal{Q}_{ad}$. From Theorem 11, we infer that the sequence $\{u_n\} \subset K(p_n)$ converges weakly in X to the unique solution $u(\overline{p}, \overline{q}) \in K(\overline{p})$ of Problem 1. Finally, from (27), we have

$$F(\overline{p}, \overline{q}, u(\overline{p}, \overline{q})) \le \liminf F(p_n, q_n, u_n) = \inf\{ F(p, q, u) \mid p \in \mathcal{P}_{ad}, q \in \mathcal{Q}_{ad}\},$$

which shows that $(\overline{p}, \overline{q})$ is a solution of the problem (22). This completes the proof. □

Similarly, we have the following results.

Theorem 19 *Assume hypotheses of Theorem 12, (24), (26) and (28) hold. Then the problem (22) has at least one solution.*

Theorem 20 *Assume hypotheses of Theorem 14, (23), (25) and (27) hold. Then the problem (22) has at least one solution.*

Theorem 21 *Assume hypotheses of Theorem 17, (24), (26) and (28) hold. Then the problem (22) has at least one solution.*

Next, we consider a special case of F.

Let $F(p, q, u) = P(p) + Q(q) + U(u)$, where $P : \mathcal{P} \to \mathbb{R}$, $Q : \mathcal{Q} \to \mathbb{R}$ and $U : X \to \mathbb{R}$. Assume that the following hypotheses hold.

$$P : \mathcal{P} \to \mathbb{R} \text{ is weakly lower semicontinuous, positive and coercive,} \tag{29}$$

$$Q : \mathcal{Q} \to \mathbb{R} \text{ is weakly lower semicontinuous, positive and coercive,} \tag{30}$$

$$U : X \to \mathbb{R} \text{ is continuous, bounded and positive.} \tag{31}$$

Consider the following optimal control problem.

Find a pair $(p^*, q^*) \in \mathcal{P} \times \mathcal{Q}$ such that

$$F(p^*, q^*, u(p^*, q^*)) = \min_{p \in \mathcal{P}, q \in \mathcal{Q}} F(p, q, u(p)). \tag{32}$$

Theorem 22 *Assume hypotheses of Theorem 17, (29)–(31) hold. Then the problem (32) has at least one solution.*

Proof Let $\{(p_n, q_n, u_n)\} \subset \mathcal{P} \times \mathcal{Q} \times X$ be a minimizing sequence of the functional F, i.e.,

$$\lim F(p_n, q_n, u_n) = \inf_{p \in \mathcal{P}, q \in \mathcal{Q}} F(p, q, u(p)),$$

where $u_n \in K(p_n)$ is the unique solution of Problem 1 that corresponds to p_n, i.e., $u_n = u(p_n, q_n)$. From (29)–(31) it follows that the sequences $\{p_n\}$ and $\{q_n\}$ are bounded. In fact, since

$$F(p_n, q_n, u_n) = P(p_n) + Q(q_n) + U(u_n) \geq P(p_n),$$

we can deduce that $F(p_n, q_n, u_n) \to +\infty$ if $\|p_n\|_{\mathcal{P}} \to +\infty$. Then there is a subsequence of $\{p_n\}$, denoted in the same way, such that $p_n \rightharpoonup \overline{p}$ in $\mathcal{P}$ with some $\overline{p} \in \mathcal{P}_{ad}$. Similarly, there is a subsequence of $\{q_n\}$, denoted in the same way, such that $q_n \rightharpoonup \overline{q}$ in $\mathcal{Q}$ with some $\overline{q} \in \mathcal{Q}_{ad}$. From Theorem 11, we infer that the sequence $\{u_n\} \subset K(p_n)$ converges weakly in X to the unique solution $u(\overline{p}, \overline{q}) \in K(\overline{p})$ of Problem 1. Finally, from (29)–(31), we have

$$F(\overline{p}, \overline{q}, u(\overline{p})) \leq \liminf F(p_n, q_n, u_n) = \inf_{p \in \mathcal{P}, q \in \mathcal{Q}} F(p, q, u(p)),$$

which shows that $(\overline{p}, \overline{q})$ is a solution of the problem (32). This completes the proof. □

6 Frictional Contact Problem

In this section, we consider an optimal control problem for a frictional contact problem from theory of elasticity to which our main results of Sects. 4 and 5 can be applied. We provide the classical formulation of the contact problem and give its variational formulation for which we obtain a result on its unique weak solvability. Then, we study the optimal control problem for the contact problem and deliver a result on its solvability.

