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Infinitely Many Solutions for Critical Degenerate Kirchhoff Type Equations Involving the Fractional p–Laplacian

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Abstract In this paper we study a class of critical Kirchhoff type equations involving the fractional p–Laplacian operator, that is

$$M\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy\right)(-\Delta)_p^s u=\lambda w(x)|u|^{q-2}u+|u|^{p_s^*-2}u, \quad x\in\mathbb{R}^N,$$

where $(-\Delta)_p^s$ is the fractional p–Laplacian operator with $0<s<1<p<\infty$, dimension $N>ps$, $1<q<p_s^*$, p_s^* is the critical exponent of the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$, λ is a positive parameter, M is a non-negative function while w is a positive weight. By exploiting Kajikiya's new version of the symmetric mountain pass lemma, we establish the existence of infinitely many solutions which tend to zero under a suitable value of λ. The main feature and difficulty of our equations is the fact that the Kirchhoff term M is zero at zero, that is the equation is *degenerate*. To our best knowledge, our results are new even in the Laplacian and p–Laplacian cases.

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1 Introduction

In the last years, the interest towards nonlinear Kirchhoff type problems has grown more and more, thanks in particular to their intriguing analytical structure due to the presence of the nonlocal Kirchhoff function M which makes the equation no longer a pointwise identity. In this paper, we consider the following critical Kirchhoff type equation involving the fractional p–Laplacian operator

$$M\left([u]_{s,p}^p\right)(-\Delta)_p^s u = \lambda w(x)|u|^{q-2}u + |u|^{p_s^*-2}u, \quad \text{in } \mathbb{R}^N,$$
$$[u]_{s,p}^p = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy, \tag{1.1}$$

where $s \in (0,1)$, $p \in (1,\infty)$, dimension $N \in (ps,\infty)$, $q \in (1,p)$, $p_s^* = Np/(N-ps)$ is the critical exponent of the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$, λ is a positive parameter and w is a positive weight whose assumption will be introduced in the sequel. Here $(-\Delta)_p^s$ denotes the fractional p–Laplace operator which, up to normalization factors, may be defined by the Riesz potential as

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}dy, \quad x \in \mathbb{R}^N,$$

along any $u \in C_0^\infty(\mathbb{R}^N)$, where $B_\varepsilon(x) = \{y \in \mathbb{R}^N : |x-y| < \varepsilon\}$. For more details about the fractional p–Laplacian, for example, we refer to [13,22,33] and the references therein.

Concerning the Kirchhoff term, we assume that $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a *continuous* function for which

(M_1) *there exists* $\theta \in (1, p_s^*/p)$ *such that* $tM(t) \le \theta\mathscr{M}(t)$ *for all* $t \in \mathbb{R}_0^+$, *where* $\mathscr{M}(t) = \int_0^t M(\tau)d\tau$;
(M_2) *for any* $\tau > 0$ *there exists* $\kappa = \kappa(\tau) > 0$ *such that* $M(t) \ge \kappa$ *for all* $t \ge \tau$;
(M_3) *there exists* $m_0 > 0$ *such that* $M(t) \ge m_0 t^{\theta-1}$ *for all* $t \in [0,1]$.

A prototype for M, due to Kirchhoff, is given by

$$M(t) = a + b\theta t^{\theta-1}, \quad a,\ b \ge 0, \ \ a+b > 0, \ \ \theta \ge 1. \tag{1.2}$$

When $M(t) \ge c > 0$ for all $t \in \mathbb{R}_0^+$, Kirchhoff equations like (1.1) are said to be *non-degenerate* and this happens for example if $a > 0$ in the model case (1.2). While, if $M(0) = 0$ but $M(t) > 0$ for all $t \in \mathbb{R}^+$, Kirchhoff equations as (1.1) are called *degenerate*. Of course, for (1.2) this occurs when $a = 0$. In the present paper, we are interested in the study of (1.1) on a degenerate setting. For this, in (M_1) we have to require $\theta > 1$, as shown in Lemma 3.1 of [5].

In the appendix of the recent paper [11], the authors provide a detailed discussion about the physical meaning underlying the fractional Kirchhoff problems and their applications. Indeed, they propose in [11] a stationary Kirchhoff variational problem, which models, as a special significant case, the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string. In this case, M measures the change of the tension on the string caused by the change of its length during the vibration. For this, the fact that $M(0) = 0$ means that the base tension of the string is zero, a very realistic model.

Several recent papers are focused both on theoretical aspects and applications related to nonlocal fractional models. Always in [11], the following critical fractional problem on Ω bounded was studied for the first time in the literature

$$\begin{cases} M\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\right)(-\Delta)^s u = \lambda f(x,u) + |u|^{2_s^*-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.3}$$

The authors prove the existence of a non-trivial non-negative solution for (1.3) on a non-degenerate setting, combining a truncation argument with a concentration compactness principle. The degenerate case of problem (1.3) is studied in [2], by introducing a new technical approach based on the asymptotic property of the critical mountain pass level. Furthermore, the existence of a solution for different critical fractional Kirchhoff problems set on the whole space $\mathbb{R}^N$ is given in [5,9,10,24,27] adapting the variational technique developed in [2]. For multiplicity results, we refer to [25], where they consider a nonhomogeneous fractional Schrödinger–Kirchhoff equation. By combining the mountain pass theorem with Ekeland's variational principle, in [25] the authors establish the existence of two solutions on a non-degenerate situation. Recently, the multiplicity result in [25] has been improved in [29], by considering weaker assumptions on the potential and on the subcritical term. With a similar approach to [25], in [31] they prove the existence of two solutions for a degenerate Kirchhoff equation in $\mathbb{R}^N$ with a concave–convex nonlinearity, while in [34] they consider a critical equation akin to (1.1) with M as in (1.2), $a = 0$ and $b = 1$. In [32], by the Fountain theorem and the dual Fountain theorem, the authors get the existence of infinitely many solutions for a symmetric subcritical Kirchhoff problem on Ω, with suitable non-degenerate assumptions for M. The existence of infinitely many solutions is still proved in [21,26] by using Krasnoselskii's genus theory, under degenerate frameworks. Moreover, to get infinitely many solutions, Krasnoselskii's genus theory is used in [8] for a critical problem similar to (1.3), but just on the non-degenerate case. In [30], applying Kajikiya's new version of the symmetric mountain pass lemma, the existence of infinitely many solutions for a critical equation similar to (1.1) is proved under a non-degenerate situation. Finally, the symmetric mountain pass theorem is applied to study both a fractional Schrödinger–Kirchhoff equation, in [23], and a subcritical degenerate Kirchhoff system on a bounded domain Ω, in [33].

