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# Dynamics of the Nonlinear Timoshenko System with Variable Delay

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## Abstract

This paper is concerned with the wellposedness of global solution and existence of global attractor to the nonlinear Timoshenko system subject to continuous variable time delay in the angular rotation of the beam filament. The waves are assumed to propagate under the same speed in the transversal and angular direction. A single mechanical damping is implemented to counter the destabilizing effect from the time delay term. By imposing appropriate assumptions on the damping term and sub-linear time delay term, we prove the existence of absorbing set and establish the quasi-stability of the gradient system generated from the solution to the system of equation. The quasi-stability property in turn implies the existence of finite dimensional global and exponential attractors that contain the unstable manifold formed from the set of equilibria.

**Keywords** Timoshenko system · Variable delay · Quasi-stability · Unstable manifold · Exponential attractor

**Mathematics Subject Classification** 35B40 · 35B41 · 35Q30 · 76D03 · 76D05

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## 1 Introduction

The vibrations of a beam or thin plate coinciding with an interval on the $x$-axis and plates could be described by the following system of partial differential equations:

$$\begin{cases} \rho A \varphi_{tt} = S_x, \\ \rho I \psi_{tt} = M_x - S, \end{cases} \tag{1.1}$$

where $\varphi = \varphi(x,t)$ is the transversal displacement of beam, $\psi = \psi(x,t)$ denotes the rotation angle for beam filament, $M$ is the bending moment, $S$ is the shear stress, $\rho$, $A$ and $I$ denote the mass density, the area and the inertial moment of the transversal section respectively.

The bending moment $M$ and shear stress $S$ in (1.1) could be further determined by the constitutive laws in the theory of mathematical elasticity given as

$$\begin{cases} M = EI\psi_x, \\ S = kA(\varphi_x + \psi), \end{cases} \tag{1.2}$$

here $EI$ represents the flexural rigidity of the beam, $k$ is a shear coefficient.

Substituting (1.2) into (1.1) yields the following classical Timoshenko system (first introduced by Timoshenko in [46]):

$$\begin{cases} \rho_1 \varphi_{tt} - (k(\varphi_x + \psi))_x = 0, \\ \rho_2 \psi_{tt} - (b\psi_x)_x + k(\varphi_x + \psi) = 0, \end{cases} \tag{1.3}$$

where $\rho_1 = \rho$, $\rho_2 = \frac{\rho I}{A}$, $k$, $b = \frac{EI}{A} > 0$ are positive constants.

Due to the physical property of material for beam or plate, the deformation might not be instantaneous, thus is subject to a delay effect. In fact, there are abundant examples in physical, chemical, biological, thermal, and economic phenomena where time delay affect the behavior of a dynamical system see for example, Datko et al. [11], Fridman [17], Nicaise and Pignotti [36]. To stabilize a hyperbolic system involving input delay terms, additional control terms will be necessary. In many cases, it was shown that delay is a source of instability and even an arbitrarily small delay may destabilize a system that is uniformly asymptotically stable in the absence of delay unless additional conditions or control terms are applied.

In this paper we consider the nonlinear Timoshenko system subject to variable time delay and internal feedback:

$$\begin{cases} \rho_1 \varphi_{tt}(x,t) - k(\varphi_x + \psi)_x(x,t) = h(x) \\ \rho_2 \psi_{tt}(x,t) - b\psi_{xx}(x,t) + k(\varphi_x + \psi)(x,t) + \mu_1 \psi_t(x,t) + \mu_2 \psi_t(x, t - \tau(t)) \\ \quad + f(\psi(x,t)) = g(x) \end{cases} \tag{1.4}$$

where $(x,t) \in (0,1) \times [0,\infty)$ and it is endowed with the following initial data

$$\begin{cases} \varphi(x,0) = \varphi_0, \ \varphi_t(x,0) = \varphi_1, \ \psi(x,0) = \psi_0, \\ \psi_t(x,0) = \psi_1, \ \psi_t(x,t-\tau(t)) = f_0(x,t-\tau(t)), (x,t) \in (0,1) \times (0,\tau) \end{cases} \tag{1.5}$$

and the Dirichlet boundary condition

$$\varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0, \quad \forall\, t > 0, \tag{1.6}$$

here $\mu_1\psi_t(x,t)$ with $\mu_1 > 0$ is the frictional damping, $h$, $g$ and the nonlinear term $f(\psi)$ are the source terms and $\mu_2\psi_t(x,t-\tau(t))$ with $\mu_2 > 0$ is the time delay to the system.

One of the main problems of analyzing the long time behavior of the nonlinear Timoshenko system is to find minimum dissipation to ensure a uniform exponential decay of energy, which is important to obtain the existence of absorbing set for semi-group from the theory of dynamical systems. We review some uniform stability results of Timoshenko system with different dissipative mechanics:

(a) For the dissipative Timoshenko system with only one locally distributed feedback (one damping added) and homogeneous boundary value conditions, one of the first results was obtained by Soufyane [43], who proved that *the Timoshenko system decays exponentially if and only it the wave propagates at the same speed* (*speed equal condition*), i.e.,

$$\frac{\rho_1}{\rho_2} = \frac{k}{b}, \tag{1.7}$$

which means the velocities of wave propagations play an important role. Almeida Júnior et al. [2] considered the Timoshenko system with one damping on the transverse displacement, the solution semigroup decays exponentially if and only if (1.7) holds. For more uniform stability results for Timoshenko system and its extended models, we refer to [3,12,15,16,20,27,31,34,45]. For nonlinear Timoshenko system, the decay results can be found in Grasselli et al. [18], Mũnoz Rivera and Racke [32,33], Messaoudi and Mustafa [29,35], Messaoudi et al. [30].

(b) The speed equal condition (1.7) could be removed if stronger damping mechanism is imposed on the Timoshenko system. For example, in [24], Kim and Renardy placed two boundary feedback controls to the Timonshenko beam system and established exponential stability for the energy functional by the multiplier method *without the speed equal condition* (1.7). For more results of exponential decay without speed equal condition but two damping on transverse displacement and rotation angle added to the system, one could consult [38,42].

(c) For the Timoshenko system coupled with other equations, such as second sound, a few other necessary and sufficient conditions for exponential stability are needed. One can see Almeida Júnior et al. [1], Apalara [5], Santos et al. [40] for more details.

(d) If the coefficients in Timoshenko system are not constant, i.e., non-uniform models, to obtain exponential decay with only one damping, one needs to assume similar conditions as (1.7), such as

$$\frac{\rho_1(x)}{k(x)} = \frac{\rho_2(x)}{b(x)}, \tag{1.8}$$

one can refer to [4,35,44].

(e) The memory or history terms added to Timoshenko system provide dissipation similar to mechanical damping to the system. The distributed or continuous delays on the other hand could not provide any dissipation. Thus, with presence of delay terms, to obtain exponential decay, damping is necessary. The exponential decay results in this direction that depend on the damping and speed equal condition, could be found in [5–8,13,14,19,21–23,25,39,41,47] and literatures therein.

Although there are fruitful works on the Timoshenko system as reviewed above, many literatures pay attentions to well-posedness and decay. There are less results on for example the existence of attractors of the Timoshenko system, one of the main goals of the work presented in this article. One related work could be found in Feng and Yang [13]. One of the main features of our work that differ from the previously available results is that we consider a *variable continuous* delay (sub-linear operator). Inspired by Marín-Rubio and Real [28], this article is concerned with the finite dimensionality and structure of attractors for the nonlinear Timoshenko system with variable delay. Due to the destabilizing effect of the delay term, the strength of the delay term has to be weaker than that of the damping. The main features and difficulties of the proof are:

(1) *Transforming system* (1.4) *into equivalent form to overcome the difficulty arising from variable delay:* The transformation is motivated by [11,13,14,36] and [43]. However, since the delay is dependent on time, the equivalent system has variable coefficients. This difficulty is circumvented by imposing boundedness and sub-linear growth conditions (2.1) on the delay function as in [28], and we use the upper and lower boundedness of variable delay to deal with it.

(2) The existence of global attractor to a dynamical system depends on establishing the invariance, attracting property and compactness of semigroup. The invariance property is satisfied if the underlying semigroup is strongly continuous. Thus, we focus on proving the other two properties *attracting and compactness*:
item[(2-I)] *Attracting:* The attracting property is obtained from the existence of a bounded absorbing set, the proof of which hinges on applying semigroup and multiplier methods and establishing a series of estimates on the source term $f(\psi)$ and the delay term $\psi_t(x, t - \tau(t))$;
item[(2-II)] *Compactness:* Using *quasi-stability method* by Lasiecka and Chueshov [9] or [10], (see the theory in Sect. 3.1), we prove the *asymptotic smoothness of the semigroup* generated by the global solution. This in turn implies the existence of finite dimensional global and exponential attractors composed of unstable manifold of equilibrium. The key step and most difficult point to the proof lies in verifying quasi-stability.

(3) If the variable delay becomes a constant, the system (1.4) reduces to the Timoshenko problem with constant delay as in [14]. This means our result is an extension of Feng and Yang's results.

(4) To obtain the estimates of the energy function defined in (3.5), we construct a Lyapunov functional which is equivalent to the energy function. With the help of this Lyapunov functional we are able to obtain the absorbing set for the energy. The Lyapunov functional is obtained by energy estimate, which can be controlled by using multiplier method (i.e., perturbed energy functional).

The rest of this article is arranged as follows: In Sect. 2, we present some preliminaries and the main results: well-posedness of global solution and existence of global attractor to (1.4). The proof of main results can be found in Sect. 3.

## 2 Main Results: Finite Fractal Dimensional Global and Exponential Attractors

### 2.1 Equivalent Initial and Boundary Value Problem

We assume that the delay function (sub-linear operator) $\tau(t)$ in (1.4) is a $C^1$ continuous function which satisfies

$$\tau(0) = \tau_0,\ 0 \leq \tau(t) \leq \tau_m,\ 0 < \tau'(t) \leq 1, \tag{2.1}$$

where $\tau_0 \geq 0$, $\tau_m > 0$.

A new dependent variable for the delay feedback term (See Datko et al. [11]) can be written as:

$$z(x, \eta, t) = \psi_t(x, t - \eta\tau), \quad \eta \in [0, 1], \quad t > 0. \tag{2.2}$$

and we have

$$\tau z_t(x, \eta, t) + (1 - \eta\tau')z_\eta(x, \eta, t) = 0 \quad \text{in} \quad (0, 1) \times (0, 1) \times (0, +\infty). \tag{2.3}$$

Using this transformation, the system (1.4) is converted to its equivalent form

$$\begin{cases} \rho_1\varphi_{tt}(x, t) - k(\varphi_x + \psi)_x(x, t) = h(x), \\ \rho_2\psi_{tt}(x, t) - b\psi_{xx}(x, t) + k(\varphi_x + \psi)(x, t) + \mu_1\psi_t(x, t) + \mu_2 z(x, 1, t) \\ \quad + f(\psi(x, t)) = g(x), \\ \tau z_t(x, \eta, t) + (1 - \eta\tau')z_\eta(x, \eta, t) = 0 \end{cases} \tag{2.4}$$

with $(x, \eta, t) \in (0, 1) \times (0, 1) \times (0, +\infty)$.