Consider the following physical setting. An elastic body occupies an open, bounded and connected set $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$. The boundary of Ω is denoted by $\Gamma = \partial\Omega$ and it is assumed to be Lipschitz continuous. We denote by $\boldsymbol{\nu} = (\nu_i)$ the outward unit normal at Γ. We suppose that Γ consists of three mutually disjoint and measurable parts $\overline{\Gamma}_1$, $\overline{\Gamma}_2$ and $\overline{\Gamma}_3$ such that *meas* $(\Gamma_1) > 0$. Moreover, the notation $\mathbb{S}^d$ stands for the space of second order symmetric tensors on $\mathbb{R}^d$. On $\mathbb{R}^d$ and $\mathbb{S}^d$ we use the inner products and the Euclidean norms defined by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \ \ \|\boldsymbol{u}\| = (\boldsymbol{u} \cdot \boldsymbol{u})^{1/2} \quad \text{for all } \boldsymbol{u} = (u_i), \boldsymbol{v} = (v_i) \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_i \tau_i, \ \ \|\boldsymbol{\sigma}\| = (\boldsymbol{\sigma} \cdot \boldsymbol{\sigma})^{1/2} \quad \text{for all } \boldsymbol{\sigma} = (\sigma_{ij}), \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d,$$

respectively. Given a vector field $\boldsymbol{u}$, notation u_ν and $\boldsymbol{u}_\tau$ represent its normal and tangential components on the boundary defined by

$$u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu} \quad \text{and} \quad \boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu \boldsymbol{\nu}.$$

For a tensor $\boldsymbol{\sigma}$, the symbols σ_ν and $\boldsymbol{\sigma}_\tau$ denote its normal and tangential components on the boundary, i.e.,

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu} \quad \text{and} \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}.$$

Sometimes, we omit the explicit dependence on $\boldsymbol{x} \in \Omega \cup \Gamma$. We also denote by $\mathcal{P}$, $\mathcal{Q}$ the normed spaces of parameters. The classical model for the contact process is the following.

Problem 23 Given $p \in \mathcal{P}$, find a displacement field $\boldsymbol{u} \colon \Omega \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$ and an interface force $\eta \colon \Gamma_3 \to \mathbb{R}$ such that

$$\boldsymbol{\sigma} = \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{u})) \ \text{ in } \ \Omega, \tag{33}$$
$$\text{Div}\boldsymbol{\sigma} + \boldsymbol{f}_0 = 0 \ \text{ in } \ \Omega, \tag{34}$$
$$\boldsymbol{u} = \boldsymbol{0} \ \text{ on } \ \Gamma_1, \tag{35}$$
$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 + Bq \ \text{ on } \ \Gamma_2, \tag{36}$$
$$u_\nu \le g(p), \ \ \sigma_\nu + \eta \le 0, \ \ (u_\nu - g(p))(\sigma_\nu + \eta) = 0, \ \ \eta \in \partial j_\nu(u_\nu) \ \text{ on } \ \Gamma_3, \tag{37}$$
$$\|\boldsymbol{\sigma}_\tau\| \le F_b(u_\nu), \quad -\boldsymbol{\sigma}_\tau = F_b(u_\nu)\frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \ \text{ if } \ \|\boldsymbol{u}_\tau\| \ne 0 \ \text{ on } \ \Gamma_3. \tag{38}$$