Motivated by the above works, in the present paper we provide the existence of infinitely many solutions for (1.1) on a degenerate setting. As far as we know, our multiplicity result is new even for degenerate Kirchhoff equations similar to (1.1), but driven by either the Laplacian or the p–Laplacian operator. Indeed, while there is a wide literature concerning the study of multiplicity results for critical Kirchhoff problems under a non-degenerate setting, see for example [6,7,12,16,18,35–37], very few attempts have been made to cover also the degenerate case. We refer to [17,19] in the Laplacian setting, where the authors just consider M like the prototype in (1.2) with $a = 0$. Furthermore, they are able to give multiplicity results either when $N = 4$ or by considering a small perturbation. Here, by using a different approach we allow M to be more general in (1.1).

Concerning the *positive* weight $w : \mathbb{R}^N \to \mathbb{R}$, we suppose that

(w) $w \in L^r(\mathbb{R}^N) \bigcap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$, with $r = p_s^*/(p_s^* - q)$.

Condition (w) is necessary, since it guaranties that the embedding $D^{s,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N, w)$ is compact, as shown in Lemma 2.1 of [5]. Indeed, the natural solution space for Eq. (1.1) is the fractional Beppo–Levi space $D^{s,p}(\mathbb{R}^N)$, that is the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $[\,\cdot\,]_{s,p}$, given by

$$[u]_{s,p} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dxdy\right)^{1/p}.$$

Denoting with $\mathcal{J}_\lambda : D^{s,p}(\mathbb{R}^N) \to \mathbb{R}$ the Euler–Lagrange functional related to variational equation (1.1), we are ready to state the main result of our paper as follows.

Theorem 1.1 *Let $M(0) = 0$, $N \in (ps, \infty)$, $q \in (1, p)$, with $s \in (0, 1)$ and $p \in (1, \infty)$. Assume that M and w satisfy assumptions (M_1)–(M_3) and (w).*

Then, there exists $\overline{\lambda} > 0$ such that for any $\lambda \in (0, \overline{\lambda})$ Eq. (1.1) admits a sequence of solutions $\{u_n\}_n$ in $D^{s,p}(\mathbb{R}^N)$ with $\mathcal{J}_\lambda(u_n) < 0$, $\mathcal{J}_\lambda(u_n) \to 0$ and $\{u_n\}_n$ converges to zero as $n \to \infty$.

The proof of Theorem 1.1 is mainly based on the application of the symmetric mountain pass lemma, introduced by Kajikiya in [14]. For this, we need a truncation argument which allow us to control from below functional $\mathcal{J}_\lambda$. Furthermore, as usual in elliptic problems involving critical nonlinearities, we must pay attention to the lack of compactness at critical level $L^{p_s^*}(\mathbb{R}^N)$. To overcome this difficulty, we fix parameter λ under a suitable threshold strongly depending on assumptions (M_2) and (M_3).

Because of the geometry of functional $\mathcal{J}_\lambda$ for (1.1), we are not able to cover the range $q \in [p, p_s^*)$. However, we can improve the result stated in Theorem 1.1 under the degenerate model case (1.2), with $a = 0$. That is, we consider the equation

$$\begin{gathered} b[u]_{s,p}^{p(\theta-1)}(-\Delta)_p^s u = \lambda w(x)|u|^{q-2}u + |u|^{p_s^*-2}u, \quad \text{in } \mathbb{R}^N, \\ [u]_{s,p}^p = \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dxdy, \end{gathered} \tag{1.4}$$

with still $s \in (0, 1)$, $p \in (1, \infty)$, dimension $N \in (ps, \infty)$, $b > 0$, $\theta \in (1, p_s^*)$ and here $q \in (1, p\theta)$. Thus, arguing similarly as in Theorem 1.1, we get the following result.

Theorem 1.2 *Let $b > 0$, $N \in (ps, \infty)$, $\theta \in (1, p_s^*/p)$, $q \in (1, p\theta)$, with $s \in (0, 1)$ and $p \in (1, \infty)$. Assume that w satisfies assumption (w).*

Then, there exists $\overline{\lambda} > 0$ such that for any $\lambda \in (0, \overline{\lambda})$ Eq. (1.4) admits a sequence of solutions $\{u_n\}_n$ in $D^{s,p}(\mathbb{R}^N)$ with $\mathcal{J}_\lambda(u_n) < 0$, $\mathcal{J}_\lambda(u_n) \to 0$ and $\{u_n\}_n$ converges to zero as $n \to \infty$.

The paper is organized as follows. In Sect. 2 we discuss the variational formulation of the Eq. (1.1) and introduce some topological notions. In Sect. 3 we prove the Palais–Smale condition for the functional $\mathcal{J}_\lambda$. In Sect. 4 we introduce a truncation argument for our functional. In Sect. 5 we prove Theorems 1.1 and 1.2.

2 Preliminaries

In this section, we first give the variational formulation of Eq. (1.1) and then provide some useful technical results, which will be used in the sequel.

Let $L^q(\mathbb{R}^N, w)$ be the weighted Lebesgue space, endowed with the norm

$$\|u\|_{q,w}^q = \int_{\mathbb{R}^N} w(x)|u(x)|^q dx.$$

By Proposition A.6 of [1] the Banach space $L^q(\mathbb{R}^N, w) = (L^q(\mathbb{R}^N, w), \|\cdot\|_{q,w})$ is uniformly convex. Furthermore, by Lemma 2.1 of [5], the embedding $D^{s,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N, w)$ is compact, with

$$\|u\|_{q,w} \le C_w[u]_{s,p} \quad \text{for all } u \in D^{s,p}(\mathbb{R}^N), \tag{2.1}$$

and $C_w = S^{-1/p}\|w\|_r^{1/q} > 0$, where $S = S(N, p, s)$ is the best fractional critical Sobolev constant, given by

$$S = \inf_{u \in D^{s,p}(\mathbb{R}^N)\setminus\{0\}} \frac{[u]_{s,p}^p}{\|u\|_{p_s^*}^p}. \tag{2.2}$$

Of course number S is positive, since Theorem 1 of [20].

