Applied Mathematics & Optimization (2020) 81:591–619
https://doi.org/10.1007/s00245-018-9513-x

Equilibrium Controls in Time Inconsistent Stochastic Linear Quadratic Problems

Tianxiao Wang[1]

Published online: 26 July 2018

Abstract
This paper deals with a class of time inconsistent stochastic linear quadratic optimal control problems in Markovian framework. Three notions, i.e., closed-loop equilibrium strategies, open-loop equilibrium controls and open-loop equilibrium strategies, are characterized in *unified* manners. These results indicate clearer and deeper distinctions among these notions. For example, in particular time consistent setting, the open-loop equilibrium controls are fully characterized by *first-order*, *second-order necessary optimality conditions*, and are not optimal in general, while the closed-loop equilibrium controls naturally reduce into *closed-loop optimal controls*.

Keywords Linear quadratic optimal control problems · Time inconsistency · Equilibrium controls · Riccati equations

Mathematics Subject Classification 93E20 · 49N10 · 91B51 · 60H10

1 Introduction

Through out this paper, $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ is a complete filtered probability space, on which one-dimensional standard Brownian motion $W(\cdot)$ is defined. Here $\mathbb{F} \equiv \{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration of $W(\cdot)$ augmented by $\mathbb{P}$-null sets.

1.1 Formulation of Time Inconsistent Optimal Control Problems

For any $t \in [0, T)$, we consider the following stochastic differential equation (SDE):

The research was supported by the NSF of China under Grant 11231007, 11401404 and 11471231, and the Fundamental Research Funds for the central Universities.

✉ Tianxiao Wang
wtxiao2014@scu.edu.cn

[1] School of Mathematics, Sichuan University, Chengdu, People's Republic of China

$$\begin{cases} dX(s) = \big[A(s)X(s) + B(s)u(s) + b(s)\big]ds \\ \qquad\qquad + \big[C(s)X(s) + D(s)u(s) + \sigma(s)\big]dW(s), \quad s \in [t,T], \\ X(t) = \xi, \end{cases} \tag{1.1}$$

and the cost functional defined by

$$\begin{aligned} J(t,\xi;u(\cdot)) = \frac{1}{2}\mathbb{E}_t\Big\{ \int_t^T \big[\langle Q(s)X(s), X(s) \rangle + 2 \langle S(s)X(s), u(s) \rangle \\ + \langle R(s)u(s), u(s) \rangle \big] ds + \langle GX(T), X(T) \rangle \Big\}. \end{aligned} \tag{1.2}$$

Here A, B, C, D, Q, S, R, G are suitable matrix-valued (deterministic) functions, b, σ are proper stochastic processes, and $\mathbb{E}_t(\cdot) := \mathbb{E}[\,\cdot\,|\mathcal{F}_t]$ stands for conditional expectation operator. In the above, $X(\cdot)$, valued in $\mathbb{R}^n$, is called the *state process*, $u(\cdot)$, valued in $\mathbb{R}^m$, is called the *control process*, and $(t,\xi) \in \mathscr{D}$ is called the *initial pair* where

$$\mathscr{D} := \Big\{(t,\xi) \;\Big|\; t \in [0,T],\ \xi \text{ is } \mathcal{F}_t\text{-measurable, } \mathbb{E}|\xi|^2 < \infty\Big\}.$$

We denote the set of all control processes by

$$\begin{aligned} \mathscr{U}[t,T] \equiv \Big\{u : [t,T] \times \Omega \to \mathbb{R}^m \;\Big|\; u \text{ is } \mathbb{F}\text{-progressively measurable,} \\ \mathbb{E}\int_t^T |u(s)|^2 ds < \infty\Big\}. \end{aligned}$$

Under some mild conditions on the coefficients, for any initial pair (t,ξ) and a control $u(\cdot) \in \mathscr{U}[t,T]$, the state equation (1.1) admits a unique solution $X(\cdot) = X(\cdot\,; t, x, u(\cdot))$, and the cost functional $J(t,\xi;u(\cdot))$ is well-defined. We pose the following stochastic linear quadratic (SLQ) optimal control problem.

Problem (SLQ) For any given (t,ξ), find a $\bar{u}(\cdot) \in \mathscr{U}[t,T]$ such that

$$J(t,\xi;\bar{u}(\cdot)) = \inf_{u(\cdot)\in\mathscr{U}[t,T]} J(t,\xi;u(\cdot)) \overset{\Delta}{=} V(t,\xi). \tag{1.3}$$

Any $\bar{u}(\cdot) \in \mathscr{U}[t,T]$ satisfying (1.3) is called an *optimal control* for the given initial pair (t,ξ), the corresponding state process $\bar{X}(\cdot)$ is called an *optimal state process* for (t,ξ), $(\bar{X}(\cdot), \bar{u}(\cdot))$ is called an *optimal pair* for (t,ξ), and $V(\cdot\,,\cdot)$ is called the *value function* of Problem (SLQ).

For above optimal control problem, it is reasonable to keep the state process stable with respect to possible variation of random factors. To this end, one effective way is to add the variation of $X(\cdot)$, i.e.

$$\text{Var}_t[X] := \mathbb{E}_t\big[X(T) - \mathbb{E}_t X(T)\big]^2 = \mathbb{E}_t|X(T)|^2 - \big[\mathbb{E}_t X(T)\big]^2$$

into the cost functional (e.g., [2,3,11–14,21,25], etc).Therefore, it is natural to propose the following general modified cost functional

$$
\begin{aligned}
J(t,\xi;u(\cdot)) = \frac{1}{2}\mathbb{E}_t\Big\{ \int_t^T \big[&\langle Q(s)X(s), X(s)\rangle + 2\langle S(s)X(s), u(s)\rangle \\
&+ \langle \widetilde{Q}(s)\mathbb{E}_t[X(s)], \mathbb{E}_t[X(s)]\rangle + 2\langle \widetilde{S}(s)\mathbb{E}_t[X(s)], \mathbb{E}_t[u(s)]\rangle \\
&+ \langle R(s)u(s), u(s)\rangle + \langle \widetilde{R}(s)\mathbb{E}_t[u(s)], \mathbb{E}_t[u(s)]\rangle \big] ds \\
&+ \langle GX(T), X(T)\rangle + \langle \widetilde{G}\mathbb{E}_t[X(T)], \mathbb{E}_t[X(T)]\rangle \Big\}.
\end{aligned} \tag{1.4}
$$

Here $\widetilde{S}$, $\widetilde{R}$, $\widetilde{G}$, $\widetilde{Q}$ are deterministic matrices-valued functions and g is a vector.

In this scenario, the optimal controls become time-inconsistent, i.e., the "optimal" control based on this moment may not keep optimality in future. We refer to [25] for some explicit examples.

1.2 Related Literature

The study on time inconsistency by economists actually dates back to Strotz [17] in the 1950s. One possible way to treat time inconsistency is to discuss the pre-committed controls for which the solutions are verified to be optimal only at the initial time.

In this paper, we shall discuss above optimal control problem from another viewpoint. More precisely, we investigate the time inconsistency within a game-theoretic framework and analyze the time-consistent equilibrium solution (e.g., [10,15]). Recently, people began to treat the equilibrium controls using the ideas of stochastic control theories, and developed several different approaches in the existing papers. These methods range from dynamic programming principles and verification procedures to maximum principles and variational techniques.

- In Björk-Murgoci [1], Björk et al [2], the authors examined a general class of time inconsistent problems under Markovian framework by equilibrium value functions. In the continuous case, they formally derived the extended HJB equations, and then rigorously proved the verification theorem by the conclusions of discrete time case, see Theorem 5.2 in [2]. They also present some special cases including a linear quadratic control problem in which equilibrium solutions are constructed. This method was also used to treat investment-reinsurance problems with mean-variance criterion, see e.g., [14,27].
- In Yong [23,25], the author discussed a class of time inconsistent optimal control problems by multi-person differential games approach, where a new kind of equilibrium HJB equations/sytems of Riccati equations were introduced. Unlike [1,2], they started the investigations in continuous time setting, made partition on time intervals and used tricks of forward-backward stochastic differential equations (FBSDEs). Further study along this can be found in [19,22], and so on.
- In Ekeland and Lazrak [8,9], they considered some financial problems such as investment and consumption model with time-inconsistency feature. They used the variational ideas to introduce certain feedback/closed-loop equilibrium controls, and spread out discussions via equilibrium value functions. Compared with the

general situation in [1,2], the particular form of equilibrium value functions were proposed according to the given cost functional, while the complex convergence arguments were avoided.

- Inspired by the ideas of stochastic maximum principles in optimal control theories, Hu et al. [11] studied a class of time inconsistent SLQ problems in Markovian setting, introduced open-loop equilibrium controls and their closed-loop representations, derived general sufficient conditions through a flow of FBSDEs or systems of backward ordinary differential equations (ODEs). Just recently, the same authors continued to discuss the uniqueness of open-loop equilibrium controls in [12]. More related details can also be found in [7,20,21].

1.3 Unified Approach and Contributions

As to Problem (SLQ), in this article we propose a unified method to characterize the open-loop equilibrium controls, open-loop equilibrium strategies, closed-loop equilibrium strategies. We combines the ideas from variational analysis, forward-backward stochastic differential equations and forward-backward decoupling procedures. In the following, we provide a brief outline of our approach.

For any $(\Theta_1, \Theta_2, \varphi) \in L^2(0, T; \mathbb{R}^{m\times n}) \times L^2(0, T; \mathbb{R}^{m\times n}) \times L^2_{\mathbb{F}}(0, T; \mathbb{R}^m)$, we start with control processes

$$u := (\Theta_1 + \Theta_2)X + \varphi, \quad u^\varepsilon := \Theta_1 X^\varepsilon + \Theta_2 X + \varphi + v I_{[t,t+\varepsilon]}. \tag{1.5}$$

They can reduce into the required equilibrium controls and perturbed controls in various settings (see Sect. 4.4).

In view of the definitions for equilibrium controls, we proceed to consider the difference of the cost functional at u, u^ε. To do so, given X and X^ε, we introduce, respectively, backward stochastic differential equations (BSDEs) with conditional expectations. We point out that the one associated with X^ε appears for the first time in the literature. As a result, we obtain two forward-backward systems in which the terminal parts and generators of backward systems rely respectively on X, X^ε.

To tackle the limit part in the definitions of both open-loop and closed-loop equilibrium controls (i.e., Definitions 2.1, 2.3 next), we continue to decouple the above two forward-backward systems. More precisely, we make conjectures on the solutions of backward systems, formally obtain a class of systems of BSDEs merely depending on given coefficients, and then verify our arguments rigorously. At last we establish our characterizations with proper convergence procedures.

At this very moment, it is worth mentioning that the previous proposed approach demonstrates several new advantages on the treatment of both open-loop equilibrium controls, closed-loop equilibrium controls/strategies. Unlike [1,2,23,25], our procedures on closed-loop equilibrium strategy in continuous time drop the reliance on complex convergence arguments from discrete time to continuous case. Comparing with [11,12], our methodology on open-loop equilibrium controls neither requires any non-definite assumptions on the involved coefficients, nor directly uses the conclusions of stochastic maximum principles. Moreover, it can be adjusted into the random coefficients case, see [21].

Even though both open-loop equilibrium controls and closed-loop equilibrium controls are widely investigated in the literature, there is no paper discussing their differences to our best. In this paper, we give a clear picture by the obtained characterizations. For example, in the classical SLQ setting, open-loop equilibrium controls are fully characterized by first-order, second-order necessary conditions. In other words, they are weaker than optimal controls (Remark 3.1). However, in the same situation, the closed-loop equilibrium controls happen to reduce exactly into closed-loop optimal controls (Remark 3.3). Eventually, we point out that the characterizations on open-loop, closed-loop equilibrium strategies, respectively, include two different *second-order equilibrium conditions*, which are absent in nearly all the relevant articles.

1.4 Outline of the Article

The remainder of this article of structured as follows. In Sect. 2, an overview of assumptions, notation used in the sequel is provided. In Sect. 3, the main conclusions of this article are gathered and some important remarks are demonstrated. In Sect. 4, the proofs of the main results in Sect. 3 are given. Section 5 concludes this article.

2 Preliminary Notations

Given $H := \mathbb{R}^n, \mathbb{R}^{n\times n}, \mathbb{S}^{n\times n}$, etc, $0 \le s \le t \le T$, we define some spaces as follows.

$$
\begin{aligned}
&L^2_{\mathcal{F}_t}(\Omega; H) := \Big\{X : \Omega \to H, \Big|\, X \text{is } \mathcal{F}_t\text{-measurable}, \mathbb{E}|X|^2 < \infty\Big\},\\
&L^2_{\mathbb{F}}(s,t; H) := \Big\{X : [s,t]\times\Omega \to H \,\Big|\, X(\cdot) \text{ is } \mathbb{F}\text{-adapted, measurable},\\
&\qquad\qquad \mathbb{E}\int_s^t |X(r)|^2 dr < \infty\Big\},\\
&L^\infty(s,t; H) := \Big\{X : [s,t] \to H \,\Big|\, X \text{ is deterministic, measurable}, \sup_{r\in[s,t]} |X(r)| < \infty\Big\},\\
&L^2_{\mathbb{F}}(\Omega; L^1(s,t; H)) := \Big\{X : [s,t]\times\Omega \to H \,\Big|\, X(\cdot) \text{ is } \mathbb{F}\text{-adapted, measurable},\\
&\qquad\qquad \mathbb{E}\Big[\int_s^t |X(r)| dr\Big]^2 < \infty\Big\},\\
&L^2_{\mathbb{F}}(\Omega; C([s,t]; H)) := \Big\{X : [s,t]\times\Omega \to H \,\Big|\, X(\cdot) \text{ is } \mathbb{F}\text{-adapted, measurable}\\
&\qquad\qquad \text{continuous} \quad \mathbb{E}\sup_{r\in[s,t]} |X(r)|^2 < \infty\Big\}.
\end{aligned}
$$

We also need the following hypotheses on coefficients of (1.1), (1.4).

(H1) Suppose $A, B, C, D, R, \widetilde{R}, Q, \widetilde{Q}, S, \widetilde{S} \in L^\infty(0,T; H), G, \widetilde{G}, g \in H$, $b \in L^2_{\mathbb{F}}(\Omega; L^1(0,T; H)), \sigma \in L^2_{\mathbb{F}}(0,T; H)$.

