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Exponential Asymptotic Stability for the Klein Gordon Equation on Non-compact Riemannian Manifolds

C. A. Bortot[1] · M. M. Cavalcanti[2] · V. N. Domingos Cavalcanti[2] · P. Piccione[3]

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Abstract The Klein Gordon equation subject to a nonlinear and locally distributed damping, posed in a complete and non compact n dimensional Riemannian manifold $(\mathcal{M}^n, \mathbf{g})$ without boundary is considered. Let us assume that the dissipative effects are effective in $(\mathcal{M}\backslash\Omega) \cup (\Omega\backslash V)$, where Ω is an arbitrary open bounded set with smooth boundary. In the present article we introduce a new class of non compact Riemannian manifolds, namely, manifolds which admit a smooth function f, such that the Hessian of f satisfies the *pinching conditions* (locally in Ω), for those ones, there exist a finite number of disjoint open subsets V_k free of dissipative effects such that $\bigcup_k V_k \subset V$ and for all $\varepsilon > 0$, $meas(V) \geq meas(\Omega) - \varepsilon$, or, in other words, the dissipative effect inside Ω possesses measure arbitrarily small. It is important to be mentioned that if the function f satisfies the pinching conditions everywhere, then it is not necessary to consider dissipative effects inside Ω.

1 Introduction

This paper addresses the well-posedness as well as sharp uniform decay rate estimates of the energy related to the Klein Gordon equation subject to a nonlinear and locally distributed damping, posed in a complete and non compact n dimensional Riemannian manifold $(\mathcal{M}^n, \mathbf{g})$ without boundary:

✉ C. A. Bortot
c.bortot@ufsc.br

[1] Technological Centre of Joinville, Federal University of Santa Catarina - Campuses Joinville, Joinville, SC 89218-035, Brazil

[2] Department of Mathematics, State University of Maringá, Maringá, PR 87020-900, Brazil

[3] Department of Mathematics, IME-Universidade de São Paulo, São Paulo, SP 05508-090, Brazil

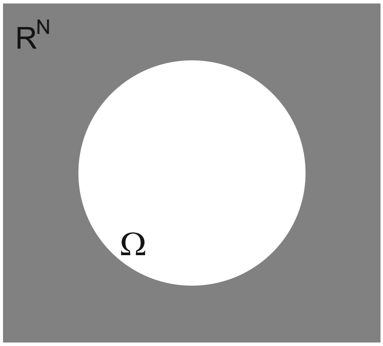

Fig. 1 When $g(s) = s$, the exponential decay is expected

$$\begin{cases} u_{tt} - \Delta u + u + a(x)g(u_t) = 0, \text{ in } \mathcal{M} \times (0, +\infty), \\ u(x,0) = u_0(x); \ u_t(x,0) = u_1(x), \ x \in \mathcal{M}, \end{cases} \tag{1.1}$$

where Δ denotes the Laplace–Beltrami operator. The non-negative and essentially bounded function $a = a(x)$, responsible by the nonlinear and localized dissipative effect, lies properly in $\mathcal{M}\backslash\overline{\Omega}$, where Ω is an arbitrary open and bounded set in $\mathcal{M}$ with smooth boundary $\partial\Omega$, that is, $a(x) \geq a_0 > 0$ a.e. in $\mathcal{M}\backslash\overline{\Omega}$.

The counterpart of the above problem (1.1) in the Euclidean setting is precisely when $\mathcal{M} = \mathbb{R}^n$ endowed with the usual Euclidean topology. In this case, it is sufficient to consider dissipative effects in $\mathbb{R}^n\backslash\overline{\Omega}$ as considered in the Fig. 1. See, for instance, the following references [17,44].

This is well-known since in the Euclidean setting the bicharacteristics (which are the graphs of unit-speed geodesics) or also called rays of the geometric optics, are straight lines so that, roughly speaking, every ray of the geometric optics that intercepts the region Ω never remains in Ω. However, if one considers a non compact Riemannian manifold $(\mathbb{R}^n, \mathbf{g})$, where $\mathbf{g}$ is a generic Riemannian metric, we have to be very careful because of the existence of complete geodesics contained in Ω, which severely violates the law of the geometric optics due to Bardos et al. [5,33,34], namely: there exists a time T_0 such that any geodesic of length $\leq T_0$ meets the open set $\{x; a(x) > 0\}$. In this case it has been established by Rauch and Taylor [34] that the energy $E(u,t) = \frac{1}{2}\int_{\mathcal{M}} \left(|\nabla_x u|^2 + |u_t|^2\right) dx$ associated to problem (1.1) (for $g(s) = s$ and compact manifolds without boundary) decays exponentially, and this result was extended to the case of compact manifolds with boundary by Barbos, Lebeau and Rauch [5]. Consequently the existence of complete geodesics contained in Ω implies that no exponential decay is expected. In other words: the existence of trapped geodesics breaks the exponential stability (see Fig. 2).

From the above comments and in order to obtain the exponential stability to problem (1.1) it is strongly necessary to consider dissipative effects inside the set Ω. The best way of doing this is to introduce damping as little as possible. Indeed, the first main goal of the present article is to show that there exists a region $V \subset \Omega$ free of dissipative effects such that $\text{meas}(V) \geq \text{meas}(\Omega) - \varepsilon$, for an arbitrary positive number ε. This result can be considered sharp in the sense that the region with damping possesses measure arbitrarily small, however totally distributed (see Fig. 3). We also show that

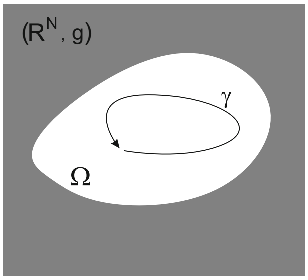

Fig. 2 The existence of trapped geodesics breaks the exponential stability

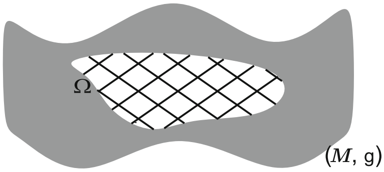

Fig. 3 The region in *black* inside Ω possesses dissipative effects and measure arbitrarily small, while the *white* region contained in Ω does not have dissipative effects and measure arbitrarily large. However, both regions are totally distributed

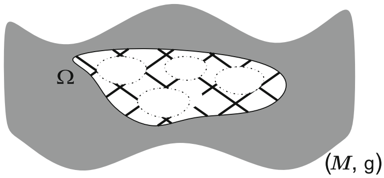

Fig. 4 No damping is necessary in open subsets of Ω when the sectional curvature of $(\mathcal{M}, g)$ satisfies $k_1 \leq sec_g \leq k_2 < 0$

there are no geodesics trapped in the interior of each region free of dissipative effects (see Appendix).

The second main task of the present paper is to prove that we can also avoid to put damping in open subsets inside Ω for non compact manifolds $(\mathcal{M}, g)$ which satisfy the condition $k_1 \leq sec_g \leq k_2 < 0$, for some negative constants k_1 and k_2, where sec_g denotes the sectional curvature of $(\mathcal{M}, g)$, according to Fig. 4. In order to achieve our goal, we combine ideas from the article [13] of the first and the third authors with new tools we shall describe in the sequel. We proceed as follows:

(1) We prove that for every $x \in \Omega$, there exist a neighborhood that can be left without damping;
(2) We prove that a very precise portion of open subsets inside Ω can be left without damping;
(3) Let $\varepsilon > 0$ and $V_1, \ldots, V_k$ be open subsets inside Ω as in (1) and (2) which closures are pairwise disjoint. We prove that there exist a $V \supset \cup_{i=1}^{k} V_i$ that can be left without damping and such that $meas(V) \geq meas(\Omega) - \varepsilon$.

For this purpose, we will construct an intrinsic multiplier that will play an important role when establishing the desired uniform decay rates of the energy, in this sense the compactness of Ω is crucial. Fix $\varepsilon > 0$. This multiplier is given by $\langle \nabla f, \nabla u \rangle$, where $f : \overline{\Omega} \to \mathbb{R}$ is a smooth function such that its Hessian, Hess(f) (sometimes also denoted by $\nabla^2 f$), is closely related to $\mathbf{g}$ on an open subset $V \subset \Omega$ that satisfies $meas(V) \geq meas(\Omega) - \varepsilon$, that is, assuming that $k_1 \leq sec_g \leq k_2 < 0$ we deduce that the Hess(f) satisfies the "pinching conditions' in each component V_i of V or, in other words, $A \leq \text{Hess}(f) \leq B$ in V_i for some positive positive constants $0 < A < B$. It is important to be mentioned that if f satisfies the pinching conditions everywhere, then we can let Ω free of dissipative effects. This construction will be clarified during the proof.

It is worth observing that the scenarios presented in Figs. 1 and 4 are distinct, i.é., the Fig. 1 refers to the Euclidean environment and Fig. 4 to a manifold with a generic Riemannian metric.

While in [12,13] the authors work with compact manifolds here we have to deal with non compact manifolds. Another technical difficulty found in the present work was the following: while in [12,13] a regular solution u to problem (1.1) lies globally in $H^2(\mathcal{M})$, in the present case, it simply exists locally in $H^2(\Omega)$ for all $\Omega \subset \mathcal{M}$. Fortunately it is enough for our purpose. However, the main technical difficulty found in the present paper was to deal with the boundary terms (see the proof of Theorem 4.1). Indeed, in [12,13] the homogeneous Dirichlet boundary condition is assumed so that the boundary terms that appear in the computations and that do not vanish are easier to be handled, when compared with the present situation. Note that in the present case we do not have any control on the terms defined on the boundary $\partial\Omega$. In the Euclidean setting Ω is assumed to be a ball centered in the origin with radius R so that its boundary possesses nice geometric properties as considered, for instance, in [44]. We employ a similar idea as considered in [44] to deal with boundary terms, however our approach is much more delicate since Ω is an arbitrary open set contained in $\mathcal{M}$. Since we are interested in the impact of the geometry in the decay rate estimates we shall assume that there exist positive constants k, K such that $k|s|^2 \leq g(s)s \leq K|s|^2$ for all $s \in \mathbb{R}$, where the real function g is assumed to be monotone increasing. It is worth mentioning that some ideas used in this work are based on works due to Yao, see for example [42].

We would like to emphasize that the main novelty in the present article is to introduce a new class of non compact Riemannian manifolds, namely, manifolds which admit a smooth function f, such that the Hessian $\nabla^2 f$ satisfies the "pinching conditions", for those ones, there exist open disjoint subsets without damping as mentioned before. In addition, analogously what has been proved in [13], for compact manifolds, it is possible, as well, by using the present approach, to let free of damping radially symmetric disjoint regions for non compact manifolds, assuming, as in [13], that we are endowing $(\mathcal{M}, \mathbf{g})$ by a radial metric. As a consequence, the present work is an expansion of [13]. It is worth mentioning that in the non compact case, we have examples of existence of functions satisfying the pinching condition, see the class of Warped Products (Sect. 5).

In the particular case when one has the non compact manifold $(\mathbb{R}^n, g_\varphi)$, endowed with the radial metric described in polar coordinates $(r, \theta) \in \left[0, +\infty\right[\times \mathbb{S}^{n-1}$ by the formula:

$$g_\varphi = \mathrm{d}r^2 + \varphi(r)^2 \mathrm{d}\theta^2,$$

where $d\theta^2$ is the standard round metric (of radius 1) of $\mathbb{S}^{n-1}$ and $\varphi :]0, +\infty[\to \mathbb{R}^+$ is a smooth function satisfying $\varphi^{(2k)}(0) = 0$ for all $k \geq 0$, $\varphi'(0) = 1$, we can avoid to put damping in every open ball Ω of radius $R > 0$ (see Fig. 1) according to the properties of the function φ. Indeed, if $\varphi'(r) \geq c > 0$ for all r, then $(\mathbb{R}^n, g_\varphi)$ admits a proper, strongly convex function which is bounded from below. This means that there exists a positive constant c such that:

$$\text{Hess}(f)(X, X) \geq c\, g(X, X), \qquad \forall X \in TM. \tag{1.2}$$

The above property plays an essential role when establishing the sharp decay rate estimates above mentioned. Indeed, in a general setting, that is, for non compact complete manifolds $(\mathcal{M}^n, \mathbf{g})$ we first prove that (1.2) is satisfied locally in Ω and from the compactness of $\overline{\Omega}$ we obtain the property globally in Ω by gluing all the finite connected components V_k of V above mentioned and putting damping between them in a region of measure arbitrary small. Note that connected components V_k can be extremely small, however the measure of $\cup V_k$ is arbitrarily large (see Figs. 3, 4). Now, for the particular case when one has $(\mathbb{R}^n, g_\varphi)$ and assuming that φ has nice properties, inequality (1.2) is now valid globally and we can avoid putting damping in the whole set Ω. This recovers the previous results in the literature in the Euclidean setting with the usual metric. Our result remains valid for the semi-linear problem $u_{tt} - \Delta u + f(u) + a(x)g(u_t) = 0$, as well, provided that: (i) we assume that $(M^n, \mathbf{g})$ possesses bounded curvature and global injectivity radius (so that we can recover the Sobolev imbedding); (ii) we assume that there exists a unique continuation property to the linear problem with potential $v_{tt} - \Delta v + V(x,t)v = 0$ where $V(x,t) \equiv f'(u(x,t))$ grows polynomially. The second condition is true for certain nonlinearities as proved in Triggiani and Yao's paper [37].

Another interesting case is the following: $(\mathbb{R}^n, g_\varphi)$ has finite volume if and only if $\displaystyle\int_0^{+\infty} \varphi(r)^{n-1}\, dr < +\infty$. In this case, we can relax the assumptions imposed on g, namely, it is sufficient to assume that $ks^2 \leq g(s)s \leq Ks^2$ for $|s| > 1$ and it is possible obtain a wide assortment of decay rate estimates different of exponential. For this purpose Jessen's inequality combined with arguments of Lasiecka and Tataru [27] play an essential role when establishing the above mentioned decay rates (see also Cavalcanti et. al [11]).

Our paper is organized as follows. Section 2 we present the preliminaries in Riemannian Geometry, in Sect. 3 is proven to well-posedness of the problem, in Sect. 4 is stated and proved the main result. The Sect. 5 is destined examples of manifolds that admit open subsets without damping and finally the Appendix aims to justify the application of unique continuation principle used in Sect. 4.

2 Preliminaries: Geometric Riemannian Tools

Let $(\mathcal{M}^n, \mathbf{g})$ be a n-dimensional complete Riemanniana manifold, $n \geq 2$ orientable, connect and without boundary, induced by the Riemannian metric $\mathbf{g}(\cdot, \cdot) = \langle \cdot, \cdot \rangle$, of class C^∞. We shall denote by $(g_{ij})_{n\times n}$ the matrix $n \times n$ in connection with the metric **g**. The tangent space at $\mathcal{M}$ em $p \in \mathcal{M}$ will be denoted by $T_p\mathcal{M} \equiv \mathbb{R}^n$.

Let $f \in C^2(\mathcal{M})$, and let us define the Laplace–Beltrami operator of f, as

$$\Delta f = div(\nabla f), \tag{2.1}$$

where ∇f denotes the gradient of f in the metric **g**, that is, for all vector field X in $\mathcal{M}$

$$\langle \nabla f, X\rangle = X(f), \tag{2.2}$$

and div denotes the divergent operator, namely, if X is a vector field in $\mathcal{M}$, $divX(p) :=$ trace of the linear map $Y(p) \mapsto \nabla_Y X(p)$, $p \in \mathcal{M}$.