The new equivalent system (2.4) is equipped with the initial condition

$$\begin{cases} \varphi(x, 0) = \varphi_0,\ \varphi_t(x, 0) = \varphi_1,\ \psi(x, 0) = \psi_0,\ \psi_t(x, 0) = \psi_1, \quad x \in (0, 1), \\ z(x, \eta, 0) = f_0(x, -\eta\tau_0), \quad (x, \eta) \in (0, 1) \times (0, 1) \end{cases} \tag{2.5}$$

and boundary condition

$$\begin{cases} \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0, \quad t > 0, \\ z(x,0,t) = \phi_t(x,t), \quad t > 0. \end{cases} \tag{2.6}$$

### 2.2 Well-Posedness

For the equivalent system (2.4)–(2.6), we give the following assumptions:

(H.1) $\dfrac{\rho_1}{\rho_2} = \dfrac{k}{b}$, which implies that both waves on the system have equal propagation speed.
(H.2) $0 < \mu_2 < \mu_1$ which is necessary to derive the energy estimates. Since some literatures shown that the system (2.4)–(2.6) (when $f(\psi) = 0$ and delay term is reduced) is exponentially stable only if $\mu_2 < \mu_1$, here we can not avoid this assumption.
(H.3) The nonlinear function $f(\psi)$ satisfies $f(0) = 0$ and

$$|f(\psi_1) - f(\psi_2)| \le k_0(|\psi_1|^\theta + |\psi_2|^\theta)|\psi_1 - \psi_2| \tag{2.7}$$

with $\theta > 0$ and $k_0 > 0$, $\psi_1, \psi_2 \in L_\infty(0,1)$ .

Based on the equivalent equation, we introduce $u = \varphi_t$ and $v = \psi_t$, and set $U = (\varphi, u, \psi, v, z)^T$, hence the operator $\mathbb{A}$ and $F$ can be defined as

$$\mathbb{A}U = \begin{pmatrix} u \\ \dfrac{k}{\rho_1}(\varphi_{xx} + \psi_x) \\ v \\ \dfrac{b}{\rho_2}\psi_{xx} - \dfrac{k}{\rho_2}(\varphi_x + \psi) - \dfrac{\mu_1}{\rho_2} v - \dfrac{\mu_2}{\rho_2} z(\cdot,1) \\ -\dfrac{1-\eta\tau'}{\tau} z_\eta \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ h \\ 0 \\ -\frac{1}{\rho_2} f(\psi) + g \\ 0 \end{pmatrix} \tag{2.8}$$

with domain

$$D(\mathbb{A}) = \{(\varphi, u, \psi, v, z)^T \in \mathbf{H} : \quad v(x,t) = z(x,0,t) \text{ for } x \in (0,1)\}, \tag{2.9}$$

where

$$\mathbf{H} = \left(H^2(0,1) \cap H_0^1(0,1)\right) \times H^1(0,1) \times \left(H^2(0,1) \cap H_0^1(0,1)\right) \times H^1(0,1) \\ \times L^2(0,1; H_0^1(0,1)). \tag{2.10}$$

Under the above definitions, the new system (2.4) can be written into the following abstract form

$$\begin{cases} \dfrac{dU(t)}{dt} = \mathbb{A}U + F, \quad t > 0, \\ U(0) = U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, f(\cdot, -\eta\tau_0))^T . \end{cases} \tag{2.11}$$

The energy space is defined as

$$\mathcal{H} := H_0^1(0,1) \times L^2(0,1) \times H_0^1(0,1) \times L^2(0,1) \times L^2((0,1)\times(0,1)). \tag{2.12}$$

The inner product of energy space $\mathcal{H}$ is defined as

$$\begin{aligned} \langle U, \bar{U} \rangle_{\mathcal{H}} = &\int_0^1 \left[\rho_1 u\bar{u} + \rho_2 v\bar{v} + k(\varphi_x + \psi)(\bar{\varphi}_x + \bar{\psi}) + b\psi_x\bar{\psi}_x\right] dx \\ &+ \xi \int_0^1 \int_0^1 z(x,\eta)\bar{z}(x,\eta)d\eta dx \end{aligned} \tag{2.13}$$

for $U = (\varphi, u, \psi, v, z)^T$, $\bar{U} = (\bar{\varphi}, \bar{u}, \bar{\psi}, \bar{v}, \bar{z})^T$.

Moreover, by assumption (H.2), we further assume for any $t > 0$, $\tau(t)$ satisfies

$$\tau' \le \frac{2(\mu_1 - \mu_2)}{2\mu_1 - \mu_2}, \tag{2.14}$$

then there is a constant $\xi > 0$, such that

$$\frac{\tau\mu_2}{1 - \tau'} \le \xi \le 2\tau\left(\mu_1 - \frac{\mu_2}{2}\right). \tag{2.15}$$

By the classical semigroup theory (see [26,37]), we obtain the following existence and uniqueness results of global solution, i.e., Hadamard well-posedness.

**Theorem 2.1** *Assume that the hypothesis* (H.1)–(H.3) *hold, and* (2.15) *is true, then for system* (2.4). *We have the following existence and uniqueness results:*

1. (*Existence and uniqueness of solution*) *Given $U_0 \in \mathcal{H}$, the abstract equivalent equation* (2.11) *possesses a unique mild solution which generates a strongly continuous semigroup $S(t)$ in the energy space $\mathcal{H}$. The global mild solution can be represented by*

$$U(t) = S(t)U_0 = e^{\mathbb{A}t} (\varphi_0, \varphi_1, \psi_0, \psi_1, f_0(\cdot, -\eta\tau_0))^T . \tag{2.16}$$

2. (*Continuous dependence on initial data*) *If $U_1$ and $U_2$ are two mild solutions of problem* (2.11)*, then there exists a constant $C_0 = C(U_1(0), U_2(0))$, such that for any $T > 0$,*

$$\|U_1(t) - U_2(t)\|_{\mathcal{H}} \le e^{C_0 T} \|U_1(0) - U_2(0)\|_{\mathcal{H}}, \quad \textit{for any } 0 \le t \le T, \tag{2.17}$$

*i.e., the solution is continuously dependent on the initial data and* $U(t) \in C(0, T; \mathcal{H})$.

3. (*Regularity*) *If* $U_0 \in D(\mathbb{A})$*, the above mild solution can be improved to a strong solution.*

***Proof*** We apply Lumer–Philips Theorem, which yields Theorem 2.1.
Dissipativity of operator $\mathbb{A}$: Given $U = (\varphi, u, \psi, v, z)^T$ from $D(\mathbb{A})$, we have

$$\begin{aligned}
\langle \mathbb{A}U, U\rangle_{\mathcal{H}} &= \int_0^1 k(\varphi_{xx} + \psi_x)u + (b\psi_{xx} - k(\varphi_x + \psi) - \mu_1 v - \mu_2 z(\cdot, 1))\, v \\
&\quad + k(u_x + v)(\varphi_x + \psi) + b v_x \psi_x \, dx + \xi \int_0^1 \int_0^1 -\frac{1 - \eta\tau'}{\tau} z_\eta z \, d\eta\, dx \\
&= -\mu_1 \int_0^1 v^2\, dx - \mu_2 \int_0^1 z(x, 1, t) v\, dx + \frac{\xi}{2\tau} \int_0^1 z^2(x, 0, t)\, dx \\
&\quad - \frac{\xi(1 - \tau')}{2\tau} \int_0^1 z^2(x, 1, t)\, dx - \frac{\xi\tau'}{2\tau} \int_0^1 \int_0^1 z^2(x, \eta, t)\, d\eta\, dx \\
&= -\left(\mu_1 - \frac{\xi}{2\tau}\right) \int_0^1 v^2\, dx - \frac{\xi(1 - \tau')}{2\tau} \int_0^1 z^2(x, 1, t)\, dx \\
&\quad - \mu_2 \int_0^1 z(x, 1, t) v\, dx - \frac{\xi\tau'}{2\tau} \int_0^1 \int_0^1 z^2(x, \eta, t)\, d\eta\, dx
\end{aligned} \tag{2.18}$$

By (2.15), we have

$$\mu_2^2 \le 4\frac{\xi(1 - \tau')}{2\tau}\left(\mu_1 - \frac{\xi}{2\tau}\right) \quad \Rightarrow \quad \mu_2 \le 2\sqrt{\frac{\xi(1 - \tau')}{2\tau}\left(\mu_1 - \frac{\xi}{2\tau}\right)} \tag{2.19}$$

Thus,

$$-\left(\mu_1 - \frac{\xi}{2\tau}\right) \int_0^1 v^2\, dx - \frac{\xi(1 - \tau')}{2\tau} \int_0^1 z^2(x, 1, t)\, dx - \mu_2 \int_0^1 z(x, 1, t) v\, dx \le 0 \tag{2.20}$$

Therefore,

$$\langle \mathbb{A}U, U\rangle_{\mathcal{H}} \le -\frac{\xi\tau'}{2\tau} \int_0^1 \int_0^1 z^2(x, \eta, t)\, d\eta\, dx < 0 \tag{2.21}$$

The dissipativity of $\mathbb{A}$ is proved.

Maximality of $I - \mathbb{A}$: we want to show that $I - \mathbb{A}$ is surjective on $D(\mathbb{A}) \to \mathcal{H}$. Given $(f_1, f_2, f_3, f_4, f_5)^T \in \mathcal{H}$, we seek a $U = (\varphi, u, \psi, v, z) \in D(\mathbb{A})$ satisfying

$$\left(I - \mathbb{A}\right)\begin{pmatrix} \varphi \\ u \\ \psi \\ v \\ z \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix} \tag{2.22}$$

That is

$$\varphi - u = f_1 \tag{2.23}$$
$$u - \frac{k}{\rho_1}\big(\varphi_{xx} + \psi_x\big) = f_2 \tag{2.24}$$
$$\psi - v = f_3 \tag{2.25}$$
$$v - \frac{1}{\rho_2}\left(b\psi_{xx} - k(\varphi_x + \psi) - \mu_1 v - \mu_2 z(\cdot, 1)\right) = f_4 \tag{2.26}$$
$$z + \frac{1 - \eta\tau'}{\tau} z_\eta = f_5 \tag{2.27}$$
$$z(\cdot, 0) = v(\cdot) \tag{2.28}$$

By standard ODE theory, we solve (2.27)–(2.28) and obtain that

$$\begin{aligned} z(x,\eta) = f_5(x) + \big[v(x) - f_5(x)\big](1-\tau'\eta)^{\frac{\tau}{\tau'}} &= \psi(x)(1-\tau'\eta)^{\frac{\tau}{\tau'}} \\ &-\big[f_3(x) + f_5(x)\big](1-\tau'\eta)^{\frac{\tau}{\tau'}} + f_5(x) \end{aligned} \tag{2.29}$$