To begin with, we look at Problem (SLQ) from an open-loop equilibrium control viewpoint. The following definition is adapted from [11,12].

Definition 2.1 Given $X^*(0) = x_0 \in \mathbb{R}^n$, a state-control pair

$$(X^*, u^*) \in L^2_{\mathbb{F}}(\Omega; C([0,T]; \mathbb{R}^n)) \times L^2_{\mathbb{F}}(0,T; \mathbb{R}^m)$$

is called an *open-loop equilibrium pair* if for any $t \in [0, T)$, $\varepsilon > 0$, $v \in L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^m)$,

$$\lim_{\varepsilon \to 0} \frac{J(t, X^*(t); u^{v,\varepsilon}(\cdot)) - J\big(t, X^*(t); u^*(\cdot)\big|_{[t,T]}\big)}{\varepsilon} \geq 0, \tag{2.1}$$

where $u^{v,\varepsilon} = u^* + vI_{[t,t+\varepsilon]}$. Here u^* and X^* are called *open-loop equilibrium control* and *open-loop equilibrium state process*.

Roughly speaking, the definition shows the *dynamic local optimality* in some sense. In this paper we will explore deeper properties of such equilibrium controls via their characterizations.

Due to our particular linear quadratic structure, we also introduce the notion of open-loop equilibrium strategy, which is independent of initial state x_0.

Definition 2.2 $(\Theta^*, \varphi^*) \in L^2(0, T; \mathbb{R}^{m \times n}) \times L^2_{\mathbb{F}}(0, T; \mathbb{R}^m)$ is called an *open-loop equilibrium strategy* of Problem (SLQ), if for any $X^*(0) = x_0 \in \mathbb{R}^n$, $u^* := \Theta^* X^* + \varphi^*$, with X^* being the associated state process, is an open-loop equilibrium control.

The open-loop equilibrium strategy enable us to capture the explicit feedback representation of open-loop equilibrium control. However, it is different from the following one.

Definition 2.3 $(\Theta^*, \varphi^*) \in L^2(0, T; \mathbb{R}^{m \times m}) \times L^2_{\mathbb{F}}(0, T; \mathbb{R}^m)$ is called a *closed-loop equilibrium strategy*, if for any initial state $x_0 \in \mathbb{R}^n$, $t \in [0, T)$, $\varepsilon > 0$, $v \in L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^m)$,

$$\lim_{\varepsilon \to 0} \frac{J(t, X^*(t); u^{\varepsilon}(\cdot)) - J\big(t, X^*(t); u^*(\cdot)\big|_{[t,T]}\big)}{\varepsilon} \geq 0, \tag{2.2}$$

where

$$u^* := \Theta^* X + \varphi^*, \;\; u^{\varepsilon} := \Theta^* X^{\varepsilon} + vI_{[t,t+\varepsilon]} + \varphi^*,$$

X^*, X^{ε} are the state process on $[0, T]$ associated with u^*, u^{ε}, respectively.

We emphasize that both open-loop equilibrium strategy and closed-loop equilibrium strategy are independent of initial state x_0. However, the perturbed control $u^{v,\varepsilon}$ in Definition 2.1 is actually different from u^{ε} in Definition 2.3. In this paper, we will demonstrate further connections between these two kinds of strategies.

In the following, let K be a generic constant which varies in different context and

$$\mathscr{R} := R + \widetilde{R}, \;\; \mathscr{Q} := Q + \widetilde{Q}, \;\; \mathscr{G} := G + \widetilde{G}, \;\; \mathscr{S} = S + \widetilde{S}. \tag{2.3}$$

3 Characterizations of Equilibrium Controls/Strategies

In this part, we state the main results of this article. To begin with, recall the notation in (1.5), we introduce the following system which is useful next,

$$\begin{cases} dY_1 = -\Big[Y_1(A + B\Theta_1 + B\Theta_2) + (C + D\Theta_1)^\top Y_1(C + D\Theta_1 + D\Theta_2) \\ \qquad + (A + B\Theta_1)^\top Y_1 + \big[Q + \Theta_1^\top S + \Theta_1^\top R(\Theta_1 + \Theta_2) + S^\top(\Theta_1 + \Theta_2)\big]\Big]ds, \\ dY_2 = -\Big\{Y_2(A + B\Theta_1 + B\Theta_2) + (A + B\Theta_1)^\top Y_2 + \big[\widetilde{Q} + \Theta_1^\top \widetilde{S} + \Theta_1^\top \widetilde{R}(\Theta_1 + \Theta_2) \\ \qquad + \widetilde{S}^\top(\Theta_1 + \Theta_2)\big]\Big\}ds, \\ dY_3 = -\Big[(A + B\Theta_1)^\top Y_3 + Y_2(B\varphi + b) + (\widetilde{S}^\top + \Theta_1^\top \widetilde{R})\varphi\Big]ds + Z_3 dW(s), \\ dY_4 = -\Big\{(A + B\Theta_1)^\top Y_4 + (C + D\Theta_1)^\top Z_4 + (C + D\Theta_1)^\top Y_1(D\varphi + \sigma) \\ \qquad + Y_1(B\varphi + b) + (S^\top + \Theta_1^\top R)\varphi\Big\}ds + Z_4 dW(s), \\ Y_1(T) = G,\ Y_2(T) = \widetilde{G},\ Y_3(T) = 0,\ Y_4(T) = 0. \end{cases} \tag{3.1}$$

It is easy to check that [18]

$$Y_1,\ Y_2 \in C([0,T];\mathbb{R}^{n\times n}),\ (Y_3, Z_3), (Y_4, Z_4) \\ \in L^2_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^n).$$

We start with the case of open-loop equilibrium controls. Recall (1.5), we choose $\Theta_1 \equiv 0$, $\Theta_2 \equiv 0$, which indicates that $u = \varphi \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$. Moreover, (3.1) reduces to

$$\begin{cases} dP_1 = -\Big[P_1 A + A^\top P_1 + C^\top P_1 C + Q\Big]ds, \\ dP_2 = -\Big\{P_2 A + A^\top P_2 + \widetilde{Q}\Big\}ds, \\ dP_3 = -\Big[A^\top P_3 + P_2 b + (P_2 B + \widetilde{S}^\top)u\Big]ds + L_3 dW(s), \\ dP_4 = -\Big\{A^\top P_4 + C^\top L_4 + C^\top P_1\sigma + P_1 b + (C^\top P_1 D \\ \qquad + P_1 B + S^\top)u\Big\}ds + L_4 dW(s), \\ P_1(T) = G,\ P_2(T) = \widetilde{G},\ P_3(T) = 0,\ P_4(T) = 0. \end{cases} \tag{3.2}$$

For later clarification, we replace (Y, Z) by (P, L), and omit the reference to the time variable for simplicity.

Above P_1, P_2 do not rely on u while P_3, P_4 do. As to (3.2), it is easy to see

$$P_1,\ P_2 \in C([0,T];\mathbb{R}^{n\times n}),\ (P_3, \Lambda_3), (P_4, \Lambda_4) \\ \in L^2_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^n).$$

Considering X in (1.1), we define

$$\begin{cases} M(s,t) := P_1(s)X(s) + P_2(s)\mathbb{E}_t X(s) + \mathbb{E}_t P_3(s) + P_4(s), \quad s \in [t,T], \\ N(s) := P_1(s)\big(C(s)X(s) + D(s)u(s) + \sigma(s)\big) + L_4(s), \quad s \in [0,T]. \end{cases} \tag{3.3}$$

Theorem 3.1 *Suppose (H1) holds, P_1 satisfies (3.2). Then $\bar{u}$ is an open-loop equilibrium control for Problem (SLQ) associated with initial state $\bar{X}(0) = x_0 \in \mathbb{R}^n$ if and only if*

(i) *the following inequality holds,*

$$\mathscr{R}(s) + D(s)^\top P_1(s)D(s) \geq 0, \qquad s \in [0, T], \ \ a.e. \tag{3.4}$$

(ii) *given $(\bar{M}, \bar{N})$ in (3.3) associated with $\bar{u}$,*

$$\mathscr{R}(s)\bar{u}(s) + \mathscr{S}(s)\bar{X}(s) + B(s)^\top \bar{M}(s, s) + D(s)^\top \bar{N}(s) = 0, \quad s \in [0, T]. \ \ a.e. \tag{3.5}$$

Above (3.4), (3.5) are named as *first-order*, *second-order equilibrium conditions* of open-loop equilibrium controls for Problem (SLQ).

Remark 3.1 As to Theorem 3.1, let us make the following comments,

(1) Above P_1 is indeed the unique solution of classical second-order adjoint equation in control theories of mean-field SDEs. That is to say, (3.4) coincides with the corresponding *second-order necessary optimality condition* [4]. To our best knowledge, (3.4) has not been discussed seriously in [11,12], and other related papers on open-loop equilibrium controls.
(2) If we denote $\widehat{v}(\cdot, t)$ the (time inconsistent) optimal control of Problem (SLQ), then the first-order adjoint equation [24] is

$$\begin{cases} d\widehat{Y}(s, t) = -\Big[A(s)^\top \widehat{Y}(s, t) + C(s)^\top \widehat{Z}(s, t) + Q(s)\widehat{X}(s, t) + S(s)^\top \widehat{v}(s, t) \\ \qquad\qquad + \widetilde{Q}(s)\mathbb{E}_t \widehat{X}(s, t) + \widetilde{S}(s)^\top \mathbb{E}_t \widehat{v}(s, t)\Big]ds + \widehat{Z}(s, t)dW(s), \\ \widehat{Y}(T, t) = GX(T, t) + \widetilde{G}\mathbb{E}_t X(T, t), \end{cases} \tag{3.6}$$

and the first-order necessary optimality condition is

$$\begin{aligned} &R(s)\widehat{v}(s, t) + \widetilde{R}(s)\mathbb{E}_t\widehat{v}(s, t) + S(s)\widehat{X}(s, t) + \widetilde{S}(s)\mathbb{E}_t\widehat{X}(s, t) \\ &\quad -B(s)^\top \widehat{Y}(s, t) - D(s)^\top \widehat{Z}(s, t) = 0, \quad s \in [t, T]. \ \ a.e. \ \ a.s. \end{aligned} \tag{3.7}$$

Let us return back to our framework. Given $(\bar{X}, \bar{u})$ in (1.1), we see that $(\bar{M}, \bar{N})$ satisfies

$$\begin{cases} d\bar{M}(s, t) = -\Big[A(s)^\top \bar{M}(s, t) + C(s)^\top \bar{N}(s) + Q(s)\bar{X}(s) + S(s)^\top \bar{u}(s) \\ \qquad\qquad + \widetilde{Q}(s)\mathbb{E}_t\bar{X}(s) + \widetilde{S}(s)^\top \mathbb{E}_t\bar{u}(s)\Big]dr + \bar{N}(s)dW(s), \\ M(T, t) = GX(T) + \widetilde{G}\mathbb{E}_t X(T). \end{cases} \tag{3.8}$$

Obviously, above (3.6), (3.7) are in general different from our (3.8), (3.5). But if there is no time-inconsistency, i.e., $\widetilde{R} = \widetilde{Q} = \widetilde{S} = \widetilde{G} = 0$, then they coincide with each other.

(3) If $\widetilde{R}=\widetilde{S}=S=0$, R, Q, G are positive definite matrices, then (3.4) is obvious to see. In this scenario, a characterization of open-loop equilibrium control, which is different yet equivalent with (3.5), was given in Theorem 3.5 of [12] without involving systems (3.2).

(4) We compare our equilibrium controls with optimal controls when $\widetilde{R}=\widetilde{Q}=\widetilde{S}=\widetilde{G}=0$.
Recall that the characterization of open-loop optimal controls includes first-order necessary condition and the following convexity condition [5,6,26]

$$\mathbb{E}_t\int_t^T u^\top\big[Ru+SX^0+B^\top Y^0+D^\top Z^0\big]dr\geq 0,\quad \forall u\in L^2_{\mathbb{F}}(t,T;\mathbb{R}^m), \tag{3.9}$$

where X^0 satisfies (1.1) with $\xi=0$, (Y^0,Z^0) solves (3.8) with $\widetilde{G}=\widetilde{S}=\widetilde{Q}=0$ and $X\equiv X^0$.
In contrast, Theorem 3.1 indicates that the open-loop equilibrium controls are fully characterized by first-order, second-order necessary optimality conditions. Therefore, when there is no time inconsistency in Problem (SLQ), the exact difference between open-loop equilibrium controls and open-loop optimal controls lies in (3.4) and (3.9).

Next we characterize the open-loop equilibrium strategy. Recall (1.5), we choose $\Theta_1\equiv 0$, which implies that $u=\Theta_2X+\varphi$. Moreover, (3.1) reduces to

$$\begin{cases} d\mathcal{P}_1=-\Big[\mathcal{P}_1A+A^\top\mathcal{P}_1+C^\top\mathcal{P}_1C+(\mathcal{P}_1B+C^\top\mathcal{P}_1D+S^\top)\Theta_2+Q\Big]ds,\\ d\mathcal{P}_2=-\Big\{\mathcal{P}_2A+A^\top\mathcal{P}_2+\widetilde{Q}+(\mathcal{P}_2B+\widetilde{S}^\top)\Theta_2]\Big\}ds,\\ d\mathcal{P}_3=-\Big[A^\top\mathcal{P}_3+(\mathcal{P}_2B+\widetilde{S}^\top)\varphi+\mathcal{P}_2b\Big]ds+\mathcal{L}_3dW(s),\\ d\mathcal{P}_4=-\Big\{A^\top\mathcal{P}_4+C^\top\mathcal{L}_4+C^\top\mathcal{P}_1\sigma+(C^\top\mathcal{P}_1D+\mathcal{P}_1B+S^\top)\varphi+\mathcal{P}_1b\Big\}ds+\mathcal{L}_4dW(s),\\ \mathcal{P}_1(T)=G,\ \mathcal{P}_2(T)=\widetilde{G},\ \mathcal{P}_3(T)=0,\ \mathcal{P}_4(T)=0, \end{cases} \tag{3.10}$$

We suppress the time variable for simplicity. We also define processes $(\mathcal{M},\mathcal{N})$ as follows,

$$\begin{cases} \mathcal{M}(s,t):=\mathcal{P}_1(s)X(s)+\mathcal{P}_2(s)\mathbb{E}_tX(s)+\mathbb{E}_t\mathcal{P}_3(s)+\mathcal{P}_4(s),\quad s\geq t,\\ \mathcal{N}(s):=\mathcal{P}_1(s)(C(s)+D(s)\Theta_2(s))X(s)+\mathcal{P}_1(s)(D(s)\varphi(s)+\sigma(s))+\mathcal{L}_4(s). \end{cases} \tag{3.11}$$

Theorem 3.2 *Suppose (H1) holds, P_1 satisfies (3.2). Then (Θ^*,φ^*) is a pair of open-loop equilibrium strategy if and only if*

(i) *condition (3.4) holds true,*
(ii) *there exist $\mathcal{P}_i^*$, $\mathcal{L}_j^*$ satisfying BSDEs (3.10) with $(\Theta_2,\varphi)\equiv(\Theta^*,\varphi^*)$ and*

$$\begin{cases} \big[\mathscr{R}+D^\top\mathcal{P}_1^*D\big]\Theta^*+B^\top\big[\mathcal{P}_1^*+\mathcal{P}_2^*\big]+D^\top\mathcal{P}_1^*C+\mathscr{S}=0,\ \ a.s.\ a.e.\\ \big[\mathscr{R}+D^\top\mathcal{P}_1^*D\big]\varphi^*+D^\top\big[\mathcal{P}_1^*\sigma+\mathcal{L}_4^*\big]+B^\top\big[\mathcal{P}_3^*+\mathcal{P}_4^*\big]=0.\ \ a.s.\ a.e. \end{cases} \tag{3.12}$$

Above (3.4), (3.12) are named as *first-order*, *second-order equilibrium conditions* of open-loop equilibrium strategy for Problem (SLQ). Different from (3.5), the conclusion (3.12) focuses on the coefficients and there are no state process or control variable involved.