From the definitions and notations above we have the following lemma:

Lemma 2.1 *Let $p \in \mathcal{M}$. Let us consider $f \in C^1(\mathcal{M})$ and H a vector field in $\mathcal{M}$. Then, the following identity hold (see [28] p. 21):*

$$\langle \nabla f, \nabla(H(f))\rangle = \nabla H(\nabla f, \nabla f) + \frac{1}{2}\left[div(|\nabla f|^2 H) - |\nabla f|^2 div H\right],$$

where ∇H is the differential covariant derivative defined by $\nabla H(X, Y) = \langle \nabla_X H, Y\rangle$.

Finally we shall define the Hessian of $f \in C^2(\mathcal{M})$ as the symmetric tensor of order two in $\mathcal{M}$, namely,

$$\text{Hess}(f)(X, Y) = \nabla^2 f(X, Y) := \nabla(\nabla f)(X, Y) = \langle \nabla_Y(\nabla f), X\rangle, \tag{2.3}$$

for all X and Y vector fields in $\mathcal{M}$.

Remark 1 In order to simplify the notation, we denote the L^2-norm, without distinguishing whether the argument of the norm is a function or tensor field of type $(0, m)$.

Let $k \in \mathbb{N}$ e $p \geq 1$. We define the space $C_k^p(\mathcal{M})$ as

$$C_k^p(\mathcal{M}) = \left\{u \in C^\infty(\mathcal{M}); \int_{\mathcal{M}} |\nabla^j u|^p \, d\mathcal{M} < \infty, \forall j = 0, 1, \ldots, k\right\}, \tag{2.4}$$

where $\nabla^j u$ denotes the jth differential covariant derivative of u, ($\nabla^0 u = u$, $\nabla^1 u = \nabla u$).

Thus, we define the Sobolev space $H_k^p(\mathcal{M})$ as the closure of $C_k^p(\mathcal{M})$ with respect to the topology

$$\|u\|^p_{H_k^p(\mathcal{M})} = \sum_{j=0}^{k} \int_{\mathcal{M}} |\nabla^j u|^p \, d\mathcal{M}. \tag{2.5}$$

From the above, we deduce:

(i) $L^2(\mathcal{M}) := H_0^2(\mathcal{M})$ is the closure of $C_0^2(\mathcal{M})$ with respect to the tolopogy

$$\|u\|^2_{L^2(M)} = \int_{\mathcal{M}} |u|^2 \, d\mathcal{M}. \tag{2.6}$$

(ii) $H^1(\mathcal{M}) := H_1^2(\mathcal{M})$ is the closure of $C_1^2(\mathcal{M})$ with respect to the topology

$$\|u\|^2_{H^1(\mathcal{M})} = \int_{\mathcal{M}} |\nabla u|^2 \, d\mathcal{M} + \int_{\mathcal{M}} |u|^2 \, d\mathcal{M}. \tag{2.7}$$

(iii) $H^2(\mathcal{M}) := H_2^2(\mathcal{M})$ is the closure of $C_2^2(\mathcal{M})$ with respect to the topology

$$\|u\|^2_{H^2(\mathcal{M})} = \int_{\mathcal{M}} |\nabla^2 u|^2 \, d\mathcal{M} + \int_{\mathcal{M}} |\nabla u|^2 \, d\mathcal{M} + \int_{\mathcal{M}} |u|^2 \, d\mathcal{M}. \tag{2.8}$$

Remark 2 From the above definitions we have the following chain of continuous embbeding

$$H^2(\mathcal{M}) \hookrightarrow H^1(\mathcal{M}) \hookrightarrow L^2(\mathcal{M}). \tag{2.9}$$

Furthermore, by Hebey ([23], Theorem 2.7, p.13), it follows that $H_0^1(\mathcal{M}) = H^1(\mathcal{M})$, where $H_0^1(\mathcal{M}) := \overline{\mathcal{D}(\mathcal{M})}^{H^1(\mathcal{M})}$, in other words, the space of infinitely differentiable functions with compact support is dense in $H^1(\mathcal{M})$.

So, from the above and making use of density arguments we can extend the formulas presented previously to Sobolev spaces. In the sequel, we shall announce three theorems that will play an important rule in the present work.

Theorem 2.1 (Gauss Divergent Theorem) *Let $\mathcal{M}^n$ a Riemannian manifold, orientable, with smooth boundary $\partial\mathcal{M}$, $X \in [H^1(\mathcal{M})]^n$ a vector filed and ν the normal unitary vector field point towards $\partial\mathcal{M}$, thus*

$$\int_{\mathcal{M}} divX \, d\mathcal{M} = \int_{\partial\mathcal{M}} \langle X, \nu\rangle \, d\partial\mathcal{M}. \tag{2.10}$$

Theorem 2.2 (Green Theorem 1) *Let $\mathcal{M}^n$ a Riemannian orientable manifold , with smooth boundary $\partial\mathcal{M}$, $X \in [H^1(\mathcal{M})]^n$ a vector field, $q \in H^1(\mathcal{M})$ and ν the normal unitary vector field point towards $\partial\mathcal{M}$, then*

$$\int_{\mathcal{M}} (divX)q \, d\mathcal{M} = -\int_{\mathcal{M}} \langle X, \nabla q\rangle \, d\mathcal{M} + \int_{\partial\mathcal{M}} (\langle X, \nu\rangle)q \, d\partial\mathcal{M}. \tag{2.11}$$

Theorem 2.3 (Green Theorem 2) *Let $\mathcal{M}^n$ a orientable Riemannian Manifold, with smooth boundary $\partial\mathcal{M}$, $f \in [H^2(\mathcal{M})]$, $q \in H^1(\mathcal{M})$ and ν the normal unitary vector field point towards $\partial\mathcal{M}$, then*

$$\int_{\mathcal{M}} (\Delta f)q \, d\mathcal{M} = -\int_{\mathcal{M}} \langle \nabla f, \nabla q\rangle \, d\mathcal{M} + \int_{\partial\mathcal{M}} (\partial_\nu f)q \, d\partial\mathcal{M}. \tag{2.12}$$

3 Existence and Uniqueness of Solutions

In what follows we shall omit some variables in order to make easier the notation and we will denote the Laplace–Beltrami operator simply by Δ . We shall study the existence and uniqueness of the following damped problem:

$$\begin{cases} u_{tt} - \Delta u + u + a(x)g(u_t) = 0 \; in \;\; \mathcal{M} \times (0,\infty) \\ u(0) = u_0 \;\; , \;\; u_t(0) = u_1 \qquad\qquad in \;\; \mathcal{M} \end{cases} \tag{3.1}$$

where $(\mathcal{M}^n, \mathbf{g})$ a n-dimensional Rimannian manifold, $n \geq 2$, simply connected, orientable and without boundary endowed by a Riemannian metric $\mathbf{g}(\cdot,\cdot) = \langle\cdot,\cdot\rangle$ complete, of class C^∞.

Assumption 3.1 Hypotheses on the function $g : \mathbb{R} \to \mathbb{R}$:

(i) $g(s)$ is continuous, monotone increasing;
(ii) $g(s)s > 0$ para $s \neq 0$;
(iii) $k|s| \leq |g(s)| \leq K|s|, \;\; \forall s \in \mathbb{R}$, where k e K are two positive constants.

Assumption 3.2 Hypotheses on the function $a : \mathcal{M} \to \mathbb{R}$:

(i) $a(x) \in L^\infty(\mathcal{M})$ is non negative function;
(ii) Let $\Omega \subset \mathcal{M}$ be an open set and bounded with smooth boundary $\partial\Omega$. We suppose that $a(x) \geq a_0 > 0$ in $\mathcal{M}\backslash\overline{\Omega}$ and on an open proper subset $\mathcal{M}_*$ of $\mathcal{M}$ ***namely:***

Let Ω^* be an open and bounded set with smooth boundary such that $\Omega \subset\subset \Omega^*$. For $\varepsilon > 0$ we shall ensure that there exist an open subset $V \subset \overline{\Omega^*}$ and smooth functions α and $f : \overline{\Omega^*} \to \mathbb{R}$ such that $\text{meas}(V) \geq \text{meas}(\overline{\Omega^*}) - \varepsilon$, $\text{meas}(V \cap \partial\overline{\Omega^*}) \geq \text{meas}(\partial\overline{\Omega^*}) - \varepsilon$, $\nabla\alpha|_V \equiv 0$ and such that α and f satisfy

$$C \int_0^T \int_V u_t^2 + |\nabla u|^2 \, d\mathcal{M}dt \leq \int_0^T \int_V \left(\frac{\Delta f}{2} - \alpha\right) u_t^2 \, d\mathcal{M}dt + \int_0^T \int_V \nabla^2 f(\nabla u, \nabla u) + \left(\alpha - \frac{\Delta f}{2}\right) |\nabla u|^2 \, d\mathcal{M}dt,$$

for some positive constant C.

Moreover, if $V_1, \ldots, V_k$ are radially symmetric or, more generally, for a wide class as in the Sect. 8, with pairwise disjoint closures, we can choose V in such a way that $V \supset (\bigcup_{i=1}^k V_i)$.

The open subset $\mathcal{M}_*$ is such that $(\overline{\Omega^*}\backslash V) \subset\subset \mathcal{M}_*$.

Remark 3 The geometric idea related to $\mathcal{M}_*$, is that, $\mathcal{M}_*$ is an open subset of $\mathcal{M}$ immediately larger than the black region that is inside the Ω as show the Fig. 3. It is worth noting that the "net" is built on $\Omega^* \supset\supset \Omega$, but out of Ω the dissipative effect is effective everywhere, so the region of greatest importance is that in the interior of Ω.

The energy associated to problem (3.1) is defined by:

$$E(t) := \frac{1}{2} \int_{\mathcal{M}} u_t^2(x,t) + |\nabla u(x,t)|^2 + u^2(x,t) \, d\mathcal{M}dt. \tag{3.2}$$

Denoting $U = \begin{pmatrix} u \\ u_t \end{pmatrix}$, $U_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$, $\mathcal{H} := H^1(\mathcal{M}) \times L^2(\mathcal{M})$,

$$\begin{aligned} F : \;\; &\mathcal{H} \longrightarrow \mathcal{H} \\ &\begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ a(x)g(v) \end{pmatrix} \end{aligned}$$

and,

$$A : D(A) \subset \mathcal{H} \longrightarrow \mathcal{H}$$
$$\begin{pmatrix} u \\ v \end{pmatrix} \quad \longmapsto A\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -v \\ -\Delta u + u \end{pmatrix},$$

then problem (3.1) can be rewritten as

$$\begin{cases} \dfrac{dU}{dt}(t) + (A + F)U(t) = 0 \\ \\ U(0) = U_0 \end{cases} \tag{3.3}$$

It is possible to prove that A is a maximal monotone operator, F is monotone, bounded and hemicontinuous. Employing Barbu ([4], Corol. 1.1, p. 39) it follows that $A + F$ is maximal monotone in $\mathcal{H}$, and making use of standard semigroup arguments we have the following result:

Theorem 3.1 (i) *Under the conditions above, the problem* (3.1) *is well posed in the space* $D(A) = \{u \in H^1(\mathcal{M});\, \Delta u \in L^2(\mathcal{M})\} \times H^1(\mathcal{M})$, *that is, for any initial data* $\{u_0, u_1\} \in D(A) = \{u \in H^1(\mathcal{M});\, \Delta u \in L^2(\mathcal{M})\} \times H^1(\mathcal{M})$ *exists a unique map* $u : [0,\infty) \longrightarrow H^1(\mathcal{M})$, *which is regular solution to problem* (3.1), *belonging to the class*

$$u \in C^1([0,\infty); L^2(\mathcal{M})) \cap C([0,\infty); H^1(\mathcal{M})); \tag{3.4}$$
$$u_t \in W^{1,\infty}(0,\infty; L^2(\mathcal{M})) \cap L^\infty(0,\infty); H^1(\mathcal{M})). \tag{3.5}$$

(ii) *Under the conditions above, the problem* (3.1) *is well posed in the space* $H^1(\mathcal{M}) \times L^2(\mathcal{M})$, *that is, for any initial data* $\{u_0, u_1\} \in H^1(\mathcal{M}) \times L^2(\mathcal{M})$ *exists a map* $u : [0,\infty) \longrightarrow H^1(\mathcal{M})$ *which is the unique weak solution to problem (3.1) belonging to the class*

$$u \in C^1([0,\infty); L^2(\mathcal{M})) \cap C([0,\infty); H^1(\mathcal{M})). \tag{3.6}$$

In addition u satisfies the additional local regularity:

Proposition 3.1 *Let* $\Omega \subset \mathcal{M}$ *be an open and bounded set with smooth boundary. Then* $u(t) \in H^2(\Omega)$, $\forall\, t \geq 0$, *where u is the regular solution to problem (3.1).*

Proof Let us fix $t \geq 0$. Defining $v := u(t)$ and $f := -u_{tt}(t) - a(\cdot)g(u_t(t)) \in L^2(\mathcal{M})$, we deduce the following identity which is valid in $L^2(\mathcal{M})$

$$-\Delta v + v = f. \tag{3.7}$$

Let $\Omega^* \subset \mathcal{M}$, be an open and bounded set with smooth boundary such that $\Omega \subset\subset \Omega^*$ e $\psi \in \mathcal{D}(\mathcal{M})$ satisfying:

(i) $\psi \equiv 1$ in $\overline{\Omega}$
(ii) $\psi \equiv 0$ in $\overline{\Omega^*}\backslash\Omega$
(iii) $0 \leq \psi \leq 1$

Multiplying equation (3.7) by ψ we infer

$$-\Delta v\psi + v\psi = f\psi.$$

We observe that

$$\Delta(v\psi) = v\Delta\psi + \psi\Delta v + 2\langle\nabla v, \nabla\psi\rangle, \tag{3.8}$$

thus

$$-\Delta(v\psi) + v\psi = f\psi - v\Delta\psi - 2\langle\nabla v, \nabla\psi\rangle \text{ in } L^2(\mathcal{M}).$$

In particular,

$$-\Delta(v\psi) + v\psi = f\psi - v\Delta\psi - 2\langle\nabla v, \nabla\psi\rangle \text{ in } L^2(\Omega^*).$$

Furthermore,

$$v\psi\ |_{\partial\Omega^*} = v\ |_{\partial\Omega^*}\ \psi\ |_{\partial\Omega^*} = 0.$$

From standard elliptic regularity results we deduce that $v\psi \in H^2(\Omega^*)$, from which we obtain $v\psi \in H^2(\Omega)$. Since $\psi \equiv 1$ in Ω we deduce that $v = u(t) \in H^2(\Omega)$. □

4 Stability Result

Before stating the main theorem of stability we shall consider an useful identity called *identity of energy* that reads as follows. Let u be a regular solution to problem (3.1), then multiplying the equation by u_t and performing an integration by parts, we infer:

$$E(t_2) - E(t_1) = -\int_{t_1}^{t_2}\int_{\mathcal{M}} a(x)g(u_t)u_t\, d\mathcal{M}dt, \tag{4.1}$$

for all $t_2 > t_1 \geq 0$.

It is worth mentioning that the identity of energy (4.1) remains true for every weak solution to problem (3.1) by using standard arguments of density.