By (2.29), we know

$$z(x, 1) = f_5(x) \tag{2.30}$$

By (2.23) and (2.25), we get $u = \varphi - f_1$ and $v = \psi - f_3$. Plugging these into (2.24) and (2.26), together with (2.30) yields

$$\rho_1\varphi - k(\varphi_{xx} + \psi_x) = \rho_1(f_1 + f_2) \tag{2.31}$$
$$\rho_2\psi - b\psi_{xx} + k(\varphi_x + \psi) + \mu_1\psi = \rho_2(f_3 + f_4) + \mu_1 f_3 + \mu_2 f_5 \tag{2.32}$$

Problem (2.31) can be reformulated as

$$\int_0^1 \rho_1\varphi w_1 - k(\varphi_x + \psi)_x w_1\,dx = \int_0^1 \rho_1(f_1 + f_2)w_1\,dx \quad w_1 \in H_0^1(0,1) \tag{2.33}$$

Problem (2.32) can be reformulated as

$$\begin{aligned} &\int_0^1 \rho_2\psi w_2 - b\psi_{xx}w_2 + k(\varphi_x + \psi)w_2 + \mu_1\psi w_2\,dx \\ &\quad = \int_0^1 \rho_2(f_3 + f_4)w_2 + \mu_1 f_3 w_2 + \mu_2 f_5 w_2\,dx \quad w_2 \in H_0^1(0,1) \end{aligned} \tag{2.34}$$

Choosing the test functions $w_1 = \varphi$ and $w_2 = \psi$ and summing up the left side of (2.33) and (2.34) gives

$$\int_0^1 \rho_1\varphi\varphi - k(\varphi_x + \psi)_x\varphi\,dx + \int_0^1 \rho_2\psi\psi - b\psi_{xx}\psi + k(\varphi_x + \psi)\psi + \mu_1\psi\psi\,dx$$

$$= \int_0^1 \rho_1\varphi^2 + k(\varphi_x + \psi)^2 + \rho_2\psi^2 + b\psi_x^2 + \mu_1\psi^2\,dx \tag{2.35}$$

So the sum of the left side of (2.33) and (2.34) is coercive for $(\varphi, \psi)$ on $H_0^1(0, 1) \times H_0^1(0, 1)$. In addition system (2.31)–(2.32) is linear. Thus there exists a solution $(\varphi, \psi)$ on $H_0^1(0, 1) \times H_0^1(0, 1)$ for system (2.31)–(2.32). Next using (2.23) and (2.25), we are able to solve for $u$ and $v$. Therefore, we have found $(\varphi, u, \psi, v, z) \in D(\mathbb{A})$ which solve system (2.23)–(2.27). The maximality of $I - \mathbb{A}$ is proved.

Thus, by Lummer–Phillips Theorem, We have proved that $\mathbb{A}$ generates a strongly continuous semigroup $S(t)$ in the energy space $\mathcal{H}$. Thus item (1) in Theorem 2.1 holds. The continuous dependence of the solution on initial data can also be obtained. Using the existence theory of global solution for the Cauchy problem for abstract evolutionary equation in [26], we can get item (3) the regularity result Theorem 2.1. □

### 2.3 Finite Dimensional Dynamic Systems: Global and Exponential Attractors

After establishing the Hadamard well-posedness, using the idea of quasi-stability in Lasiecka and Chueshov [9] or [10], we could establish the existence of the finite dimensional global and exponential attractors.

**Theorem 2.2** *Assume* (H.1)–(H.3). *If* $g,\ h \in L^2(0, 1)$*, for any initial condition* $U_0 \in \mathcal{H}$*, then*

(1) *The dynamical system* $(S(t), \mathcal{H})$ *generated by the abstract system* (2.11) *has a compact finite dimensional global attractor* $\mathcal{A}$ *in* $\mathcal{H}$.
(2) *The global attractor* $\mathcal{A}$ *in* $\mathcal{H}$ *has the structure:*

$$\mathcal{A} = M_+(\mathcal{N}) \tag{2.36}$$

*where* $\mathcal{N} = \{y \in \mathcal{H},\ S(t)y = y\}$ *for all* $t > 0$ *is the set of stationary points and* $M_+(\mathcal{N})$ *is the unstable manifold from the set emanating from the set* $\mathcal{N}$.
(3) *Moreover, the gradient system has a generalized exponential attractor* $\mathcal{A}^{exp} \subset \mathcal{H}$ *with finite fractal dimension in* $\mathcal{H}$.

***Proof*** These results are established in Sect. 3 via deriving existence of the absorbing set and verifying quasi-stability for the semigroup. □

## 3 Proof of Main Result

### 3.1 Preliminaries: The Preliminary Theory of Quasi-Stability and Attractors

The existence of a global attractor requires three sufficient properties: continuity property (continuous-semigroup), dissipative property (absorbing set) and compactness property (asymptotic compactness). In this section, we will first briefly review the basic definitions and theory of global attractor and then review the quasi-stability theory as shown in Lasiecka and Chueshov [9,10]. More details could be found in the original papers [9,10].

- **Some Definitions:**

We shall give some preliminary definitions.

**Definition 3.1** (a) (Dissipation) A set $B_0 \subset X$ is called an absorbing set for the semigroup $S(t)$ $(t \geq 0)$ if for any bounded set $B \subset X$ there exists a time $t_1 = t_1(B) > 0$ such that for all $t > t_1$, $S(t)B \subseteq B_0$.

(b) (Asymptotic smoothness) The semigroup $S(t)$ $(t \geq 0)$ is said to be asymptotically smooth in $X$ if for any closed bounded subset $B \subset X$ satisfying $S(t)B \subset B$, there exists a nonempty compact set $K = K(B) \subset X$ such that $dist(S(t)B, K(B)) \to 0$ as $t \to \infty$.

(c) (Asymptotic compactness) A dynamical system $(X, S(t))$ is asymptotically compact if for any bounded set $B \subset X$, and sequence $\{x_k\} \subset B$, the sequence $\{S(t_k)x_k\}$ has convergent subsequence as $t_k \to \infty$.

**Definition 3.2** A compact set $\mathcal{A} \subset X$ is called a global attractor of the semigroup $S(t)$ if

(i) $\mathcal{A}$ is strictly invariant with respect to $S(t)$, i.e., for all $t \geq 0$, $S(t)\mathcal{A} = \mathcal{A}$

(ii) $\mathcal{A}$ attracts any bounded set $B \subset X$: for any $\varepsilon > 0$ there exists a time $t_1 = t_1(\varepsilon, B) > 0$ such that for all $t \geq t_1(\varepsilon, B)$, $S(t)B \subseteq \mathcal{O}_\varepsilon(\mathcal{A})$, where $\mathcal{O}_\varepsilon(\mathcal{A})$ is an $\varepsilon$-neighborhood of $\mathcal{A}$ in $X$.

**Definition 3.3** Given a compact set $M$ in a metric space $X$, the fractal dimension of $M$ is defined by

$$\dim_f^X M = \limsup_{\varepsilon \to 0} \frac{\ln N(M, \varepsilon)}{\ln(1/\varepsilon)},$$

where $N(M, \varepsilon)$ is the minimal number of closed balls with radius $\varepsilon > 0$ which cover $M$.

- **Quasi-Stability and Global Attractors:**

**Definition 3.4** $(X, S(t))$ is called a gradient system if it admits a strict Lyapunov function, i.e., a functional $\Phi : X \to \mathbb{R}$ is a strict Lyapunov function for the system $(X, S(t))$ if

(i) the map $t \to \Phi(S(t)z)$ is non-increasing for any $z \in X$;

(ii) if $\Phi(S(t)z) = \Phi(z)$ for all $t$, then $z$ is a stationary point of $S(t)$.

**Definition 3.5** The continuous semigroup generated by a dynamic system (or gradient system) possesses a global attractor $\mathcal{A}$ if

(1) there exists an absorbing set for semigroup,

(2) the semigroup (or gradient system) is asymptotically smooth or compact.

The asymptotic smoothness and compactness of semigroup are difficult to verify, so some other criteria, such as condition-(C) method, contractive function technique, quasi-stability, are used instead. In this article, we apply the quasi-stability theory, which we briefly review below.

**Definition 3.6** The unstable manifold $M_+(\mathcal{N})$ is defined as the family of $y \in X$ such that there exists a full trajectory $u(t)$ satisfying

$$u(0) = y, \quad \text{and} \quad \lim_{t \to -\infty} dist_X(u(t), \mathcal{N}) = 0, \tag{3.1}$$

here $\mathcal{N}$ is the set of equilibrium for $S(t)$.

**Theorem 3.1** (See Chueshov and Lasiecka [10]) *Assume that the gradient system $(S(t), X)$ with corresponding Lyapunov functional $\Phi$ is asymptotically compact. Moreover, assume that*

(I) $\Phi(S(t)z) \to \infty$ *if and only if* $\|z\|_X \to \infty$,
(II) *the set of equilibrium $\mathcal{N}$ is bounded.*

*Then, the gradient system $(S(t), X)$ possesses a compact global attractor $\mathcal{A} \subset X$ which has the structure $\mathcal{A} = M_+(\mathcal{N})$.*

**Definition 3.7** (See Chueshov and Lasiecka [9,10]) The dynamical system $(S(t), X)$ is quasi-stable on a set $B \subset X$ if there exists a compact semi-norm $n_Y$ on $Y$, the subspace of $X$ and nonnegative scalar functions $a(t)$ and $c(t)$, locally bounded on $[0, \infty)$ and $b(t) \in L^1(\mathbb{R}^+)$ with $\lim_{t \to \infty} b(t) = 0$, such that for $U_1,\ U_2 \in B$

$$\|S(t)U_1 - S(t)U_2\|_X^2 \le a(t)\|U_1 - U_2\|_X^2, \tag{3.2}$$

$$\|S(t)U_1 - S(t)U_2\|_X^2 \le b(t)\|U_1 - U_2\|_X^2 + c(t) \sup_{0<s<t} \left[n_Y(y_1(s) - y_2(s))\right]^2. \tag{3.3}$$

Inequality (3.3) is usually called stabilizability inequality.

**Theorem 3.2** (See Chueshov and Lasiecka [9,10]) *Based on the quasi-stability property of gradient system, we have*

(a) *Let $(X, S(t))$ be a dynamical system and suppose that the system is quasi-stable on every bounded positively invariant set $B \subset X$. Then $(X, S(t))$ is asymptotically compact.*
(b) *Suppose that the dynamical system has a global attractor $\mathcal{A}$ and it is quasi-stable. Then, the global attractor $\mathcal{A}$ has finite fractal dimension.*

- **Fractal Dimensional Exponential Attractors:**

Quasi-stability also implies the existence of finite fractal dimensional exponential attractors.

**Theorem 3.3** (See Chueshov and Lasiecka [9,10]) *Assume* $(X, S(t))$ *is a dissipative dynamical system satisfying quasi-stable property on some bounded absorbing set* $\mathcal{B}$*, and there exists an external space* $\tilde{X}$ *with* $X \subset \tilde{X}$*, such that for every* $T > 0$*,*

$$\|S(t_1)y - S(t_2)y\|_{\tilde{X}} \leq C_{\mathcal{B}T}|t_1 - t_2|^{\eta},\ t_1, t_2 \in [0, T],\ y \in \mathcal{B}, \tag{3.4}$$

*where* $C_{\mathcal{B}T}$ *and* $\eta \in (0, 1]$ *are positive constants. Then this system has a generalized finite fractal dimensional exponential attractor* $\mathcal{A}^{exp}$ *in* $\tilde{X}$*.*

### 3.2 Dissipation: Lyapunov Functional and Existence of Absorbing Set

In this section, we shall use multiplier method to establish the existence of absorbing set for our semigroup.