Note that Problem 23 was first investigated in [14] where results on its unique weak solvability, continuous dependence on the data and penalty method were obtained. In this problem (33) represents the elastic constitutive law in which $\mathcal{A}$ is the elasticity operators and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the linearized strain tensor defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}) \ \text{ in } \Omega.$$

Equation (34) is the equation of equilibrium, where $\boldsymbol{f}_0$ denotes the density of the body forces, (35) is the displacement homogeneous boundary condition which means that the body is fixed on Γ_1, and (36) is the traction boundary condition with surface tractions of density $\boldsymbol{f}_2 + Bp$ acting on Γ_2, where p is the control parameter. Finally, conditions (37) and (38) given on the contact surface Γ_3, represent the contact and the friction law, respectively. Here g denotes the thickness of the elastic layer which depends on p, F_b is the friction bound and ∂j_ν represents the Clarke subdifferential of a given function j_ν. A detailed description of these conditions together with mechanical interpretations can be found in [14]. We restrict ourselves to a comment that the friction law (38) is a variant of the Coulomb law of dry friction in which the friction bound F_b depends on the normal displacement u_ν and (37) is the multivalued normal compliance contact condition with unilateral constraints of Signorini type. More details on static contact models with elastic materials can be found in [13,16,20].

In order to study the variational formulation of Problem 23, we use the spaces V and $\mathcal{H}$ defined by

$$V = \{ \boldsymbol{v} \in H^1(\Omega; \mathbb{R}^d) \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1 \}, \quad \mathcal{H} = L^2(\Omega; \mathbb{S}^d).$$

Here and below we denote by $\boldsymbol{v}$ the trace on the boundary of an element $\boldsymbol{v} \in H^1(\Omega; \mathbb{R}^d)$. On the space V we consider the inner product and the corresponding norm given by

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \quad \|\boldsymbol{v}\|_V = \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}} \text{ for all } \boldsymbol{u}, \boldsymbol{v} \in V.$$

Recall (see e.g. [13]) that, since $meas(\Gamma_1) > 0$, it follows that V is a Hilbert space. Moreover, by the Sobolev trace theorem, we have

$$\|\boldsymbol{v}\|_{L^2(\Gamma;\mathbb{R}^d)} \le \|\gamma\| \|\boldsymbol{v}\|_V \text{ for all } \boldsymbol{v} \in V, \tag{39}$$

where $\|\gamma\|$ is the norm of the trace operator $\gamma \colon V \to L^2(\Gamma; \mathbb{R}^d)$. The space $\mathcal{H}$ is a Hilbert space endowed with the inner product

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}(\boldsymbol{x})\tau_{ij}(\boldsymbol{x})\, dx,$$

and the associated norm $\| \cdot \|_{\mathcal{H}}$.

Our hypotheses on Problem 23 read as follows.

$$\left\{ \begin{array}{l} \mathcal{A} \colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d \text{ is such that} \\ \quad \text{(a) } \mathcal{A}(\cdot, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \quad \text{(b) there exists } L_{\mathcal{A}} > 0 \text{ such that} \\ \qquad\qquad \|\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\ \qquad \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \quad \text{(c) there exists } m_{\mathcal{A}} > 0 \text{ such that} \\ \qquad\qquad (\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge \alpha_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \\ \qquad \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \quad \text{(d) } \mathcal{A}(\boldsymbol{x}, \boldsymbol{0}) = \boldsymbol{0} \text{ for a.e. } \boldsymbol{x} \in \Omega. \end{array} \right. \tag{40}$$

$$\begin{cases} F_b\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R} \text{ is such that} \\ \text{(a) } F_b(\cdot, r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R}. \\ \text{(b) there exists } L_{F_b} > 0 \text{ such that} \\ \qquad |F_b(\boldsymbol{x}, r_1) - F_b(\boldsymbol{x}, r_2)| \le L_{F_b}|r_1 - r_2| \\ \qquad \text{for all} r_1, r_2 \in \mathbb{R}, \ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) } F_b(\boldsymbol{x}, r) = 0 \text{ for all } r \le 0,\ F_b(\boldsymbol{x}, r) \ge 0 \text{ for all } r \ge 0 \\ \qquad \text{for a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases} \tag{41}$$