Theorem 4.1 *Let u be a weak solution to problem* (3.1)*, with the energy defined as in* (3.2)*. Then, under Assumption* 3.1 *and Assumption* 3.2 *there exist positive constants* T_0*,* C_0 *and* λ_0 *such that*

$$E(t) \leq C_0 e^{-\lambda_0 t} E(0);\quad \forall t \geq T_0.$$

Our main task is to prove the following inequality inequality:

$$\int_0^T E(t)dt \le C_1 E(T) + C_2 \int_0^T \int_{\mathcal{M}} a(x)[u_t^2(x,t) + g^2(u_t(x,t))]\, d\mathcal{M}dt, \quad (4.2)$$

where C_1 and C_2 are positive constants and C_1 doe not depend on T. As we have already mentioned, it is sufficient to work with regular solutions to problem (3.1), since the exponential decay rate can be recovered for weak solutions by using density approach.

4.1 Recovering the Energy Outside the Region Ω^*

Let $\Omega^* \subset \mathcal{M}$ be an open and bounded set with smooth boundary $\partial\Omega^*$, such that $\Omega \subset\subset \Omega^*$. Let us consider $\varphi \in C^\infty(\mathcal{M})$ satisfying:

(i) $\varphi \equiv 1$ in $\mathcal{M}\backslash\Omega^*$;
(ii) $\varphi \equiv 0$ in $\overline{\Omega}$;
(iii) $0 \le \varphi \le 1$ in $\mathcal{M}$.

Multiplying equation (3.1) by φu and integrating over $[0, T] \times \mathcal{M}$, we obtain

$$\int_0^T \int_{\mathcal{M}} [u_{tt} - \Delta u + a(x)g(u_t) + u]\varphi u\, d\mathcal{M}dt = 0. \quad (4.3)$$

We observe that

$$\begin{aligned} \frac{d}{dt}\left[\int_{\mathcal{M}} u_t\varphi u\, d\mathcal{M}\right] &= \int_{\mathcal{M}} u_{tt}\varphi u\, d\mathcal{M} + \int_{\mathcal{M}} u_t\varphi u_t\, d\mathcal{M} \\ &= \int_{\mathcal{M}} u_{tt}\varphi u\, d\mathcal{M} + \int_{\mathcal{M}} u_t^2\varphi\, d\mathcal{M}. \end{aligned}$$

Integrating over $[0, T]$ we conclude that

$$\int_0^T \int_{\mathcal{M}} u_{tt}\varphi u\, d\mathcal{M}dt = \left[\int_{\mathcal{M}} u_t\varphi u\, d\mathcal{M}\right]_0^T - \int_{\mathcal{M}} u_t^2\varphi\, d\mathcal{M}dt. \quad (4.4)$$

In addition,

$$\begin{aligned} \int_0^T \int_{\mathcal{M}} -\Delta u(\varphi u)\, d\mathcal{M}dt &= \int_0^T \int_{\mathcal{M}} \langle \nabla u, \nabla(\varphi u)\rangle\, d\mathcal{M}dt \\ &= \int_0^T \int_{\mathcal{M}} \langle \nabla u, \nabla\varphi\rangle u + \varphi|\nabla u|^2\, d\mathcal{M}dt. \quad (4.5) \end{aligned}$$

Combining (4.3), (4.4) and (4.5), we deduce

$$
\begin{aligned}
0 = & \left[\int_{\mathcal{M}} u_t \varphi u \, d\mathcal{M}\right]_0^T - \int_{\mathcal{M}} u_t^2 \varphi \, d\mathcal{M} dt \\
& + \int_0^T \int_{\mathcal{M}} \langle \nabla u, \nabla \varphi \rangle u + \varphi |\nabla u|^2 \, d\mathcal{M} dt \\
& + \int_0^T \int_{\mathcal{M}} \varphi u^2 \, d\mathcal{M} dt + \int_0^T \int_{\mathcal{M}} a(x) g(u_t) \varphi u \, d\mathcal{M} dt. \qquad (4.6)
\end{aligned}
$$

Taking into account the properties of function φ, we can write

$$
\begin{aligned}
& \int_0^T \int_{\Omega^* \setminus \Omega} \varphi [u^2 + |\nabla u|^2] \, d\mathcal{M} dt + \int_0^T \int_{\mathcal{M} \setminus \Omega^*} u^2 + |\nabla u|^2 \, d\mathcal{M} dt \\
& = - \left[\int_{\mathcal{M}} u_t \varphi u \, d\mathcal{M}\right]_0^T + \int_{\Omega^* \setminus \Omega} \varphi u_t^2 \, d\mathcal{M} dt + \int_{\mathcal{M} \setminus \Omega^*} u_t^2 \, d\mathcal{M} dt \\
& - \int_0^T \int_{\mathcal{M}} \langle \nabla u, \nabla \varphi \rangle u \, d\mathcal{M} dt - \int_0^T \int_{\mathcal{M} \setminus \Omega} a(x) g(u_t) \varphi u \, d\mathcal{M} dt. \qquad (4.7)
\end{aligned}
$$

Observing that $a(x) \geq a_0 > 0$ in $\mathcal{M} \setminus \overline{\Omega}$, it holds that

$$
\begin{aligned}
& \int_0^T \int_{\mathcal{M} \setminus \Omega^*} u^2 + |\nabla u|^2 \, d\mathcal{M} dt \leq - \left[\int_{\mathcal{M}} u_t \varphi u \, d\mathcal{M}\right]_0^T \\
& + a_0^{-1} \int_0^T \int_{\Omega^* \setminus \Omega} a(x) \varphi u_t^2 \, d\mathcal{M} dt + a_0^{-1} \int_0^T \int_{\mathcal{M} \setminus \Omega^*} a(x) u_t^2 \, d\mathcal{M} dt \\
& - \int_0^T \int_{\mathcal{M}} \langle \nabla u, \nabla \varphi \rangle u \, d\mathcal{M} dt - \int_0^T \int_{\mathcal{M} \setminus \Omega} a(x) g(u_t) \varphi u \, d\mathcal{M} dt. \qquad (4.8)
\end{aligned}
$$

Adding $\int_0^T \int_{\mathcal{M} \setminus \Omega^*} u_t^2 \, d\mathcal{M} dt$ in the previous inequality and using the fact that $0 \leq \varphi \leq 1$, we obtain

$$
\begin{aligned}
& \int_0^T \int_{\mathcal{M} \setminus \Omega^*} u^2 + |\nabla u|^2 + u_t^2 \, d\mathcal{M} dt \leq - \left[\int_{\mathcal{M}} u_t \varphi u \, d\mathcal{M}\right]_0^T \\
& + a_0^{-1} \int_0^T \int_{\mathcal{M}} a(x) u_t^2 \, d\mathcal{M} dt - \int_0^T \int_{\mathcal{M}} \langle \nabla u, \nabla \varphi \rangle u \, d\mathcal{M} dt \\
& - \int_0^T \int_{\mathcal{M} \setminus \Omega} a(x) g(u_t) \varphi u \, d\mathcal{M} dt + \int_0^T \int_{\mathcal{M} \setminus \Omega^*} u_t^2 \, d\mathcal{M} dt \qquad (4.9)
\end{aligned}
$$

Next, we shall estimate the term $\int_0^T \int_{\mathcal{M}} \langle \nabla u, \nabla \varphi \rangle u \, d\mathcal{M} dt$.

Observe that $\langle \nabla u, \nabla \varphi \rangle = 0$ in $\mathcal{M} \setminus \Omega^*$.

We shall make use of a generalized Green's identity, that makes sense in our case, because $u \in H^1(\Omega^*)$, then $u^2 \in W^{1,1}(\Omega^*)$, thus $u^2\big|_{\partial \Omega^*} \in L^1(\partial \Omega^*)$ (see [6]) and therefore by using the dense and continuous immersion $\mathcal{D}(\overline{\Omega^*}) \hookrightarrow W^{1,1}(\Omega^*)$ we can show the desired. Note that in our case $\varphi \in C^\infty(\overline{\Omega^*})$ and $\varphi \equiv 1$ in $\partial\Omega^*$.

So we can write

$$\begin{aligned}\int_{\mathcal{M}} \langle \nabla u, \nabla\varphi\rangle u \, d\mathcal{M} &= \int_{\Omega^*} \langle \nabla u, \nabla\varphi\rangle u \, d\mathcal{M} \\ &= \frac{1}{2}\int_{\Omega^*} \langle \nabla(u^2), \nabla\varphi\rangle \, d\mathcal{M} \\ &= -\frac{1}{2}\int_{\Omega^*} \Delta\varphi \, u^2 \, d\mathcal{M} + \int_{\partial\Omega^*} \partial_\nu\varphi \, u^2 \, d\Gamma \\ &= -\frac{1}{2}\int_{\Omega^*} \Delta\varphi \, u^2 \, d\mathcal{M} \end{aligned} \tag{4.10}$$

Therefore,

$$\left|\int_0^T \int_{\mathcal{M}} \langle \nabla u, \nabla\varphi\rangle u \, d\mathcal{M}dt\right| \le \frac{c}{2}\int_0^T \int_{\Omega^*} u^2 \, d\mathcal{M}dt, \tag{4.11}$$

where $c := max_{x\in\overline{\Omega^*}} |\Delta\varphi(x)|$.

We conclude so, from (4.9) and (4.11) that

$$\begin{aligned}\int_0^T \int_{\mathcal{M}\setminus\,\Omega^*} u^2 + |\nabla u|^2 + u_t^2 \, d\mathcal{M}dt &\le -\left[\int_{\mathcal{M}} u_t\varphi u \, d\mathcal{M}\right]_0^T \\ &+ 3a_0^{-1}\int_0^T \int_{\mathcal{M}} a(x)u_t^2 \, d\mathcal{M}dt + \frac{c}{2}\int_0^T \int_{\Omega^*\setminus\Omega} u^2 \, d\mathcal{M}dt \\ &- \int_0^T \int_{\mathcal{M}\setminus\,\Omega} a(x)g(u_t)\varphi u \, d\mathcal{M}dt. \end{aligned} \tag{4.12}$$

From (4.12), it remains to estimate $\int_0^T \int_{\Omega^*} u^2 + |\nabla u|^2 + u_t^2 \, d\mathcal{M}dt$ in terms of "good terms". Indeed, we observe that Ω^* is a submanifold with smooth boundary $\partial\Omega^*$.

4.2 Recovering the Energy Inside Ω^*

Let q be a vector field in $\overline{\Omega^*}$ of class C^1. Multiplying equation (3.1) by $\langle \nabla u, q\rangle$ and integrating over $[0, T] \times \Omega^*$ we infer

$$0 = \int_0^T \int_{\Omega^*} (u_{tt} - \Delta u + u + a(x)g(u_t))\langle \nabla u, q\rangle d\mathcal{M}dt. \tag{4.13}$$

- Estimate for $I_1 := \int_0^T \int_{\Omega^*} u_{tt}\langle \nabla u, q\rangle d\mathcal{M}dt$.

We note that

$$\frac{d}{dt}\int_{\Omega^*} u_t\langle\nabla u, q\rangle d\mathcal{M} = \int_{\Omega^*} u_{tt}\langle\nabla u, q\rangle d\mathcal{M} + \int_{\Omega^*} u_t\langle\nabla u_t, q\rangle d\mathcal{M}.$$

Thus

$$I_1 = \left[\int_{\Omega^*} u_t\langle\nabla u, q\rangle d\mathcal{M}\right]_0^T - \int_0^T\int_{\Omega^*} u_t\langle\nabla u_t, q\rangle d\mathcal{M}dt.$$

Recall that

$$u_t\langle q, \nabla u_t\rangle = \frac{1}{2}\langle q, \nabla u_t^2\rangle. \tag{4.14}$$

Then, taking (4.14) into account and making use of Theorem 2.2, we can write

$$\begin{aligned} I_1 &= \left[\int_{\Omega^*} u_t\langle\nabla u, q\rangle d\mathcal{M}\right]_0^T - \int_0^T\int_{\Omega^*} \frac{1}{2}\langle q, \nabla u_t^2\rangle\, d\mathcal{M}dt \\ &= \left[\int_{\Omega^*} u_t\langle\nabla u, q\rangle d\mathcal{M}\right]_0^T + \frac{1}{2}\int_0^T\int_{\Omega^*} div(q)u_t^2\, d\mathcal{M}dt \\ &\quad -\frac{1}{2}\int_0^T\int_{\partial\Omega^*}\langle q, \nu\rangle u_t^2\, d\Gamma dt. \end{aligned} \tag{4.15}$$

• Estimate for $I_2 := \int_0^T\int_{\Omega^*} -\Delta u\langle q, \nabla u\rangle\, d\mathcal{M}dt$.
Employing Green's theorem before mentioned and from Lemma 2.1, we deduce

$$\begin{aligned} I_2 &= \int_0^T\int_{\Omega^*}\langle\nabla u, \nabla(\langle q, \nabla u\rangle)\rangle\, d\mathcal{M}dt - \int_0^T\int_{\partial\Omega^*}\partial_\nu u\langle q, \nabla u\rangle\, d\Gamma dt \\ &= \int_0^T\int_{\Omega^*}\nabla q(\nabla u, \nabla u)\, d\mathcal{M}dt \\ &\quad + \int_0^T\int_{\Omega^*}\frac{1}{2}\left[div(|\nabla u|^2 q) - div(q)|\nabla u|^2\right] d\mathcal{M}dt \\ &\quad - \int_0^T\int_{\partial\Omega^*}\partial_\nu u\langle q, \nabla u\rangle\, d\Gamma dt \\ &= \int_0^T\int_{\Omega^*}\nabla q(\nabla u, \nabla u)\, d\mathcal{M}dt + \int_0^T\int_{\Omega^*}\frac{1}{2}\langle q, \nabla(|\nabla u|^2)\rangle\, d\mathcal{M}dt \\ &\quad - \int_0^T\int_{\partial\Omega^*}\partial_\nu u\langle q, \nabla u\rangle\, d\Gamma dt \\ &= \int_0^T\int_{\Omega^*}\nabla q(\nabla u, \nabla u)\, d\mathcal{M}dt - \frac{1}{2}\int_0^T\int_{\Omega^*} div(q)|\nabla u|^2\, d\mathcal{M}dt \\ &\quad + \frac{1}{2}\int_0^T\int_{\partial\Omega^*}\langle q, \nu\rangle|\nabla u|^2\, d\Gamma dt - \int_0^T\int_{\partial\Omega^*}\partial_\nu u\langle q, \nabla u\rangle\, d\Gamma dt. \end{aligned} \tag{4.16}$$

• Estimate for $I_3 := \int_0^T \int_{\Omega^*} u\langle q, \nabla u\rangle \, d\mathcal{M}dt$.

$$
\begin{aligned}
I_3 &= \frac{1}{2}\int_0^T \int_{\Omega^*} \langle q, \nabla u^2\rangle \, d\mathcal{M}dt \\
&= -\frac{1}{2}\int_0^T \int_{\Omega^*} div(q)u^2 \, d\mathcal{M}dt + \frac{1}{2}\int_0^T \int_{\partial\Omega^*} \langle q, \nu\rangle u^2 \, d\Gamma dt
\end{aligned} \tag{4.17}
$$

Combining (4.13), (4.15), (4.16) and (4.17), we can state the following lemma:

Lemma 4.1 *Let q be a vector field of class C^1 in $\overline{\Omega^*}$. For all regular solution u to problem (3.1), we have the following identity:*

$$
\begin{aligned}
&\left[\int_{\Omega^*} u_t\langle \nabla u, q\rangle d\mathcal{M}\right]_0^T + \frac{1}{2}\int_0^T \int_{\Omega^*} div(q)[u_t^2 - |\nabla u|^2 - u^2] \, d\mathcal{M}dt \\
&+ \int_0^T \int_{\Omega^*} \nabla q(\nabla u, \nabla u) \, d\mathcal{M}dt + \int_0^T \int_{\Omega^*} a(x)g(u_t)\langle q, \nabla u\rangle \, d\mathcal{M}dt \\
&= \int_0^T \int_{\partial\Omega^*} \partial_\nu u\langle q, \nabla u\rangle \, d\Gamma dt + \frac{1}{2}\int_0^T \int_{\partial\Omega^*} \langle q, \nu\rangle[u_t^2 - |\nabla u|^2 - u^2] \, d\Gamma dt.
\end{aligned}
$$

Exploiting Lemma 4.1 com $q = \nabla f$, where $f : \overline{\Omega^*} \longrightarrow \mathbb{R}$ is a C^∞ map to be determined later on, we obtain

$$
\begin{aligned}
&\left[\int_{\Omega^*} u_t\langle \nabla u, \nabla f\rangle d\mathcal{M}\right]_0^T + \frac{1}{2}\int_0^T \int_{\Omega^*} \Delta f[u_t^2 - |\nabla u|^2 - u^2] \, d\mathcal{M}dt \\
&\quad + \int_0^T \int_{\Omega^*} \nabla^2 f(\nabla u, \nabla u) \, d\mathcal{M}dt + \int_0^T \int_{\Omega^*} a(x)g(u_t)\langle \nabla f, \nabla u\rangle \, d\mathcal{M}dt \\
&= \frac{1}{2}\int_0^T \int_{\partial\Omega^*} \langle \nabla f, \nu\rangle[u_t^2 - |\nabla u|^2 - u^2] \, d\Gamma dt \\
&\quad + \int_0^T \int_{\partial\Omega^*} \partial_\nu u\langle \nabla f, \nabla u\rangle \, d\Gamma dt.
\end{aligned} \tag{4.18}
$$

Lemma 4.2 *Let u be a regular solution to problem (3.1) and $\alpha \in C^1(\overline{\Omega^*})$. Thus*

$$
\begin{aligned}
\left[\int_{\Omega^*} u_t\alpha u \, d\mathcal{M}\right]_0^T &= \int_0^T \int_{\Omega^*} \alpha[u_t^2 - |\nabla u|^2 - u^2] \, d\mathcal{M}dt - \int_0^T \int_{\Omega^*} \langle \nabla u, \nabla\alpha\rangle u \, d\mathcal{M}dt \\
&\quad - \int_0^T \int_{\Omega^*} a(x)g(u_t)\alpha u \, d\mathcal{M}dt + \int_0^T \int_{\partial\Omega^*} \partial_\nu u\alpha u \, d\Gamma dt.
\end{aligned}
$$

Proof Multiplying equation (3.1) by αu and performing an integration by parts we obtain the desired result. □

Adding (4.18) and the identity given in Lemma 4.2 we deduce

$$\begin{aligned}
&\int_0^T \int_{\Omega^*} \left(\frac{\Delta f}{2} - \alpha\right) u_t^2 \, d\mathcal{M}dt + \int_0^T \int_{\Omega^*} \nabla^2 f\,(\nabla u, \nabla u)\; d\mathcal{M}dt \\
&+ \int_0^T \int_{\Omega^*} \left(\alpha - \frac{\Delta f}{2}\right) |\nabla u|^2 \, d\mathcal{M}dt = \int_0^T \int_{\Omega^*} \left(\frac{\Delta f}{2} - \alpha\right) u^2 \, d\mathcal{M}dt \\
&- \left[\int_{\Omega^*} u_t \langle \nabla f, \nabla u\rangle \, d\mathcal{M}\right]_0^T - \left[\int_{\Omega^*} u_t \alpha u \, d\mathcal{M}\right]_0^T \\
&- \int_0^T \int_{\Omega^*} a(x) g(u_t) \alpha u \, d\mathcal{M}dt - \int_0^T \int_{\Omega^*} a(x) g(u_t) \langle \nabla f, \nabla u\rangle \, d\mathcal{M}dt \\
&+ \int_0^T \int_{\partial\Omega^*} \partial_\nu u \langle \nabla f, \nabla u\rangle \, d\Gamma dt \\
&+ \frac{1}{2} \int_0^T \int_{\partial\Omega^*} \langle \nabla f, \nu\rangle \left[u_t^2 - |\nabla u|^2 - u^2\right] \, d\Gamma dt \\
&- \int_0^T \int_{\Omega^*} \langle \nabla u, \nabla \alpha\rangle u \, d\mathcal{M}dt + \int_0^T \int_{\partial\Omega^*} \partial_\nu u \alpha u \, d\Gamma dt. \qquad (4.19)
\end{aligned}$$

• **Cut-off**:
Let $\varepsilon > 0$ be sufficiently small such that the tubular neighbourhood

$$\omega_\varepsilon = \left\{x \in \mathcal{M};\ d(x, \partial\Omega^*) \le \frac{\varepsilon}{2}\right\}$$

is contained in $\mathcal{M}\backslash\overline{\Omega}$. Let us consider $\psi \in \mathcal{D}(\mathcal{M})$ such that (see Fig. 5)

(i) $\psi = 1 \; in \; \omega_{\frac{\varepsilon}{2}} := \omega_\varepsilon \backslash \Omega^*$;
(ii) $\psi = 0 \; in \; \overline{\Omega^*} \backslash int(\omega'_{\frac{\varepsilon}{2}})$, where $\omega'_{\frac{\varepsilon}{2}} := \omega_\varepsilon \backslash \omega_{\frac{\varepsilon}{2}}$;
(iii) $0 \le \psi \le 1 \; in \; \mathcal{M}$.

Taking $q = \psi \nabla f$ in Lemma 4.1, we infer,

$$\begin{aligned}
&\left[\int_{\Omega^*} u_t \psi \langle \nabla u, \nabla f\rangle d\mathcal{M}\right]_0^T + \frac{1}{2} \int_0^T \int_{\Omega^*} div(\psi \nabla f) \left[u_t^2 - |\nabla u|^2 - u^2\right] \, d\mathcal{M}dt \\
&+ \int_0^T \int_{\Omega^*} \nabla(\psi \nabla f)(\nabla u, \nabla u) \, d\mathcal{M}dt + \int_0^T \int_{\Omega^*} a(x) g(u_t) \langle \psi \nabla f, \nabla u\rangle \, d\mathcal{M}dt \\
&= \int_0^T \int_{\partial\Omega^*} \partial_\nu u \langle \psi \nabla f, \nabla u\rangle \, d\Gamma dt + \frac{1}{2} \int_0^T \int_{\partial\Omega^*} \langle \psi \nabla f, \nu\rangle \left[u_t^2 - |\nabla u|^2 - u^2\right] \, d\Gamma dt,
\end{aligned}$$

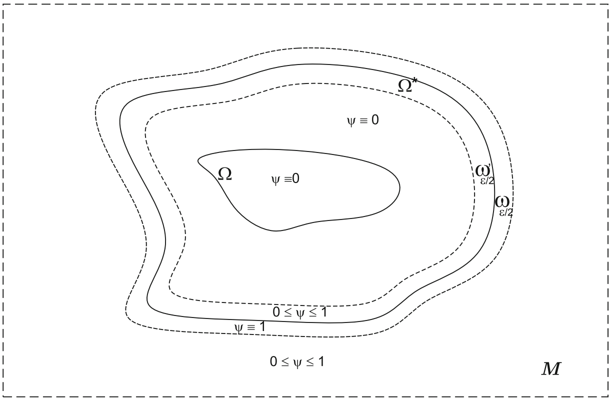

Fig. 5 Properties of function ψ

that is,

$$\begin{aligned}
&\left[\int_{\omega'_{\frac{\varepsilon}{2}}} u_t\psi\langle\nabla u,\nabla f\rangle d\mathcal{M}\right]_0^T+\frac{1}{2}\int_0^T\int_{\omega'_{\frac{\varepsilon}{2}}}\psi\Delta f[u_t^2-|\nabla u|^2-u^2]\,d\mathcal{M}dt\\
&\quad+\frac{1}{2}\int_0^T\int_{\omega'_{\frac{\varepsilon}{2}}}\langle\nabla f,\nabla\psi\rangle\left[u_t^2-|\nabla u|^2-u^2\right]\,d\mathcal{M}dt\\
&\quad+\int_0^T\int_{\omega'_{\frac{\varepsilon}{2}}}\psi\nabla^2 f(\nabla u,\nabla u)\,d\mathcal{M}dt+\int_0^T\int_{\omega'_{\frac{\varepsilon}{2}}}\langle\nabla u,\nabla\psi\rangle\langle\nabla f,\nabla u\rangle\,d\mathcal{M}dt\\
&\quad+\int_0^T\int_{\omega'_{\frac{\varepsilon}{2}}}a(x)g(u_t)\psi\langle\nabla f,\nabla u\rangle\,d\mathcal{M}dt\\
&=\frac{1}{2}\int_0^T\int_{\partial\Omega^*}\langle\nabla f,\nu\rangle\left[u_t^2-|\nabla u|^2-u^2\right]\,d\Gamma dt\\
&\quad+\int_0^T\int_{\partial\Omega^*}\partial_\nu u\langle\nabla f,\nabla u\rangle\,d\Gamma dt.
\end{aligned}\tag{4.20}$$

Substituting (4.20) in (4.19) it results that

$$\begin{aligned}
&\int_0^T\int_{\Omega^*}\left(\frac{\Delta f}{2}-\alpha\right)u_t^2\,d\mathcal{M}dt+\int_0^T\int_{\Omega^*}\nabla^2 f(\nabla u,\nabla u)\,d\mathcal{M}dt+\int_0^T\int_{\Omega^*}\left(\alpha-\frac{\Delta f}{2}\right)|\nabla u|^2\,d\mathcal{M}dt\\
&=\int_0^T\int_{\Omega^*}\left(\frac{\Delta f}{2}-\alpha\right)u^2\,d\mathcal{M}dt-\left[\int_{\Omega^*}u_t\langle\nabla f,\nabla u\rangle\,d\mathcal{M}\right]_0^T-\left[\int_{\Omega^*}u_t\alpha u\,d\mathcal{M}\right]_0^T\\
&-\int_0^T\int_{\Omega^*}a(x)g(u_t)\alpha u\,d\mathcal{M}dt-\int_0^T\int_{\Omega^*}a(x)g(u_t)\langle\nabla f,\nabla u\rangle\,d\mathcal{M}dt\\
&-\int_0^T\int_{\Omega^*}\langle\nabla u,\nabla\alpha\rangle u\,d\mathcal{M}dt+\frac{1}{2}\int_0^T\int_{\omega'_{\frac{\varepsilon}{2}}}\psi\Delta f\left[u_t^2-|\nabla u|^2-u^2\right]\,d\mathcal{M}dt
\end{aligned}$$

$$+\frac{1}{2}\int_0^T\int_{\omega'_{\frac{\varepsilon}{2}}}\langle\nabla f,\nabla\psi\rangle\left[u_t^2-|\nabla u|^2-u^2\right]d\mathcal{M}dt+\left[\int_{\omega'_{\frac{\varepsilon}{2}}}u_t\psi\langle\nabla u,\nabla f\rangle d\mathcal{M}\right]_0^T$$
$$+\int_0^T\int_{\omega'_{\frac{\varepsilon}{2}}}\psi\nabla^2 f(\nabla u,\nabla u)\,d\mathcal{M}dt+\int_0^T\int_{\omega'_{\frac{\varepsilon}{2}}}\langle\nabla u,\nabla\psi\rangle\langle\nabla f,\nabla u\rangle\,d\mathcal{M}dt$$
$$+\int_0^T\int_{\omega'_{\frac{\varepsilon}{2}}}a(x)g(u_t)\psi\langle\nabla f,\nabla u\rangle\,d\mathcal{M}dt+\int_0^T\int_{\partial\Omega^*}\partial_\nu u\,\alpha\,u\,d\Gamma dt. \tag{4.21}$$

Remark 4 This is the precise moment when the properties of the function f will be essential. Note that we need to find a subset $V\subset\overline{\Omega^*}$ with regular smooth boundary $\overline{\partial_1 V}$ which intercepts $\partial\overline{\Omega^*}$ transversally, in such way that $meas(V)\geq meas(\overline{\Omega^*})-\varepsilon$ and $meas(V\cap\partial\overline{\Omega^*})\geq meas(\partial\overline{\Omega^*})-\varepsilon$, for all $\varepsilon>0$. In addition, we need to find regular functions $\alpha, f:\overline{\Omega^*}\longrightarrow\mathbb{R}$, $\alpha\geq 0$ and $\nabla\alpha\mid_V\equiv 0$ such that

$$C\int_0^T\int_V u_t^2+|\nabla u|^2\,d\mathcal{M}dt\leq\int_0^T\int_V(\frac{\Delta f}{2}-\alpha)u_t^2\,d\mathcal{M}dt$$
$$+\int_0^T\int_V\nabla^2 f(\nabla u,\nabla u)+(\alpha-\frac{\Delta f}{2})|\nabla u|^2\,d\mathcal{M}dt, \tag{4.22}$$

for some positive constant C.

The construction of a smooth function f satisfying the Remark 4 can be found in [13]. The general idea is construct this function locally. Afterwards we glue them. The compactness of $\overline{\Omega^*}$ is a crucial ingredient. We can put radially symmetric open sets satisfying some conditions inside V (and outside the damping region), remembering that we say that an open set $V\subset\overline{\Omega^*}$ is radially symmetric with respect to $p\in V$ if the expression of the metric in polar coordinates $(r,\theta)=(r,\theta_1,\theta_2,\ldots,\theta_{n-1})$ centered in p is given by $ds^2=dr^2+Q^2(r)d\theta^2$.

Thus we will omit the construction and quote the present theorem that guarantees the existence of such a function whose proof is given in detail in [13].

Theorem 4.2 *Let $(\mathcal{M}^n,\mathbf{g})$ be a n-dimensional Riemannian manifold and let $\overline{\Omega^*}\subset\mathcal{M}$ the compact subset mentioned before. Fix $\epsilon>0$. Then there exist an open subset $V\subset\overline{\Omega^*}$ and smooth functions $\alpha, f:\overline{\Omega^*}\to\mathbb{R}$ such that $meas(V)\geq meas(\overline{\Omega^*})-\epsilon$, $meas(V\cap\partial\overline{\Omega^*})\geq meas(\partial\overline{\Omega^*})-\epsilon$, $\alpha\geq 0$, $\nabla\alpha\mid_V\equiv 0$ and*

$$C\int_0^T\int_V\left[u_t^2+|\nabla u|^2\right]dMdt\leq\int_0^T\int_V\left(\frac{\Delta f}{2}-\alpha\right)u_t^2\,d\mathcal{M}dt$$
$$+\int_0^T\int_V\left[\nabla^2 f(\nabla u,\nabla u)+\left(\alpha-\frac{\Delta f}{2}\right)|\nabla u|^2\right]d\mathcal{M}dt,$$

for some positive constant C.

Moreover if $\overline{\Omega^*}$ contains radially symmetric subsets, then we can choose V in such a way that a precise part of these radially symmetric subsets is contained in V, *namely, the radially symmetric regions that can get rid of dissipative effects are those that satisfy the following condition*

$$Q'(r) \in \left[\frac{2}{n}(\alpha + C), \frac{\alpha - C}{\frac{n}{2} - 1}\right] \tag{4.23}$$

(if $n = 2$, $Q'(r)$ does not need to satisfy any upper bound), where Q is function at the metric expression $ds^2 = dr^2 + Q^2(r)d\theta^2$.