We say that $u \in D^{s,p}(\mathbb{R}^N)$ is a (weak) solution of Eq. (1.1), if u satisfies

$$M([u]_{s,p}^p)\langle u, \varphi\rangle_{s,p} = \lambda\langle u, \varphi\rangle_{q,w} + \langle u, \varphi\rangle_{p_s^*},$$

for all $\varphi \in D^{s,p}(\mathbb{R}^N)$, where

$$\langle u, \varphi\rangle_{s,p} = \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}[u(x) - u(y)] \cdot [\varphi(x) - \varphi(y)]}{|x - y|^{N+sp}} dxdy,$$

$$\langle u, \varphi\rangle_{q,w} = \int_{\mathbb{R}^N} w(x)|u(x)|^{q-2}u(x)\varphi(x)dx, \ \langle u, \varphi\rangle_{p_s^*} = \int_{\mathbb{R}^N} |u(x)|^{p_s^*-2}u(x)\varphi(x)dx.$$

Equation (1.1) has a variational structure and $\mathcal{J}_\lambda : D^{s,p}(\mathbb{R}^N) \to \mathbb{R}$, defined by

$$\mathcal{J}_\lambda(u) = \frac{1}{p}\mathscr{M}([u]_{s,p}^p) - \frac{\lambda}{q}\|u\|_{q,w}^q - \frac{1}{p_s^*}\|u\|_{p_s^*}^{p_s^*},$$

is the underlying functional associated with (1.1). Essentially, as shown in Lemma 4.2 of [5], the functional $\mathcal{J}_\lambda$ is of class $C^1(D^{s,p}(\mathbb{R}^N))$.

In order to handle the degenerate Kirchhoff coefficient we need appropriate lower and upper bounds for M, given by (M_1) and (M_2). Indeed, condition (M_2) implies that $M(t) > 0$ for any $t > 0$ and consequently by (M_1) for all $t \in (0, 1]$ we have $M(t)/\mathscr{M}(t) \leq \theta/t$. Thus, integrating on $[t, 1]$, with $0 < t < 1$, we get

$$\mathscr{M}(t) \geq \mathscr{M}(1)t^\theta, \tag{2.3}$$

and (2.3) holds for all $t \in [0, 1]$ by continuity. Hence, (M_3) is a stronger request. Furthermore (2.3) is compatible with (M_3), since integrating (M_3) we have $\mathscr{M}(t) \geq m_0 t^\theta/\theta$ for any $t \in [0, 1]$, from which $\mathscr{M}(1) \geq m_0/\theta$.

Similarly, for any $\varepsilon > 0$ there exists $\delta_\varepsilon = \mathscr{M}(\varepsilon)/\varepsilon^\theta > 0$ such that

$$\mathscr{M}(t) \leq \delta_\varepsilon t^\theta \quad \text{for any } t \geq \varepsilon. \tag{2.4}$$

To prove the multiplicity result stated in Theorem 1.1, we will use some topological results introduced by Krasnoselskii in [15]. For the sake of completeness and for reader's convenience, we recall here some basic notions on the Krasnoselskii's genus. Let X be a Banach space and let us denote by Σ the class of all closed subsets $A \subset X \setminus \{0\}$ that are symmetric with respect to the origin, that is, $u \in A$ implies $-u \in A$.

Definition 2.1 Let $A \in \Sigma$. The Krasnoselskii's genus $\gamma(A)$ of A is defined as being the least positive integer n such that there is an odd mapping $\phi \in C(A, \mathbb{R}^n)$ such that $\phi(x) \neq 0$ for any $x \in A$. If n does not exist, we set $\gamma(A) = \infty$. Furthermore, we set $\gamma(\emptyset) = 0$.

In the sequel we will recall only the properties of the genus that will be used throughout this work. More information on this subject may be found in the references [14,15,28].

Proposition 2.1 *Let A and B be closed symmetric subsets of X which do not contain the origin. Then the following hold.*

(1) If there exists an odd continuous mapping from A to B, then $\gamma(A) \leq \gamma(B)$;
(2) If there is an odd homeomorphism from A to B, then $\gamma(A) = \gamma(B)$;
(3) If $\gamma(B) < \infty$, then $\gamma(\overline{A \setminus B}) \geq \gamma(A) - \gamma(B)$;
(4) Then n-dimensional sphere S^n has a genus of $n+1$ by the Borsuk–Ulam Theorem;
(5) If A is compact, then $\gamma(A) < \infty$ and there exists $\delta > 0$ such that $N_\delta(A) \subset \Sigma$ and $\gamma(N_\delta(A)) = \gamma(A)$, with $N_\delta(A) = \{x \in X : \ dist(x, A) \leq \delta\}$.

We conclude this section recalling the symmetric mountain-pass lemma introduced by Kajikiya in [14]. The proof of Theorem 1.1 is based on the application of the following result.

Lemma 2.1 *Let E be an infinite-dimensional space and $J \in C^1(E, \mathbb{R})$ and suppose the following conditions hold.*

- (J_1) $J(u)$ is even, bounded from below, $J(0) = 0$ and $J(u)$ satisfies the local Palais-Smale condition, i.e. for some $\bar{c} > 0$, in the case when every sequence $\{u_n\}_n$ in E satisfying $\lim_{n\to\infty} J(u_n) = c < \bar{c}$ and $\lim_{n\to\infty} \|J'(u_n)\|_{E'} = 0$ has a convergent subsequence;
- (J_2) For each $n \in \mathbb{N}$, there exists an $A_n \in \Sigma_n$ such that $\sup_{u\in A_n} J(u) < 0$.

Then either (i) or (ii) below holds.

- (i) There exists a sequence $\{u_n\}_n$ such that $J'(u_n) = 0$, $J(u_n) < 0$ and $\{u_n\}_n$ converges to zero.
- (ii) There exist two sequences $\{u_n\}_n$ and $\{v_n\}_n$ such that $J'(u_n) = 0$, $J(u_n) = 0$, $u_n \neq 0$, $\lim_{n\to\infty} u_n = 0$, $J'(v_n) = 0$, $J(v_n) < 0$, $\lim_{n\to\infty} J(v_n) = 0$, and $\{v_n\}_n$ converges to a non-zero limit.

3 The Palais–Smale Condition

Throughout this paper, we consider $N > ps$ with $s \in (0, 1)$ and $p \in (1, \infty)$, $M(0) = 0$ and we assume M and w satisfy (M_1)–(M_3) and (w), without further mentioning .