Remark 3.2 As to Theorem 3.2, we point out two useful facts.

(1) Given (Θ^*, φ^*), it is easy to check that $(\mathcal{M}^*, \mathcal{N}^*)$ solves (3.8) with $u^* := \Theta^* X^* + \varphi^*$. Since u^* is an open-loop equilibrium control, and one can define (M^*, N^*) as in (3.3). By the uniqueness of BSDEs, we end up with $(\mathcal{M}^*, \mathcal{N}^*) \equiv (M^*, N^*)$. In other words, the unique solution of (3.8) has two different forms of representations.

(2) From (3.12), there exists $\theta' \in L^2(0,T;\mathbb{R}^{n\times n})$, $\varphi' \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$ s.t.

$$\begin{cases} \Theta^* = -\big[\mathscr{R} + D^\top \mathcal{P}_1^* D\big]^\dagger \big[B^\top(\mathcal{P}_1^* + \mathcal{P}_2^*) + D^\top \mathcal{P}_1^* C + \mathscr{S}\big] \\ \qquad\quad + \Big\{I - \big[\mathscr{R} + D^\top \mathcal{P}_1^* D\big]^\dagger \big[\mathscr{R} + D^\top \mathcal{P}_1^* D\big]\Big\}\theta', \\ \varphi^* = -\big[\mathscr{R} + D^\top \mathcal{P}_1^* D\big]^\dagger \big[B^\top[\mathcal{P}_4^* + \mathcal{P}_3^*] + D^\top[\mathcal{P}_1^*\sigma + \mathcal{L}_4^*]\big] \\ \qquad\quad + \Big\{I - \big[\mathscr{R} + D^\top \mathcal{P}_1^* D\big]^\dagger \big[\mathscr{R} + D^\top \mathcal{P}_1^* D\big]\Big\}\varphi'. \end{cases} \tag{3.13}$$

Moreover,

$$\begin{cases} \mathcal{R}\Big(B^\top(\mathcal{P}_1^* + \mathcal{P}_2^*) + D^\top \mathcal{P}_1^* C + \mathscr{S}\Big) \subset \mathcal{R}\Big(\mathscr{R} + D^\top \mathcal{P}_1^* D\Big), \ \ a.e. \\ \Big[B^\top[\mathcal{P}_4^* + \mathcal{P}_3^*] + D^\top[\mathcal{P}_1^*\sigma + \mathcal{L}_4^*]\Big] \in \mathcal{R}\Big(\mathscr{R} + D^\top \mathcal{P}_1^* D\Big), \ \ a.e.\ a.s. \\ \big[\mathscr{R} + D^\top \mathcal{P}_1^* D\big]^\dagger \big[B^\top(\mathcal{P}_1^* + \mathcal{P}_2^*) + D^\top \mathcal{P}_1^* C + \mathscr{S}\big] \in L^2(0,T;\mathbb{R}^{m\times n}), \\ \Big[\big[\mathscr{R} + D^\top \mathcal{P}_1^* D\big]^\dagger \big[B^\top[\mathcal{P}_4^* + \mathcal{P}_3^*] + D^\top[\mathcal{P}_1^*\sigma + \mathcal{L}_4^*]\big] \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m). \end{cases} \tag{3.14}$$

In the above, $\mathcal{R}(A)$, $A^\dagger$ are the range, pseudo-inverse of matrix A, respectively. As a result, we obtain the explicit forms of (Θ^*, φ^*), as well as some intrinsic relations among coefficients.

At last, we give the characterizations of closed-loop equilibrium strategies. Recall (1.5), we choose $\Theta_2 \equiv 0$ which implies that $u = \Theta_1 X + \varphi$. Moreover, (3.1) reduces to

$$\begin{cases} d\mathscr{P}_1 = -\Big[\mathscr{P}_1(A + B\Theta_1) + (A + B\Theta_1)^\top \mathscr{P}_1 + (C + D\Theta_1)^\top \mathscr{P}_1(C + D\Theta_1) \\ \qquad\qquad + \big[Q + \Theta_1^\top S + \Theta_1^\top R\Theta_1 + S^\top\Theta_1\big]\Big]ds, \\ d\mathscr{P}_2 = -\Big\{\mathscr{P}_2(A + B\Theta_1) + (A + B\Theta_1)^\top \mathscr{P}_2 + \big[\widetilde{Q} + \Theta_1^\top \widetilde{S} + \Theta_1^\top \widetilde{R}\Theta_1 + \widetilde{S}^\top\Theta_1\big]\Big\}ds, \\ d\mathscr{P}_3 = -\Big[(A + B\Theta_1)^\top \mathscr{P}_3 + \mathscr{P}_2 b + (\mathscr{P}_2 B + \widetilde{S}^\top + \Theta_1^\top \widetilde{R})\varphi\Big]ds + \mathscr{L}_3 dW(s), \\ d\mathscr{P}_4 = -\Big\{(A + B\Theta_1)^\top \mathscr{P}_4 + (C + D\Theta_1)^\top \mathscr{L}_4 + (C + D\Theta_1)^\top \mathscr{P}_1(D\varphi + \sigma) \\ \qquad\qquad + \mathscr{P}_1(B\varphi + b) + (S^\top + \Theta_1^\top R)\varphi\Big\}ds + \mathscr{L}_4 dW(s), \\ \mathscr{P}_1(T) = G, \ \mathscr{P}_2(T) = \widetilde{G}, \ \mathscr{P}_3(T) = 0, \ \mathscr{P}_4(T) = 0. \end{cases} \tag{3.15}$$

We also define two processes $\mathscr{M}$, $\mathscr{N}$ as follows,

$$\begin{cases}\mathscr{M}(s,t) := \mathscr{P}_1(s)X(s) + \mathscr{P}_2(s)\mathbb{E}_t X(s) + \mathbb{E}_t \mathscr{P}_3(s) + \mathscr{P}_4(s), \quad s \geq t,\\ \mathscr{N}(s) := \mathscr{P}_1(s)(C(s)+D(s)\Theta_1(s))X(s)+\mathscr{P}_1(s)(D(s)\varphi(s)+\sigma(s))+\mathscr{L}_4(s).\end{cases} \tag{3.16}$$

Theorem 3.3 *A pair of* $(\Theta^*, \varphi^*) \in L^2(0,T;\mathbb{R}^{m\times n}) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$ *is a closed-loop equilibrium strategy if and only if there exists* $\mathscr{P}_i^*$ *satisfies (3.15) with* $(\Theta_1, \varphi) \equiv (\Theta^*, \varphi^*)$ *such that*

$$\begin{cases}\mathscr{R} + D^\top \mathscr{P}_1^* D \geq 0,\\ (\mathscr{R} + D^\top \mathscr{P}_1^* D)\Theta^* + B^\top(\mathscr{P}_1^* + \mathscr{P}_2^*) + D^\top \mathscr{P}_1^* C + \mathscr{S} = 0,\\ (\mathscr{R} + D^\top \mathscr{P}_1^* D)\varphi^* + B^\top(\mathscr{P}_3^* + \mathscr{P}_4^*) + D^\top \mathscr{P}_1^* \sigma + D^\top \mathscr{L}_4^* = 0.\end{cases} \tag{3.17}$$

For the closed-loop equilibrium strategy (Θ^*, φ^*), the first inequality in (3.17) is referred as the *second-order equilibrium condition*, while the other two conditions are named as *first-order equilibrium condition*.

Remark 3.3 If $\widetilde{G} = \widetilde{S} = \widetilde{Q} = \widetilde{R} = 0$, above (3.17) reduces to

$$\begin{aligned}&R + D^\top \mathscr{P}_1^* D \geq 0, \quad (R + D^\top \mathscr{P}_1^* D)\Theta^* + B^\top \mathscr{P}_1^* + D^\top \mathscr{P}_1^* C + S = 0,\\ &(R + D^\top \mathscr{P}_1^* D)\varphi^* + B^\top \mathscr{P}_4^* + D^\top \mathscr{P}_1^* \sigma + D^\top \mathscr{L}_4^* = 0,\end{aligned} \tag{3.18}$$

where $(\mathscr{P}_1^*, \mathscr{P}_4^*, \mathscr{L}_4^*)$ are described as,

$$\begin{cases}d\mathscr{P}_1^* = -\Big[\mathscr{P}_1^*(A + B\Theta^*) + (A + B\Theta^*)^\top \mathscr{P}_1^* + (C + D\Theta^*)^\top \mathscr{P}_1^*(C + D\Theta^*)\\ \qquad\qquad + \big[Q + [\Theta^*]^\top S + [\Theta^*]^\top R\Theta^* + S^\top \Theta^*\big]\Big]ds,\\ d\mathscr{P}_4^* = -\Big\{(A + B\Theta^*)^\top \mathscr{P}_4^* + (C + D\Theta^*)^\top \mathscr{L}_4^* + (C + D\Theta^*)^\top \mathscr{P}_1^*(D\varphi^* + \sigma)\\ \qquad\qquad + \mathscr{P}_1^*(B\varphi^* + b) + (S^\top + [\Theta^*]^\top R)\varphi^*\Big\}ds + \mathscr{L}_4^* dW(s),\\ \mathscr{P}_1^*(T) = G, \quad \mathscr{P}_4(T) = 0.\end{cases} \tag{3.19}$$

According to [17,18], (3.18) is equivalent to the optimality of strategy pair (Θ^*, φ^*). Therefore, we find the following two useful aspects.

(1) Our defined closed-loop equilibrium controls/strategies are natural extension of closed-loop optimal controls/strategies.
(2) From the optimality viewpoint, closed-loop equilibrium controls/strategies are essentially different and stronger than open-loop equilibrium controls/strategies.

To conclude this section, we clarify the relations among above three characterizations in the following three manners.

Firstly, we make comparisons among (3.2), (3.10), (3.15). On the one hand, they are basically the same in the sense that all of them are particular cases of system (3.1). On the other hand, they also differ from each other in the following three ways. In the first place, the solutions of the first two equations in (3.2), (3.15) are symmetric, while the analogue of (3.10) are non-symmetric [25]. In the second place, the first two equations in (3.2) merely depends on given coefficients, while the counterparts in (3.10) and (3.15) are determined by Θ_1 or Θ_2. In the third place, the last two equations in (3.2) rely on control process u, while the analogue equations in (3.10) and (3.15) are determined by φ.

Secondly, we make the following comments on the second-order equilibrium conditions. For both open-loop equilibrium controls and open-loop equilibrium strategies, we use $\mathscr{R} + D^\top P_1 D \geq 0$, where P_1 satisfies the second-order adjoint equation in LQ optimal control problems of mean-field SDEs. This condition was missing in [11,12,20,21]. As to closed-loop equilibrium strategies, we introduce $\mathscr{R}+D^\top \mathscr{P}_1^* D \geq 0$ where $\mathscr{P}_1^*$ satisfies one backward ordinary differential equation that contains Riccati equation as special case. Notice that this condition has not been discussed in [1,2,23,25].

Thirdly, let us give three examples.

First we consider the relations between open-loop equilibrium controls and open-loop equilibrium strategies. Given open-loop equilibrium strategy (Θ^*, φ^*), for any initial state $x_0 \in \mathbb{R}^n$, Problem (SLQ) admits an open-loop equilibrium control $u^* := \Theta^* X^* + \varphi^*$. Conversely, the conclusion is not true, even when there is no time consistency feature in Problem (SLQ). To see it, we look at the following example.

Example 3.1 Suppose $m = n = 1$, function B is continuous, B^{-1} exists and is bounded, and

$$D = 0,\quad R = \widetilde{R} = 0,\quad Q = \widetilde{Q} = 0,\quad \widetilde{S} = S = 0,\quad \widetilde{G} = 0,\quad G > 0,\quad b = \sigma = 0.$$

By introducing

$$\begin{cases} d\Phi(t) = A(t)\Phi(t)dt + C(t)\Phi(t)dW(t), & t \in [0, T], \\ \Phi(0) = 1, \end{cases}$$

we can represent $X(\cdot)$ by

$$X(t) = \Phi(t)x_0 + \Phi(t)\int_0^t B(s)u(s)\Phi^{-1}(s)ds,\quad t \in [0, T].$$

Since $G > 0$, for any $x_0 \in \mathbb{R}$, we see that $\bar{u}$ is an optimal control as along as the corresponding $\bar{X}(T) = 0$. To this end, we set

$$\bar{u}(\cdot) := -\frac{\Phi(\cdot)B^{-1}(\cdot)}{T}x_0 \in L^2_{\mathbb{F}}(0, T; \mathbb{R}).$$

Moreover, $\bar{u}$ is also an open-loop equilibrium control satisfying (3.4), (3.5).