Once established that the Remark 4 is valid, we are able to complete the stabilization energy.

Taking into account that $0 \le \psi \le 1$, we conclude, from (4.21), that

$$\begin{aligned} C\int_0^T\int_{\Omega^*} u_t^2 + |\nabla u|^2\, d\mathcal{M}dt &\le C^*\int_0^T\int_{\Omega^*\setminus V} u_t^2 + |\nabla u|^2\, d\mathcal{M}dt \\ &+ \left|\left[\int_{\Omega^*} u_t\langle\nabla f, \nabla u\rangle d\mathcal{M}\right]_0^T\right| + C_2\left|\left[\int_{\Omega^*} u_t u\, d\mathcal{M}\right]_0^T\right| \\ &+ C_1\int_0^T\int_{\Omega^*} u^2\, d\mathcal{M}dt + C_2\int_0^T\int_{\Omega^*} a(x)|g(u_t)||u|\, d\mathcal{M}dt \\ &+ \int_0^T\int_{\Omega^*} a(x)|g(u_t)||\nabla f||\nabla u|\, d\mathcal{M}dt + \int_0^T\int_{\Omega^*\setminus V} |\langle\nabla u, \nabla\alpha\rangle||u|\, d\mathcal{M}dt \\ &+ \left|\left[\int_{\omega'_{\frac{\varepsilon}{2}}} u_t\langle\nabla f, \nabla u\rangle\, d\mathcal{M}\right]_0^T\right| + C_3\int_0^T\int_{\omega'_{\frac{\varepsilon}{2}}} u_t^2 + |\nabla u|^2 + u^2\, d\mathcal{M}dt \\ &+ C_4\int_0^T\int_{\omega'_{\frac{\varepsilon}{2}}} |\nabla u|^2\, d\mathcal{M}dt \\ &+ C_5\int_0^T\int_{\omega'_{\frac{\varepsilon}{2}}} |\nabla u|^2\, d\mathcal{M}dt + \frac{C_4}{2}\int_0^T\int_{\omega'_{\frac{\varepsilon}{2}}} u_t^2 + |\nabla u|^2 + u^2\, d\mathcal{M}dt \\ &+ \int_0^T\int_{\omega'_{\frac{\varepsilon}{2}}} a(x)|g(u_t)||\nabla u||\nabla f|\, d\mathcal{M}dt + C_6\left|\int_0^T\int_{\partial\Omega^*} \partial_\nu u\, u\, d\Gamma dt\right|, \end{aligned} \tag{4.24}$$

where $C_1 := \max_{x\in\overline{\Omega^*}}\left|\left(\alpha - \frac{\Delta f}{2}\right)(x)\right|$, $C_2 := \max_{x\in\overline{\Omega^*}} \alpha(x)$, $C_3 := \max_{x\in\overline{\Omega^*}} |\Delta f(x)|$,

$C_4 := \max_{x\in\overline{\Omega^*}}(|\nabla f(x)||\nabla\psi(x)|)$, C_5 is such that $|\nabla^2 f(\nabla u, \nabla u)(x)| \le C_5|\nabla u(x)|^2$,

$\forall\, x \in \overline{\Omega^*}$, $C_6 := \max_{x\in\partial\Omega^*} \alpha(x)$.

Adding $C\int_0^T\int_{\Omega^*} u^2\, d\mathcal{M}dt$ in (4.24), we conclude that

$$
\begin{aligned}
C\int_0^T\int_{\Omega^*} u_t^2+|\nabla u|^2+u^2\, d\mathcal{M}dt &\leq C^*\int_0^T\int_{\Omega^*\setminus V} u_t^2+|\nabla u|^2\, d\mathcal{M}dt\\
&+\int_0^T\int_{\Omega^*\setminus V} |\langle\nabla u,\nabla\alpha\rangle||u|\, d\mathcal{M}dt+\widetilde{C}a_0^{-1}\int_0^T\int_{\omega'_{\frac{\varepsilon}{2}}} a(x)u_t^2\, d\mathcal{M}dt\\
&+\widetilde{C}\int_0^T\int_{\omega'_{\frac{\varepsilon}{2}}} |\nabla u|^2\, d\mathcal{M}dt+\left|\left[\int_{\Omega^*} u_t\langle\nabla f,\nabla u\rangle\, d\mathcal{M}\right]_0^T\right|\\
&+\widetilde{C}\left|\left[\int_{\Omega^*} u_t u\, d\mathcal{M}\right]_0^T\right|+\widetilde{C}\int_0^T\int_{\Omega^*} u^2\, d\mathcal{M}dt+\widetilde{C}\int_0^T\int_{\Omega^*} a(x)|g(u_t)||u|\, d\mathcal{M}dt\\
&+\widetilde{C}\int_0^T\int_{\Omega^*} a(x)|g(u_t)||\nabla u|\, d\mathcal{M}dt+\frac{\widetilde{C}}{2}\int_0^T\int_{\omega'_{\frac{\varepsilon}{2}}} u^2\, d\mathcal{M}dt\\
&+\left|\left[\int_{\omega'_{\frac{\varepsilon}{2}}} u_t\langle\nabla f,\nabla u\rangle\, d\mathcal{M}\right]_0^T\right|\\
&+\widetilde{C}\int_0^T\int_{\omega'_{\frac{\varepsilon}{2}}} a(x)|g(u_t)||\nabla u|\, d\mathcal{M}dt+\widetilde{C}\left|\int_0^T\int_{\partial\Omega^*} \partial_\nu u\, u\, d\Gamma dt\right|. \qquad (4.25)
\end{aligned}
$$

In the sequel, let us analyze some terms in (4.25). We shall use Hölder inequality as well as the inequality $ab\leq\frac{a^2}{4\beta}+\beta b^2$, where β is an arbitrary positive number.

- Analysis of $J_1:=\int_0^T\int_{\Omega^*} a(x)|g(u_t)||u|\, d\mathcal{M}dt$.

$$
\begin{aligned}
J_1 &\leq \int_0^T\left(\int_{\Omega^*} a(x)|g(u_t)|^2\, d\mathcal{M}\right)^{\frac12}\left(\int_{\Omega^*} a(x)|u|^2\, d\mathcal{M}\right)^{\frac12} dt\\
&\leq \int_0^T\left[\frac{1}{4\beta}\int_{\Omega^*} a(x)|g(u_t)|^2\, d\mathcal{M}+\beta\int_{\Omega^*} a(x)|u|^2\, d\mathcal{M}\right] dt\\
&\leq \frac{1}{4\beta}\int_0^T\int_{\Omega^*} a(x)|g(u_t)|^2\, d\mathcal{M}dt+\|a\|_\infty 2\beta\int_0^T\int_{\Omega^*}\frac12|u|^2\, d\mathcal{M}dt\\
&\leq \frac{1}{4\beta}\int_0^T\int_{\mathcal{M}} a(x)|g(u_t)|^2\, d\mathcal{M}dt+\widehat{C}\beta\int_0^T E(t)\, dt, \qquad (4.26)
\end{aligned}
$$

where $E(t)$ is the energy defined in (3.2).

- Analysis of $J_2:=\int_0^T\int_{\omega'_{\frac{\varepsilon}{2}}} a(x)|u_t|^2\, d\mathcal{M}dt$.

$$
J_2\leq\int_0^T\int_{\mathcal{M}} a(x)|u_t|^2\, d\mathcal{M}dt \qquad (4.27)
$$

- Analysis of $J_3:=\int_0^T\int_{\Omega^*} a(x)|g(u_t)||\nabla u|\, d\mathcal{M}dt+\int_0^T\int_{\omega'_{\frac{\varepsilon}{2}}} a(x)|g(u_t)||\nabla u|\, d\mathcal{M}dt$.

$$J_3 \le 2\int_0^T \int_{\Omega^*} a(x)|g(u_t)||\nabla u|\, d\mathcal{M}dt$$
$$\le \frac{1}{2\beta}\int_0^T \int_{\mathcal{M}} a(x)|g(u_t)|^2\, d\mathcal{M}dt + \widehat{C}\beta \int_0^T E(t)\, dt \tag{4.28}$$

• Analysis of $J_4 := \left|\int_0^T \int_{\partial\Omega^*} \partial_\nu u\, u\, d\Gamma dt\right|$.
Taking $\alpha = \psi$ in Lemma 4.2 we obtain

$$\left[\int_{\omega'_{\frac{\varepsilon}{2}}} u_t \psi u\, d\mathcal{M}\right]_0^T = \int_0^T \int_{\omega'_{\frac{\varepsilon}{2}}} \psi \left[u_t^2 - |\nabla u|^2 - u^2\right] d\mathcal{M}dt$$
$$- \int_0^T \int_{\omega'_{\frac{\varepsilon}{2}}} \langle \nabla u, \nabla \psi\rangle u\, d\mathcal{M}dt$$
$$- \int_0^T \int_{\omega'_{\frac{\varepsilon}{2}}} a(x) g(u_t)\psi u\, d\mathcal{M}dt + \int_0^T \int_{\partial\Omega^*} \partial_\nu u\, u\, d\Gamma dt. \tag{4.29}$$

Thus,

$$J_4 \le \left|\left[\int_{\omega'_{\frac{\varepsilon}{2}}} u_t u\, d\mathcal{M}\right]_0^T\right| + \int_0^T \int_{\omega'_{\frac{\varepsilon}{2}}} u_t^2 + |\nabla u|^2\, d\mathcal{M}dt$$
$$+ \widetilde{C}\int_0^T \int_{\omega'_{\frac{\varepsilon}{2}}} \frac{|\nabla u|^2}{2} + \frac{u^2}{2}\, d\mathcal{M}dt + \int_0^T \int_{\omega'_{\frac{\varepsilon}{2}}} a(x)|g(u_t)||u|\, d\mathcal{M}dt$$
$$+ \int_0^T \int_{\omega'_{\frac{\varepsilon}{2}}} u^2\, d\mathcal{M}dt. \tag{4.30}$$

Employing the same procedure we have followed in (4.26) yields:

$$J_4 \le \left|\left[\int_{\omega'_{\frac{\varepsilon}{2}}} u_t u\, d\mathcal{M}\right]_0^T\right| + a_0^{-1}\int_0^T \int_{\Omega^*} a(x) u_t^2\, d\mathcal{M}dt$$
$$+ \widehat{C}\int_0^T \int_{\omega'_{\frac{\varepsilon}{2}}} |\nabla u|^2\, d\mathcal{M}dt + \widehat{C}\int_0^T \int_{\Omega^*} u^2\, d\mathcal{M}dt$$
$$+ \frac{1}{4\beta}\int_0^T \int_{\mathcal{M}} a(x)|g(u_t)|^2\, d\mathcal{M}dt + \widehat{C}\beta\int_0^T E(t)\, dt \tag{4.31}$$

Therefore, from (4.25), (4.26), (4.27), (4.28) and (4.31), setting

$$\chi := \left|\left[\int_{\Omega^*} u_t\langle\nabla f,\nabla u\rangle\,d\mathcal{M}\right]_0^T\right| + \left|\left[\int_{\Omega^*} u_t u\,d\mathcal{M}\right]_0^T\right| + \left|\left[\int_{\omega'_{\frac{\varepsilon}{2}}} u_t\langle\nabla f,\nabla u\rangle\,d\mathcal{M}\right]_0^T\right| + \left|\left[\int_{\omega'_{\frac{\varepsilon}{2}}} u_t u\,d\mathcal{M}\right]_0^T\right|, \tag{4.32}$$

we conclude that

$$\begin{aligned} C\int_0^T\int_{\Omega^*} u_t^2+|\nabla u|^2+u^2\,d\mathcal{M}dt &\le C'|\chi| + C'\beta\int_0^T E(t)dt \\ &+ C'\int_0^T\int_{\Omega^*\backslash V}|\nabla u|^2+u^2\,d\mathcal{M}dt + C^*\int_0^T\int_{\Omega^*\backslash V} u_t^2+|\nabla u|^2\,d\mathcal{M}dt \\ &+ C'\int_0^T\int_{\mathcal{M}} a(x)[|g(u_t)|^2+u_t^2]\,d\mathcal{M}dt + C'\int_0^T\int_{\omega'_{\frac{\varepsilon}{2}}}|\nabla u|^2\,d\mathcal{M}dt \\ &+ C'\int_0^T\int_{\Omega^*} u^2\,d\mathcal{M}dt, \end{aligned} \tag{4.33}$$

where $C' = C'(\|a\|_\infty, f, \psi, \alpha, a_0^{-1})$.

In the sequel, we shall estimate the term $\int_0^T\int_{\omega'_{\frac{\varepsilon}{2}}}|\nabla u|^2\,d\mathcal{M}dt$. For this purpose, we shall construct a "cut-off" η_δ defined in an specific neighborhood of $\omega'_{\frac{\varepsilon}{2}}$ [see [13], Sect. 7, p. 955].

Initially let $\Omega^{**}\subset\mathcal{M}$ be an open bounded set with smooth boundary $\partial\Omega^{**}$ such that $\Omega\subset\subset\Omega^*\subset\subset\Omega^{**}$ and $\tilde{\eta}:\mathbb{R}\longrightarrow\mathbb{R}$ satisfying

$$\tilde{\eta}(x) = \begin{cases} 1 & if\ x\le 0 \\ (x-1)^2 & if\ x\in[1/2,1] \\ 0 & if\ x>1 \end{cases}$$

and it is defined on $(0, 1/2)$ in such way that $\tilde{\eta}$ is a non-increasing function of class C^1. For $\delta>0$, we set, $\tilde{\eta}_\delta(x) := \tilde{\eta}(\frac{x}{\delta})$. We observe that exists a constant M which does not depend on δ, such that

$$\frac{|\tilde{\eta}'_\delta(x)|^2}{\tilde{\eta}_\delta(x)} \le \frac{M}{\delta^2} \quad \text{para todo } x<\delta$$

Now, let $\delta>0$ sufficiently small such that

$$\tilde{\omega}_\delta := \left\{x\in\Omega^{**};\ d\left(x,\partial\omega'_{\frac{\varepsilon}{2}}\right)<\delta\right\}$$

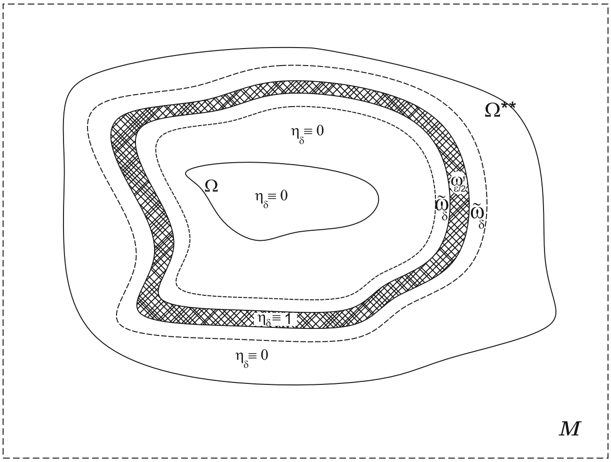

Fig. 6 Properties of function η_δ

is totally contained in $\Omega^{**}\backslash\Omega$, that is, $\tilde{\omega}_\delta$ is a tubular neighborhood of $\omega'_{\frac{\varepsilon}{2}}$ totally contained in $\Omega^{**}\backslash\Omega$.