To apply Lemma 2.1, we discuss now the compactness property for the functional $\mathcal{J}_\lambda$, given by the Palais–Smale condition. We recall that $\{u_n\}_n \subset D^{s,p}(\mathbb{R}^N)$ is a Palais–Smale sequence for $\mathcal{J}_\lambda$ at level $c \in \mathbb{R}$ if

$$\mathcal{J}_\lambda(u_n) \to c \quad \text{and} \quad \mathcal{J}'_\lambda(u_n) \to 0 \quad \text{in } (D^{s,p}(\mathbb{R}^N))' \text{ as } n \to \infty. \tag{3.1}$$

We say that $\mathcal{J}_\lambda$ satisfies the Palais–Smale condition at level c if any Palais–Smale sequence $\{u_n\}_n$ at level c admits a convergent subsequence in $D^{s,p}(\mathbb{R}^N)$.

Lemma 3.1 *Let $c < 0$. Then, there exists $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$, the functional $\mathcal{J}_\lambda$ satisfies $(PS)_c$.*

Proof Let us consider $\lambda_0 > 0$ sufficiently small such that

$$\left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)^{-\frac{p_s^*}{p_s^*-q}} \left[\lambda_0 \left(\frac{1}{q} - \frac{1}{p\theta}\right) \|w\|_r\right]^{\frac{p_s^*}{p_s^*-q}} < \min\left\{\left(m_0^{1/\theta} S\right)^{\frac{p_s^*\theta}{p_s^*-p\theta}}, (\kappa S)^{\frac{p_s^*}{p_s^*-p}}\right\} \tag{3.2}$$

where $q < p < p\theta < p_s^*$, m_0 comes from (M_3), $\kappa = \kappa(1)$ is defined in (M_2) with $\tau = 1$, while S is given in (2.2).

Let $\lambda \in (0, \lambda_0)$ and let $\{u_n\}_n$ be a $(PS)_c$ sequence in $D^{s,p}(\mathbb{R}^N)$. Due to the degenerate nature of (1.1), two situations must be considered: either $\inf_{n\in\mathbb{N}}[u_n]_{s,p} = d > 0$ or $\inf_{n\in\mathbb{N}}[u_n]_{s,p} = 0$. For this, we divide the proof in two cases.

• *Case* $\inf_{n\in\mathbb{N}}[u_n]_{s,p} = d > 0$. We first show that $\{u_n\}_n$ is bounded. By (M_2), with $\tau = d^p$, there exists $\kappa = \kappa(d^p) > 0$ such that

$$M([u_n]_{s,p}^p) \geq \kappa \quad \text{for any } n \in \mathbb{N}. \tag{3.3}$$

Furthermore, from (M_1), (2.1) and (3.3) we get

$$\begin{aligned}\mathcal{J}_\lambda(u_n) - \frac{1}{p_s^*}\langle \mathcal{J}'_\lambda(u_n), u_n\rangle &= \frac{1}{p}\mathscr{M}([u_n]_{s,p}^p) - \frac{1}{p_s^*}M([u_n]_{s,p}^p)[u_n]_{s,p}^p \\ &\quad - \left(\frac{1}{q} - \frac{1}{p_s^*}\right)\lambda\|u_n\|_{q,w}^q \\ &\geq \left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)\kappa[u_n]_{s,p}^p - \left(\frac{1}{q} - \frac{1}{p_s^*}\right)\lambda C_w^q[u_n]_{s,p}^q.\end{aligned}$$

Thus, by (3.1) there exists $\sigma > 0$ such that as $n \to \infty$

$$c + \sigma[u_n]_{s,p}^q + o(1) \geq \left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)\kappa[u_n]_{s,p}^p,$$

being $q < p < p\theta < p_s^*$. This implies at once that $\{u_n\}_n$ is bounded in $D^{s,p}(\mathbb{R}^N)$.

Therefore, using arguments similar to Lemma 2.1 of [5], there exist a subsequence, still denoted by $\{u_n\}_n$, and a function $u \in D^{s,p}(\mathbb{R}^N)$ such that

$$\begin{aligned}&u_n \rightharpoonup u \text{ in } D^{s,p}(\mathbb{R}^N), \quad [u_n]_{s,p} \to \mu, \\ &u_n \rightharpoonup u \text{ in } L^{p_s^*}(\mathbb{R}^N), \quad \|u_n - u\|_{p_s^*} \to \ell, \\ &u_n \to u \text{ in } L^q(\mathbb{R}^N, w), \; u_n \to u \text{ a.e. in } \mathbb{R}^N\end{aligned} \tag{3.4}$$

as $n \to \infty$. Clearly $\mu > 0$ since we are in the case in which $d > 0$.

Furthermore, as shown in the proof of Lemma 2.4 of [5], by (3.4) the sequence $\{\mathcal{U}_n\}_n$, defined in $\mathbb{R}^{2N} \setminus \mathrm{Diag}\,\mathbb{R}^{2N}$ by

$$(x, y) \mapsto \mathcal{U}_n(x, y) = \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x - y|^{(N+ps)/p'}},$$

is bounded in $L^{p'}(\mathbb{R}^{2N})$ as well as $\mathcal{U}_n \to \mathcal{U}$ a.e. in $\mathbb{R}^{2N}$, where

$$\mathcal{U}(x, y) = \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{(N+ps)/p'}}.$$

Thus, up to a subsequence, we get $\mathcal{U}_n \rightharpoonup \mathcal{U}$ in $L^{p'}(\mathbb{R}^{2N})$, and so as $n \to \infty$

$$\langle u_n, \varphi\rangle_{s,p} \to \langle u, \varphi\rangle_{s,p} \tag{3.5}$$

for any $\varphi \in D^{s,p}(\mathbb{R}^N)$, since $|\varphi(x)-\varphi(y)| \cdot |x-y|^{-(N+ps)/p} \in L^p(\mathbb{R}^{2N})$. Similarly, (3.4) and Proposition A.8 of [1] imply that $|u_n|^{q-2}u_n \rightharpoonup |u|^{q-2}u$ in $L^{q'}(\mathbb{R}^N, w)$ and $|u_n|^{p_s^*-2}u_n \rightharpoonup |u|^{p_s^*-2}u$ in $L^{p_s^{*\prime}}(\mathbb{R}^N)$, from which as $n \to \infty$

$$\langle u_n, \varphi\rangle_{q,w} \to \langle u, \varphi\rangle_{q,w}, \quad \langle u_n, \varphi\rangle_{p_s^*} \to \langle u, \varphi\rangle_{p_s^*}, \tag{3.6}$$

for any $\varphi \in D^{s,p}(\mathbb{R}^N)$, Then, (3.1), (3.4), (3.5) and (3.6) give

$$M(\mu^p)\langle u, \varphi\rangle_{s,p} = \lambda\langle u, \varphi\rangle_{q,w} + \langle u, \varphi\rangle_{p_s^*},$$

for any $\varphi \in D^{s,p}(\mathbb{R}^N)$. Hence, u is a critical point of the $C^1(D^{s,p}(\mathbb{R}^N))$ functional

$$\widetilde{\mathcal{J}}_\lambda(u) = \frac{1}{p}M(\mu^p)[u]_{s,p}^p - \frac{\lambda}{q}\|u\|_{q,w}^q - \frac{1}{p_s^*}\|u\|_{p_s^*}^{p_s^*}. \tag{3.7}$$