Next we claim that the open-loop equilibrium strategy does not exists. Actually, if there exists (Θ^*, φ^*), it then follows from (3.12) that $B\mathcal{P}_1^* \equiv 0$. a.s. a.e. On the other hand, $\mathcal{P}_1^*(T) = G > 0$, by the continuity of $\mathcal{P}_1^*$ and B, as well as the existence of B^{-1}, there exists $T_1 < T$ such that for any $t \in [T_1, T]$, $B(t)\mathcal{P}_1^*(t) \neq 0$. A contradiction arises.

Now let us turn to the connections between open-loop equilibrium strategies and closed-loop equilibrium strategy. The following example shows that open-loop equilibrium strategy equals to closed-loop equilibrium strategy.

Example 3.2 Suppose $m = n = 1$, $G > 0$, $Q(\cdot) \geq 0$,

$$\begin{aligned} &C = 0, \quad B = D = 1, \quad S = \widetilde{S} = 0, \ \widetilde{R} = R = 0, \\ &Q(\cdot) + \widetilde{Q}(\cdot) = 0, \quad G + \widetilde{G} = 0, \quad b = \sigma = 0. \end{aligned} \tag{3.20}$$

From Theorem 3.2, we have $\mathcal{P}_1^* \Theta^* + \mathcal{P}_1^* + \mathcal{P}_2^* = 0$ where

$$\begin{cases} d\mathcal{P}_1^*(s) = -\Big[(2A(s) + \Theta^*(s))\mathcal{P}_1^*(s) + Q(s)\Big]ds, \quad s \in [0, T], \\ d\mathcal{P}_2^*(s) = -\Big\{(2A(s) + \Theta^*(s))\mathcal{P}_2^*(s) + \widetilde{Q}(s)\Big\}ds, \quad s \in [0, T], \\ \mathcal{P}_1^*(T) = G, \ \mathcal{P}_2^*(T) = \widetilde{G}. \end{cases} \tag{3.21}$$

It is easy to see that $(\mathcal{P}_1^*, \mathcal{P}_2^*) := (P, -P)$ satisfies (3.21) with $\Theta \equiv 0$, where

$$\begin{cases} dP(s) = -\Big[2A(s)P(s) + Q(s)\Big]ds, \quad s \in [0, T], \\ \mathcal{P}_1(T) = G. \end{cases} \tag{3.22}$$

Suppose there is another Θ' and $(\mathcal{P}_1', \mathcal{P}_2') \in C([0, T]; \mathbb{R}^2)$ such that (3.21) is satisfied and

$$\mathcal{P}_1'\Theta' + \widehat{\mathcal{P}}' = 0, \widehat{\mathcal{P}}' := \mathcal{P}_1' + \mathcal{P}_2'.$$

Notice that

$$\begin{cases} d\widehat{\mathcal{P}}'(s) = -(2A(s) + \Theta'(s))\widehat{\mathcal{P}}'(s)ds, \quad s \in [0, T], \\ \widehat{\mathcal{P}}'(T) = 0. \end{cases} \tag{3.23}$$

By uniqueness, $\widehat{\mathcal{P}}' \equiv 0$. By (3.20), $\frac{1}{\mathcal{P}_1'}$ exists and is bounded. Hence $\Theta' = 0$.

Due to $b = \sigma = 0$, it is easy to check there exists a unique $\varphi^* \equiv 0$. Moreover, condition (3.4) holds.

To sum up, Problem (SLQ) admits a unique pair of open-loop equilibrium strategy $(\Theta^*, \varphi^*) \equiv (0, 0)$ under condition (3.20).

Now we look at the closed-loop equilibrium strategies (Ξ^*, ϕ^*). Here we change the notation for later comparisons. From Theorem 3.3, one has $\mathscr{P}_1^* \Xi^* + (\mathscr{P}_1^* + \mathscr{P}_2^*) = 0$, where

$$\begin{cases} d\mathscr{P}_1^*(s) = -\Big[2\mathscr{P}_1^*(s)(A(s) + \Xi^*(s)) + |\Xi^*(s)|^2\mathscr{P}_1^*(s) + Q(s)\Big]ds, \\ d\mathscr{P}_2^*(s) = -\Big\{2\mathscr{P}_2^*(s)(A(s) + \Xi^*(s)) + \widetilde{Q}(s)\Big\}ds, \\ \mathscr{P}_1^*(T) = G, \ \mathscr{P}_2^*(T) = \widetilde{G}. \end{cases} \tag{3.24}$$

For P in (3.22), we see that $(\mathscr{P}_1^*, \mathscr{P}_2^*) \equiv (P, -P)$ satisfies (3.24) with $\Xi^* \equiv 0$. Moreover, $\mathscr{P}_1^* \geq 0$. Suppose there is another Ξ' and $(\mathscr{P}_1', \mathscr{P}_2')$ such that (3.24) is satisfied and

$$\mathscr{P}_1'\Xi' + \widehat{\mathscr{P}'} = 0, \quad \widehat{\mathscr{P}'} := \mathscr{P}_1' + \mathscr{P}_2'.$$

Notice that

$$\begin{cases} d\widehat{\mathscr{P}'}(s) = -\Big[(2A(s) + \Xi'(s))\widehat{\mathscr{P}'}(s) + |\Xi'(s)|^2\mathscr{P}_1'(s)\Big]ds, \quad s \in [0, T], \\ \widehat{\mathcal{P}}'(T) = 0. \end{cases} \tag{3.25}$$

By (3.20), $\frac{1}{\mathscr{P}_1'}$ exists and is bounded. Hence $\Xi' = -\frac{\widehat{\mathscr{P}'}}{\mathscr{P}_1'}$ and (3.25) can be rewritten as,

$$\begin{cases} d\widehat{\mathscr{P}'}(s) = -\Big[2A(s)\widehat{\mathscr{P}'}(s) - (1 + \frac{1}{\mathscr{P}_1'})\widehat{\mathscr{P}'}(s)\Big]ds, \\ \widehat{\mathcal{P}}'(T) = 0. \end{cases} \tag{3.26}$$

By uniqueness, $\widehat{\mathscr{P}'} \equiv 0$, which yields $\Xi' = 0$.

Due to $b = \sigma = 0$, it is easy to check there exists a unique $\phi^* \equiv 0$.

To sum up, Problem (SLQ) admits a unique pair of closed-loop equilibrium strategy $(\Xi^*, \phi^*) \equiv (0, 0)$ under condition (3.20), which is the same as open-loop equilibrium strategy.

The following example shows that open-loop equilibrium strategy are also distinctive from closed-loop equilibrium strategy in some cases.

Example 3.3 Suppose $m = n = 1$,

$$\begin{aligned} &C = 0, \quad B = D = 1, \quad S = \widetilde{S} = 0, \quad \widetilde{R} = R = 0, \\ &Q(\cdot) = \widetilde{Q}(\cdot) = 0, \quad G \geq \widetilde{G} > 0, \quad b = \sigma = 0. \end{aligned} \tag{3.27}$$

As to open-loop equilibrium strategy (Θ^*, φ^*), from Theorem 3.2, $\mathcal{P}_1^*\Theta^* + \mathcal{P}_1^* + \mathcal{P}_2^* = 0$, where $(\mathcal{P}_1^*, \mathcal{P}_2^*)$ satisfies (3.21) with $Q = \widetilde{Q} = 0$. As to closed-loop equilibrium strategies (Ξ^*, ϕ^*), from Theorem 3.3, one has $\mathscr{P}_1^*\Xi^* + (\mathscr{P}_1^* + \mathscr{P}_2^*) = 0$, where $(\mathscr{P}_1^*, \mathscr{P}_2^*)$ satisfies (3.24) with $Q = \widetilde{Q} = 0$. We illustrate out point by three steps.

Step 1 Under (3.27), we look at the solvability of system (3.21).
Consider an ODE of

$$\begin{cases} dP(s) = -(2A(s) - 1 - \frac{\widetilde{G}}{G})P(s)ds, \quad s \in [0, T], \\ P(T) = 1. \end{cases}$$

It is easy to see that $(\mathcal{P}_1^*(\cdot), \mathcal{P}_2^*(\cdot)) \equiv (GP(\cdot), \widetilde{G}P(\cdot))$ is a solution of (3.27).

Step 2 Under (3.27), we claim that system (3.24) is solvable with $\mathscr{P}_1^* > \mathscr{P}_2^*$. By a simplification, it is suffice to consider the regularity of the following system

$$\begin{cases} P_1(t) = G\exp\Big[\int_t^T \big[2A(s) - 1 + \frac{|P_2(s)|^2}{|P_1(s)|^2}\big]ds\Big], \quad t \in [0,T], \\ P_2(t) = \widetilde{G}\exp\Big[\int_t^T 2\big\{A(s) - 1 - \frac{P_2(s)}{P_1(s)}\big\}ds\Big], \quad t \in [0,T]. \end{cases} \tag{3.28}$$

For later usefulness, we make the following conventions,

$$\begin{cases} \|p\|_{[\tau_1,\tau_2]} := \sup\limits_{t\in[\tau_1,\tau_2]} |p(t)|, \quad \tau_1,\ \tau_2 \in [0,T], \\ L_1 := \widetilde{G}e^{-2T(\|A\|_{[0,T]}+2)}, \quad L_2 := Ge^{2T(\|A\|_{[0,T]}+2)}, \quad K_1 := L_2 e^{2(\|A\|_{[0,T]}+1)T}, \\ C_{L_1,L_2}([0,T];\mathbb{R}^2) := \big\{(x_1,x_2)\in C([0,T];\mathbb{R}^2),\ L_1 \le x_i(\cdot) \le L_2,\ \ x_1(\cdot) \ge x_2(\cdot)\big\}. \end{cases}$$

For $i = 1, 2$, we choose $(p_1^{(i)}, p_2^{(i)}) \in C_{L_1,L_2}([0,T];\mathbb{R}^2)$, and define

$$\begin{cases} P_2^{(i)}(t) := \widetilde{G}\exp\Big[\int_t^T 2\big\{A(s) - 1 - \frac{p_2^{(i)}(s)}{p_1^{(i)}(s)}\big\}ds\Big], \quad t \in [0,T], \\ P_1^{(i)}(t) := G\exp\Big[\int_t^T \big[2A(s) - 1 + \frac{|p_2^{(i)}(s)|^2}{|p_1^{(i)}(s)|^2}\big]ds\Big], \quad t \in [0,T]. \end{cases}$$

Under (3.27), it is easy to see that $(P_1^{(i)}, P_2^{(i)}) \in C_{L_1,L_2}([0,T];\mathbb{R}^2)$. We denote by

$$\widehat{k}_1(s) := k_1^{(1)}(s) - k_1^{(2)}(s), \quad \widehat{k}_2(s) := k_2^{(1)}(s) - k_2^{(2)}(s), \quad s \in [0,T], \quad k := P,\ p.$$

After some calculation, one has

$$\begin{cases} \big|\widehat{P}_2(t)\big| \le 4K_1 e^{2\frac{L_2}{L_1}(T-t)} \dfrac{L_2(\|\widehat{p}_1\|_{[T-t,T]} + \|\widehat{p}_2\|_{[T-t,T]})}{L_1^2}(T-t), \\ \big|\widehat{P}_1(t)\big| \le 2K_1 e^{2\frac{L_2^2}{L_1^2}(T-t)} \dfrac{L_2^2(\|\widehat{p}_1\|_{[T-t,T]} + \|\widehat{p}_2\|_{[T-t,T]})}{L_1^3}(T-t). \end{cases}$$

We can choose T_1 such that for $\delta := T - T_1$,

$$2K_1\frac{L_2}{L_1^2}\Big[2e^{2\frac{L_2}{L_1}T} + e^{2\frac{L_2^2}{L_1^2}T}\frac{L_2}{L_1}\Big]\delta = \frac{1}{2}.$$

By contraction, one has the existence and uniqueness of (P_1, P_2) satisfying (3.28) on $[T_1, T]$.

Now let us look at the case of $[T_1-\delta, T_1]$, where

$$\begin{cases} P_2(t) = P_2(T_1)\exp\Big[\int_t^{T_1} 2\big\{A(s)-1-\frac{P_2(s)}{P_1(s)}\big\}ds\Big], & t\in[0,T_1],\\ P_1(t) = P_1(T_1)\exp\Big[\int_t^{T_1} \big[2A(s)-1+\frac{|P_2(s)|^2}{|P_1(s)|^2}\big]ds\Big], & t\in[0,T_1]. \end{cases}$$

Given $(p_1^{(i)}, p_2^{(i)}) \in C_{L_1,L_2}([0,T_1];\mathbb{R}^2)$, we see that $(P_1^{(i)}, P_2^{(i)}) \in C_{L_1,L_2}([0,T_1];\mathbb{R}^2)$, and

$$\begin{cases} \big|\widehat{P}_2(t)\big| \le 4K_1 e^{2\frac{L_2}{L_1}(T_1-t)}\frac{L_2(\|\widehat{p}_1\|_{[T_1-t,T_1]}+\|\widehat{p}_2\|_{[T_1-t,T_1]})}{L_1^2}(T_1-t), & t\in[0,T_1],\\ \big|\widehat{P}_1(t)\big| \le 2K_1 e^{2\frac{L_2^2}{L_1^2}(T_1-t)}\frac{L_2^2(\|\widehat{p}_1\|+\|\widehat{p}_2\|)}{L_1^3}(T_1-t), & t\in[0,T_1]. \end{cases}$$

By the choice of δ, we obtain the solvability in $[T-2\delta, T_1]$. By induction, one has the conclusion in $[0,T]$.

Step 3 We claim that $\Theta^* \neq \Xi^*$.