We define $\eta_\delta : \overline{\Omega^{**}} \longrightarrow \mathbb{R}$ where (see Fig. 6)

$$\eta_\delta(x) = \begin{cases} 1 & \text{if } x \in \omega'_{\frac{\varepsilon}{2}} \\ \tilde{\eta}_\delta(d(x, \omega'_{\frac{\varepsilon}{2}})) & \text{if } x \in \tilde{\omega}_\delta \\ 0 & \text{otherwise,} \end{cases}$$

We have that η_δ is a C^1 function defined in $\overline{\Omega^{**}}$, because $\partial\omega'_{\frac{\varepsilon}{2}}$ and $\partial(\tilde{\omega}_\delta \cup \omega'_{\frac{\varepsilon}{2}})$ are regular. We note also, that

$$\nabla\eta_\delta(x) = \nabla(\tilde{\eta}'_\delta(d(x, \omega'_{\frac{\varepsilon}{2}}))) = \tilde{\eta}'_\delta(d(x, \omega'_{\frac{\varepsilon}{2}}))\nabla d(x, \omega'_{\frac{\varepsilon}{2}}), \tag{4.34}$$

soon

$$\frac{|\nabla\eta_\delta(x)|^2}{\eta_\delta(x)} = \frac{|\tilde{\eta}'_\delta(d(x, \omega'_{\frac{\varepsilon}{2}})|^2}{\tilde{\eta}_\delta(d(x, \omega'_{\frac{\varepsilon}{2}}))} \leq \frac{M}{\delta^2} \tag{4.35}$$

for all $x \in \tilde{\omega}_\delta$. Particularly, $\dfrac{|\nabla\eta_\delta(x)|^2}{\eta_\delta(x)} \in L^\infty(\tilde{\omega}_\delta \cup \omega'_{\frac{\varepsilon}{2}})$.

Using Lemma 4.2 with $\eta_\delta = \alpha$ and $\Omega^* = \Omega^{**}$, we infer,

$$\begin{aligned}\left[\int_{\Omega^{**}} u_t\eta_\delta u \, d\mathcal{M}\right]_0^T &= \int_0^T \int_{\Omega^{**}} \eta_\delta[u_t^2 - |\nabla u|^2 - u^2] \, d\mathcal{M}dt - \int_0^T \int_{\Omega^{**}} \langle\nabla u, \nabla\eta_\delta\rangle u \, d\mathcal{M}dt \\ &\quad - \int_0^T \int_{\Omega^{**}} a(x)g(u_t)\eta_\delta u \, d\mathcal{M}dt + \int_0^T \int_{\partial\Omega^{**}} \partial_\nu u\eta_\delta u \, d\Gamma dt.\end{aligned}$$

Thus, defining, $V_\delta := \tilde{\omega}_\delta \cup \omega'_{\frac{\varepsilon}{2}}$ it holds that

$$\left[\int_{V_\delta} u_t \eta_\delta u \, d\mathcal{M}\right]_0^T = \int_0^T \int_{V_\delta} \eta_\delta [u_t^2 - |\nabla u|^2 - u^2] \, d\mathcal{M}dt - \int_0^T \int_{V_\delta} \langle \nabla u, \nabla \eta_\delta \rangle u \, d\mathcal{M}dt - \int_0^T \int_{V_\delta} a(x) g(u_t) \eta_\delta u \, d\mathcal{M}dt. \tag{4.36}$$

- Estimate for $K_1 := \int_0^T \int_{V_\delta} \eta_\delta |u_t|^2 \, d\mathcal{M}dt$

$$|K_1| \le a_0^{-1} \int_0^T \int_{V_\delta} a(x)|u_t|^2 \, d\mathcal{M}dt \le a_0^{-1} \int_0^T \int_{\mathcal{M}} a(x)|u_t|^2 \, d\mathcal{M}dt. \tag{4.37}$$

- Estimate for $K_2 := \int_0^T \int_{V_\delta} a(x) g(u_t) \eta_\delta u \, d\mathcal{M}dt$

$$|K_2| \le \frac{C}{4\beta} \int_0^T \int_{\mathcal{M}} a(x)|g(u_t)|^2 \, d\mathcal{M}dt + 2\beta \int_0^T E(t)dt. \tag{4.38}$$

- Estimate for $K_3 := \int_0^T \int_{V_\delta} u \langle \nabla u, \nabla \eta_\delta \rangle \, d\mathcal{M}dt$

$$\begin{aligned} |K_3| &\le \int_0^T \int_{V_\delta} |u||\nabla u||\nabla \eta_\delta| \, d\mathcal{M}dt \\ &\le \int_0^T \int_{V_\delta} \frac{1}{2} \eta_\delta |\nabla u|^2 \, d\mathcal{M}dt + \int_0^T \int_{V_\delta} \frac{1}{2} \frac{|\nabla \eta_\delta|^2}{\eta_\delta} |u|^2| \, d\mathcal{M}dt \\ &\le \frac{1}{2} \int_0^T \int_{V_\delta} \eta_\delta |\nabla u|^2 \, d\mathcal{M}dt + \frac{M}{2\delta^2} \int_0^T \int_{V_\delta} |u|^2 \, d\mathcal{M}dt. \end{aligned} \tag{4.39}$$

Denoting $\chi_1 := -\left[\int_{V_\delta} u_t \eta_\delta u \, d\mathcal{M}\right]_0^T$ it results that, according to (4.36), (4.37), (4.38) and (4.39), that

$$\frac{1}{2} \int_0^T \int_{V_\delta} \eta_\delta |\nabla u|^2 \, d\mathcal{M}dt \le |\chi_1| + \tilde{C} \int_0^T \int_{\mathcal{M}} a(x) u_t^2 \, d\mathcal{M}dt + \frac{\tilde{C}}{4\beta} \int_0^T \int_{\mathcal{M}} a(x)|g(u_t)|^2 \, d\mathcal{M} + \tilde{C} \int_0^T \int_{\Omega^{**}} |u|^2 \, d\mathcal{M}dt + 2\beta \int_0^T E(t)dt. \tag{4.40}$$

We observe that

$$\int_0^T \int_{\omega'_{\frac{\varepsilon}{2}}} |\nabla u| \, d\mathcal{M}dt \le \int_0^T \int_{V_\delta} \eta_\delta |\nabla u|^2 \, d\mathcal{M}dt, \tag{4.41}$$

remembering that $V_\delta := \tilde{\omega}_\delta \cup \omega'_{\frac{\varepsilon}{2}}$.

Consequently, substituting (4.40) in (4.33), taking (4.41) into account, grouping some terms and noting that

$$\int_0^T \int_{\Omega^*\setminus V} u^2 \, d\mathcal{M}dt \leq \int_0^T \int_{\Omega^{**}} u^2 \, d\mathcal{M}dt$$

we conclude that

$$\begin{aligned} C\int_0^T \int_{\Omega^*} u_t^2 + |\nabla u|^2 + u^2 \, d\mathcal{M}dt &\leq C'|\chi| + \widehat{C}|\chi_1| \\ &+ \widehat{C}\int_0^T \int_{\mathcal{M}} a(x)[g(u_t)|^2 + u_t^2] \, d\mathcal{M}dt + \widehat{C}\int_0^T \int_{\Omega^{**}} u^2 \, d\mathcal{M}dt \\ &+ C_1^*\int_0^T \int_{\Omega^*\setminus V} |\nabla u|^2 + u_t^2 \, d\mathcal{M}dt + \widehat{C}\beta \int_0^T E(t)dt. \end{aligned} \tag{4.42}$$

where $\widehat{C} = \widehat{C}(\|a\|_\infty, a_0^{-1}, C', \frac{M}{\delta^2}, \psi, f, \alpha)$.

We will now estimate the term $\int_0^T \int_{\Omega^*\setminus V} |\nabla u|^2 \, d\mathcal{M}dt$.

Let $\mathcal{M}_* \subset \mathcal{M}$ a open subset, so that $\overline{\Omega^*}\setminus V \subset \mathcal{M}_* \subset\subset \Omega^{**}$. Note that by hypothesis $a(x) \geq a_0 > 0$ in $\mathcal{M}_*$.

Consider $\theta > 0$ sufficiently small such that

$$\widetilde{\omega}_\theta := \{x \in \Omega^{**}; d(x, \partial V) < \theta\} \subset \mathcal{M}_*,$$

$\widetilde{\omega}_\theta$ is a tubular neighborhood of ∂V that has only one connected component.

Proceeding similarly to what was done for estimating the term $\int_0^T \int_{\omega'_{\frac{\epsilon}{2}}} |\nabla u|^2 \, d\mathcal{M}dt$, we conclude that

$$\begin{aligned} \frac{1}{2}\int_0^T \int_{\Omega^*\setminus V} |\nabla u|^2 \, d\mathcal{M}dt &\leq |\chi_2| + C_2\int_0^T \int_{\mathcal{M}} a(x)u_t^2 \, d\mathcal{M}dt \\ &+ \frac{C_2}{4\beta}\int_0^T \int_{\mathcal{M}} a(x)|g(u_t)|^2 \, d\mathcal{M} + C_2\int_0^T \int_{\Omega^{**}} |u|^2 \, d\mathcal{M}dt + 2\beta \int_0^T E(t)dt. \end{aligned} \tag{4.43}$$

where $\omega_\theta := \widetilde{\omega}_\theta \cup \Omega^*\setminus V \subset \mathcal{M}_*$ and $\chi_2 := -\left[\int_{\omega_\theta} u_t \eta_\theta u \, d\mathcal{M}\right]_0^T$.

Therefore, from (4.43) and (4.42), we can write

$$\begin{aligned} C\int_0^T \int_{\Omega^*} u_t^2 + |\nabla u|^2 + u^2 \, d\mathcal{M}dt &\leq C'|\chi| + \widehat{C}|\chi_1| + C_3|\chi_2| \\ &+ C_3\int_0^T \int_{\mathcal{M}} a(x)[g(u_t)|^2 + u_t^2] \, d\mathcal{M}dt + C_3\int_0^T \int_{\Omega^{**}} u^2 \, d\mathcal{M}dt \\ &+ C_3\beta \int_0^T E(t)dt. \end{aligned} \tag{4.44}$$

The estimate (4.44) is almost what we want. It is necessary to estimate the term $\int_0^T \int_{\Omega^{**}} u^2 \, d\mathcal{M}dt$ as a function of "good terms", this is the next step, already with the total energy of the system recovered.

4.3 Recovering the Total Energy

Multiplying (4.12) by C and adding with (4.44), we obtain the following inequality:

$$\begin{aligned} C\int_0^T \int_{\mathcal{M}} u_t^2 + |\nabla u|^2 + u^2 \, d\mathcal{M}dt &\le C\left|\left[\int_{\mathcal{M}} u_t\varphi u \, d\mathcal{M}\right]_0^T\right| \\ &+ C'|\chi| + \widehat{C}|\chi_1| + C_3|\chi_2| + C_4\int_0^T \int_{\mathcal{M}} a(x)(|g(u_t)|^2 + u_t^2) \, d\mathcal{M}dt \\ &+ C\int_0^T \int_{\mathcal{M}\setminus\Omega} a(x)|g(u_t)||u|\varphi \, d\mathcal{M}dt + C_4\int_0^T \int_{\Omega^{**}} u^2 \, d\mathcal{M}dt \\ &+ C_3\beta\int_0^T E(t)dt. \end{aligned} \tag{4.45}$$

• Estimate for the term $\int_0^T \int_{\mathcal{M}\setminus\Omega} a(x)|g(u_t)||u|\varphi \, d\mathcal{M}dt$.

$$\begin{aligned} \int_0^T \int_{\mathcal{M}\setminus\Omega} a(x)|g(u_t)||u|\varphi \, d\mathcal{M}dt &\le \int_0^T \left(\int_{\mathcal{M}} a(x)|g(u_t)|^2 \, d\mathcal{M}\right)^{\frac{1}{2}} \left(\int_{\mathcal{M}} a(x)|u|^2 \, d\mathcal{M}\right)^{\frac{1}{2}} dt \\ &\le \frac{1}{4\beta}\int_0^T \int_{\mathcal{M}} a(x)|g(u_t)|^2 \, d\mathcal{M}dt + C_1\beta\int_0^T E(t)dt \end{aligned} \tag{4.46}$$

Thus by (4.45) it follows that

$$\begin{aligned} 2C\int_0^T E(t)dt &\le C\left|\left[\int_{\mathcal{M}} u_t\varphi u \, d\mathcal{M}\right]_0^T\right| + C'|\chi| + \widehat{C}|\chi_1| + C_3|\chi_2| \\ &+ \tilde{C}\int_0^T \int_{\mathcal{M}} a(x)(|g(u_t)|^2 + u_t^2) \, d\mathcal{M}dt + \tilde{C}\int_0^T \int_{\Omega^{**}} u^2 \, d\mathcal{M}dt \\ &+ \tilde{C}\beta\int_0^T E(t)dt. \end{aligned} \tag{4.47}$$

where $\tilde{C} = \tilde{C}(\|a\|_\infty, a_0^{-1}, C, C', C_3, C_4, \frac{M}{\delta^2}, \psi, f, \alpha, \varphi)$.

Next, we shall estimate $|\chi|$, $|\chi_1|$, $|\chi_2|$ and $\left|\left[\int_{\mathcal{M}} u_t\varphi u \, d\mathcal{M}\right]_0^T\right|$.

Using the identity (4.1), we have that

$$
\begin{aligned}
|\chi_1| &= \left| \left[\int_{V_\delta} u_t \eta_\delta u \, d\mathcal{M} \right]_0^T \right| \\
&\leq C_1 \left[\int_{\mathcal{M}} |u_t(T,x)||u(T,x)| \, d\mathcal{M} + \int_{\mathcal{M}} |u_t(0,x)||u(0,x)| \, d\mathcal{M} \right] \\
&\leq C_1(E(T) + E(0)) \\
&= C_1 \left(2E(T) + \int_0^T \int_{\mathcal{M}} a(x) g(u_t) u_t \, d\mathcal{M} dt \right) \\
&\leq 2C_1 E(T) + C_2 \int_0^T \int_{\mathcal{M}} a(x) \left[|g(u_t)|^2 + |u_t|^2 \right] \, d\mathcal{M} dt.
\end{aligned}
$$

Making use of analogous arguments we conclude that

$$
\begin{aligned}
C'|\chi| + \widehat{C}|\chi_1| + C_3|\chi_2| + C \left| \left[\int_{\mathcal{M}} u_t \varphi u \, d\mathcal{M} \right]_0^T \right| &\leq C_0 E(T) \\
+ C_0 \int_0^T \int_{\mathcal{M}} a(x) \left[|g(u_t)|^2 + |u_t|^2 \right] \, d\mathcal{M} dt. &
\end{aligned} \tag{4.48}
$$

where C_0 does not depend on T.