**Step 1: The estimate of energy functional** $E(t)$.
We define the energy functional for system (2.4)–(2.6) as

$$\begin{aligned} E(t) =& \frac{1}{2}\int_0^1 \left[\rho_1\varphi_t^2 + \rho_2\psi_t^2 + k(\varphi_x + \psi)^2 + b\psi_x^2\right] dx \\ &+ \frac{\xi}{2}\int_0^1\int_0^1 z^2(x, \eta, t)\, d\eta\, dx + \int_0^1 \hat{f}(\psi(t))\, dx - \int_0^1 (h\varphi + g\psi)\, dx, \end{aligned} \tag{3.5}$$

where $\xi$ satisfies (2.15). Moreover, we define $\hat{f}(\psi(t)) = \int_0^{\psi(t)} f(z)\, dz$ throughout the remaining of the paper.

**Lemma 3.4** *The energy functional satisfies the following estimate*

$$\begin{aligned} E(t) \geq& C_E\left(\int_0^1 \varphi_t^2\, dx + \int_0^1 \psi_t^2\, dx + \int_0^1 |\varphi_x + \psi|^2\, dx + \int_0^1 \psi_x^2\, dx \right. \\ &\left. + \int_0^1 \hat{f}(\psi(t))\, dx\right) + \frac{\xi}{2}\int_0^1\int_0^1 z^2(x, \eta, t)\, d\eta\, dx - C_{h,g}\big(\|h\|_2^2 + \|g\|_2^2\big). \end{aligned} \tag{3.6}$$

***Proof*** See, e.g., Feng and Yang [14]. □

**Lemma 3.5** *The derivative of the energy functional satisfies the following estimate*

$$E'(t) \leq -C\int_0^1 \psi_t^2(x, t)\, dx - C\int_0^1 z^2(x, 1, t)\, dx - C\int_0^1\int_0^1 z^2(x, \eta, t)\, d\eta\, dx. \tag{3.7}$$

***Proof***

$$\frac{dE(t)}{dt} = \int_0^1 [\rho_1\varphi_{tt}\varphi_t + \rho_2\psi_{tt}\psi_t + k(\varphi_x+\psi)(\varphi_{xt}+\psi_t) + b\psi_x\psi_{xt}]\,dx$$
$$+\xi\int_0^1\int_0^1 z(x,\eta,t)z_t(x,\eta,t)\,d\eta\,dx + \int_0^1 f(\psi(t))\psi'(t)\,dx$$
$$-\int_0^1 (h\varphi_t + g\psi_t)\,dx. \tag{3.8}$$

Since $\rho_1\varphi_{tt} = k(\varphi_x+\psi)_x + h$, $\rho_2\psi_{tt} = b\psi_{xx} - k(\varphi_x+\psi) - \mu_1\psi_t - \mu_2 z(x,1,t) - f(\psi(t)) + g$, thus it follows

$$\int_0^1 [\rho_1\varphi_{tt}\varphi_t + \rho_2\psi_{tt}\psi_t]\,dx$$
$$= \int_0^1 [k(\varphi_x+\psi)_x + h]\varphi_t + [b\psi_{xx} - k(\varphi_x+\psi) - \mu_1\psi_t - \mu_2 z_t(x,1,t)$$
$$-f(\psi(t)) + g]\psi_t\,dx$$
$$= \int_0^1 -k(\varphi_x+\psi)\varphi_{xt} - b\psi_x\psi_{xt} - k(\varphi_x+\psi)\psi_t - \mu_1\psi_t^2 - \mu_2 z(x,1,t)\psi_t\,dx$$
$$+\int_0^1 h\varphi_t + g\psi_t\,dx - \int f(\psi(t))\psi_t\,dx$$
$$= \int_0^1 -k(\varphi_x+\psi)(\varphi_{xt}+\psi_t) - b\psi_x\psi_{xt}\,dx + \int_0^1 h\varphi_t + g\psi_t\,dx$$
$$-\int f(\psi(t))\psi_t\,dx - \mu_1\int_0^1 \psi_t^2\,dx - \mu_2\int_0^1 z(x,1,t)\psi_t\,dx. \tag{3.9}$$

Moreover,

$$\int_0^1\int_0^1 z(x,\eta,t)z_t(x,\eta,t)\,d\eta\,dx$$
$$= -\int_0^1\int_0^1 \frac{1-\eta\tau'}{\tau} z(x,\eta,t)z_\eta(x,\eta,t)\,d\eta\,dx$$
$$= -\frac{1}{2\tau}\int_0^1 z^2(x,1,t) - z^2(x,0,t)\,dx + \frac{\tau'}{2\tau}\int_0^1\int_0^1 \eta\frac{\partial}{\partial\eta}z^2(x,\eta,t)d\eta\,dx$$
$$= -\frac{1}{2\tau}\int_0^1 z^2(x,1,t) - z^2(x,0,t)\,dx + \frac{\tau'}{2\tau}\int_0^1 z^2(x,1,t)\,dx$$
$$-\frac{\tau'}{2\tau}\int_0^1\int_0^1 z^2(x,\eta,t)d\eta\,dx$$

$$= \frac{1}{2\tau}\int_0^1 z^2(x,0,t)\,dx - \frac{1-\tau'}{2\tau}\int_0^1 z^2(x,1,t)\,dx$$
$$-\frac{\tau'}{2\tau}\int_0^1\int_0^1 z^2(x,\eta,t)d\eta\,dx. \tag{3.10}$$

Plugging the above results into (3.8), we get

$$\frac{dE(t)}{dt} = -\mu_1\int_0^1 \psi_t^2\,dx - \mu_2\int_0^1 z(x,1,t)\psi_t\,dx + \frac{\xi}{2\tau}\int_0^1 z^2(x,0,t)\,dx$$
$$-\frac{\xi(1-\tau')}{2\tau}\int_0^1 z^2(x,1,t)\,dx - \frac{\xi\tau'}{2\tau}\int_0^1\int_0^1 z^2(x,\eta,t)d\eta\,dx$$
$$\le \left(-\mu_1 + \frac{\mu_2}{2} + \frac{\xi}{2\tau}\right)\int_0^1 \psi_t^2\,dx + \left(-\frac{\xi(1-\tau')}{2\tau} + \frac{\mu_2}{2}\right)\int_0^1 z^2(x,1,t)\,dx$$
$$-\frac{\xi\tau'}{2\tau}\int_0^1\int_0^1 z^2(x,\eta,t)d\eta\,dx. \tag{3.11}$$

Using the condition (2.15), we can derive (3.7). □

**Step 2: Estimating the perturbed energy functional $\mathcal{L}$**
The perturbed energy functional is defined as

$$\mathcal{L} = ME(t) + \frac{1}{8}I_1(t) + NI_2(t) + J(t) + \frac{\varepsilon}{k}\int_0^1 \rho_1 q\varphi_t\varphi_x\,dx + \frac{\rho_2 b}{4\varepsilon}\int_0^1 q\psi_t\psi_x\,dx + I_3(t), \tag{3.12}$$

where

$$I_1(t) = -\int_0^1 \rho_1\varphi\varphi_t + \rho_2\psi\psi_t\,dx - \frac{\mu_1}{2}\int_0^1 \psi^2\,dx, \tag{3.13}$$
$$I_2(t) = \int_0^1 (\rho_2\psi_t\psi + \rho_1\varphi_t\widehat{\psi})\,dx + \frac{\mu_1}{2}\int_0^1 \psi^2\,dx, \tag{3.14}$$
$$J(t) = \rho_2\int_0^1 \psi_t(\varphi_x + \psi)\,dx + \rho_2\int_0^1 \psi_x\varphi_t\,dx, \tag{3.15}$$
$$I_3(t) = \int_0^1\int_0^1 e^{-2\tau\eta}z^2(x,\eta,t)\,d\eta\,dx. \tag{3.16}$$

**Lemma 3.6** *Let $(\varphi, \varphi_t, \psi, \psi_t, z)$ be the solution to system (2.4)–(2.6), then the auxiliary functional $I_1$ satisfies the following estimate: for any $\varepsilon > 0$*

$$\frac{d}{dt}I_1(t) \le -\int_0^1 (\rho_1\varphi_t^2 + \rho_2\psi_t^2)\,dx + (k + 2\lambda_1\varepsilon)\int_0^1 (\varphi_x + \psi)^2\,dx$$
$$+\left(b + \mu_2\varepsilon + 3\varepsilon + \frac{1}{\varepsilon}\right)\int_0^1 \psi_x^2\,dx + \frac{\mu_2}{4\lambda_1\varepsilon}\int_0^1 z^2(x,1,t)\,dx + \frac{1}{4\lambda_1^2\varepsilon}\int_0^1 h^2\,dx + \frac{1}{4\lambda_1\varepsilon}\int_0^1 g^2\,dx$$
$$+\frac{\varepsilon}{4\lambda_1}\|\psi\|_{\theta+1}^{\theta+1}, \tag{3.17}$$

*where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ in $H_0^1(0,1)$.*

***Proof*** Differentiating $I_1$, using Green's first identity together with the zero boundary condition for $\varphi$ and $\psi$, we derive that

$$\frac{dI_1}{dt} = -\int_0^1 \rho_1\varphi_{tt}\varphi + \rho_2\psi_{tt}\psi\, dx - \int_0^1 \rho_1\varphi_t^2 + \rho_2\psi_t^2\, dx - \mu_1\int_0^1 \psi\psi_t\, dx. \tag{3.18}$$

By virtue of (2.4), we have

$$\begin{aligned}\frac{dI_1}{dt} &= -\int_0^1 [k(\varphi_x+\psi)_x + h]\varphi + [b\psi_{xx} - k(\varphi_x+\psi) - \mu_1\psi_t - \mu_2 z(x,1,t) \\ &\quad - f(\psi) + g]\psi\, dx - \int_0^1 \rho_1\varphi_t^2 + \rho_2\psi_t^2\, dx - \mu_1\int_0^1 \psi\psi_t\, dx \\ &= -\int_0^1 \rho_1\varphi_t^2 + \rho_2\psi_t^2\, dx + \int_0^1 k(\varphi_x+\psi)^2 + b\psi_x^2\, dx + \int_0^1 \mu_2 z(x,1,t)\psi\, dx \\ &\quad + \int_0^1 f(\psi)\psi\, dx - \int_0^1 h\varphi + g\psi\, dx. \end{aligned} \tag{3.19}$$