Thanks to (3.4) it results

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} w(x)(|u_n(x)|^{q-2}u_n(x) - |u(x)|^{q-2}u(x))(u_n(x)-u(x))dx = 0. \tag{3.8}$$

Furthermore, using again (3.4) and the celebrated Brézis and Lieb lemma of [4]

$$[u_n]_{s,p}^p = [u_n - u]_{s,p}^p + [u]_{s,p}^p + o(1), \quad \|u_n\|_{p_s^*}^{p_s^*} = \|u_n - u\|_{p_s^*}^{p_s^*} + \|u\|_{p_s^*}^{p_s^*} + o(1) \tag{3.9}$$

as $n \to \infty$. Consequently, we deduce from (3.1), (3.4), (3.7), (3.8) and (3.9) that as $n \to \infty$

$$\begin{aligned}
o(1) &= \langle \mathcal{J}_\lambda'(u_n) - \widetilde{\mathcal{J}}_\lambda'(u), u_n - u\rangle \\
&= M([u_n]_{s,p}^p)[u_n]_{s,p}^p + M(\mu^p)[u]_{s,p}^p - \langle u_n, u\rangle_{s,p}[M([u_n]_{s,p}^p) + M(\mu^p)] \\
&\quad - \lambda\int_{\mathbb{R}^N} w(x)(|u_n(x)|^{q-2}u_n(x) - |u(x)|^{q-2}u(x))(u_n(x)-u(x))dx \\
&\quad - \int_{\mathbb{R}^N}(|u_n(x)|^{p_s^*-2}u_n(x) - |u(x)|^{p_s^*-2}u(x))(u_n(x)-u(x))dx \\
&= M(\mu^p)(\mu^p - [u]_{s,p}^p) - \|u_n\|_{p_s^*}^{p_s^*} + \|u\|_{p_s^*}^{p_s^*} + o(1) \\
&= M(\mu^p)[u_n - u]_{s,p}^p - \|u_n - u\|_{p_s^*}^{p_s^*} + o(1).
\end{aligned}$$

Therefore, we have proved the crucial formula

$$M(\mu^p)\lim_{n\to\infty}[u_n - u]_{s,p}^p = \lim_{n\to\infty}\|u_n - u\|_{p_s^*}^{p_s^*}. \tag{3.10}$$

By (2.2), the notation in (3.4) and (3.10), we get

$$\ell^{p_s^*} \geq S\, M(\mu^p)\ell^p. \tag{3.11}$$

When $\ell = 0$, since $\mu > 0$ and M admits a unique zero at 0, then (3.10) yields $u_n \to u$ in $D^{s,p}(\mathbb{R}^N)$, concluding the proof.

Thus, let us assume by contradiction that $\ell > 0$. Noting that (3.10) implies in particular that

$$M(\mu^p)\big(\mu^p - [u]_{s,p}^p\big) = \ell^{p_s^*},$$

using (3.11), it follows that

$$\big(\ell^{p_s^*}\big)^{ps/N} = M(\mu^p)^{ps/N}\big(\mu^p - [u]_{s,p}^p\big)^{ps/N} \geq S\,M(\mu^p). \tag{3.12}$$

Since we do not know the exact behavior of M, we must consider other two situations: either $\mu \in (0,1)$ or $\mu > 1$. For this, we divide the proof of the first case in two subcases.

• *Subcase* $\mu \in (0,1)$. By (3.12) and (M_3), we obtain

$$\mu^{p^2 s/N} \geq \big(\mu^p - [u]_{s,p}^p\big)^{ps/N} \geq S\,M(\mu^p)^{\frac{N-ps}{N}} \geq m_0^{\frac{N-ps}{N}}\,S\,\mu^{\frac{p(\theta-1)(N-ps)}{N}}$$

and considering $N < ps\theta/(\theta-1) = ps\theta'$, it follows that

$$\mu^p \geq \left(m_0^{\frac{N-ps}{N}}\,S\right)^{\frac{N}{ps\theta - N(\theta-1)}}. \tag{3.13}$$

Indeed, the restriction $N/(p\theta') < s$ follows directly from the fact that $1 < \theta < p_s^*/p = N/(N-ps)$. By using (M_3), (3.12) and (3.13), we obtain

$$\ell^{p_s^*} \geq \big(S\,M(\mu^p)\big)^{N/ps} \geq \left(S\,m_0\,\mu^{p(\theta-1)}\right)^{N/ps} \geq \left(m_0^{1/\theta}\,S\right)^{\frac{N\theta}{ps\theta - N(\theta-1)}}. \tag{3.14}$$

Now, by (M_1) for any $n \in \mathbb{N}$ we have

$$\begin{aligned}
\mathcal{J}_\lambda(u_n) - \frac{1}{p\theta}\big\langle \mathcal{J}_\lambda'(u_n), u_n\big\rangle &= \frac{1}{p}\mathscr{M}([u_n]_{s,p}^p) - \frac{1}{p\theta}M([u_n]_{s,p}^p)[u_n]_{s,p}^p \\
&\quad + \lambda\left(\frac{1}{p\theta} - \frac{1}{q}\right)\|u_n\|_{q,w}^q + \left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)\|u_n\|_{p_s^*}^{p_s^*} \\
&\geq \lambda\left(\frac{1}{p\theta} - \frac{1}{q}\right)\|u_n\|_{q,w}^q + \left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)\|u_n\|_{p_s^*}^{p_s^*}.
\end{aligned}$$

From this, as $n \to \infty$, by (3.1), (3.4), (3.9), (w), the Hölder inequality and Young inequality

$$\begin{aligned} c &\geq \left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)\left(\ell^{p_s^*} + \|u\|_{p_s^*}^{p_s^*}\right) - \lambda\left(\frac{1}{q} - \frac{1}{p\theta}\right)\|u\|_{q,w}^q \\ &\geq \left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)\left(\ell^{p_s^*} + \|u\|_{p_s^*}^{p_s^*}\right) - \lambda\left(\frac{1}{q} - \frac{1}{p\theta}\right)\|w\|_r\|u\|_{p_s^*}^q \\ &\geq \left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)\left(\ell^{p_s^*} + \|u\|_{p_s^*}^{p_s^*}\right) - \left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)\|u\|_{p_s^*}^{p_s^*} \\ &\quad - \left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)^{-\frac{q}{p_s^*-q}}\left[\lambda\left(\frac{1}{q} - \frac{1}{p\theta}\right)\|w\|_r\right]^{\frac{p_s^*}{p_s^*-q}}. \end{aligned} \tag{3.15}$$