It remains in (4.47), to estimate the term $\int_0^T \int_{\Omega^{**}} u^2 \, d\mathcal{M} dt$. For this purpose, we need the following lemma:

Lemma 4.3 *Let u be a regular solution to problem (3.1). Thus, for all $T > T_0$, where T_0 is a positive constant large enough, there exists a positive constant $C(T_0, E(0))$ such that the following inequality holds*

$$
\int_0^T \int_{\overline{\Omega^{**}}} u^2 \, d\mathcal{M} dt \leq C(T_0, E(0)) \left\{ \int_0^T \int_{\mathcal{M}} a(x) \left[|g(u_t)|^2 + |u_t|^2 \right] \, d\mathcal{M} dt \right\}. \tag{4.49}
$$

Proof We argue by contradiction. To ease notations, we will set $u' := u_t$. Assume that (4.49) is not verified. Then there exists a sequence of initial data $\{u_k(0), u_k'(0)\}$, such that the corresponding regular solutions $\{u_k\}$ of (3.1) with $E_k(0)$ assumed uniformly bounded in k, verifies

$$
\lim_{k \to +\infty} \frac{\int_0^T \int_{\overline{\Omega^{**}}} u_k^2 \, d\mathcal{M} dt}{\int_0^T \int_{\mathcal{M}} a(x)(g^2(u_k') + u_k'^2) \, d\mathcal{M} dt} = +\infty, \tag{4.50}
$$

this is,

$$
\lim_{k \to +\infty} \frac{\int_0^T \int_{\mathcal{M}} a(x)(g^2(u_k') + u_k'^2) \, d\mathcal{M} dt}{\int_0^T \int_{\overline{\Omega^{**}}} u_k^2 \, d\mathcal{M} dt} = 0 \tag{4.51}
$$

Since the energy is a non-increasing function, it holds that $E_k(t) \leq E_k(0) \leq L$, consequently,

$$\int_{\mathcal{M}} u'^2_k(x,t) + |\nabla u_k(x,t)|^2 + u_k^2(x,t)\, d\mathcal{M} \leq 2L,$$

that is,

$$\int_0^T \int_{\mathcal{M}} u'^2_k(x,t) + |\nabla u_k(x,t)|^2 + u_k^2(x,t)\, d\mathcal{M} dt \leq 2LT,$$

thus,

$$\|u_k\|^2_{H^1(\Sigma_T)} \leq 2LT, \tag{4.52}$$

where $\Sigma_T = (0,T) \times \mathcal{M}$.

In addition,

$$\int_{\mathcal{M}} |\nabla u_k(x,t)|^2 + u_k^2(x,t)\, d\mathcal{M} \leq 2L \text{ and } \int_{\mathcal{M}} u'^2_k(x,t) d\mathcal{M} \leq 2L,$$

consequently

$$\|u_k\|^2_{L^\infty(0,T;H^1(\mathcal{M}))} \leq 2L; \tag{4.53}$$

$$\|u'_k\|^2_{L^\infty(0,T;L^2(\mathcal{M}))} \leq 2L. \tag{4.54}$$

Thus, there exists a subsequence of $\{u_k\}$, still denote by the same index, which verifies the following convergence

$$u_k \rightharpoonup u \text{ in } H^1(\Sigma_T); \tag{4.55}$$

$$u_k \overset{\star}{\rightharpoonup} u \text{ in } L^\infty(0,T;H^1(\mathcal{M})); \tag{4.56}$$

$$u'_k \overset{\star}{\rightharpoonup} u' \text{ in } L^\infty(0,T;L^2(\mathcal{M})). \tag{4.57}$$

From the fact that $H^1(\Omega^{**}) \overset{c}{\hookrightarrow} L^2(\Omega^{**})$ and both spaces are reflexive, it holds from (4.53), (4.54) and employing Aubin–Lions Theorem, that

$$u_k \longrightarrow u \text{ in } L^2(0,T;L^2(\Omega^{**})). \tag{4.58}$$

We note that u does not depend on Ω^{**} in view of (4.55).

At this point we shall divide the proof into two cases, namely: $u \neq 0$ e $u = 0$.

(i) $u \neq 0$ em $(0,T) \times \mathcal{M}$.

We observe that taking (4.58) into account, we have

$$\|u_k\|^2_{L^2(0,T;L^2(\Omega^{**}))} \leq C.$$

Thus, considering (4.51) we obtain

$$\lim_{k\to+\infty}\int_0^T\int_{\mathcal{M}} a(x)(g^2(u_k')+u_k'^2)\,d\mathcal{M}dt=0 \tag{4.59}$$

and from (4.59) we conclude that

$$\begin{aligned}\lim_{k\to+\infty}&\int_0^T\int_{\mathcal{M}} |a(x)g(u_k')|^2\,d\mathcal{M}dt\\ &\le \|a\|_\infty \lim_{k\to+\infty}\int_0^T\int_{\mathcal{M}} a(x)(g^2(u_k')+u_k'^2)\,d\mathcal{M}dt=0,\end{aligned}$$

that is

$$ag(u_k')\longrightarrow 0 \text{ in } L^2(0,T;L^2(\mathcal{M})). \tag{4.60}$$

Our aim is taking limit in the equation

$$\begin{cases} u_k''-\Delta u_k+u_k+a(x)g(u_k')=0 \; in \;\; \mathcal{M}\times(0,\infty)\\ u_k(0)=u_{0,k}\;\;,\;\; u_k'(0)=u_{1,k} \qquad in \;\; \mathcal{M}\end{cases} \tag{4.61}$$

We observe that according to (4.59) we infer

$$\begin{aligned}\int_0^T\int_{(\mathcal{M}\backslash\Omega)\cup\mathcal{M}_*} u_k'^2\,d\mathcal{M}dt &\le a_0^{-1}\int_0^T\int_{(\mathcal{M}\backslash\Omega)\cup\mathcal{M}_*} a(x)u_k'^2\,d\mathcal{M}dt\\ &\le a_0^{-1}\int_0^T\int_{\mathcal{M}} a(x)u_k'^2\,d\mathcal{M}dt\to 0,\end{aligned}$$

when $k\to\infty$.

Thus, from (4.57) it follows, for a. e. $t\in[0,T]$, $u'(t)$ satisfies

$$u'(t)=\begin{cases} u'(t) \; in \;\; \overline{\Omega}\\ 0 \qquad in \;\; (\mathcal{M}\backslash\Omega)\cup\mathcal{M}_*\end{cases} \tag{4.62}$$

Passing to the limit, we deduce

$$\begin{cases} u''-\Delta u+u=0 \; in \;\; L^\infty(0,T;H^{-1}(\mathcal{M}))\\ u'=0 \qquad\qquad\quad in \;\; (\mathcal{M}\backslash\Omega)\cup\mathcal{M}_*\end{cases} \tag{4.63}$$

with $u\in L^\infty(0,T;H^1(\mathcal{M}))$ e $u'\in L^\infty(0,T;L^2(\mathcal{M}))$.

Considering $\Omega^*\supset\supset\Omega$ satisfying the conditions mentioned before and setting $v=u'$, from (4.63) we obtain the following problem

$$\begin{cases} v''-\Delta v+v=0 \; in \;\; \Omega^*\times(0,T)\\ v=0 \qquad\qquad\quad in \;\; (\overline{\Omega^*}\backslash\Omega)\cup\mathcal{M}_*\times(0,T)\end{cases} \tag{4.64}$$

in the distributional sense, with $v \in L^\infty(0,T;L^2(\mathcal{M}))$ e $v' \in L^\infty(0,T;H^{-1}(\mathcal{M}))$. In addition, employing Lions-Magenes ([30], Chap. 3, Theor. 9.3, p. 288), we have

$$v \in C([0,T];L^2(\mathcal{M})) \cap C^1([0,T];H^{-1}(\mathcal{M})).$$

From Appendix, we deduce that $v = u' = 0$ em $\mathcal{M}$.

It follows that (4.63) can be written as

$$\begin{cases} -\Delta u + u = 0 \; in \;\; \mathcal{M} \times (0,\infty) \\ u' = 0 \qquad\qquad in \;\; \mathcal{M} \end{cases} \tag{4.65}$$

Since $u \in L^\infty(0,T;H^1(\mathcal{M}))$ we conclude that $\Delta u \in L^\infty(0,T;H^1(\mathcal{M}))$, then $u(t) \in H^1(\mathcal{M})$ and $\Delta u(t) \in H^1(\mathcal{M}) \hookrightarrow L^2(\mathcal{M})$. Multiplying (4.65) by $u(t)$ and integrating over $\mathcal{M}$, we deduce

$$\int_{\mathcal{M}} -\Delta u(t)u(t)\,d\mathcal{M} + \int_{\mathcal{M}} |u(t)|^2\,d\mathcal{M} = 0,$$

that is,

$$\int_{\mathcal{M}} |\nabla u(t)|^2\,d\mathcal{M} + \int_{\mathcal{M}} |u(t)|^2\,d\mathcal{M} = 0,$$

Consequently $u(t) = 0$ in $H^1(\mathcal{M})$ for a. e. $t \in [0,T]$, that is $u = 0$, which is a contradiction.

(ii) $u = 0$ em $(0,T) \times \mathcal{M}$.

We define

$$c_k := \left[\int_0^T \int_{\overline{\Omega^{**}}} |u_k|^2\,d\mathcal{M}dt\right]^{\frac{1}{2}} \tag{4.66}$$

$$\bar{u}_k := \frac{1}{c_k} u_k \tag{4.67}$$

Thus, we deduce

$$\|\bar{u}_k\|^2_{L^2(0,T;L^2(\Omega^{**}))} = 1 \tag{4.68}$$

If one considers

$$\overline{E}_k(t) := \frac{1}{2}\int_{\mathcal{M}} \left|\bar{u}'_k\right|^2 + |\bar{u}_k|^2 + |\nabla \bar{u}_k|^2\,d\mathcal{M}, \tag{4.69}$$

then

$$\overline{E}_k(t) = \frac{E_k(t)}{c_k^2} \tag{4.70}$$

According to (4.47) and (4.48) it holds, for T sufficiently large that

$$E(T) \leq \widehat{C}\left[\int_0^T \int_{\mathcal{M}} a(x)(g^2(u') + u'^2)\, d\mathcal{M}dt + \int_0^T \int_{\overline{\Omega^{**}}} u^2\, d\mathcal{M}dt\right], \tag{4.71}$$

and according to the identity of energy that follows

$$\begin{aligned} E(0) &= E(T) + \int_0^T \int_{\mathcal{M}} a(x)g(u')u'\, d\mathcal{M}dt \\ &\leq E(T) + \int_0^T \int_{\mathcal{M}} a(x)(g^2(u') + u'^2)\, d\mathcal{M}dt. \end{aligned}$$

Thus,

$$E(t) \leq E(0) \leq \tilde{C}\left[\int_0^T \int_{\mathcal{M}} \left(a(x)g^2(u') + a(x)u'^2\right) d\mathcal{M}dt + \int_0^T \int_{\overline{\Omega^{**}}} |u|^2 d\mathcal{M}dt\right]$$

for all $t \in (0, T)$, with T sufficiently large.

From the last inequality and taking (4.70) into account, we obtain,

$$\overline{E}_k(t) = \frac{E_k(t)}{c_k^2} \leq \left[\frac{\int_0^T \int_{\mathcal{M}} \left(a(x)g^2(u_k') + a(x)u_k'^2\right) d\mathcal{M}dt}{\int_0^T \int_{\overline{\Omega^{**}}} |u_k|^2 d\mathcal{M}dt} + 1\right] \tag{4.72}$$

Then, from (4.51) and (4.72), we can guarantee the existence of a constant $L_1 > 0$ such that

$$\overline{E}_k(t) \leq L_1\,,\ \forall t \in [0, T],\ \forall k \in \mathbb{N}$$

Analogously we have done to the case (i), we obtain a subsequence such that

$$\bar{u}_k \rightharpoonup \bar{u} \text{ in } H^1(\Sigma_T); \tag{4.73}$$

$$\bar{u}_k \overset{\star}{\rightharpoonup} \bar{u} \text{ in } L^\infty(0, T; H^1(\mathcal{M})); \tag{4.74}$$

$$\bar{u}_k' \overset{\star}{\rightharpoonup} \bar{u}' \text{ in } L^\infty(0, T; L^2(\mathcal{M})), \tag{4.75}$$

and, therefore, making use of compactness arguments, it holds that

$$\bar{u}_k \longrightarrow \bar{u} \text{ in } L^2(0, T; L^2(\Omega^{**})) \tag{4.76}$$

Furthermore, we observe that from (4.51)

$$\lim_{k\to+\infty} \frac{\int_0^T \int_{\mathcal{M}} a(x)g^2(u_k')\, d\mathcal{M}dt}{c_k^2} = 0 \tag{4.77}$$

and

$$\int_0^T \int_{(\mathcal{M}\backslash\Omega)\cup\mathcal{M}_*} \bar{u}_k'^2 \, d\mathcal{M}dt \le a_0^{-1} \int_0^T \int_{(\mathcal{M}\backslash\Omega)\cup\mathcal{M}_*} a(x)\bar{u}_k'^2 \, d\mathcal{M}dt$$
$$\le a_0^{-1} \int_0^T \int_{\mathcal{M}} a(x)\bar{u}_k'^2 \, d\mathcal{M}dt \longrightarrow 0 \qquad (4.78)$$

We have that $\bar{u}_k$ satisfies

$$\bar{u}_k'' - \Delta \bar{u}_k + \bar{u}_k + a(x)\frac{g(u_k')}{c_k} = 0 \ \text{ in } \ \mathcal{M} \times (0, T) \qquad (4.79)$$

Thus, passing to the limit in (4.79) when $k \to \infty$ we obtain, from (4.73), (4.74), (4.75), (4.76), (4.77) and (4.78), that

$$\begin{cases} \bar{u}'' - \Delta \bar{u} + \bar{u} = 0 \ \text{ in } \mathcal{M} \times (0, T) \\ \qquad\qquad \bar{u}' = 0 \ \text{ in } (\mathcal{M}\backslash\Omega) \cup \mathcal{M}_* \times (0, T). \end{cases} \qquad (4.80)$$

Using the same arguments employed in the first case (i),we conclude that $\bar{u} = 0$ in $\mathcal{M} \times (0, T)$, which is a contradiction having in mind (4.68) and (4.76).

This is finishes the proof of the lemma. □

Thus, from (4.47), (4.48) and Lemma 4.3 we obtain the following

$$2C \int_0^T E(t)dt \le C_0 E(T) + \tilde{C}_1 \int_0^T \int_{\mathcal{M}} a(x)(|g(u_t)|^2 + u_t^2) \, d\mathcal{M}dt$$
$$+\tilde{C}\beta \int_0^T E(t)dt. \qquad (4.81)$$

for all $T \ge T_0$, T_0 sufficiently large such that $T_0(2C - \tilde{C}\beta) - C_0 > 0$. Note that C_0 not depend on T.

Considering β sufficiently small so that $\tilde{C}_2 := 2C - \tilde{C}\beta > 0$ we obtain

$$\tilde{C}_2 \int_0^T E(t)dt \le C_0 E(T) + \tilde{C}_1 \int_0^T \int_{\mathcal{M}} a(x)(|g(u_t)|^2 + u_t^2) \, d\mathcal{M}dt, \qquad (4.82)$$

for all $T \ge T_0$.