By Poincare's and Young's inequality, we have for any $\varepsilon > 0$

$$\begin{aligned}\int_0^1 |z(x,1,t)\psi|\, dx &\le \varepsilon\lambda_1\int_0^1 \psi^2\, dx + \frac{1}{4\varepsilon\lambda_1}\int_0^1 z^2(x,1,t)\, dx \\ &\le \varepsilon\int \psi_x^2\, dx + \frac{1}{4\varepsilon\lambda_1}\int_0^1 z^2(x,1,t)\, dx, \end{aligned} \tag{3.20}$$

$$\begin{aligned}\int_0^1 |f(\psi)\psi|\, dx &\le \int_0^1 |\psi|^\theta|\psi||\psi|\, dx \\ &\le \frac{\varepsilon}{4\lambda_1}\int_0^1 \psi^{\theta+1}\, dx + \frac{\lambda_1}{\varepsilon}\int_0^1 \psi^2\, dx \\ &\le \frac{\varepsilon}{4\lambda_1}\|\psi\|_{\theta+1}^{\theta+1} + \frac{1}{\varepsilon}\int_0^1 \psi_x^2\, dx, \end{aligned} \tag{3.21}$$

$$\begin{aligned}\int_0^1 |(h\varphi + g\psi)|\, dx &\le \lambda_1^2\varepsilon\int_0^1 \varphi^2\, dx + \frac{1}{4\varepsilon\lambda_1^2}\int_0^1 h^2\, dx + \lambda_1\varepsilon\int_0^1 \psi^2\, dx \\ &\quad + \frac{1}{4\varepsilon\lambda_1}\int_0^1 g^2\, dx \\ &\le \lambda_1\varepsilon\int_0^1 \varphi_x^2\, dx + \frac{1}{4\varepsilon\lambda_1^2}\int_0^1 h^2\, dx + \varepsilon\int_0^1 \psi_x^2\, dx \\ &\quad + \frac{1}{4\varepsilon\lambda_1}\int_0^1 g^2\, dx. \end{aligned} \tag{3.22}$$

Also noting that

$$\int_0^1 \varphi_x^2\, dx \le 2\int_0^1 (\varphi_x+\psi)^2\, dx + 2\int_0^1 \psi^2\, dx \tag{3.23}$$

and combining with the above estimates, we get (3.17). □

**Lemma 3.7** *Let $\widehat{\psi}$ be the solution of the following boundary value problem*

$$\widehat{\psi}_{xx} = -\psi_x, \quad \widehat{\psi}(0,t) = \widehat{\psi}(1,t) = 0, \tag{3.24}$$

*then the auxiliary functional $I_2(t)$ satisfies the following estimates for any $\alpha_1 > 0$ and $\alpha_2 > 0$:*

$$\begin{aligned} \frac{dI_2(t)}{dt} \le& \big((\mu_2+3)\alpha_1 - b\big)\int_0^1 \psi_x^2\,dx + (\rho_2+\rho_1\alpha_2)\int_0^1 \psi_t^2\,dx + \frac{\rho_1}{4\lambda_1\alpha_2}\int_0^1 \varphi_t^2\,dx \\ &+ \frac{\mu_2}{4\lambda_1\alpha_1}\int_0^1 z^2(x,1,t)\,dx + \frac{1}{4\lambda_1\alpha_1}\int_0^1 (g^2+h^2)\,dx + \frac{1}{4\lambda_1\alpha_1}\|\psi\|_{\theta+1}^{\theta+1}. \end{aligned} \tag{3.25}$$

***Proof*** Integrating by parts, using Green's first identity and zero boundary condition for $\varphi$ and $\psi$, we derive that

$$\begin{aligned} \frac{dI_2(t)}{dt} =& \int_0^1 (\rho_2\psi_{tt}\psi + \rho_2\psi_t^2 + \rho_1\varphi_{tt}\widehat{\psi} + \rho_1\varphi_t\widehat{\psi}_t)\,dx + \mu_1\int_0^1 \psi\psi_t\,dx \\ =& -b\int_0^1 \psi_x^2\,dx - k\int_0^1 \psi^2\,dx + k\int_0^1 \widehat{\psi}_x^2\,dx - \mu_2\int_0^1 z(x,1,t)\psi\,dx \\ &- \int_0^1 f(\psi)\psi\,dx + \rho_2\int_0^1 \psi_t^2\,dx + \rho_1\int_0^1 \varphi_t\widehat{\psi}_t\,dx + \int_0^1 g\psi + h\widehat{\psi}\,dx. \end{aligned} \tag{3.26}$$

From (3.24), we know that

$$\int_0^1 \widehat{\psi}_x^2\,dx \le \int_0^1 \psi^2\,dx \le \int_0^1 \psi_x^2\,dx. \tag{3.27}$$

Applying Young's inequality and Poincere's inequality, for positive constants $\alpha_1, \alpha_2$,

$$\begin{aligned} \int_0^1 |z(x,1,t)\psi|\,dx \le& \lambda_1\alpha_1\int_0^1 \psi^2\,dx + \frac{1}{4\lambda_1\alpha_1}\int_0^1 z^2(x,1,t)\,dx \\ \le& \alpha_1\int_0^1 \psi_x^2\,dx + \frac{1}{4\lambda_1\alpha_1}\int_0^1 z^2(x,1,t)\,dx. \end{aligned} \tag{3.28}$$

Similarly, we have

$$\int_0^1 |\varphi_t\widehat{\psi}_t|\,dx \le \alpha_2\int_0^1 \psi_t^2\,dx + \frac{1}{4\lambda_1\alpha_2}\int_0^1 \varphi_t^2\,dx, \tag{3.29}$$

$$\int_0^1 |g\psi|\,dx \le \alpha_1\int_0^1 \psi_x^2\,dx + \frac{1}{4\lambda_1\alpha_1}\int_0^1 g^2\,dx, \tag{3.30}$$

$$\int_0^1 |h\widehat{\psi}|\,dx \le \alpha_1 \int_0^1 \widehat{\psi}_x^2\,dx + \frac{1}{4\lambda_1\alpha_1}\int_0^1 h^2\,dx$$
$$\le \alpha_1 \int_0^1 \psi_x^2\,dx + \frac{1}{4\lambda_1\alpha_1}\int_0^1 h^2\,dx, \tag{3.31}$$
$$\int_0^1 |f(\psi)\psi|\,dx \le \alpha_1 \int_0^1 \psi_x^2 + \frac{1}{4\lambda_1\alpha_1}\|\psi\|_{\theta+1}^{\theta+1}. \tag{3.32}$$

Incorporating (3.27)–(3.31) into (3.26), we obtain (3.25). □

The functional $J(t)$ satisfies the following lemma:

**Lemma 3.8** *$J(t)$ satisfies the following estimates:*

$$\frac{dJ(t)}{dt} \le b[\psi_x\varphi_x]_{x=0}^{x=1} - \frac{k}{2}\int_0^1 (\varphi_x+\psi)^2\,dx + \left(\left(1+\frac{\rho_2}{\rho_1}\right)\varepsilon + \frac{k}{8\lambda_1}\right)\int_0^1 \psi_x^2\,dx$$
$$+ \left(\rho_2 + \frac{2\mu_1^2}{k}\right)\int_0^1 \psi_t^2\,dx + \frac{2\mu_2^2}{k}\int_0^1 z^2(x,1,t)\,dx$$
$$+ \frac{2}{k}\int_0^1 g^2\,dx + \frac{\rho_2}{4\rho_1\varepsilon}\int_0^1 h^2\,dx + \frac{4}{k}\|\psi\|_{\theta+1}^{\theta+1}. \tag{3.33}$$

***Proof*** Integrating by parts, using Green's first identity and zero boundary condition for $\varphi$ and $\psi$, we have

$$\frac{d}{dt}J(t) = \rho_2\int_0^1 \psi_{tt}(\varphi_x+\psi)\,dx + \rho_2\int_0^1 \psi_t(\varphi_{xt}+\psi_t)\,dx + \rho_2\int_0^1 \psi_{xt}\varphi_t\,dx$$
$$+\rho_2\int_0^1 \psi_x\varphi_{tt}\,dx. \tag{3.34}$$

Recalling that $\dfrac{\rho_1}{\rho_2} = \dfrac{k}{b}$, we have

$$\frac{dJ(t)}{dt} = b\,[\psi_x\varphi_x]_{x=0}^{x=1} + \rho_2\int_0^1 \psi_t^2\,dx - k\int_0^1 (\varphi_x+\psi)^2\,dx - \mu_1\int_0^1 \psi_t(\varphi_x+\psi)\,dx$$
$$- \mu_2\int_0^1 (\varphi_x+\psi)z(x,1,t)\,dx - \int_0^1 \varphi_x f(\psi)\,dx - \int_0^1 f(\psi)\psi\,dx$$
$$+ \int_0^1 g(\varphi_x+\psi)\,dx + \frac{\rho_2}{\rho_1}\int_0^1 h\psi_x\,dx. \tag{3.35}$$

By Young's inequality and Poincare's inequality, we have

$$\begin{aligned}\int_0^1 |\varphi_x f(\psi)|\,dx &\le \int_0^1 |\varphi_x||\psi|^\theta|\psi|\,dx \le \frac{k}{16}\int_0^1 \varphi_x^2\,dx + \frac{4}{k}\|\psi\|_{\theta+1}^{\theta+1}\\ &\le \frac{k}{8}\int_0^1(\varphi_x+\psi)^2\,dx + \frac{k}{8}\int_0^1 \psi^2\,dx + \frac{4}{k}\|\psi\|_{\theta+1}^{\theta+1}\\ &\le \frac{k}{8}\int_0^1(\varphi_x+\psi)^2\,dx + \frac{k}{8\lambda_1}\int_0^1 \psi_x^2\,dx + \frac{4}{k}\|\psi\|_{\theta+1}^{\theta+1} \qquad (3.36)\end{aligned}$$

and

$$\int_0^1 |\psi_t(\varphi_x+\psi)|\,dx \le \frac{k}{8\mu_1}\int_0^1(\varphi_x+\psi)^2\,dx + \frac{2\mu_1}{k}\int_0^1 \psi_t^2\,dx, \qquad (3.37)$$

$$\int_0^1 |(\varphi_x+\psi)z(x,1,t)|\,dx \le \frac{k}{8\mu_2}\int_0^1(\varphi_x+\psi)^2\,dx + \frac{2\mu_2}{k}\int_0^1 z^2(x,1,t)\,dx, \qquad (3.38)$$

$$\int_0^1 |g(\varphi_x+\psi)|\,dx \le \frac{k}{8}\int_0^1(\varphi_x+\psi)^2\,dx + \frac{2}{k}\int_0^1 g^2\,dx, \qquad (3.39)$$

$$\int_0^1 |h\psi_x|\,dx \le \varepsilon\int_0^1 \psi_x^2\,dx + \frac{1}{4\varepsilon}\int_0^1 h^2\,dx. \qquad (3.40)$$

Incorporate (3.36)–(3.40) and (3.21) into (3.34), we will obtain estimate (3.33) in Lemma 3.8. □

We now need to deal with the boundary term $b\,[\psi_x\varphi_x]_{x=0}^{x=1}$. Setting

$$q(x) = -4x + 2, \quad x \in (0,1), \qquad (3.41)$$

then $|q(x)| \le 2$. Hence, the following lemma is obtained.