To prove this result, we first recall that $\widehat{\mathcal{P}}^* := \mathcal{P}_1^* + \mathcal{P}_2^*$, $\widehat{\mathscr{P}}^* := \mathscr{P}_1^* + \mathscr{P}_2^*$ satisfy

$$\begin{cases} d\widehat{\mathcal{P}}^*(s) = -\big[(2A(s)+\Theta^*(s))\widehat{\mathcal{P}}^*(s)\big]ds, & s\in[0,T],\\ d\widehat{\mathscr{P}}^*(s) = -\Big[(2A(s)+\Xi^*(s))\widehat{\mathscr{P}}^*(s)\Big]ds, & s\in[0,T],\\ \widehat{\mathcal{P}}^*(T) = (\widetilde{G}+G), \quad \widehat{\mathscr{P}}^*(T) = \widetilde{G}+G. \end{cases} \tag{3.29}$$

If $\Theta^* \equiv \Xi^*$, then by the uniqueness, $\widehat{\mathcal{P}}^* \equiv \widehat{\mathscr{P}}^*$. According to above two steps, $\frac{1}{\mathcal{P}_1^*}$, $\frac{1}{\mathscr{P}_1^*}$ exist. Therefore, due to the definitions of Θ^*, Ξ^*, one has $\mathscr{P}_1^* \equiv \mathcal{P}_1^*$ which implies that

$$\Theta^*\mathcal{P}_1^* = \Xi^*\mathscr{P}_1^* = 2\mathscr{P}_1^*\Xi^* + |\Xi^*|^2\mathscr{P}_1^*.$$

Hence $\Xi^*\mathscr{P}_1^*(\Xi^*+1) = 0$, which leads to $\Xi^* = -1$ or $\Xi^* = 0$. This indicates that $\widehat{\mathscr{P}}^* \equiv 0$ or $\mathscr{P}_2^* \equiv 0$.

Since $\widehat{\mathscr{P}}^*(T) = G + \widetilde{G} > 0$, by the continuity of $\widehat{\mathscr{P}}^*$ and the fact of $\Xi^* = -\frac{\widehat{\mathscr{P}}^*}{\mathscr{P}_1^*}$, we see that $\Xi^* \equiv 0$ does not hold.

Similarly, since $\widetilde{G} \neq 0$, by the continuity of $\mathscr{P}_2^*$, we see that $\mathscr{P}_2^* \equiv 0$ does not hold as well. We finish Step 3 by contradiction.

4 Proofs of the Main Results

In this section, we prove Theorems 3.1–3.3.

For $(\Theta_1, \Theta_2, \varphi) \in L^2(0,T;\mathbb{R}^{m\times m}) \times L^2(0,T;\mathbb{R}^{m\times m}) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$, we consider

$$
\begin{cases} dX = \big[AX + B(\Theta_1 + \Theta_2)X + B\varphi + b\big]ds \\ \qquad\quad + \big[CX + D(\Theta_1 + \Theta_2)X + D\varphi + \sigma\big]dW(s), \quad s \in [0, T], \\ X(0) = x_0. \end{cases} \tag{4.1}
$$

For $t \in [0, T)$, $\varepsilon > 0$, $v \in L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^m)$, let X^ε solve the following perturbed system:

$$
\begin{cases} dX^\varepsilon = \big[(A + B\Theta_1)X^\varepsilon + B\Theta_2 X + BvI_{[t,t+\varepsilon]} + B\varphi + b\big]ds \\ \qquad\quad + \big[(C + D\Theta_1)X^\varepsilon + D\Theta_2 X + DvI_{[t,t+\varepsilon]} + D\varphi + \sigma\big]dW(s), \\ X^\varepsilon(0) = x_0, \end{cases} \tag{4.2}
$$

with $s \in [0, T]$. Hence we see that $X_0^\varepsilon := X^\varepsilon - X$ satisfies

$$
\begin{cases} dX_0^\varepsilon = \big[(A + B\Theta_1)X_0^\varepsilon + BvI_{[t,t+\varepsilon]}\big]ds \\ \qquad\quad + \big[(C + D\Theta_1)X_0^\varepsilon + DvI_{[t,t+\varepsilon]}\big]dW(s), \\ X_0^\varepsilon(0) = 0. \end{cases} \tag{4.3}
$$

By Proposition 2.1 in [18], we have the following estimate of X_0^ε

$$
\mathbb{E}_t \sup_{r\in[t,t+\varepsilon]} |X_0^\varepsilon(r)|^2 \le K\varepsilon, \quad a.s., \quad t \in [0, T).
$$

We also define

$$
u := (\Theta_1 + \Theta_2)X + \varphi, \quad u^\varepsilon := \Theta_1 X^\varepsilon + \Theta_2 X + \varphi + vI_{[t,t+\varepsilon]}. \tag{4.4}
$$

Lemma 4.1 *Suppose (H1) holds,* $(\Theta_1, \Theta_2, \varphi)$ *are given as above,* u, u^ε *are defined in* (4.4). *Then we have*

$$
\begin{aligned} &J(t, x, u^\varepsilon(\cdot)) - J(t, x, u(\cdot)) \\ &\quad = J_1(t, x) + J_2(t, x) + \mathbb{E}_t \int_t^{t+\varepsilon} \langle (\mathscr{S}^\top + \Theta_1^\top \mathscr{R})v, X_0^\varepsilon \rangle\, ds, \end{aligned} \tag{4.5}
$$

where $\mathscr{R}$, $\mathscr{S}$ *are defined in* (2.3),

$$
\begin{cases} J_1(t) := \mathbb{E}_t \displaystyle\int_t^T \big[\langle F_1, X_0^\varepsilon \rangle + \langle F_2, vI_{[t,t+\varepsilon)} \rangle\big]ds + \mathbb{E}_t \langle GX(T) + \widetilde{G}\mathbb{E}_t X(T), X_0^\varepsilon(T) \rangle, \\ J_2(t) := \dfrac{1}{2}\mathbb{E}_t \displaystyle\int_t^T \langle F_1^\varepsilon, X_0^\varepsilon \rangle\, ds + \dfrac{1}{2}\mathbb{E}_t \langle GX_0^\varepsilon(T) + \widetilde{G}\mathbb{E}_t X_0^\varepsilon(T), X_0^\varepsilon(T) \rangle, \\ F_1 := \big[Q + \Theta_1^\top S + \Theta_1^\top R(\Theta_1 + \Theta_2) + S^\top(\Theta_1 + \Theta_2)\big]X + (S^\top + \Theta_1^\top R)\varphi \\ \quad + \big[\widetilde{Q} + \Theta_1^\top \widetilde{S} + \Theta_1^\top \widetilde{R}(\Theta_1 + \Theta_2) + \widetilde{S}^\top(\Theta_1 + \Theta_2)\big]\mathbb{E}_t X + (\widetilde{S}^\top + \Theta_1^\top \widetilde{R})\mathbb{E}_t\varphi, \\ F_2 := \dfrac{1}{2}\mathscr{R}v + \big[S + R(\Theta_1 + \Theta_2)\big]X + R\varphi + \big[\widetilde{S} + \widetilde{R}(\Theta_1 + \Theta_2)\big]\mathbb{E}_t X + \widetilde{R}\mathbb{E}_t\varphi, \\ F_1^\varepsilon := \big[Q + S^\top\Theta_1 + \Theta_1^\top S + \Theta_1^\top R\Theta_1\big]X_0^\varepsilon + \big[\widetilde{Q} + \widetilde{S}^\top\Theta_1 + \Theta_1^\top \widetilde{S} + \Theta_1^\top \widetilde{R}\Theta_1\big]\mathbb{E}_t X_0^\varepsilon. \end{cases}
$$

Proof By above definitions of X, X^ε and X_0^ε, we deal with the terms in the cost functional one by one. First let us treat the term associated with Q,

$$\langle QX^\varepsilon, X^\varepsilon \rangle - \langle QX, X \rangle = 2\langle QX, X_0^\varepsilon \rangle + \langle QX_0^\varepsilon, X_0^\varepsilon \rangle .$$

Then we look at the one with S. From the definitions of u and u^ε, we have

$$\begin{aligned}\langle SX^\varepsilon, u^\varepsilon \rangle - \langle SX, u \rangle \; &= \langle S^\top \Theta_1 X_0^\varepsilon, X_0^\varepsilon \rangle + \langle X_0^\varepsilon, S^\top \big[(\Theta_1 + \Theta_2)X + vI_{[t,t+\varepsilon]} + \varphi\big] \rangle \\ &\quad + \langle X_0^\varepsilon, \Theta_1^\top SX \rangle + \langle SX, vI_{[t,t+\varepsilon]} \rangle .\end{aligned}$$

We also have

$$\begin{aligned}&\langle Ru^\varepsilon, u^\varepsilon \rangle - \langle Ru, u \rangle \\ &\quad = \langle \Theta_1^\top R\Theta_1 X_0^\varepsilon, X_0^\varepsilon \rangle + 2\langle RvI_{[t,t+\varepsilon]}, \Theta_1 X_0^\varepsilon \rangle + \langle Rv, vI_{[t,t+\varepsilon]} \rangle \\ &\qquad + 2\langle R\Theta_1 X_0^\varepsilon, (\Theta_1+\Theta_2)X + \varphi \rangle + 2\langle RvI_{[t,t+\varepsilon]}, (\Theta_1+\Theta_2)X + \varphi \rangle .\end{aligned}$$

Similarly one can obtain the terms involving $\widetilde{Q}$, $\widetilde{S}$, $\widetilde{R}$ as,

$$\begin{cases} \langle \widetilde{Q}\mathbb{E}_t X^\varepsilon, \mathbb{E}_t X^\varepsilon \rangle - \langle \widetilde{Q}\mathbb{E}_t X, \mathbb{E}_t X \rangle = 2\langle \widetilde{Q}\mathbb{E}_t X, \mathbb{E}_t X_0^\varepsilon \rangle + \langle \widetilde{Q}\mathbb{E}_t X_0^\varepsilon, \mathbb{E}_t X_0^\varepsilon \rangle, \\ \langle \widetilde{S}\mathbb{E}_t X^\varepsilon, \mathbb{E}_t u^\varepsilon \rangle - \langle \widetilde{S}\mathbb{E}_t X, \mathbb{E}_t u \rangle \\ \quad = \Big\langle \widetilde{S}^\top \Theta_1 \mathbb{E}_t X_0^\varepsilon, \mathbb{E}_t X_0^\varepsilon \rangle + \langle \mathbb{E}_t X_0^\varepsilon, \widetilde{S}^\top \big[(\Theta_1+\Theta_2)\mathbb{E}_t X + vI_{[t,t+\varepsilon]} + \mathbb{E}_t\varphi\big]\Big\rangle \\ \qquad + \langle \mathbb{E}_t X_0^\varepsilon, \Theta_1^\top \widetilde{S}\mathbb{E}_t X \rangle + \langle \widetilde{S}\mathbb{E}_t X, vI_{[t,t+\varepsilon]} \rangle, \\ \langle \widetilde{R}\mathbb{E}_t u^\varepsilon, \mathbb{E}_t u^\varepsilon \rangle - \langle \widetilde{R}\mathbb{E}_t u, \mathbb{E}_t u \rangle \\ \quad = \Big\langle \Theta_1^\top \widetilde{R}\Theta_1 \mathbb{E}_t X_0^\varepsilon, \mathbb{E}_t X_0^\varepsilon \rangle + 2\langle \widetilde{R}vI_{[t,t+\varepsilon]}, \Theta_1 \mathbb{E}_t X_0^\varepsilon \rangle + \langle \widetilde{R}v, vI_{[t,t+\varepsilon]}\Big\rangle \\ \qquad + 2\langle \widetilde{R}\Theta_1 \mathbb{E}_t X_0^\varepsilon, (\Theta_1+\Theta_2)\mathbb{E}_t X + \mathbb{E}_t\varphi \rangle + 2\langle \widetilde{R}vI_{[t,t+\varepsilon]}, (\Theta_1+\Theta_2)\mathbb{E}_t X + \mathbb{E}_t\varphi \rangle . \end{cases}$$

At last we have the follows results on the terms associated with G and $\widetilde{G}$,

$$\begin{cases} \langle GX^\varepsilon(T), X^\varepsilon(T) \rangle - \langle GX(T), X(T) \rangle \\ \quad = 2\langle GX(T), X_0^\varepsilon(T) \rangle + \langle GX_0^\varepsilon(T), X_0^\varepsilon(T) \rangle, \\ \langle \widetilde{G}\mathbb{E}_t X^\varepsilon(T), \mathbb{E}_t X^\varepsilon(T) \rangle - \langle \widetilde{G}\mathbb{E}_t X(T), \mathbb{E}_t X(T) \rangle \\ \quad = 2\langle \widetilde{G}\mathbb{E}_t X(T), \mathbb{E}_t X_0^\varepsilon(T) \rangle + \langle \widetilde{G}\mathbb{E}_t X_0^\varepsilon(T), \mathbb{E}_t X_0^\varepsilon(T) \rangle . \end{cases}$$

To sum up, we deduce above (4.5). □

Next we carry out further study on $J_1(t)$ and $J_2(t)$ by making some equivalent transformations. In fact, from the definitions of equilibrium controls it is unavoidable to take certain convergence arguments. Fortunately, in above we derive the important and useful structure of $\mathbb{E}_t \int_t^{t+\varepsilon} \langle F_2(r), v \rangle \, dr$. Consequently, we will derive similar expressions for other terms in $J_1(t)$, $J_2(t)$. This is the starting point for our later investigations.

4.1 A New Decoupling Result

Inspired by the decoupling techniques in the literature (e.g., [11,24], etc), we present one conclusion which serves our purpose of this paper. It is interesting in its own right and may be potentially useful for (among others) various problems.