Finally, by (3.14) we get

$$0 > c \geq \left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)\left(m_0^{1/\theta}\, S\right)^{\frac{p_s^*\theta}{p_s^*-p\theta}} - \left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)^{-\frac{q}{p_s^*-q}}\left[\lambda\left(\frac{1}{q} - \frac{1}{p\theta}\right)\|w\|_r\right]^{\frac{p_s^*}{p_s^*-q}} > 0,$$

where last inequality follows from (3.2). We obtain our contradiction concluding the proof of the first subcase.

• *Subcase* $\mu \geq 1$. Here, by (3.12) and (M_2) with $\tau = 1$, we obtain

$$\ell^{p_s^*} \geq (\kappa\, S)^{N/ps},$$

with $\kappa = \kappa(1) > 0$. Thus, (3.15) yields

$$0 > c \geq \left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)(\kappa\, S)^{\frac{p_s^*}{p_s^*-p}} - \left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)^{-\frac{q}{p_s^*-q}}\left[\lambda\left(\frac{1}{q} - \frac{1}{p\theta}\right)\|w\|_r\right]^{\frac{p_s^*}{p_s^*-q}} > 0,$$

where again last inequality follows from (3.2). We still have a contradiction which concludes the proof of the first case.

• *Case* $\inf_{n\in\mathbb{N}}[u_n]_{s,p} = 0$. Here, either 0 is an accumulation point for the real sequence $\{[u_n]_{s,p}\}_n$ and so there is a subsequence of $\{u_n\}_n$ strongly converging to $u = 0$, or 0 is an isolated point of $\{[u_n]_{s,p}\}_n$. The first case can not occur since it implies that the trivial solution is a critical point at level c. This is impossible, being $0 = \mathcal{J}_\lambda(0) = c < 0$. Hence only the latter case can occur, so that there is a subsequence, denoted by $\{[u_{n_k}]_{s,p}\}_k$, such that $\inf_{k\in\mathbb{N}}[u_{n_k}]_{s,p} = d > 0$ and we can proceed as before. This completes the proof of the second case and of the lemma. □

4 A Truncation Argument

We note that our functional $\mathcal{J}_\lambda$ is not bounded from below in $D^{s,p}(\mathbb{R}^N)$. Indeed, by fixing $\varepsilon > 0$ in (2.4) we see that for any $u \in D^{s,p}(\mathbb{R}^N)$

$$\mathcal{J}_\lambda(tu) \le t^{p\theta}\frac{\delta_\varepsilon}{p}[u]_{s,p}^{p\theta} - t^q\frac{\lambda}{q}\,\|u\|_{q,w}^q - t^{p_s^*}\frac{1}{p_s^*}\,\|u\|_{p_s^*}^{p_s^*} \to -\infty \quad \text{as } t \to \infty,$$

since $q < p < p\theta < p_s^*$.

For this in the sequel we introduce a truncation like in [1], to get a special lower bound which will be worth to construct critical values for $\mathcal{J}_\lambda$. Let us denote

$$\mathcal{G}_\lambda(t) = \frac{\mathscr{M}(1)}{p}t^{p\theta} - \frac{\lambda}{q}C_w^q\,t^q - \frac{1}{p_s^*\,S}t^{p_s^*}$$

where C_w comes from (2.1), while S is defined in (2.2). Denoting $\kappa = \kappa(1)$ the constant given by (M_2) with $\tau = 1$, we can take $R_1 \in (0,1)$ sufficiently small such that

$$\frac{\kappa}{p\theta}R_1^p > \frac{\mathscr{M}(1)}{p}R_1^{p\theta} > \frac{1}{p_s^*\,S}R_1^{p_s^*}, \tag{4.1}$$

since $p < p\theta < p_s^*$, and we define

$$\lambda^* = \frac{q}{2\,C_w^q\,R_1^q}\left(\frac{\mathscr{M}(1)}{p}R_1^{p\theta} - \frac{1}{p_s^*\,S}R_1^{p_s^*}\right), \tag{4.2}$$

so that $\mathcal{G}_{\lambda^*}(R_1) > 0$. From this, we consider

$$R_0 = \max\left\{t \in (0, R_1):\ \mathcal{G}_{\lambda^*}(t) \le 0\right\}.$$

Since by $q < p$ we have $\mathcal{G}_\lambda(t) \le 0$ for t near to 0 and since also $\mathcal{G}_{\lambda^*}(R_1) > 0$, it easily follows that $\mathcal{G}_{\lambda^*}(R_0) = 0$.

We can choose $\phi \in C_0^\infty([0,\infty),[0,1])$ such that $\phi(t) = 1$ if $t \in [0, R_0]$ and $\phi(t) = 0$ if $t \in [R_1, \infty)$. Thus, we consider the truncated functional

$$\mathcal{I}_\lambda(u) = \frac{1}{p}\mathscr{M}([u]_{s,p}^p) - \frac{\lambda}{q}\,\|u\|_{q,w}^q - \phi([u]_{s,p})\frac{1}{p_s^*}\|u\|_{p_s^*}^{p_s^*}.$$

It immediately follows that $\mathcal{I}_\lambda(u) \to \infty$ as $[u]_{s,p} \to \infty$, by (M_1) and (M_2). Hence, $\mathcal{I}_\lambda$ is coercive and bounded from below.

Now, we prove a local Palais–Smale and a topological result for the truncated functional $\mathcal{I}_\lambda$.

Lemma 4.1 *There exists $\overline{\lambda} > 0$ such that, for any $\lambda \in (0, \overline{\lambda})$*

(i) *if $\mathcal{I}_\lambda(u) \le 0$ then $[u]_{s,p} < R_0$ and also $\mathcal{J}_\lambda(v) = \mathcal{I}_\lambda(v)$ for any v in a sufficiently small neighborhood of u;*
(ii) *$\mathcal{I}_\lambda$ satisfies a local Palais–Smale condition for $c < 0$.*

Proof Considering λ_0 and λ^* given respectively by Lemma 3.1 and (4.2), we choose $\overline{\lambda}$ sufficiently small such that $\overline{\lambda} \le \min\{\lambda_0, \lambda^*\}$. Let $\lambda < \overline{\lambda}$.