**Lemma 3.9** *$b\,[\psi_x\varphi_x]_{x=0}^{x=1}$ satisfies the following estimates*

$$\begin{aligned}b[\psi_x\varphi_x]_{x=0}^{x=1} \le &-\frac{\rho_1\varepsilon}{k}\frac{d}{dt}\int_0^1 q\varphi_t\varphi_x\,dx - \frac{b\rho_2}{4\varepsilon}\frac{d}{dt}\int_0^1 q\psi_t\psi_x\,dx\\ &+\left(\frac{k^2\varepsilon}{4}+8\varepsilon\right)\int_0^1(\varphi_x+\psi)^2\,dx\\ &+\left(\varepsilon\left(\frac{8}{\lambda_1}+1\right)+\frac{b^2}{2\varepsilon}+\frac{b^2}{\varepsilon^3}\right)\int_0^1 \psi_x^2\,dx + \frac{2\rho_1\varepsilon}{k}\int_0^1 \varphi_t^2\,dx\\ &+\left(\frac{\rho_2 b}{2\varepsilon}+\frac{\mu_1^2}{4}\right)\int_0^1 \psi_t^2\,dx + \frac{\mu_2^2}{2}\int_0^1 z^2(x,1,t)\,dx + \frac{\varepsilon}{k^2}\int_0^1 h^2\,dx + \frac{1}{4}\int_0^1 g^2\,dx.\end{aligned} \qquad (3.42)$$

***Proof*** Firstly, it is obvious to obtain the following estimate:

$$b\,[\psi_x\varphi_x]_{x=0}^{x=1} \le \varepsilon[\varphi_x^2(1)+\varphi_x^2(0)] + \frac{b^2}{4\varepsilon}[\psi_x^2(1)+\psi_x^2(0)]. \qquad (3.43)$$

Next we consider

$$\frac{d}{dt}\int_0^1 b\rho_2 q\psi_t\psi_x\,dx = \int_0^1 b\rho_2 q\psi_{tt}\psi_x\,dx + \int_0^1 b\rho_2 q\psi_t\psi_{xt}\,dx, \tag{3.44}$$

where

$$\begin{aligned}
&\int_0^1 b\rho_2 q\psi_{tt}\psi_x\,dx \\
&\quad = \int_0^1 b^2 q\psi_x\psi_{xx}\,dx - bk\int_0^1 q\psi_x(\varphi_x+\psi)\,dx \\
&\qquad - b\mu_1\int_0^1 q\psi_x\psi_t\,dx - b\mu_2\int_0^1 q\psi_x z(x,1,t)\,dx \\
&\qquad - b\int_0^1 q\psi_x f(\psi)\,dx + b\int_0^1 q\psi_x g\,dx \\
&\quad \le -b^2[\psi_x^2(1)+\psi_x^2(0)] + 2b^2\int_0^1 \psi_x^2\,dx + \varepsilon^2 k^2\int_0^1(\varphi_x+\psi)^2\,dx + \frac{b^2}{\varepsilon^2}\int_0^1\psi_x^2\,dx \\
&\qquad + \frac{3b^2}{\varepsilon}\int_0^1 \psi_x\,dx + \varepsilon\mu_1^2\int_0^1\psi_t^2\,dx + \varepsilon\mu_2^2\int_0^1 z(x,1,t)^2\,dx + \varepsilon\int_0^1 g^2\,dx.
\end{aligned} \tag{3.45}$$

Furthermore, because $\psi(0,t)=\psi(1,t)\equiv 0$, we have

$$\int_0^1 b\rho_2 q\psi_t\psi_{xt}\,dx = \frac{1}{2}b\rho_2 q(x)\psi_t^2(x)|_{x=0}^{x=1} + 2b\rho_2\int_0^1\psi_t^2\,dx = 2b\rho_2\int_0^1\psi_t^2\,dx. \tag{3.46}$$

Thus,

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 b\rho_2 q\psi_t\psi_x\,dx \\
&\quad \le -b^2[\psi_x^2(0)+\psi_x^2(1)] + 2b^2\int_0^1\psi_x^2\,dx + 2\rho_2 b\int_0^1\psi_t^2\,dx + \frac{4b^2}{\varepsilon}\int_0^1\psi_x^2\,dx \\
&\qquad + \varepsilon k^2\int_0^1(\varphi_x+\psi)^2\,dx + \mu_1^2\varepsilon\int_0^1\psi_t^2\,dx + \mu_2^2\varepsilon\int_0^1 z(x,1,t)^2\,dx + \varepsilon\int_0^1 g^2\,dx.
\end{aligned} \tag{3.47}$$

Similarly, we have

$$\begin{aligned}
\frac{d}{dt}\int_0^1 \rho_1 q\varphi_t\varphi_x\,dx &= \int_0^1 \rho_1 q\varphi_{tt}\varphi_x\,dx + \int_0^1 \rho_1 q\varphi_t\varphi_{xt}\,dx \\
&= \int_0^1 kq\varphi_x\varphi_{xx}\,dx + \int_0^1 kq\psi_x\varphi_x\,dx + \int_0^1 qh\varphi_x\,dx \\
&\quad + \int_0^1 \rho_1 q\varphi_t\varphi_{xt}\,dx
\end{aligned}$$

$$
\begin{aligned}
&\leq -k[\varphi_x^2(0)+\varphi_x^2(1)] + 4k\int_0^1 \varphi_x^2\,dx + k\int_0^1 \psi_x^2\,dx \\
&\quad + 2\rho_1\int_0^1 \varphi_t^2\,dx + \frac{1}{k}\int_0^1 h^2\,dx \\
&\leq -k[\varphi_x^2(0)+\varphi_x^2(1)] + 8k\int_0^1 (\varphi_x+\psi)^2\,dx + k\left(\frac{8}{\lambda_1}+1\right)\int_0^1 \psi_x^2\,dx \\
&\quad + 2\rho_1\int_0^1 \psi_t^2\,dx + \frac{1}{k}\int_0^1 h^2\,dx. \qquad (3.48)
\end{aligned}
$$

Combining (3.47) and (3.48), we obtain

$$
\begin{aligned}
&\varepsilon[\varphi_x^2(0)+\varphi_x^2(1)] + \frac{b^2}{4\varepsilon}[\psi_x^2(0)+\psi_x^2(1)] \\
&\leq -\frac{\rho_1\varepsilon}{k}\frac{d}{dt}\int_0^1 q\varphi_t\varphi_x\,dx + 8\varepsilon\int_0^1 (\varphi_x+\psi)^2\,dx + \varepsilon\left(\frac{8}{\lambda_1}+1\right)\int_0^1 \psi_x^2\,dx \\
&\quad + \frac{2\rho_1\varepsilon}{k}\int_0^1 \varphi_t^2\,dx + \frac{\varepsilon}{k^2}\int_0^1 h^2\,dx - \frac{b\rho_2}{4\varepsilon}\frac{d}{dt}\int_0^1 q\psi_t\psi_x\,dx + \frac{b^2}{2\varepsilon}\int_0^1 \psi_x^2\,dx + \frac{\rho_2 b}{2\varepsilon}\int_0^1 \psi_t^2\,dx + \frac{b^2}{\varepsilon^3}\int_0^1 \psi_x^2\,dx \\
&\quad + \frac{k^2\varepsilon}{4}\int_0^1 (\varphi_x+\psi)^2\,dx + \frac{\mu_1^2}{4}\int_0^1 \psi_t^2\,dx + \frac{\mu_2^2}{4}\int_0^1 z^2(x,1,t)\,dx + \frac{1}{4}\int_0^1 g^2\,dx, \qquad (3.49)
\end{aligned}
$$

plugging (3.49) into (3.43), we obtain the estimate (3.42). □

**Lemma 3.10** *The functional $I_3$ satisfies the following estimate:*

$$
\frac{d}{dt}I_3(t) \leq -2I_3 - \frac{c_\tau}{\tau}\int_0^1 z^2(x,1,t)\,dx + \frac{1}{\tau}\int_0^1 \psi_t^2\,dx. \qquad (3.50)
$$

***Proof*** Differentiating $I_3$, and then using integration by parts, we have

$$
\begin{aligned}
\frac{dI_3}{dt} &= \frac{d}{dt}\int_0^1\int_0^1 e^{-2\tau\eta}z^2(x,\eta,t)\,d\eta\,dx \\
&= \int_0^1\int_0^1 (-2\tau'\eta)e^{-2\tau\eta}z^2(x,\eta,t)\,d\eta\,dx + \int_0^1\int_0^1 e^{-2\tau\eta}2zz_t\,d\eta\,dx \\
&= \int_0^1\int_0^1 (-2\tau'\eta)e^{-2\tau\eta}z^2(x,\eta,t)\,d\eta\,dx + \int_0^1\int_0^1 \frac{\eta\tau'-1}{\tau}e^{-2\tau\eta}2zz_\eta\,d\eta\,dx \\
&= -2\tau'\int_0^1\int_0^1 \eta e^{-2\tau\eta}z^2\,d\eta\,dx + \frac{\tau'}{\tau}\int_0^1\int_0^1 \eta e^{-2\tau\eta}\,dz^2\,dx \\
&\quad - \frac{1}{\tau}\int_0^1\int_0^1 e^{-2\eta\tau}\,dz^2\,dx, \qquad (3.51)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\tau'}{\tau}\int_0^1\int_0^1 \eta e^{-2\tau\eta}\,dz^2\,dx &= \frac{\tau'}{\tau}\int_0^1\int_0^1 \eta e^{-2\tau\eta}z^2\Big|_{\eta=0}^{\eta=1}dx \\
&\quad - \frac{\tau'}{\tau}\int_0^1\int_0^1 (1-2\tau\eta)e^{-2\tau\eta}z^2\,d\eta\,dx, \qquad (3.52)
\end{aligned}
$$

$$\frac{1}{\tau}\int_0^1\int_0^1 e^{-2\tau\eta}\,dz^2\,dx = \frac{1}{\tau}\left(\int_0^1 e^{-2\tau\eta}z^2\Big|_{\eta=0}^{\eta=1}dx - \int_0^1\int_0^1 (-2\tau)e^{-2\tau\eta}z^2\,d\eta\,dx\right). \tag{3.53}$$

The above three equations yield

$$\begin{aligned}\frac{dI_3}{dt} &= \frac{d}{dt}\int_0^1\int_0^1 e^{-2\tau\eta}z^2(x,\eta,t)\,d\eta\,dx\\ &= \int_0^1\int_0^1 -2\tau'\eta e^{-2\tau\eta}z^2\,d\eta\,dx + \frac{\tau'}{\tau}\int_0^1 e^{-2\tau}z^2(x,1,t)\,dx\\ &\quad - \frac{\tau'}{\tau}\int_0^1\int_0^1 e^{-2\tau\eta}z^2\,d\eta\,dx + \int_0^1\int_0^1 2\tau'\eta e^{-2\tau\eta}z^2\,d\eta\,dx - \frac{1}{\tau}\int_0^1 e^{-2\tau}z^2(x,1,t) - \psi_t^2(x,t)\,dx\\ &\quad - 2\int_0^1\int_0^1 e^{-2\tau\eta}z^2\,d\eta\,dx = \frac{\tau'-1}{\tau}\int_0^1 e^{-2\tau}z^2(x,1,t)\,dx\\ &\quad - \left(\frac{\tau'}{\tau}+2\right)\int_0^1\int_0^1 e^{-2\tau\eta}z^2\,d\eta\,dx + \frac{1}{\tau}\int_0^1 \psi_t^2(x,t)\,dx,\end{aligned} \tag{3.54}$$

which implies that (3.50) holds. □

**Step 3: Using the perturbed energy functional $\mathcal{L}$ to control $E(t)$.** By some delicate estimates, we could establish the following lemma.