Given $t \in [0, T]$, we consider

$$\begin{cases} dX = \big[A_1 X + A_2\big]dr + \big[B_1 X + B_2\big]dW(r), \quad r \in [t, T], \\ dY = -\Big[C_1 Y + C_2 Z + C_3 X + C_4 \mathbb{E}_t X + C_5 + \mathbb{E}_t C_6\Big]dr + Z dW(r), \\ X(0) = x, \quad Y(T, t) = D_1 X(T) + D_2 \mathbb{E}_t X(T) + D_3. \end{cases} \tag{4.6}$$

(H1) For $H := \mathbb{R}^m, \ \mathbb{R}^n, \ \mathbb{R}^{n\times n}$, etc, suppose

$$\begin{aligned} &A_1, \ B_1, \ C_i \in L^2(0, T; H), \quad A_2, \ C_5 \in L^2(\Omega; L^1(0, T; H)), \\ &B_2 \in L^2_{\mathbb{F}}(0, T; H), \quad D_1, \ D_2, \ D_3, \ x \in H. \end{aligned}$$

For $t \in [0, T]$ and $s \in [t, T]$, suppose that

$$Y(s, t) = P_1(s)X(s) + P_2(s)\mathbb{E}_t X(s) + \mathbb{E}_t P_3(s) + P_4(s), \tag{4.7}$$

where $P_1, \ P_2$ are deterministic, $P_3, \ P_4$ are stochastic processes satisfying

$$\begin{cases} dP_i(s) = \Pi_i(s)ds, \quad i = 1, 2, \quad P_1(T) = D_1, \quad P_2(T) = D_2, \\ dP_j(s) = \Pi_j(s)ds + \mathcal{L}_j(s)dW(s), \quad j = 3, 4, \quad P_3(T) = 0, \quad P_4(T) = D_3. \end{cases}$$

Here Π_i are to be determined. It is easy to see

$$d\mathbb{E}_t X = \big[A_1 \mathbb{E}_t X + \mathbb{E}_t A_2\big]dr.$$

Using Itô's formula, we derive that

$$\begin{cases} d\big[P_1 X\big] = \Big[\Pi_1 X + P_1(A_1 X + A_2)\Big]ds + P_1\big(B_1 X + B_2\big)dW(s), \\ d\big[P_2 \mathbb{E}_t X\big] = \Big\{\Pi_2 \mathbb{E}_t X + P_2\big[A_1 \mathbb{E}_t X + \mathbb{E}_t A_2\big]\Big\}ds. \end{cases}$$

As a result, we have

$$\begin{aligned} dY = &\Big\{\big[\Pi_1 + P_1 A_1\big]X + (\Pi_2 + P_2 A_1)\mathbb{E}_t X \\ &+ \mathbb{E}_t\big[\Pi_3 + P_2 A_2\big] + \Pi_4 + P_1 A_2\Big\}ds + \Big[P_1 B_1 X + P_1 B_2 + L_4\Big]dW(s). \end{aligned}$$

Consequently, it is necessary to have

$$Z = P_1 B_1 X + P_1 B_2 + L_4. \tag{4.8}$$

In this case, from (4.7), (4.8), we see that

$$\begin{cases} \mathbb{E}_t Y = (P_1 + P_2)\mathbb{E}_t X + \mathbb{E}_t\big[P_3 + P_4\big], \\ \mathbb{E}_t Z = P_1 B_1 \mathbb{E}_t X + \mathbb{E}_t\big[P_1 B_2 + L_4\big]. \end{cases}$$

On the other hand,

$$
\begin{aligned}
&-\Big[C_1Y + C_2Z + C_3X + C_4\mathbb{E}_tX + C_5 + \mathbb{E}_tC_6\Big]\\
&\quad = -C_1\Big\{P_1X + P_2\mathbb{E}_tX + \mathbb{E}_tP_3 + P_4\Big\} - C_2\Big[P_1B_1X + P_1B_2 + L_4\Big]\\
&\qquad - C_3X - C_4\mathbb{E}_tX - C_5 - \mathbb{E}_tC_6.
\end{aligned}
$$

At this moment, we can choose $\Pi_i(\cdot)$ in the following ways,

$$
\begin{cases}
0 = \Pi_1 + P_1A_1 + C_1P_1 + C_2P_1B_1 + C_3,\\
0 = \Pi_2 + P_2A_1 + C_1P_2 + C_4,\\
0 = \Pi_4 + P_1A_2 + C_1P_4 + C_2\big[P_1B_2 + L_4\big] + C_5,\\
0 = \Pi_3 + P_2A_2 + C_1P_3 + C_6.
\end{cases}
$$

Next we make above arguments rigorous. Given the notations in (2.3), for $s \in [0, T]$, we consider the following systems of equations

$$
\begin{cases}
dP_1 = -\Big[P_1A_1 + C_1P_1 + C_3P_1B_1 + C_3\Big]ds,\\
dP_2 = -\Big\{P_2A_1 + C_1P_2 + C_4\Big\}ds,\\
dP_3 = -\Big[C_1P_3 + P_2A_2 + C_6\Big]ds + L_3dW(s),\\
dP_4 = -\Big\{C_1P_4 + C_2L_4 + C_2P_1B_2 + P_1A_2 + C_5\Big\}ds + L_4dW(s),\\
P_1(T) = D_1,\ P_2(T) = D_2,\ P_3(T) = 0,\ P_4(T) = D_3.
\end{cases} \tag{4.9}
$$

From Proposition 2.1 in [18], under (H1) we see the following regularities,

$$
\begin{aligned}
P_1,\ P_2 \in C([0, T]; \mathbb{R}^{n\times n}),\ (P_3, L_3), (P_4, L_4)\\
\in L^2_{\mathbb{F}}(\Omega; C([0, T]; \mathbb{R}^n)) \times L^2_{\mathbb{F}}(0, T; \mathbb{R}^n).
\end{aligned}
$$

At this moment, for $s \in [0, T]$, and $t \in [0, s]$, we define a pair of processes

$$
M := P_1X + P_2\mathbb{E}_tX + \mathbb{E}_tP_3 + P_4,\quad N := P_1B_1X + P_1B_2 + L_4. \tag{4.10}
$$

By the results of P_i, we can conclude that

$$
(M_d, N) \in L^2_{\mathbb{F}}(\Omega; C([0, T]; \mathbb{R}^n)) \times L^2_{\mathbb{F}}(0, T; \mathbb{R}^n)
$$

where $M_d(s) \equiv M(s, s)$ with $s \in [0, T]$. We present the following result.

Lemma 4.2 *Given* $(\Theta, \varphi) \in L^2(0, T; \mathbb{R}^{m\times n}) \times L^2_{\mathbb{F}}(0, T; \mathbb{R}^m)$*, suppose* (X, Y, Z) *is the unique solution of* (4.6) *and* (M, N) *is defined in* (4.10)*. Then for any* $t \in [0, T]$*,*

$$
\begin{aligned}
&\mathbb{P}\Big\{\omega \in \Omega;\ Y(s, t) = M(s, t),\quad \forall s \in [t, T]\Big\} = 1,\\
&\mathbb{P}\Big\{\omega \in \Omega;\ Z(s, t) = N(s)\Big\} = 1,\quad s \in [t, T].\ a.e.
\end{aligned}
$$

Proof Given (4.10), it is easy to see that

$$\mathbb{E}_t M = (P_1 + P_2)\mathbb{E}_t X + \mathbb{E}_t[P_3 + P_4], \quad \mathbb{E}_t N = P_1 B_1 \mathbb{E}_t X + P_1 \mathbb{E}_t B_2 + \mathbb{E}_t L_4.$$

Using Itô's formula, we know that

$$\begin{cases} d\big[P_1 X\big] = \Big[-(C_1 P_1 + C_2 P_1 B_1 + C_3)X + P_1 A_2\Big]ds + P_1\big(B_1 X + B_2\big)dW(s), \\ d\big[P_2 \mathbb{E}_t X\big] = \Big\{ -\Big[C_1 P_2 + C_4\Big]\mathbb{E}_t X + P_2 \mathbb{E}_t A_2\Big\}ds. \end{cases}$$

Consequently, after some calculations one has

$$dM = -\Big[C_1 M + C_2 N + C_3 X + C_4 \mathbb{E}_t X + C_5 + \mathbb{E}_t C_6\Big]dr + N dW(r).$$

Considering $P_i(T)$ in (4.9), we see that for any $t \in [0, T]$, $(M, N) \in L^2_{\mathbb{F}}(\Omega; C([t, T]; \mathbb{R}^n)) \times L^2_{\mathbb{F}}(0, T; \mathbb{R}^n)$ satisfies the backward equation in (4.6). The conclusion is followed by the uniqueness of BSDEs. □

4.2 A New Expression of J_1

In this part, we deal with $J_1(t)$ in Lemma 4.1. For convenience, we rewrite the equation of $X_0^\varepsilon := X^\varepsilon - X$ as

$$\begin{cases} dX_0^\varepsilon = \big[A_\theta X_0^\varepsilon + BvI_{[t,t+\varepsilon]}\big]ds + \big[C_\theta X_0^\varepsilon + DvI_{[t,t+\varepsilon]}\big]dW(s), \\ X_0^\varepsilon(0) = 0, \end{cases} \tag{4.11}$$

where $s \in [0, T]$, and

$$A_\theta := A + B\Theta_1, \quad C_\theta := C + D\Theta_1.$$

We introduce

$$\begin{cases} dY = -\Big[A_\theta^\top Y + C_\theta^\top Z + F_1\Big]dr + ZdW(r), \quad r \in [t, T], \\ Y(T, t) = GX(T) + \widetilde{G}\mathbb{E}_t X(T), \end{cases} \tag{4.12}$$

where X satisfies (4.1), F_1 is in Lemma 4.1. From Proposition 2.1 in [18], (4.12) is solvable with

$$(Y, Z) \in L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n), \quad t \in [0,T).$$

By Itô's formula on $[t, T]$, we have

$$\begin{aligned} d\langle Y, X_0^\varepsilon\rangle = & -\langle A_\theta^\top Y + C_\theta^\top Z + F_1, X_0^\varepsilon\rangle dr + \langle Z, X_0^\varepsilon\rangle dW(r) \\ & + \langle Y, A_\theta X_0^\varepsilon + BvI_{[t,t+\varepsilon]}\rangle dr + \langle Y, C_\theta X_0^\varepsilon + DvI_{[t,t+\varepsilon]}\rangle dW(r) \\ & + \langle Z, C_\theta X_0^\varepsilon + DvI_{[t,t+\varepsilon]}\rangle dr. \end{aligned}$$

From (4.12) we then arrive at

$$\begin{aligned} & \mathbb{E}_t \langle GX(T) + \widetilde{G}\mathbb{E}_t X(T) + g, X_0^\varepsilon(T)\rangle + \mathbb{E}_t \int_t^T \langle F_1, X_0^\varepsilon\rangle dr \\ & \quad = \mathbb{E}_t \int_t^{t+\varepsilon} \langle B^\top Y + D^\top Z, v\rangle dr. \end{aligned} \tag{4.13}$$

Inspired by Lemma 4.2, we introduce

$$\begin{cases} d\mathcal{P}_1 = -\Big[\mathcal{P}_1(A + B\Theta_1 + B\Theta_2) + (C + D\Theta_1)^\top \mathcal{P}_1(C + D\Theta_1 + D\Theta_2) \\ \qquad + (A + B\Theta_1)^\top \mathcal{P}_1 + \big[Q + \Theta_1^\top S + \Theta_1^\top R(\Theta_1 + \Theta_2) + S^\top(\Theta_1 + \Theta_2)\big]\Big] ds, \\ d\mathcal{P}_2 = -\Big\{\mathcal{P}_2(A + B\Theta_1 + B\Theta_2) + (A + B\Theta_1)^\top \mathcal{P}_2 + \big[\widetilde{Q} + \Theta_1^\top \widetilde{S} \\ \qquad + \Theta_1^\top \widetilde{R}(\Theta_1 + \Theta_2) + \widetilde{S}^\top(\Theta_1 + \Theta_2)\big]\Big\} ds, \\ d\mathcal{P}_3 = -\Big[(A + B\Theta_1)^\top \mathcal{P}_3 + \mathcal{P}_2(B\varphi + b) + (\widetilde{S}^\top + \Theta_1^\top \widetilde{R})\varphi\Big] ds + \mathcal{L}_3 dW(s), \\ d\mathcal{P}_4 = -\Big\{(A + B\Theta_1)^\top \mathcal{P}_4 + (C + D\Theta_1)^\top \mathcal{L}_4 + (C + D\Theta_1)^\top \mathcal{P}_1(D\varphi + \sigma) \\ \qquad + \mathcal{P}_1(B\varphi + b) + (S^\top + \Theta_1^\top R)\varphi\Big\} ds + \mathcal{L}_4 dW(s), \\ \mathcal{P}_1(T) = G, \ \mathcal{P}_2(T) = \widetilde{G}, \ \mathcal{P}_3(T) = 0, \ \mathcal{P}_4(T) = 0. \end{cases} \tag{4.14}$$

Moreover, the following equalities hold on $[t, T]$,

$$\begin{aligned} Y &= \mathcal{P}_1 X + \mathcal{P}_2 \mathbb{E}_t X + \mathbb{E}_t \mathcal{P}_3 + \mathcal{P}_4, \\ Z &= \mathcal{P}_1(C + D\Theta_1 + D\Theta_2)X + \mathcal{P}_1(D\varphi + \sigma) + \mathcal{L}_4. \end{aligned}$$

Consequently,

$$\begin{aligned} B^\top Y + D^\top Z = & \big[B^\top \mathcal{P}_1 + D^\top \mathcal{P}_1(C + D\Theta_1 + D\Theta_2)\big]X + B^\top \mathcal{P}_2 \mathbb{E}_t X \\ & + B^\top \mathbb{E}_t \mathcal{P}_3 + B^\top \mathcal{P}_4 + D^\top \mathcal{P}_1(D\varphi + \sigma) + D^\top \mathcal{L}_4. \end{aligned}$$

This shows that

$$\begin{aligned} & \mathbb{E}_t \int_t^{t+\varepsilon} \langle B^\top Y + D^\top Z, v\rangle dr \\ & \quad = \mathbb{E}_t \int_t^{t+\varepsilon} \Big\langle \big[B^\top(\mathcal{P}_1 + \mathcal{P}_2) + D^\top \mathcal{P}_1(C + D\Theta_1 + D\Theta_2)\big]X \\ & \qquad\qquad + B^\top(\mathcal{P}_3 + \mathcal{P}_4) + D^\top \mathcal{P}_1(D\varphi + \sigma) + D^\top \mathcal{L}_4, v\Big\rangle dr. \end{aligned}$$

By the definition of $J_1(t)$ and above (4.13), we see that

$$J_1(t) = \mathbb{E}_t \int_t^{t+\varepsilon} \Big\langle \Big[\mathscr{S} + \mathscr{R}(\Theta_1+\Theta_2) + \big[B^\top(\mathcal{P}_1+\mathcal{P}_2) + D^\top \mathcal{P}_1(C + D\Theta_1 + D\Theta_2)\big]\Big] X + \frac{1}{2}\mathscr{R}v + \mathscr{R}\varphi + B^\top(\mathcal{P}_3+\mathcal{P}_4) + D^\top\mathcal{P}_1(D\varphi+\sigma) + D^\top\mathcal{L}_4, v\Big\rangle dr. \tag{4.15}$$

Lemma 4.3 *Suppose (H1) holds, X solves* (4.1) *associated with* $(\Theta_1, \Theta_2, \varphi)$*, and* $J_1(t)$ *is defined in Lemma* 4.1*. Then* (4.15) *is true, where* $\mathcal{P}_i$ *satisfies* (4.14)*.*