$$\begin{cases} j_\nu\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R} \text{ is such that} \\ \text{(a) } j_\nu(\cdot, r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R} \\ \qquad \text{and there exists } \overline{e} \in L^2(\Gamma_3) \text{ such that } j_\nu(\cdot, \overline{e}(\cdot)) \in L^1(\Gamma_3). \\ \text{(b) } j_\nu(\boldsymbol{x}, \cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) there exist } \overline{c}_0, \overline{c}_1 \ge 0 \text{ such that} \\ \qquad |\partial j_\nu(\boldsymbol{x}, r)| \le \overline{c}_0 + \overline{c}_1 |r| \\ \qquad \text{for all } r \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(d) there exists } \alpha_{j_\nu} \ge 0 \text{ such that} \\ \qquad j_\nu^0(\boldsymbol{x}, r_1; r_2 - r_1) + j_\nu^0(\boldsymbol{x}, r_2; r_1 - r_2) \le \alpha_{j_\nu} |r_1 - r_2|^2 \\ \qquad \text{for all } r_1, r_2 \in \mathbb{R}, \ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases} \tag{42}$$

Finally, we assume that

$$\begin{aligned} &\boldsymbol{f}_0 \in L^2(\Omega; \mathbb{R}^d),\ \ \boldsymbol{f}_2 \in L^2(\Gamma_2; \mathbb{R}^d),\ \ g \in C(\Gamma_3 \times \mathcal{P}; \mathbb{R}_+), \\ &B \in \mathcal{L}(\mathcal{Q}; L^2(\Gamma_2; \mathbb{R}^d)). \end{aligned} \tag{43}$$

We introduce the set of admissible displacement fields $U(p)$ defined by

$$U(p) = \{\, \boldsymbol{v} \in V \mid v_\nu \le g(p) \text{ on } \Gamma_3 \,\}, \quad p \in \mathcal{P}.$$

Moreover, we define an element $\boldsymbol{f} \in V^*$ by

$$\langle \boldsymbol{f}, \boldsymbol{v} \rangle_V = \langle \boldsymbol{f}_0, \boldsymbol{v} \rangle_{L^2(\Omega;\mathbb{R}^d)} + \langle \boldsymbol{f}_2, \boldsymbol{v} \rangle_{L^2(\Gamma_2;\mathbb{R}^d)} \tag{44}$$

and an operator $B_1 \in \mathcal{L}(\mathcal{Q}; V^*)$ by

$$\langle B_1 q, \boldsymbol{v} \rangle_V = \langle B q, \boldsymbol{v} \rangle_{L^2(\Gamma_2;\mathbb{R}^d)} \tag{45}$$

for all $\boldsymbol{v} \in V$.

Using (33)–(38) and (44)–(45), by a standard argument (see Sect. 7 in [13]), we derive the following variational formulation of Problem 23.

Problem 24 Given $p \in \mathcal{P}$, $q \in \mathcal{Q}$, find $\boldsymbol{u} \in U(p)$ such that

$$\begin{aligned} &\langle \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}) \rangle_{\mathcal{H}} \\ &\quad + \int_{\Gamma_3} F_b(u_\nu)(\|\boldsymbol{v}_\tau\| - \|\boldsymbol{u}_\tau\|)\, d\Gamma + \int_{\Gamma_3} j_\nu^0(u_\nu; v_\nu - u_\nu)\, d\Gamma \ge \langle \boldsymbol{f} + B_1 q, \boldsymbol{v} - \boldsymbol{u} \rangle_V \end{aligned}$$

for all $\boldsymbol{v} \in U(p)$.

The following result concerns the well posedness of Problem 24.