**Lemma 3.11** *For $M$ large enough, there exists two positive constants $\gamma_1$ and $\gamma_2$, depending on $M$, $N$, and $\varepsilon$ such that for any $t \geq 0$,*

$$\gamma_1 E(t) - C_1(\|h\|^2 + \|g\|^2) \leq \mathcal{L}(t) \leq \gamma_2 E(t) + C_1(\|h\|^2 + \|g\|^2). \tag{3.55}$$

***Proof*** Integrating by parts, we have

$$\begin{aligned}\left|\mathcal{L}(t) - ME(t)\right| &\leq \frac{1}{8}\left|\int_0^1 \rho_1\varphi\varphi_t + \rho_2\psi\psi_t\,dx + \frac{\mu_1}{2}\int_0^1 \psi^2\,dx\right|\\ &\quad + N\left|\int_0^1 (\rho_2\psi_t\psi + \rho_1\varphi_t\widehat{\psi})\,dx + \frac{\mu_1}{2}\int_0^1 \psi^2\,dx\right|\\ &\quad + \left|\rho_2\int_0^1 \psi_t(\varphi_x+\psi)\,dx + \rho_2\int_0^1 \psi_x\varphi_t\,dx\right| + \frac{\varepsilon}{k}\left|\int_0^1 \rho_1 q\varphi_t\varphi_x\,dx\right|\\ &\quad + \frac{\rho_2 b}{4\varepsilon}\left|\int_0^1 q\psi_t\psi_x\,dx\right| + \left|\int_0^1\int_0^1 z^2(x,\eta,t)\,d\eta\,dx\right|.\end{aligned} \tag{3.56}$$

Noting that

$$\int_0^1 \varphi^2\,dx \leq \int_0^1 \varphi_x^2\,dx \leq 2\int_0^1 (\varphi_x+\psi)^2\,dx + 2\int_0^1 \psi_x^2\,dx, \quad \int_0^1 \psi^2\,dx \leq \int_0^1 \psi_x^2\,dx, \tag{3.57}$$

we have

$$|\mathcal{L}(t) - ME(t)| \le \beta_1 \int_0^1 \varphi_t^2\,dx + \beta_2 \int_0^1 \psi_t^2\,dx + \beta_3 \int_0^1 (\varphi_x + \psi)^2\,dx + \beta_4 \int_0^1 \psi_x^2\,dx + \int_0^1\int_0^1 z^2(x,\eta,t)\,d\eta\,dx, \tag{3.58}$$

where

$$\begin{cases} \beta_1 = \frac{1}{16}\rho_1 + \frac{1}{2}\rho_1 N + \frac{1}{2}\rho_2 + \frac{\varepsilon}{k}, \\ \beta_2 = \frac{1}{16}\rho_2 + \frac{1}{2}\rho_2 N + \frac{1}{2}\rho_2 + \frac{\rho_2 b}{4\varepsilon}, \\ \beta_3 = \frac{1}{8}\rho_1 + \frac{1}{2}\rho_2 + \frac{2\varepsilon}{k}, \\ \beta_4 = (\frac{1}{16} + \frac{N}{2})\rho_2 + (\frac{1}{16} + \frac{N}{2})\mu_1 + \frac{1}{2}\rho_1 N + \frac{\rho_2}{2} + \frac{\rho_2 b}{4\varepsilon}. \end{cases} \tag{3.59}$$

On the other hand, by Lemma 3.4, we have

$$E(t) \ge \min\{C_E, \xi\}\left(\int_0^1 \varphi_t^2\,dx + \int_0^1 \psi_t^2\,dx + \int_0^1 (\varphi_x + \psi)^2\,dx + \int_0^1 \psi_x^2\,dx + \int_0^1\int_0^1 z^2(x,\eta,t)d\eta\,dx + \int_0^1 \hat{f}(\psi)\,dx\right) - C_{h,g}(\|h\|^2 + \|g\|^2). \tag{3.60}$$

Thus, combining (3.58) with (3.60), gives that there exists $\gamma_1$ and $\gamma_2$,

$$\gamma_1 E(t) - C_1(\|h\|^2 + \|g\|^2) \le \mathcal{L}(t) \le \gamma_2 E(t) + C_1(\|h\|^2 + \|g\|^2), \tag{3.61}$$

This finishes the proof of Lemma 3.11. □

**Step 4: Proof of dissipation-Existence of absorbing set.**
By Lemmas 3.5, 3.6, 3.7, 3.8, 3.9, 3.10, and 3.11, we have the following estimates:

$$\frac{d}{dt}\mathcal{L}(t) \le \left[-MC - \frac{\rho_2}{8} + N(\rho_2 + \rho_1\alpha_2) + \left(\rho_2 + \frac{2\mu_1^2}{k}\right) + \left(\frac{\rho_2 b}{2\varepsilon} + \frac{\mu_1^2}{4}\right) + \frac{1}{\tau}\right)\right] \times \int_0^1 \psi_t^2\,dx + \left[-MC + \frac{\mu_2}{32\lambda_1\varepsilon} + \frac{N\mu_2}{4\lambda_1\alpha_1} + \frac{2\mu_2^2}{k} + \frac{\mu_2^2}{2} - \frac{c_\tau}{\tau}\right]\int_0^1 z^2(x,1,t)\,dx$$

$$
\begin{aligned}
&+\left[-MC-e^{-2}\right]\int_0^1\int_0^1 z^2(x,\eta,t)\,d\eta\,dx\\
&+\left[-\frac{\rho_1}{8}+\frac{N\rho_1}{4\lambda_1\alpha_2}+\frac{2\rho_1\varepsilon}{k}\right]\int_0^1\varphi_t^2\,dx\\
&+\left[-\frac{3k}{8}+\frac{\lambda_2\varepsilon}{4}+\frac{k^2\varepsilon}{4}+8\varepsilon\right]\int_0^1(\varphi_x+\psi)^2\,dx\\
&+\left[\frac{b+\mu_2\varepsilon+3\varepsilon+\frac{1}{\varepsilon}}{8\lambda_1}+N((\mu_2+2)\alpha_1+\varepsilon-b)+\left(\left(1+\frac{\rho_2}{\rho_1}\right)\varepsilon+\frac{k}{8\lambda_1}\right)\right.\\
&\left.+\left(\varepsilon\left(\frac{8}{\lambda_1}+1\right)+\frac{b^2}{2\varepsilon}+\frac{b^2}{\varepsilon^3}\right)\right]\int_0^1\psi_x^2\,dx\\
&+\left(\frac{1}{32\lambda_1^2\varepsilon}+\frac{N}{4\lambda_1\alpha_1}+\frac{\rho_2}{4\rho_1\varepsilon}+\frac{\varepsilon}{k^2}\right)\int_0^1 h^2\,dx\\
&+\left(\frac{1}{32\lambda_1\varepsilon}+\frac{N}{4\lambda_1\alpha_1}+\frac{2}{k}+\frac{1}{4}\right)\int_0^1 g^2\,dx\\
&+\left(\frac{\varepsilon}{32\lambda_1}+\frac{N}{4\lambda_1\alpha_1}+\frac{4}{k}\right)\|\psi\|_{\theta+1}^{\theta+1}.
\end{aligned}\tag{3.62}
$$

Let the positive constant $\varepsilon < \dfrac{k}{32}$ small enough such that

$$
-\frac{3k}{8}+\frac{\lambda_1\varepsilon}{4}+\frac{k^2\varepsilon}{4}+8\varepsilon<0. \tag{3.63}
$$

Setting $\alpha_1 < \dfrac{b}{2(\mu_2+3)}$ and letting $N$ large enough, it follows

$$
\frac{b+\mu_2\varepsilon+3\varepsilon+\frac{1}{\varepsilon}}{8\lambda_1}+N\big((\mu_2+3)\alpha_1-b\big)+\left(1+\frac{\rho_2}{\rho_1}\right)\varepsilon+\frac{k}{8\lambda_1}+\varepsilon\left(\frac{8}{\lambda_1}+1\right)+\frac{b^2}{2\varepsilon}+\frac{b^2}{\varepsilon^3}<0. \tag{3.64}
$$

Combining with $\varepsilon < \frac{k}{32}$, setting $\alpha_2$ large enough such that

$$
-\frac{\rho_1}{8}+\frac{N\rho_1}{4\lambda_1\alpha_2}+\frac{2\rho_1\varepsilon}{k}<0. \tag{3.65}
$$

Lastly, choosing $M$ large enough, there exists a $\delta$, such that

$$
\begin{aligned}
\frac{d}{dt}\mathcal{L}(t)\le&-\delta\int_0^1\left(\varphi_t^2+\psi_t^2+(\varphi_x+\psi)^2+z^2(x,1,t)+\psi_x^2\right)dx\\
&-\delta\int_0^1\int_0^1 z^2(x,\eta,t)\,d\eta\,dx+C_2\|g\|^2+C_3\|h\|^2+C_4\|\psi\|_{\theta+1}^{\theta+1}.
\end{aligned}\tag{3.66}
$$

Then by Lemma 3.4, there exists a $\sigma > 0$ such that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\sigma E(t) + C_2\|g\|^2 + C_3\|h\|^2 + C_4\|\psi\|_{\theta+1}^{\theta+1}. \tag{3.67}$$

Combining with Lemma 3.11, we have

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{\sigma}{\gamma_2}\mathcal{L}(t) + D_2\|g\|^2 + D_3\|h\|^2 + D_4\|\psi\|_{\theta+1}^{\theta+1}. \tag{3.68}$$

Therefore, it yields

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-\frac{\sigma}{\gamma_2}t} + E_2\|g\|^2 + E_3\|h\|^2 + E_4\|\psi\|_{\theta+1}^{\theta+1}. \tag{3.69}$$

By (3.55) in Lemma 3.11, we have

$$E(t) \leq \frac{1}{\gamma_1}(\gamma_2 E(0) + C_1\|g\|^2 + C_1\|h\|^2)e^{-\frac{\sigma}{\gamma_2}t} + F_2\|g\|^2 + F_3\|h\|^2 + F_4\|\psi\|_{\theta+1}^{\theta+1}. \tag{3.70}$$

That is,

$$\|(\varphi, \varphi_t, \psi, \psi_t, z)\|_{\mathcal{H}}^2 \leq C_0 e^{-\frac{\sigma}{\gamma_2}t} + C_2'\|g\|^2 + C_3'\|h\|^2 + C_4'\|\psi\|_{\theta+1}^{\theta+1}, \tag{3.71}$$

which implies there exists an absorbing ball $B(0, R)$ with radius

$$R = 1 + \sqrt{C_2'\|g\|^2 + C_3'\|h\|^2 + C_4'\|\psi\|_{\theta+1}^{\theta+1}} \tag{3.72}$$

for the dynamical system $(S(t), \mathcal{H})$.