4.3 A New Expression of J_2

In the following, we turn to treating J_2. To this end, we introduce

$$\begin{cases} dY_0^\varepsilon = -\Big[A_\theta^\top Y_0^\varepsilon + C_\theta^\top Z_0^\varepsilon + F_1^\varepsilon\Big]dr + Z_0^\varepsilon dW(r), \quad r \in [t,T], \\ Y_0^\varepsilon(T,t) = GX_0^\varepsilon(T) + \widetilde{G}\mathbb{E}_t X_0^\varepsilon(T), \end{cases}$$

where F_1^ε is defined in Lemma 4.1. From Proposition 2.1 in [18], we see that

$$(Y_0^\varepsilon, Z_0^\varepsilon) \in L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n), \quad t \in [0,T).$$

Recall X_0^ε in (4.11), we obtain the following by Itô's formula,

$$\begin{aligned} d\langle Y_0^\varepsilon, X_0^\varepsilon\rangle = &-\langle A_\theta^\top Y_0^\varepsilon + C_\theta^\top Z_0^\varepsilon + F_1^\varepsilon, X_0^\varepsilon\rangle dr + \langle Z_0^\varepsilon, X_0^\varepsilon\rangle dW(r) \\ &+ \langle Y_0^\varepsilon, A_\theta X_0^\varepsilon + BvI_{[t,t+\varepsilon]}\rangle dr + \langle Y_0^\varepsilon, C_\theta X_0^\varepsilon + DvI_{[t,t+\varepsilon]}\rangle dW(r) \\ &+ \langle Z_0^\varepsilon, C_\theta X_0^\varepsilon + DvI_{[t,t+\varepsilon]}\rangle dr. \end{aligned}$$

As a result, we then have

$$\begin{aligned} &\mathbb{E}_t\langle GX_0^\varepsilon(T) + \widetilde{G}\mathbb{E}_t X_0^\varepsilon(T), X_0^\varepsilon(T)\rangle + \mathbb{E}_t\int_t^T \langle F_1^\varepsilon, X_0^\varepsilon\rangle dr \\ &= \mathbb{E}_t\int_t^{t+\varepsilon}\langle B^\top Y_0^\varepsilon + D^\top Z_0^\varepsilon, v\rangle dr. \end{aligned} \tag{4.16}$$

By the decoupling tricks in Lemma 4.2, we introduce

$$\begin{cases} d\bar{\mathcal{P}}_1 = -\Big[\bar{\mathcal{P}}_1(A+B\Theta_1) + (A+B\Theta_1)^\top\bar{\mathcal{P}}_1 + (C+D\Theta_1)^\top\bar{\mathcal{P}}_1(C+D\Theta_1) \\ \qquad\quad + \big[Q + S^\top\Theta_1 + \Theta_1^\top S + \Theta_1^\top R\Theta_1\big]\Big]ds, \\ d\bar{\mathcal{P}}_2 = -\Big\{\bar{\mathcal{P}}_2(A+B\Theta_1) + (A+B\Theta_1)^\top\bar{\mathcal{P}}_2 + \big[\widetilde{Q} + \widetilde{S}^\top\Theta_1 + \Theta_1^\top\widetilde{S} + \Theta_1^\top\widetilde{R}\Theta_1\big]\Big\}ds, \\ d\bar{\mathcal{P}}_3 = -\Big[(A+B\Theta_1)^\top\bar{\mathcal{P}}_3 + \bar{\mathcal{P}}_2 BvI_{[t,t+\varepsilon]}\Big]ds + \bar{\mathcal{L}}_3 dW(s), \\ d\bar{\mathcal{P}}_4 = -\Big\{(A+B\Theta_1)^\top\bar{\mathcal{P}}_4 + \big[(C+D\Theta_1)^\top\bar{\mathcal{P}}_1 D + \bar{\mathcal{P}}_1 B\big]vI_{[t,t+\varepsilon]} \\ \qquad\quad + (C+D\Theta_1)^\top\bar{\mathcal{L}}_4\Big\}ds + \bar{\mathcal{L}}_4 dW(s), \\ \bar{\mathcal{P}}_1(T) = G,\ \bar{\mathcal{P}}_2(T) = \widetilde{G},\ \bar{\mathcal{P}}_3(T) = 0,\ \bar{\mathcal{P}}_4(T) = 0. \end{cases}$$

Moreover, from Lemma 4.2, the following holds on $[t, T]$,

$$\begin{aligned} Y_0^\varepsilon &= \bar{\mathcal{P}}_1 X_0^\varepsilon + \bar{\mathcal{P}}_2 \mathbb{E}_t X_0^\varepsilon + \mathbb{E}_t \bar{\mathcal{P}}_3 + \bar{\mathcal{P}}_4, \\ Z_0^\varepsilon &= \bar{\mathcal{P}}_1 (C + D\Theta_1) X_0^\varepsilon + \bar{\mathcal{P}}_1 D v I_{[t,t+\varepsilon]} + \bar{\mathcal{L}}_4. \end{aligned}$$

At this moment, we take a closer look at $(\bar{\mathcal{P}}_3, \bar{\mathcal{L}}_3)$, $(\bar{\mathcal{P}}_4, \bar{\mathcal{L}}_4)$. By the uniqueness of BSDEs in Proposition 2.1 of [18], we have the following equalities

$$\bar{\mathcal{P}}_3(s) = \widetilde{\mathcal{P}}_3(s) v, \quad \bar{\mathcal{L}}_3(s) = 0, \quad \bar{\mathcal{P}}_4(s) = \widetilde{\mathcal{P}}_4(s) v, \quad \bar{\mathcal{L}}_4(s) = 0, \quad s \in [t, T],$$

where

$$\begin{cases} d\widetilde{\mathcal{P}}_3 = -\Big[(A + B\Theta_1)^\top \widetilde{\mathcal{P}}_3 + \bar{\mathcal{P}}_2 B I_{[t,t+\varepsilon]}\Big] ds, \quad s \in [t, T], \\ d\widetilde{\mathcal{P}}_4 = -\Big\{(A + B\Theta_1)^\top \widetilde{\mathcal{P}}_4 + \big[(C + D\Theta_1)^\top \bar{\mathcal{P}}_1 D + \bar{\mathcal{P}}_1 B\big] I_{[t,t+\varepsilon]}\Big\} ds, \quad s \in [t, T], \\ \widetilde{\mathcal{P}}_3(T) = \widetilde{\mathcal{P}}_4(T) = 0. \end{cases}$$

Consequently, on $[t, T]$ we conclude that

$$\begin{aligned} B^\top Y_0^\varepsilon + D^\top Z_0^\varepsilon = &\big[B^\top \bar{\mathcal{P}}_1 + D^\top \bar{\mathcal{P}}_1 (C + D\Theta_1)\big] X_0^\varepsilon + B^\top \bar{\mathcal{P}}_2 \mathbb{E}_t X_0^\varepsilon \\ &+ B^\top \mathbb{E}_t \widetilde{\mathcal{P}}_3 + B^\top \widetilde{\mathcal{P}}_4 + D^\top \bar{\mathcal{P}}_1 D v I_{[t,t+\varepsilon]}. \end{aligned}$$

As a result,

$$\begin{aligned} &\mathbb{E}_t \int_t^{t+\varepsilon} \langle B^\top Y_0^\varepsilon + D^\top Z_0^\varepsilon, v \rangle \, dr \\ &= \mathbb{E}_t \int_t^{t+\varepsilon} \langle B^\top \big[\bar{\mathcal{P}}_1 + \bar{\mathcal{P}}_2 + D^\top \bar{\mathcal{P}}_1 (C + D\Theta_1)\big] X_0^\varepsilon + B^\top [\widetilde{\mathcal{P}}_3 + \widetilde{\mathcal{P}}_4] + D^\top \bar{\mathcal{P}}_1 D v, v \rangle \, dr. \end{aligned}$$

By the estimate of X_0^ε, for almost $t \in [0, T)$,

$$\mathbb{E}_t \int_t^{t+\varepsilon} \Big\langle B^\top \big[\bar{\mathcal{P}}_1 + \bar{\mathcal{P}}_2 + D^\top \bar{\mathcal{P}}_1 (C + D\Theta_1)\big] X_0^\varepsilon, v \Big\rangle dr = o(\varepsilon).$$

From the equations of $(\widetilde{\mathcal{P}}_3, \widetilde{\mathcal{P}}_4)$,

$$\sup_{t \in [t, t+\varepsilon]} \big[|\widetilde{\mathcal{P}}_3(t)|^2 + |\widetilde{\mathcal{P}}_4(t)|^2\big] = o(\varepsilon).$$

To sum up, by the definition of J_2 and (4.16), for almost $t \in [0, T)$ we deduce that

$$J_2(t) = \frac{\varepsilon}{2} \langle D(t)^\top \bar{\mathcal{P}}_1(t) D(t) v, v \rangle + o(\varepsilon). \tag{4.17}$$

Lemma 4.4 *Suppose (H1) holds, X_0^ε is in (4.11) associated with $(\Theta_1, \Theta_2, \varphi)$, and $J_2(t)$ is defined in Lemma 4.1. Then (4.17) is true.*

4.4 Proofs of the Main Results

We are in the position to give the proofs of the main results in Sect. 3.

To begin with, we give the proof of Theorem 3.1.

Proof In Lemmas 4.1, 4.3, and 4.4, we take $\Theta_1 \equiv \Theta_2 \equiv 0$. Hence for the notations in (4.4), $u \equiv \varphi$ and

$$\begin{cases} J_1(t) = \mathbb{E}_t \int_t^{t+\varepsilon} \Big\langle \Big[\mathscr{S} + \big[B^\top(P_1+P_2) + D^\top P_1 C\big]\Big] X + \frac{1}{2}\mathscr{R} v \\ \qquad + \mathscr{R} u + B^\top(P_3+P_4) + D^\top P_1(Du+\sigma) + D^\top L_4, v \Big\rangle dr, \\ J_2(t) = \frac{\varepsilon}{2}\langle D(t)^\top P_1(t) D(t) v, v\rangle + o(\varepsilon), \end{cases}$$

where P_i, $i = 1, 2$, (P_j, L_j), $j = 3, 4$, satisfies (3.2). Moreover, for any $t \in [0, T)$, by the estimate of X_0^ε,

$$\mathbb{E}_t \int_t^{t+\varepsilon} \langle \mathscr{S}(s)^\top v, X_0^\varepsilon(s)\rangle\, ds = o(\varepsilon).$$

We set out to define $\bar{X}$ the state process associated with $\bar{u}$, $u^{v,\varepsilon} := \bar{u} + vI_{[t,t+\varepsilon]}$, and for any $t \in [0, T)$

$$\begin{cases} \mathscr{D}_0(t) := \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_t^{t+\varepsilon} \big[\mathscr{R}(s) + D(s)^\top P_1(s) D(s)\big] ds, \\ \mathscr{H}_0(t) := \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\mathbb{E}_t \int_t^{t+\varepsilon} \Big[\mathscr{S}(s)\bar{X}(s) + \mathscr{R}(s)\bar{u}(s) + B(s)^\top \bar{M}(s,s) + D(s)^\top \bar{N}(s)\Big] ds \end{cases} \tag{4.18}$$

with $(\bar{M}, \bar{N})$ in (3.3) corresponding to $\bar{u}$. To sum up, $u \equiv \bar{u} = \bar{\varphi}$ is an equilibrium control associated with x_0 if and only if for any $t \in [0, T)$, $v \in L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^m)$,

$$0 \le \lim_{\varepsilon\to 0} \frac{J(t, \bar{X}(t); u^{v,\varepsilon}(\cdot)) - J\big(t, \bar{X}(t); \bar{u}(\cdot)\big)}{\varepsilon} = \langle \mathscr{D}_0(t) v, v\rangle + \langle \mathscr{H}_0(t), v\rangle.$$

Given $t \in [0, T)$, this holds if and only if both $\mathscr{H}_0(t) = 0$ and $\mathscr{D}_0(t) \ge 0$. Since both $\mathscr{R}$ and P_1 are bounded and deterministic, we thus know that

$$0 \le \mathscr{R}(t) + D(t)^\top P_1(t) D(t), \quad t \in [0, T]. \ a.e.$$

If $\mathscr{H}_0(t) = 0$, then by Lemma 3.4 in [12], above (3.5) holds. Conversely, if (3.5) is true, we immediately obtain $\mathscr{H}_0(t) = 0$. □

Next we present the proof of Theorem 3.2.