For proving (i) we assume that $\mathcal{I}_\lambda(u) \leq 0$. When $[u]_{s,p} \geq 1$, by using (M_1), (M_2) with $\tau = 1$, (2.1) and $\lambda < \lambda^*$, we see that

$$\mathcal{I}_\lambda(u) \geq \frac{\kappa}{p\theta}[u]_{s,p}^p - \frac{\lambda^*}{q}C_w^q[u]_{s,p}^q > 0$$

where the last inequality follows by $q < p$ and because by $\mathcal{G}_{\lambda^*}(R_1) > 0$ and (4.1) we have

$$\frac{\kappa}{p\theta}R_1^p - \frac{\lambda^*}{q}C_w^q R_1^q > 0.$$

Thus, we get the contradiction $0 \geq \mathcal{I}_\lambda(u) > 0$. Similarly, when $R_1 \leq [u]_{s,p} < 1$, by using (2.1), (2.3) and $\lambda < \lambda^*$, we obtain

$$\mathcal{I}_\lambda(u) \geq \frac{\mathscr{M}(1)}{p}[u]_{s,p}^{p\theta} - \frac{\lambda^*}{q}C_w^q[u]_{s,p}^q > 0$$

where the last inequality follows by $q < p < p\theta$ and because by $\mathcal{G}_{\lambda^*}(R_1) > 0$ we have

$$\frac{\mathscr{M}(1)}{p}R_1^{p\theta} - \frac{\lambda^*}{q}C_w^q R_1^q > 0.$$

We get again the contradiction $0 \geq \mathcal{I}_\lambda(u) > 0$. When $[u]_{s,p} < R_1$, since $\phi(t) \leq 1$ for any $t \in [0, \infty)$ and $\lambda < \lambda^*$, we have

$$0 \geq \mathcal{I}_\lambda(u) \geq \mathcal{G}_\lambda([u]_{s,p}) \geq \mathcal{G}_{\lambda^*}([u]_{s,p}),$$

and this yields $[u]_{s,p} \leq R_0$, by definition of R_0. Furthermore, for any $u \in B(0, R_0/2)$ we have $\mathcal{I}_\lambda(u) = \mathcal{J}_\lambda(u)$.

To prove a local Palais–Smale condition for $\mathcal{I}_\lambda$ at level $c < 0$, we first observe that any Palais–Smale sequences for $\mathcal{I}_\lambda$ must be bounded, since $\mathcal{I}_\lambda$ is coercive. Thus, since $\lambda < \lambda_0$ by Lemma 3.1 we have a local Palais–Smale condition for $\mathcal{J}_\lambda \equiv \mathcal{I}_\lambda$ at any level $c < 0$. □

Here, in order to get the next technical result, we need a finite dimensional subspace of $D^{s,p}(\mathbb{R}^N)$. For this, since $D^{s,p}(\mathbb{R}^N)$ is a separable and reflexive Banach space, see for example [26], there exists $\{\varphi_n\}_n \subset D^{s,p}(\mathbb{R}^N)$. Then, for any $n \in \mathbb{N}$ we can set $X_n = span\,\{\varphi_n\}$ and $Y_n = \oplus_{i=1}^n X_i$.

Lemma 4.2 *For any $\lambda > 0$ and $n \in \mathbb{N}$, there exists $\varepsilon = \varepsilon(\lambda, n) > 0$ such that*

$$\gamma(\mathcal{I}_\lambda^{-\varepsilon}) \geq n,$$

where $\mathcal{I}_\lambda^{-\varepsilon} = \left\{u \in D^{s,p}(\mathbb{R}^N) : \mathcal{I}_\lambda(u) \leq -\varepsilon\right\}$.

Proof Fix $\lambda > 0, n \in \mathbb{N}$. Since Y_n is finite dimensional, there exists a positive constant $c(n)$ such that

$$c(n)[u]_{s,p}^q \le \|u\|_{q,w}^q ,$$

for any $u \in Y_n$. Thus, for any $u \in Y_n$ with $[u]_{s,p} \le R_0$ we get

$$\begin{aligned}\mathcal{I}_\lambda(u) &\le \frac{M^*}{p}[u]_{s,p}^p - \frac{\lambda}{q}\|u\|_{q,w}^q - \frac{1}{p_s^*}\|u\|_{p_s^*}^{p_s^*}\\ &\le \frac{M^*}{p}[u]_{s,p}^p - \frac{\lambda}{q}c(n)[u]_{s,p}^q,\end{aligned}$$

with $M^* = \max_{\tau\in[0,R_0]} M(\tau) < \infty$, by continuity of M. Finally, let ρ and R be two positive constants with

$$\rho < R < \min\left\{R_0, \left[\frac{\lambda\, c(n)\, p}{q\, M^*}\right]^{\frac{1}{p-q}}\right\}, \tag{4.3}$$

and let

$$\mathbb{S}_n = \left\{u \in Y_n : \ [u]_{s,p} = \rho\right\}.$$

Of course, $\mathbb{S}_n$ is homeomorphic to the $n-1$–dimensional sphere S^{n-1}. Moreover for any $u \in \mathbb{S}_n$

$$\begin{aligned}\mathcal{I}_\lambda(u) &\le \rho^q\left(\frac{M^*}{p}\rho^{p-q} - \frac{\lambda\, c(n)}{q}\right)\\ &\le R^q\left(\frac{M^*}{p}R^{p-q} - \frac{\lambda\, c(n)}{q}\right) < 0\end{aligned}$$

where the last inequality follows by (4.3). Thus, we can find a constant $\varepsilon > 0$ such that $\mathcal{I}_\lambda(u) < -\varepsilon$ for any $u \in \mathbb{S}_n$. Hence $\mathbb{S}_n \subset \mathcal{I}_\lambda^{-\varepsilon}$, by parts (2) and (4) of Proposition 2.1 we get $\gamma(\mathcal{I}_\lambda^{-\varepsilon}) \ge \gamma(\mathbb{S}_n) = n$. □

5 Main Results

Here we define for any $n \in \mathbb{N}$ the sets

$$\begin{aligned}\Sigma_n &= \left\{A \subset D^{s,p}(\mathbb{R}^N)\setminus\{0\} : \ A \text{ is closed, } A = -A \text{ and } \gamma(A) \ge n\right\},\\ K_c &= \left\{u \in D^{s,p}(\mathbb{R}^N) : \ \mathcal{I}_\lambda'(u) = 0 \text{ and } \mathcal{I}_\lambda(u) = c\right\},\end{aligned}$$

and the number

$$c_n = \inf_{A \in \Sigma_n} \sup_{u \in A} \mathcal{I}_\lambda(u). \tag{5.1}$$

Before proving our main results, we state some crucial properties of the family of numbers $\{c_n\}_n$.