Theorem 25 *Assume that* (40)–(43) *hold and the following smallness condition is satisfied*

$$(L_{F_b} + \alpha_{j_\nu})\|\gamma\|^2 < \alpha_{\mathcal{A}}. \tag{46}$$

Then

(i) *for all* $p \in \mathcal{P}$, *Problem* 24 *has a unique solution* $\boldsymbol{u} = \boldsymbol{u}(p, q) \in U(p)$.
(ii) $p_n \to \overline{p}$ *in* $\mathcal{P}$, $q_n \to \overline{q}$ *in* $\mathcal{Q} \Rightarrow \boldsymbol{u}(p_n, q_n) \to \boldsymbol{u}(\overline{p}, \overline{q})$ *in* X *as* $n \to \infty$.

Proof We will apply Theorems 10 and 14 in the following functional framework: $X = V$, $K(p) = U(p)$ and

$$\begin{aligned} A\colon V \to V^*, &\quad \langle A\boldsymbol{u}, \boldsymbol{v}\rangle = \langle \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_{\mathcal{H}} \text{ for } \boldsymbol{u}, \boldsymbol{v} \in V, \\ \varphi\colon V \times V \to \mathbb{R}, &\quad \varphi(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} F_b(u_\nu)\|\boldsymbol{v}_\tau\|\, d\Gamma \text{ for } \boldsymbol{u}, \boldsymbol{v} \in V, \\ j\colon V \to \mathbb{R}, &\quad j(\boldsymbol{v}) = \int_{\Gamma_3} j_\nu(v_\nu)\, d\Gamma \quad \text{for } \boldsymbol{v} \in V \end{aligned}$$

for all $p \in \mathcal{P}$.

We will check that the set $K(p)$, the operator A, functions φ and j satisfy all the hypotheses of Theorems 9 and 14.

It is clear that the set $K(p) = U(p)$ is a nonempty closed and convex subset of V for every $p \in \mathcal{P}$ and $K(p) = c(p)K$, where $c(p) = g(p)$ and

$$K = \{\, \boldsymbol{v} \in V \mid v_\nu \le 1 \text{ on } \Gamma_3 \,\}.$$

Then (16) holds.

From the proof of Theorem 32 in [14], it follows that (2)–(5) are satisfied. The existence and uniqueness of solution to Problem 24 follows from Theorem 32 in [14].

Next, we establish (17). From (41) and (39), for $\boldsymbol{u}, \boldsymbol{v}_1, \boldsymbol{v}_2 \in V$ we obtain

$$\begin{aligned} \varphi(\boldsymbol{u}, \boldsymbol{v}_1) - \varphi(\boldsymbol{u}, \boldsymbol{v}_2) &= \int_{\Gamma_3} F_b(u_\nu)(\|\boldsymbol{v}_{1\tau}\| - \|\boldsymbol{v}_{2\tau}\|)\, d\Gamma \\ &\le L_{F_b}\|\boldsymbol{u}\|_{L^2(\Gamma_3;\mathbb{R}^d)}(\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{L^2(\Gamma_3;\mathbb{R}^d)}) \\ &\le L_{F_b}\|\gamma\|^2\|\boldsymbol{u}\|_V\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V. \end{aligned}$$

This implies condition (17) with $c_\varphi(\|\boldsymbol{u}\|_V) = L_{F_b}\|\gamma\|^2\|\boldsymbol{u}\|_V$.

The proof of the theorem is complete. □

Consider now the following inverse problem for the contact problem (33)–(38). Let $\mathcal{P}_{ad} \subset \mathcal{P}$ and $\mathcal{Q}_{ad} \subset \mathcal{Q}$ be the admissible subsets of parameters and $F\colon \mathcal{P} \times \mathcal{Q} \times V \to \mathbb{R}$ be a cost functional. We look for a pair $(p^*, q^*) \in \mathcal{P}_{ad} \times \mathcal{Q}_{ad}$ to the following minimization problem

$$F(p^*, q^*, \boldsymbol{u}(p^*, q^*)) = \min_{p \in \mathcal{P}_{ad}, q \in \mathcal{Q}_{ad}} F(p, q, \boldsymbol{u}(p, q)), \tag{47}$$

where $\boldsymbol{u} = \boldsymbol{u}(p, q) \in U(p)$ denotes the unique solution of Problem 24 corresponding to (p, q). As a corollary from Theorems 20 and 25, we deduce the following result.