Proof In Lemmas 4.1, 4.3, and 4.4, we take $\Theta_1 \equiv 0$. Hence for the notations in (4.4), we have $u \equiv \Theta_2 X + \varphi$ and

$$\begin{cases} J_1(t) = \mathbb{E}_t \int_t^{t+\varepsilon} \Big\langle \Big[\mathscr{S} + \mathscr{R}\Theta_2 + \big[B^\top (\mathcal{P}_1 + \mathcal{P}_2) + D^\top \mathcal{P}_1 (C + D\Theta_2) \big] \Big] X + \frac{1}{2}\mathscr{R} v \\ \qquad +\mathscr{R}\varphi + B^\top (\mathcal{P}_3 + \mathcal{P}_4) + D^\top \mathcal{P}_1 (D\varphi + \sigma) + D^\top \mathcal{L}_4, v \Big\rangle dr, \\ J_2(t) = \frac{\varepsilon}{2} \langle D(t)^\top P_1(t) D(t) v, v \rangle + o(\varepsilon), \end{cases}$$

where $\mathcal{P}_i$, $i = 1, 2$, $(\mathcal{P}_j, \mathcal{L}_j)$, $j = 3, 4$, satisfies (3.10). Moreover, by the estimate of X_0^ε,

$$\mathbb{E}_t \int_t^{t+\varepsilon} \langle \mathscr{S}(s)^\top v, X_0^\varepsilon(s) \rangle ds = o(\varepsilon), \quad t \in [0, T).$$

For open-loop equilibrium strategy pair (Θ^*, φ^*) and associated equilibrium control u^*, we define X^* the corresponding state process as,

$$\begin{cases} dX^* = \big[(A + B\Theta^*)X^* + B\varphi^* + b\big]ds + \big[(C + D\Theta^*)X^* + D\varphi^* + \sigma\big]dW(s), \\ X^*(0) = x_0, \end{cases}$$

and perturbed control $u^{v,\varepsilon} := \Theta^* X^* + \varphi^* + v I_{[t,t+\varepsilon]}$. Moreover, for $(\mathcal{M}^*, \mathcal{N}^*)$ in (3.11) corresponding to u^*, let

$$\mathscr{H}_1(t) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{E}_t \int_t^{t+\varepsilon} \Big[\mathscr{S}(s) X^*(s) + \mathscr{R}(s) u^*(s) + B^\top \mathcal{M}^*(s, s) + D^\top \mathcal{N}^*(s) \Big] ds.$$

To sum up, $u^* = \Theta^* X^* + \varphi^*$ is an equilibrium control associated with $x_0 \in \mathbb{R}^n$ if and only if for any $t \in [0, T]$, $v \in L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^m)$,

$$0 \leq \langle \mathscr{D}_0(t) v, v \rangle + \langle \mathscr{H}_1(t), v \rangle, \tag{4.19}$$

where $\mathscr{D}_0$ is in (4.18). Given $t \in [0, T)$, this holds if and only if both $\mathscr{H}_1(t) = 0$ and $\mathscr{D}_0(t) \geq 0$. Since both $\mathscr{R}$ and P_1 are bounded and deterministic,

$$0 \leq \mathscr{R}(t) + D(t)^\top P_1(t) D(t), \quad t \in [0, T]. \ a.e.$$

$\Longrightarrow$ If $\mathscr{H}_1(t) = 0$, then by Lemma 3.4 in [12], for almost $s \in [0, T]$, we have

$$\begin{aligned} 0 &= \mathscr{S} X^* + \mathscr{R} u^* + B^\top \mathcal{M}^* + D^\top \mathcal{N}^* \\ &= \Big[\mathscr{S} + \mathscr{R}\Theta^* + \big[B^\top (\mathcal{P}_1^* + \mathcal{P}_2^*) + D^\top \mathcal{P}_1^* (C + D\Theta^*) \big] \Big] X^* \\ &\quad + \mathscr{R}\varphi^* + B^\top (\mathcal{P}_3^* + \mathcal{P}_4^*) + D^\top \mathcal{P}_1^* (D\varphi^* + \sigma) + D^\top \mathcal{L}_4^*. \end{aligned} \tag{4.20}$$

Notice that (4.20) holds for any $x_0 \in \mathbb{R}^n$. We choose $x_0 = 0$, and denote the state process by X_0^*. As a result,

$$\Big[\big[\mathscr{R} + D^\top \mathcal{P}_1^* D\big]\Theta^* + B^\top\big[\mathcal{P}_1^* + \mathcal{P}_2^*\big] + D^\top \mathcal{P}_1^* C + \mathscr{S}\Big](X^* - X_0^*) = 0.$$

At this moment, given $I \in \mathbb{R}^{n\times n}$ the unit matrix, we consider the following equation

$$\begin{cases} d\mathscr{X} = (A + B\Theta^*)\mathscr{X} ds + (C + D\Theta^*)\mathscr{X} dW(s), \quad s \in [0, T], \\ \mathscr{X}(0) = I, \end{cases} \tag{4.21}$$

the solvability of which is easy to see. Moreover, $\mathscr{X}^{-1}$ also exists. By the standard theory of SDEs,

$$\mathbb{P}\big\{\omega \in \Omega;\ \mathscr{X}(t,\omega)x = X^*(t,\omega) - X_0^*(t,\omega),\ \forall t \in [0, T]\big\} = 1.$$

Using the existence of $\mathscr{X}^{-1}$, it is easy to see above (3.12).

$\Longleftarrow$ In this case, it is easy to see (4.20) with $u^* := \Theta^* X^* + \varphi^*$. Consequently, the conclusion is followed by (4.19), (3.4) and the fact of $\mathscr{H}_1(t) = 0$. □

At last, we show the proof of Theorem 3.3.

Proof In Lemmas 4.1, 4.3, and 4.4, we take $\Theta_2 \equiv 0$. Hence for the notations in (4.4), $u \equiv \Theta_1 X + \varphi$ and

$$\begin{cases} J_1(t) = \mathbb{E}_t \displaystyle\int_t^{t+\varepsilon} \Big\langle \Big[\mathscr{S} + \mathscr{R}\Theta_1 + \big[B^\top(\mathscr{P}_1 + \mathscr{P}_2) + D^\top \mathscr{P}_1(C + D\Theta_1)\big]\Big]X + \frac{1}{2}\mathscr{R}v \\ \qquad + \mathscr{R}\varphi + B^\top(\mathscr{P}_3 + \mathscr{P}_4) + D^\top \mathscr{P}_1(D\varphi + \sigma) + D^\top \mathscr{L}_4, v\Big\rangle dr, \\ J_2(t) = \dfrac{\varepsilon}{2}\langle D(t)^\top \mathscr{P}_1(t) D(t)v, v\rangle + o(\varepsilon), \end{cases}$$

where $\mathscr{P}_i$, $i = 1, 2$, $(\mathscr{P}_j, \mathscr{L}_j)$, $j = 3, 4$, satisfies (3.15). Moreover, in view of the estimate of X_0^ε, it is straightforward to see

$$\mathbb{E}_t \int_t^{t+\varepsilon} \langle (\mathscr{S}(s)^\top + \Theta_1(s)^\top \mathscr{R}(s))v, X_0^\varepsilon(s)\rangle ds = o(\varepsilon), \quad t \in [0, T).$$

For closed-loop equilibrium strategy pair (Θ^*, φ^*) in the sense of Definition 2.3 and associated equilibrium control $u^* := \Theta^* X^* + \varphi^*$, we define X^* the corresponding state process as,

$$\begin{cases} dX^* = \big[(A + B\Theta^*)X^* + B\varphi^* + b\big]ds + \big[(C + D\Theta^*)X^* + D\varphi^* + \sigma\big]dW(s), \\ X^*(0) = x_0, \end{cases}$$

and perturbed control variable $u^{v,\varepsilon} := \Theta^* X^{v,\varepsilon} + \varphi^* + v I_{[t,t+\varepsilon]}$. In addition, for $(\mathscr{M}^*, \mathscr{N}^*)$ in (3.16) corresponding to u^*, we denote by

$$\begin{cases} \mathscr{H}_2(t) := \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\mathbb{E}_t \int_t^{t+\varepsilon} \Big[\mathscr{S}(s)X^*(s) + \mathscr{R}(s)u^*(s) + B^\top \mathscr{M}^*(s,s) + D^\top \mathscr{N}^*(s)\Big]ds, \\ \mathscr{D}_1(t) := \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_t^{t+\varepsilon} \big[\mathscr{R}(s) + D(s)^\top \mathscr{P}_1^*(s)D(s)\big]ds. \end{cases}$$

To sum up, $u^* = \Theta^* X^* + \varphi^*$ is a closed-loop equilibrium control associated with $x_0 \in \mathbb{R}^n$ if and only if for any $t \in [0,T]$, $v \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^m)$,

$$0 \le \langle \mathscr{D}_1(t)v, v\rangle + \langle \mathscr{H}_2(t), v\rangle. \tag{4.22}$$

Given $t \in [0,T)$, this holds if and only if both $\mathscr{H}_2(t) = 0$ and $\mathscr{D}_1(t) \ge 0$.

$\Longrightarrow$ Given equilibrium strategy pair (Θ^*, φ^*), we conclude that $\mathscr{P}_1^*$ is bounded and deterministic. Recall the requirement on $\mathscr{R}$, it is clear that

$$0 \le \mathscr{R}(t) + D(t)^\top \mathscr{P}_1^*(t)D(t), \quad t \in [0,T].\ a.e. \tag{4.23}$$

If $\mathscr{H}_2(t) = 0$, then by Lemma 3.4 in [12], for almost $s \in [0,T]$, we have

$$\begin{aligned} 0 &= \mathscr{S}X^* + \mathscr{R}u^* + B^\top \mathscr{M}^* + D^\top \mathscr{N}^* \\ &= \Big[\mathscr{S} + \mathscr{R}\Theta^* + \big[B^\top(\mathscr{P}_1^* + \mathscr{P}_2^*) + D^\top \mathscr{P}_1^*(C + D\Theta^*)\big]\Big]X^* \\ &\quad + \mathscr{R}\varphi^* + B^\top(\mathscr{P}_3^* + \mathscr{P}_4^*) + D^\top \mathscr{P}_1^*(D\varphi^* + \sigma) + D^\top \mathscr{L}_4^*. \end{aligned} \tag{4.24}$$

Notice that (4.24) holds for any $x_0 \in \mathbb{R}^n$. We choose $x_0 = 0$, and denote the state process by X_0^*. As a result,

$$\Big[\big[\mathscr{R} + D^\top \mathscr{P}_1^* D\big]\Theta^* + B^\top\big[\mathscr{P}_1^* + \mathscr{P}_2^*\big] + D^\top \mathscr{P}_1^* C + \mathscr{S}\Big](X^* - X_0^*) = 0.$$

As in Theorem 3.2, we introduce $\mathscr{X}$ satisfying (4.21), and therefore obtain (3.17) by following the same spirit of that in Theorem 3.2.

$\Longleftarrow$ In this case, it is easy to see (4.20) with $u^* := \Theta^* X^* + \varphi^*$. Consequently, the conclusion is followed by (4.22), (4.23) and the fact of $\mathscr{H}_1(t) = 0$. ◻

5 Concluding Remarks

In the Markovian setting, a unified approach by variational technique is developed to build the characterizations for three notions, i.e., closed-loop equilibrium controls/strategies, open-loop equilibrium controls, as well as the closed-loop representations of open-loop equilibrium controls. The intrinsic differences among different equilibrium controls are also revealed clearly and deeply. Related studies with random coefficients or in mean-field setting are under consideration. We hope to do some relevant research in future.

References

1. Björk, T., Murgoci, A.: A theory of Markovian time-inconsistent stochasitic control in discrete time. Financ. Stoch. **18**, 545–592 (2014)
2. Björk, T., Khapko, M., Murgoci, A.: On time-inconsistent stochastic control in continuous time. Financ. Stoch. **21**, 331–360 (2017)
3. Björk, T., Murgoci, A., Zhou, X.: Mean-variance portfolio optimization with state-dependent risk aversion. Math. Financ. **24**, 1–24 (2014)
4. Buckdahn, R., Djehiche, B., Li, J.: A general stochastic maximum principle of SDEs of mean-field type. Appl. Math. Optim. **64**, 197–216 (2011)
5. Chen, S., Li, X., Zhou, X.: Stochastic linear quadratic regulators with indefnite control weight costs. SIAM J. Control Optim. **36**, 1685–1702 (1998)
6. Chen, S., Yong, J.: Stochastic linear quadratic opitmal control problems. Appl. Math. Optim. **43**, 21–45 (2001)
7. Djehiche, B., Huang, M.: A characterization of sub-game perfect equilibria for SDEs of mean-field type. Dyn. Games Appl. **55**, 55–81 (2016)
8. Ekeland, I., Pirvu, T.: Investment and consumption without commitment. Math. Financ. Econ. **2**, 57–86 (2008)
9. Ekeland, I., Mbodji, O., Pirvu, T.: Time-consistent portfolio management. SIAM J. Financ. Math. **3**, 1–32 (2012)
10. Goldman, S.: Consistent plans. Rev. Econ. Stud. **47**, 533–537 (1980)
11. Hu, Y., Jin, H., Zhou, X.: Time-inconsistent stochastic linear-quadratic control. SIAM J. Control Optim. **50**, 1548–1572 (2012)
12. Hu, Y., Jin, H., Zhou, X.: Time-inconsistent stochastic linear-quadratic control: characterization and uniqueness of equilibrium. SIAM J. Control Optim. **55**, 1261–1279 (2017)
13. Huang, J., Li, X., Wang, T.: Characterizations of closed-loop equilibrium solutions for dynamic mean-variance optimization problems. Syst. Control Lett. **110**, 15–20 (2017)
14. Li, Y., Li, Z.: Optimal time-consistent investment and reinsurance strategies for mean-variance insurers with state dependent risk aversion. Insurance Math. Econ. **53**, 86–97 (2013)
15. Peleg, B., Yaari, M.: On the existence of a consistent course of action when tastes are changing. Rev. Econ. Stud. **40**, 391–401 (1973)
16. Strotz, R.: Myopia and inconsistency in dynamic utility maximization. Rev. Econ. Stud. **23**, 165–180 (1955)
17. Sun, J., Li, X., Yong, J.: Open-loop and closed-loop solvabilities for stochastic linear quadratic optimal control problems. SIAM J. Control Optim. **54**, 2274–2308 (2016)
18. Sun, J., Yong, J.: Linear quadratic stochastic differential games: open-loop and closed-loop saddle points. SIAM J. Control Optim. **52**, 4082–4121 (2014)
19. Wang, H., Wu, Z.: Time-inconsistent optimal control problem with random coefficients and stochastic equilibrium HJB equation. Math. Control Relat. Fields **3**, 651–678 (2015)
20. Wang, H., Wu, Z.: Partially observed time-inconsistency recursive optimization problem and application. J. Optim. Theory Appl. **161**, 664–687 (2014)
21. Wei, J., Wang, T.: Time-consistent mean-variance asset-liability management with random coefficients. Insurance Math. Econ. **77**, 84–96 (2017)
22. Wei, Q., Yong, J., Yu, Z.: Time-inconsistent recursive stochastic optimal control problems. SIAM J. Control Optim. **55**, 4156–4201 (2017)
23. Yong, J.: Time-inconsistent optimal control problem and the equilibrium HJB equation. Math. Control Relat. Fields **2**, 271–329 (2012)
24. Yong, J.: Linear-quadratic optimal control problems for mean-field stochastic differential equations. SIAM J. Control Optim. **51**, 2809–2838 (2013)
25. Yong, J.: Linear-quadratic optimal control problems for mean-field stochastic differential equations-time-consistent solutions. Trans. Am. Math. Soc. **369**, 5467–5523 (2017)
26. Yong, J., Zhou, X.: Stochstic Control: Hamiltonian Systems and HJB Equations. Springer, New York (1999)
27. Zeng, Y., Li, Z.: Optimal time-consistent investment and reinsurance policies for mean-variance insurers. Insurance Math. Econ. **49**, 145–154 (2011)