Lemma 5.1 *For any $\lambda > 0$ and $n \in \mathbb{N}$, the number c_n is negative.*

Proof Let $\lambda > 0$ and $n \in \mathbb{N}$. By Lemma 4.2, there exists $\varepsilon > 0$ such that $\gamma(\mathcal{I}_\lambda^{-\varepsilon}) \geq n$. Since also $\mathcal{I}_\lambda$ is continuous and even, $\mathcal{I}_\lambda^{-\varepsilon} \in \Sigma_n$. From $\mathcal{I}_\lambda(0) = 0$ we have $0 \notin \mathcal{I}_\lambda^{-\varepsilon}$. Furthermore $\sup_{u \in \mathcal{I}_\lambda^{-\varepsilon}} \mathcal{I}_\lambda(u) \leq -\varepsilon$. In conclusion, remembering also that $\mathcal{I}_\lambda$ is bounded from below, we get

$$-\infty < c_n = \inf_{A \in \Sigma_n} \sup_{u \in A} \mathcal{I}_\lambda(u) \leq \sup_{u \in \mathcal{I}_\lambda^{-\varepsilon}} \mathcal{I}_\lambda(u) \leq -\varepsilon < 0.$$

Hence the proof is complete. □

Lemma 5.2 *Let $\lambda \in (0, \overline{\lambda})$, with $\overline{\lambda}$ given in Lemma 4.1.*

Then, all c_n given by (5.1) are critical values for $\mathcal{I}_\lambda$ and $c_n \to 0$ as $n \to \infty$.

Proof It is clear that $c_n \leq c_{n+1}$, By Lemma 5.1 we have $c_n < 0$. Hence $c_n \to \overline{c} \leq 0$. Moreover, by Lemma 4.1 the functional $\mathcal{I}_\lambda$ satisfies the Palais–Smale condition at c_n. Thus, it follows from standard arguments as in [28] that all c_n are critical values of $\mathcal{I}_\lambda$. We claim that $\overline{c} = 0$. If $\overline{c} < 0$, then still by by Lemma 4.1, we have $K_{\overline{c}}$ is compact. From part (5) of Proposition 2.1 it follows that $\gamma(K_{\overline{c}}) = n_0 < \infty$ and there exists $\delta > 0$ such that $\gamma(K_{\overline{c}}) = \gamma(N_\delta(K_{\overline{c}})) = n_0$.

By Theorem 3.4 of [3] there exist $\varepsilon \in (0, \overline{c})$ and an odd homeomorphism $\eta : D^{s,p}(\mathbb{R}^N) \to D^{s,p}(\mathbb{R}^N)$ such that

$$\eta(\mathcal{I}_\lambda^{\overline{c}+\varepsilon} \setminus N_\delta(K_{\overline{c}})) \subset \mathcal{I}_\lambda^{\overline{c}-\epsilon}. \tag{5.2}$$

Since c_n is increasing and converges to $\overline{c}$, there exists $n \in \mathbb{N}$ such that $c_n > \overline{c} - \varepsilon$ and $c_{n+n_0} \leq \overline{c}$. Choose $A \in \Sigma_{n+n_0}$ such that $\sup_{u \in A} \mathcal{I}_\lambda(u) < \overline{c} + \varepsilon$. By part (3) of Proposition 2.1, we have

$$\gamma(\overline{A \setminus N_\delta(K_{\overline{c}})}) \geq \gamma(A) - \gamma(N_\delta(K_{\overline{c}})), \quad \gamma(\eta(\overline{A \setminus N_\delta(K_{\overline{c}})})) \geq n. \tag{5.3}$$

Therefore, we have

$$\eta(\overline{A \setminus N_\delta(K_{\overline{c}})}) \in \Sigma_n.$$

Consequently

$$\sup_{u \in \eta(\overline{A \setminus N_\delta(K_{\overline{c}})})} \mathcal{I}_\lambda(u) \geq c_n > \overline{c} - \epsilon. \tag{5.4}$$

On the other hand, by (5.2) and (5.3), we have

$$\eta(\overline{A \setminus N_\delta(K_{\bar{c}})}) \subset \eta(\mathcal{I}_\lambda^{\bar{c}+\epsilon} \setminus N_\delta(K_{\bar{c}})) \subset \mathcal{I}_\lambda^{\bar{c}-\epsilon}, \tag{5.5}$$

which contradicts (5.4). Hence $c_n \to 0$ as $n \to \infty$. □

Now, we are ready to give the proof of Theorem 1.1, as follows.

Proof of Theorem 1.1 By Lemma 4.1, $\mathcal{I}_\lambda(u) = \mathcal{J}_\lambda(u)$ if $\mathcal{I}_\lambda(u) < 0$. Then, by Lemmas 4.1, 4.2, 5.1 and 5.2, one can see that all the assumptions of Lemma 2.1 are satisfied. This completes the proof. □

We conclude proving our second main result.

Proof of Theorem 1.2 The proof is substantially similar to the one of Theorem 1.1. We just observe that, to prove Lemma 3.1 we must consider $\lambda \in (0, \lambda_0)$ with λ_0 sufficiently small such that

$$\left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)^{-\frac{p_s^*}{p_s^*-q}} \left[\lambda_0 \left(\frac{1}{q} - \frac{1}{p\theta}\right) \|w\|_r\right]^{\frac{p_s^*}{p_s^*-q}} < \left(b^{1/\theta} S\right)^{\frac{p_s^*\theta}{p_s^*-p\theta}},$$

where $q < p\theta < p_s^*$, b comes from (1.4) and S is given in (2.2). While, in order to state our truncation argument as in Sect. 4, we must take $R_1 \in (0, 1)$ sufficiently small and $\lambda^* > 0$ as follows

$$0 < \frac{b}{p\theta} R_1^{p\theta} - \frac{1}{p_s^* S} R_1^{p_s^*} \quad \text{and} \quad \lambda^* = \frac{q}{2\, C_w^q\, R_1^q} \left(\frac{b}{p\theta} R_1^{p\theta} - \frac{1}{p_s^* S} R_1^{p_s^*}\right).$$

Hence, by considering $q \in (1, p\theta)$ we can argue as in Theorem 1.1. □

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