Theorem 26 *Assume that* (23), (40)–(43) *and* (46) *hold and the cost functional* $F : \mathcal{P} \times V \to \mathbb{R}$ *is l.s.c. on* $\mathcal{P}_{ad} \times V$. *Then the problem* (47) *has at least one solution.*

We conclude with the following two examples of the cost functionals which satisfy hypothesis of Theorem 26.

Example 27 Let $F : \mathcal{P} \times \mathcal{Q} \times V \to \mathbb{R}$ be of the form

$$F_1(p, q, \boldsymbol{u}) = \int_\Omega L_1(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}))\, dx + h(p) + k(q).$$

Assume the following hypotheses.

$$\begin{cases} L_1 : \Omega \times \mathbb{R}^d \to \mathbb{R} \text{ is such that} \\ \text{(a) } L_1(\cdot, \boldsymbol{\xi}) \text{ is measurable on } \Omega \text{ for all } \boldsymbol{\xi} \in \mathbb{R}^d, \\ \text{(b) } L_1(\boldsymbol{x}, \cdot) \text{ is l.s.c. on } \mathbb{R}^d \text{ for a.e. } \boldsymbol{x} \in \Omega, \\ \text{(c) } L_1(\boldsymbol{x}, \boldsymbol{\xi}) \geq -a_1(\boldsymbol{x}) + b\|\boldsymbol{\xi}\| \text{ for all } \boldsymbol{\xi} \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Omega \\ \qquad \text{with } a_1 \in L^1(\Omega),\ b \in \mathbb{R} \end{cases}$$

and $h : \mathcal{P}_{ad} \to \mathbb{R}$ is l.s.c. on $\mathcal{P}_{ad}$, $k : \mathcal{Q}_{ad} \to \mathbb{R}$ is l.s.c. on $\mathcal{Q}_{ad}$. Then F is l.s.c. on $\mathcal{P}_{ad} \times \mathcal{Q}_{ad} \times V$.

Example 28 Let $F : \mathcal{P} \times V \to \mathbb{R}$ be of the form

$$F_2(p, q, \boldsymbol{u}) = \int_\Gamma L_2(\boldsymbol{x}, \gamma \boldsymbol{u}(\boldsymbol{x}))\, d\Gamma + h(p) + k(q)$$

Assume the following hypotheses.

$$\begin{cases} L_2 : \Gamma \times \mathbb{R}^d \to \mathbb{R} \text{ is such that} \\ \text{(a) } L_2(\cdot, \boldsymbol{\xi}) \text{ is measurable on } \Gamma \text{ for all } \boldsymbol{\xi} \in \mathbb{R}^d, \\ \text{(b) } L_2(\boldsymbol{x}, \cdot) \text{ is l.s.c. on } \mathbb{R}^d \text{ for a.e. } \boldsymbol{x} \in \Gamma, \\ \text{(c) } L_2(\boldsymbol{x}, \boldsymbol{\xi}) \geq -a_2(\boldsymbol{x}) + b\|\boldsymbol{\xi}\| \text{ for all } \boldsymbol{\xi} \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Gamma \\ \qquad \text{with } a_2 \in L^1(\Gamma),\ b \in \mathbb{R} \end{cases}$$

and $h : \mathcal{P}_{ad} \to \mathbb{R}$ is l.s.c. on $\mathcal{P}_{ad}$, $k : \mathcal{Q}_{ad} \to \mathbb{R}$ is l.s.c. on $\mathcal{Q}_{ad}$. Then F is l.s.c. on $\mathcal{P}_{ad} \times V$.

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