### 3.3 Asymptotic Compactness of Gradient System: Quasi-Stability

Inspired by the idea of Chueshov and Lasiecka [9,10], we only need to verify quasi-stability for the gradient system, which implies asymptotic smoothness for our semigroup.

**Theorem 3.12** *Assume* (H.1)–(H.3) *and* $h$, $g$ *are in* $L_2(0, 1)$, *then there exists functions* $b(t)$ *and* $c(t)$, *such that the semigroup defined in* (2.16) *satisfies the following quasi-stability condition for initial conditions* $U_0^i = (\varphi_0^i, \varphi_1^i, \psi_0^i, \psi_1^i, f_0^i) \in B(0, R)$ *defined in* (3.72):

$$\|S(t)U_0^1 - S(t)U_0^2\|_{\mathcal{H}}^2 \leq b(t)\|U_0^1 - U_0^2\|_{\mathcal{H}}^2 + c(t) \sup_{0<s<t}\left[\|\psi_1(s) - \psi_2(s)\|_{\theta+1}^{\theta+1}\right]^2, \tag{3.73}$$

*where* $\psi_i(t) = S(t)\psi_0^i$, $b(t)$ *and* $c(t)$ *satisfy the conditions in Definition* 3.7.

*Moreover, the dynamical system* $(S(t), \mathcal{H})$ *is quasi-stable on the absorbing set defined in* (3.72).

***Proof*** For any initial condition $(\varphi_0^i, \varphi_1^i, \psi_0^i, \psi_1^i, f_0^i) \in B$, let $(\varphi^i, \varphi_t^i, \psi^i, \psi_t^i, z^i)$ be the corresponding solutions with respect to the initial condition $(\varphi_0^i, \varphi_1^i, \psi_0^i, \psi_1^i, f^i)$, $i = 1, 2$. Letting

$$\begin{aligned} W(t) &:= (\Phi, \Phi_t, \Psi, \Psi_t, \mathcal{Z})^T = (\varphi^1 - \varphi^2, (\varphi^1 - \varphi^2)_t, \psi^1 - \psi^2, (\psi^1 - \psi^2)_t, z^1 - z^2) \\ &= S(t)U_0^1 - S(t)U_0^2 = U^1(t) - U^2(t), \end{aligned} \tag{3.74}$$

then $W(t)$ satisfies

$$\begin{cases} \rho_1 \Phi_{tt}(x,t) - k(\Phi_x + \Psi)_x(x,t) = 0, \\ \rho_2 \Psi_{tt}(x,t) - b\Psi_{xx}(x,t) + k(\Phi_x + \Psi)(x,t), \\ \quad + \mu_1 \Psi_t(x,t) + \mu_2 \mathcal{Z}(x,1,t) + f(\psi^1(t)) - f(\psi^2(t)) = 0, \\ \tau \mathcal{Z}(x,\eta,t) + (1 - \eta\tau')\mathcal{Z}_\eta(x,\eta,t) = 0 \end{cases} \tag{3.75}$$

with initial and boundary conditions

$$\begin{cases} \Phi(x,0) = \varphi_0^1 - \varphi_0^2, \quad \Phi_t(x,0) = \varphi_1^1 - \varphi_1^2, \\ \Psi(x,0) = \psi_0^1 - \psi_0^2, \quad \Psi_t(x,0) = \psi_1^1 - \psi_1^2, \\ \mathcal{Z}(x,1,0) = f_0^1(x, -\eta\tau) - f_0^2(x, -\eta\tau), \\ \Phi(0,t) = \Phi(1,t) = \Psi(0,t) = \Psi(1,t) = 0, \\ \mathcal{Z}(x,0,t) = \Psi_t(x,t). \end{cases} \tag{3.76}$$

Then we can define

$$\begin{aligned} F(t) &= \|U^1(t) - U^2(t)\|_{\mathcal{H}}^2 \\ &= \int_0^1 (\rho_1 \Phi_t^2 + \rho_2 \Psi_t^2 + k(\Phi_x + \Psi)^2 + b\Psi_x^2)\,dx + \xi \int_0^1 \int \mathcal{Z}^2(x,\eta,t)\,d\eta\,dx. \end{aligned} \tag{3.77}$$

There exists a constant $c > 0$, such that

$$\begin{aligned} \frac{d}{dt}F(t) &= -2\mu_1 \int_0^1 \Psi_t^2\,dx - 2\mu_2 \int_0^1 \Psi_t \mathcal{Z}(x,1,t)\,dx - \int_0^1 \big[f(\psi^1) - f(\psi^2)\big]\Psi_t\,dx \\ &\le -c \int_0^1 \Psi_t^2 + \mathcal{Z}^2(x,1,t)\,dx - \int_0^1 \big[f(\psi^1) - f(\psi^2)\big]\Psi_t\,dx \\ &\le -c \int_0^1 \Psi_t^2 + \mathcal{Z}^2(x,1,t)\,dx + C_B \left(\|\Psi(t)\|_{\theta+1}^{\theta+1}\right)^2, \end{aligned} \tag{3.78}$$

The last term could be estimated by

$$\int_0^1 |(f(\psi^1) - f(\psi^2))\Psi_t|\,dx \le \int_0^1 (|\psi^1|^\theta + |\psi^2|^\theta)|\Psi||\Psi_t|\,dx \tag{3.79}$$

$$\le (\|\psi^1\|_{\theta+1}^\theta + \|\psi^2\|_{\theta+1}^\theta)\|\Psi\|_{\theta+1}\|\Psi_t\| \tag{3.80}$$

$$\le \varepsilon\|\Psi_t\|^2 + C_B\left(\|\Psi\|_{\theta+1}^{\theta+1}\right)^2. \tag{3.81}$$

Now define the following functionals:

$$A_1(t) = -\int_0^1 \rho_1\Phi\Phi_t + \rho_2\Psi\Psi_t\,dx - \frac{\mu_1}{2}\int_0^1 \Psi^2\,dx, \tag{3.82}$$

$$A_2(t) = \int_0^1 (\rho_2\Psi_t\Psi + \rho_1\Phi_t\widehat{\Phi})\,dx + \frac{\mu_1}{2}\int_0^1 \Psi^2\,dx, \tag{3.83}$$

$$A_3(t) = \rho_2\int_0^1 \Psi_t(\Phi_x + \Psi)\,dx + \rho_2\int_0^1 \Psi_x\Phi_t\,dx, \tag{3.84}$$

$$A_4(t) = \int_0^1\int_0^1 e^{-2\tau\rho}\mathcal{Z}(x,\eta,t)\,d\eta\,dx, \tag{3.85}$$

$$\mathcal{F}(t) := MF(t) + \frac{1}{8}A_1(t) + NA_2(t) + A_3(t) + \frac{\varepsilon}{k}\int_0^1 \rho_1 q\Phi_t\Phi_x\,dx$$
$$+\frac{\rho_2 b}{4\varepsilon}\int_0^1 a\Psi_t\Psi_x\,dx + A_4(t). \tag{3.86}$$

Using the similar procedure as in Lemma 3.11, we could show that there exist $\gamma_1 > 0$, $\gamma_2 > 0$, such that

$$\gamma_1 F(t) \le \mathcal{F}(t) \le \gamma_2 F(t). \tag{3.87}$$

Then using the similar technique as in the proof of Theorem 2.2, we obtain that there exists a $\sigma > 0$ such that

$$\frac{d}{dt}\mathcal{F}(t) \le -\sigma F(t) + C_B\left(\|\Psi(t)\|_{\theta+1}^{\theta+1}\right)^2 \le -\frac{\sigma}{\gamma_2}\mathcal{F}(t) + C_B\left(\|\Psi(t)\|_{\theta+1}^{\theta+1}\right)^2. \tag{3.88}$$

By Gronwall's inequality, we have

$$F(t) \le F(0)e^{-\frac{\sigma}{\gamma_2}t} + C_B'\int_0^t e^{-\frac{\sigma}{\gamma_2}(t-s)}\left(\|\Psi\|_{\theta+1}^{\theta+1}\right)^2 ds$$
$$\le e^{-\frac{\sigma}{\gamma_2}t}\|U_0^1 - U_0^2\|^2 + C_B'\int_0^t e^{-\frac{\sigma}{\gamma_2}(t-s)}\,ds \sup_{0<s<t}\left[\|\psi^1(s) - \psi^2(s)\|_{\theta+1}^{\theta+1}\right]^2. \tag{3.89}$$

Setting $b(t) = e^{-\frac{\sigma}{\gamma_2}t}$ and $c(t) = C_B'\int_0^t e^{-\frac{\sigma}{\gamma_2}(t-s)}\,ds$. Then $b(t) \in L^1(\mathbb{R}^+)$ with $\lim_{t\to\infty} b(t) = 0$ and $c(t)$ is locally bounded for $t \in [0, \infty)$. Hence $b(t)$ and $c(t)$ satisfy Definition 3.7, which means Theorem 3.12 is proved. □

### 3.4 Proof of Main Result: Theorem 2.2

From Sect. 3.2, the dynamical system $(S(t), \mathcal{H})$ possesses an absorbing set $B(0, R)$ with $R$ defined in (3.72). By Theorem 3.12, if the initial conditions $U_0^1$ and $U_0^2$ are from the absorbing ball $B(0, R)$, then the trajectories $S(t)U_0^1$ and $S(t)U_0^2$ satisfies the quasi-stability inequality (3.73). Moreover, by Theorem 2.1, $S(t)$ is a strongly continuous semigroup on the energy space $\mathcal{H}$, thus

$$\|S(t)U_0^1 - S(t)U_0^2\|_{\mathcal{H}} \leq e^{C_0 t}\|U_0^1 - U_0^2\|_{\mathcal{H}}. \tag{3.90}$$

Therefore by Definition 3.7, the dynamical system $(S(t), \mathcal{H})$ is quasi-stable on the absorbing set $B(0, R)$. Then by Theorem 3.1 in Sect. 3.1, the global attractor $B(0, R)$ has a finite fractal dimension. In addition, the existence of finite dimensional exponential attractor also can be obtained. This means Theorem 2.2 is established.

## 4 Conclusion and Further Research

In this paper, we establish the well-posedness of global solution and existence of global attractor for a 1D Timoshenko system subject to a single mechanical damping and a continuous variable sub-linear time delay in the angular direction of beam filament's movement. The result depends on an interplay between the strength of the damping and the time delay and suitable physical assumptions, such as speed equal condition in the transversal and angular directions. A natural question to investigate next is the convergence of corresponding attractors as delay disappears. In this context, most of the results obtained in the literature has focused on constant delay. We would like to investigate a similar question for the 1D Timoshenko system subject to a variable delay, i.e. the upper semi-continuity of attractors as the variable delay approaches zero in the future.

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