Observe that, for all $T \ge T_0$ and $t \in [0, T_0]$ we have

$$E(T) \le E(T_0) \le E(t) \Longrightarrow T_0 E(T) \le \int_0^{T_0} E(t)dt \le \int_0^T E(t)dt$$

Then

$$\tilde{C_2} T_0 E(T) \le \tilde{C_2} \int_0^T E(t)dt$$
$$\le C_0 E(T) + \tilde{C_1} \int_0^T \int_{\mathcal{M}} a(x)(|g(u_t)|^2 + u_t^2) d\mathcal{M} dt \quad (4.83)$$

Therefore

$$E(T) \le \frac{\tilde{C_1}}{\tilde{C_2} T_0 - C_0} \int_0^T \int_{\mathcal{M}} a(x)(|g(u_t)|^2 + u_t^2)\, d\mathcal{M} dt,$$

which leads us to the following result:

Proposition 4.1 *For $T \ge T_0$, T_0 sufficiently large, the solution u to problem* (3.1) *satisfies*

$$E(T) \le C \int_0^T \int_{\mathcal{M}} \big[a(x)|u_t|^2 + a(x)|g(u_t)|^2\big] d\mathcal{M} dt \quad (4.84)$$

where the constant $C = C\big(T_0, E(0), \|a\|_{L^\infty(\mathcal{M})}, \alpha, f, \psi, a_0, M/\delta^2\big)$.

Proof of Theorem 4.1 Let us suppose initially that u is a regular solution to problem (3.1). By hypotheses, we have that $|g(s)| \ge k|s|$, $\forall s \in \mathbb{R}$.

From the identity of the energy it holds that

$$E(0) = E(T) + \int_0^T \int_{\mathcal{M}} a(x) g(u_t) u_t \, d\mathcal{M} dt$$
$$\ge E(T) + k \int_0^T \int_{\mathcal{M}} a(x) u_t^2 \, d\mathcal{M} dt,$$

that is,

$$E(T) - E(0) \le -k \int_0^T \int_{\mathcal{M}} a(x) u_t^2 \, d\mathcal{M} dt \quad (4.85)$$

In addition, from Proposition (4.1) and from the fact that $|g(s)| \le K|s|$, $\forall s \in \mathbb{R}$ we obtain

$$E(T) \le C \int_0^T \int_{\mathcal{M}} a(x)(g^2(u_t) + u_t^2)\, d\mathcal{M} dt$$
$$\le C_1 \int_0^T \int_{\mathcal{M}} a(x) u_t^2 \, d\mathcal{M} dt, \quad \forall\, T \ge T_0.$$

Consequently,

$$-E(T) \geq -C_1 \int_0^T \int_{\mathcal{M}} a(x) u_t^2 \, d\mathcal{M} dt, \quad \forall\, T \geq T_0. \tag{4.86}$$

Multiplying (4.85) by C_1 and considering (4.86) we obtain

$$\begin{aligned} C_1 \left[E(T) - E(0) \right] &\leq -k C_1 \int_0^T \int_{\mathcal{M}} a(x) u_t^2 \, d\mathcal{M} dt \\ &\leq -k E(T), \qquad \forall\, T \geq T_0, \end{aligned}$$

that is,

$$E(T) \leq \frac{C_1}{C_1 + k} E(0) = \frac{1}{1+C} E(0), \quad \forall\, T \geq T_0,$$

where $C = \frac{k}{C_1}$.

Repeating the process for $2T$, we arrive at

$$E(2T) \leq \frac{1}{1+C} E(T) \leq \frac{1}{(1+C)^2} E(0), \quad \forall\, T \geq T_0.$$

In general, we conclude that

$$E(nT) \leq \frac{1}{(1+C)^n} E(0), \quad \forall\, T \geq T_0. \tag{4.87}$$

Now, let $t \geq T_0$, then $t = nT_0 + r$, $0 \leq r < T_0$. Since the energy is a non-increasing function, we obtain

$$E(t) \leq E(t-r) = E(nT_0) \leq \frac{1}{(1+C)^n} E(0) = \frac{1}{(1+C)^{\frac{t-r}{T_0}}} E(0)$$

Considering $C_0 = e^{\frac{r}{T_0} ln(1+C)}$ and $\lambda_0 = \frac{ln(1+C)}{T_0} > 0$ it follows the desired for regular solutions, or still,

$$E(t) \leq C_0 e^{-\lambda_0 t} E(0); \quad \forall t \geq T_0. \tag{4.88}$$

By density arguments the exponential decay is recovered for weak solutions. □

5 Manifolds that Admit Open Subsets Without Damping

5.1 The Manifold $(\mathbb{R}^n, g_\varphi)$

Consider the manifold $\mathcal{M} = \mathbb{R}^n$, $n \geq 2$, endowed with the radial metric described in polar coordinates $(r, \theta) \in \big[0, +\infty\big[\times \mathbb{S}^{n-1}$ by the formula:

$$g_\varphi = \mathrm{d}r^2 + \varphi(r)^2 \mathrm{d}\theta^2, \tag{5.1}$$

where $\mathrm{d}\theta^2$ is the standard round metric (of radius 1) of $\mathbb{S}^{n-1}$ and $\varphi : \big]0, +\infty\big[\to \mathbb{R}^+$ is a smooth function satisfying:

- $\varphi^{(2k)}(0) = 0$ for all $k \geq 0$;
- $\varphi'(0) = 1$.

The metric g_φ is complete. Discreteness, existence of strongly convex functions, as well as many other spectral and geometric properties of the manifold $\mathbb{R}^n$ endowed with the metric (5.1) can be given in terms of (asymptotic) properties of the function φ. For instance:

Lemma 5.1 *The following statements hold true.*

(1) $(\mathbb{R}^n, g_\varphi)$ *has finite volume if and only if* $\int_0^{+\infty} \varphi(r)^{n-1}\, \mathrm{d}r < +\infty$.

(2) *If* $\lim_{r\to+\infty} \left| \frac{\varphi'(r)}{\varphi(r)} \right| = +\infty$, *then* $(\mathbb{R}^n, g_\varphi)$ *is discrete.*

(3) *If* $\varphi'(r) \geq c > 0$ *for all* r, *then* $(\mathbb{R}^n, g_\varphi)$ *admits a proper, strongly convex function which is bounded from below.*

(4) $(\mathbb{R}^n, g_\varphi)$ *is* stochastically complete[1] *if and only if*

$$\int_a^{+\infty} \frac{\int_0^r \varphi^{n-1}(s)\, \mathrm{d}s}{\varphi^{n-1}(r)}\, \mathrm{d}r = +\infty.$$

(5) $(\mathbb{R}^n, g_\varphi)$ *is* parabolic[2] *if and only if* $\int_a^{+\infty} \varphi(r)^{-n+1}\, \mathrm{d}r = +\infty$ *for some* $a \geq 0$.

Proof The volume of $(\mathbb{R}^n, g_\varphi)$ is easily computed as

$$c_{n-1} \int_0^{+\infty} \varphi(r)^{n-1}\, \mathrm{d}r,$$

[1] Recall that a Riemannian manifold (M, g) is said to be stochastically complete if for all $(x, t) \in M \times \big]0, +\infty\big[$, $\int_M p(x, y, t)\, \mathrm{d}M = 1$, where p is the heat kernel of the Laplacian Δ_g. It is well known (see [32]), that stochastic completeness is equivalent to the weak maximum principle at infinity. The weak maximum principles says that, given a function $u : M \to \mathbb{R}$ of class $\mathcal{C}^2$ with $\sup u = u_* < +\infty$, then there exists a sequence $(x_k)_k$ in M such that $u(x_k) \geq u_* - \frac{1}{k}$ and $\Delta_g u(x_k) < \frac{1}{k}$ for all $k \geq 1$.

[2] A complete Riemannian manifold is said to be parabolic if it does not admit a global positive Green function.

where c_{n-1} is the volume of the unit sphere of dimension $n-1$. This settles (1).

For part (3), set $f(r) = \int_0^r \varphi(t)\,\mathrm{d}t$. The Hessian of f is computed as:

$$\begin{aligned}\mathrm{Hess}(f) &= f''(r)\,\mathrm{d}r \otimes \mathrm{d}r + f'(r)\mathrm{Hess}(r)\\ &= \varphi'(r)\,\mathrm{d}r \otimes \mathrm{d}r + \varphi(\varphi'/\varphi)(g_\varphi - \mathrm{d}r \otimes \mathrm{d}r) = \varphi'(r) g_\varphi.\end{aligned}$$

Thus, f is strongly convex when $\varphi' \geq c > 0$. In this situation, such a function is proper, and positive.

For part (2), (4) and (5), see for instance [32]. □

Let us assume that $\mathbb{R}^n$ is endowed by the metric g_φ described in the example (5.1), so that $\varphi'(r) \geq c > 0$, $\forall r$. According to Lemma 5.1, item (3), it holds that $(\mathbb{R}^n, g_\varphi)$ admits a regular function $f : \mathbb{R}^n \to \mathbb{R}$ proper, strongly convex and bounded from below.

In addition, let us assume that the set $\Omega \subset \mathbb{R}^n$, is an open bounded set with smooth boundary, and φ satisfies

$$\varphi'(r) \in \left[\frac{2}{n}(\alpha + C), \frac{\alpha - C}{\frac{n}{2} - 1}\right],$$

where $C \in (0, \frac{1}{2}]$ and $\alpha \in \left[\frac{n}{2} - 1 + C, \frac{n}{2} - C\right]$, $n > 2$. If $n = 2$, it is not necessary to have an upper bound to φ'.

According to the proof of item (3), $f(r, \theta) = \int_0^t \varphi(t) dt$. Consequently, we deduce

$$\begin{aligned}\nabla^2 f\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) &= \frac{\partial^2 f}{\partial r^2} = \varphi'(r),\\ \nabla^2 f\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right) &= 0,\\ \nabla^2 f\left(\frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j}\right) &= 0, \end{aligned} \tag{5.2}$$

if $i \neq j$ and

$$\nabla^2 f\left(\frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_i}\right) = \frac{\partial f}{\partial r}\,\varphi(r)\,\varphi'(r) = \varphi^2(r)\,\varphi'(r), \tag{5.3}$$

thus $\nabla^2 f$ is proportional to the metric g_φ.

Thus, from the Theorem 4.2 remains true in the whole Ω, which allow us to avoid putting dissipative effects totally in Ω.

Remark 5 From Lemma 5.1, item (1), the manifold $(\mathbb{R}^n, g_\varphi)$ has finite volume if and only if $\int_0^\infty \varphi(r)^{n-1} dr < \infty$. Then, assuming this can occur, we can consider a nonlinear dissipation g satisfying

$$ks^2 \leq g(s)s \leq Ks^2$$

for $|s| > 1$. In other words, it is not necessary to consider the previous property for $\forall s$, since, in this case, we can prove analogously what we have done before: the existence of solutions as well as very general decay rate estimates by employing the method developed in Lasiecka-Tataru (ver[27]) (see also Cavalcanti et. al. [11]).

5.2 Manifolds with Curvature Conditions

Let $\Omega^* \subset \mathcal{M}$ be as defined before. We denote $K = \overline{\Omega^*}$.

The main goal of the present subsection is to determine an important class of Riemannian manifolds, namely, Riemannian manifolds $(\mathcal{M}, \mathbf{g})$ with sectional curvature verifying $k_1 \leq sec_g \leq k_2 < 0$, such that it is possible to guarantee the existence of open and disjoint subsets, $V_1, \ldots, V_k$, with $\bigcup_{i=1}^k V_i \subset V$, where $V \subset K$ can be free of dissipative effects and, in addition, $meas(V) > meas(K) - \epsilon$, $meas(V \cap \partial K) > meas(\partial K) - \epsilon$, for all $\epsilon > 0$, more precisely, Remark 4 occurs for such class of Riemannian manifolds with $V \supset \bigcup_{i=1}^k V_i$.

We shall determine precisely an open subset $V_i \subset K$ and function α_i, $f_i : V_i \to \mathbb{R}$, where the inequality (4.22) holds, and consequently such an open can stay free of dissipative effects.

Initially we start with the following result:

Theorem 5.1 *Let α, $C \in \mathbb{R}$ such that $\alpha > (n-1)C$, $n \in \mathbb{N}$ and $n \geq 2$. Let us consider $W \subset \mathcal{M}^n$ and $f : \mathcal{M} \to \mathbb{R}$ verifying*

$$\frac{2}{n}(\alpha + C)|v|_p^2 \leq \nabla^2 f(v, v) \leq \left[\frac{4}{n^2}(\alpha + C) + \frac{2}{n}(\alpha - C)\right]|v|_p^2,$$

$\forall\, p \in W$ e $\forall\, v \in T_p\mathcal{M}$. Then, the following inequality holds

$$\begin{aligned} C\int_0^T \int_W u_t^2 + |\nabla u|^2 \, d\mathcal{M}dt \leq & \int_0^T \int_W (\frac{\Delta f}{2} - \alpha)u_t^2 \, d\mathcal{M}dt \\ & + \int_0^T \int_W \nabla^2 f(\nabla u, \nabla u) + (\alpha - \frac{\Delta f}{2})|\nabla u|^2 \, d\mathcal{M}dt, \end{aligned} \tag{5.4}$$

for all regular solution u to problem (3.1)

Proof Initially we observe that

$$\Delta f(p) = \sum_{i=1}^n \nabla^2 f(e_i, e_i),$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis in $T_p\mathcal{M}$. Thus, we obtain,

$$2(\alpha + C) \leq \Delta f(p) \leq \frac{4}{n}(\alpha + C) + 2(\alpha - C), \ \forall p \in W.$$

Therefore,

$$\frac{1}{2}\Delta f(p)|\nabla u|^2 \leq \left[\frac{2}{n}(\alpha + C) + (\alpha - C)\right]|\nabla u|^2,$$

for all $p \in W$.

Consequently

$$\begin{aligned}
&\nabla^2 f(p)(\nabla u, \nabla u) + \left(\alpha - \frac{1}{2}\Delta f(p)\right)|\nabla u|^2 \geq \frac{2}{n}(\alpha + C)|\nabla u|^2 \\
&+ \alpha|\nabla u|^2 - \left[\frac{2}{n}(\alpha + C) + (\alpha - C)\right]|\nabla u|^2 \\
&= C|\nabla u|^2
\end{aligned} \tag{5.5}$$

for all $p \in W$.

On the other hand,

$$\left(\frac{1}{2}\Delta f(p) - \alpha\right)u_t^2 \geq (\alpha + C) - \alpha = Cu_t^2 \tag{5.6}$$

for all $p \in W$. Integrating the inequalities (5.5) and (5.6) over $W \times (0, T)$ we deduce the desired. □

5.2.1 Manifolds Admitting Smooth Functions with Bounds on the Hessian

Given a smooth function $f : (\mathcal{M}, \mathbf{g}) \to \mathbb{R}$ on a Riemannian manifold, let us denote by $\mathrm{Hess}(f)$ its Hessian, which is the tensor field of symmetric $(0, 2)$-tensors on $\mathcal{M}$ defined by:

$$\mathrm{Hess}(f)(v, w) = \mathbf{g}(\nabla_v(\nabla f), w),$$

for all $p \in \mathcal{M}$ and all $v, w \in T_p\mathcal{M}$. Given a function $h : \mathcal{M} \to \mathbb{R}$, we write $\mathrm{Hess}(f) \geq h$ (resp., $\mathrm{Hess}(f) \leq h$) if $\mathrm{Hess}(f)_p(v, v) \geq h(p)\mathbf{g}_p(v, v)$ (resp., $\mathrm{Hess}(f)_p(v, v) \leq h(p)\mathbf{g}_p(v, v)$) for all $p \in \mathcal{M}$ and all $v \in T_p\mathcal{M}$.

We would like to describe a class of manifolds admitting smooth functions f that satisfy an inequality of the form:

$$A \leq \mathrm{Hess}(f) \leq B \tag{5.7}$$

for some prescribed constants $B > A > 0$. Our motivation is to satisfy (5.4), which amounts to determining positive constants α, C and a smooth function f such that the following inequalities:

$$\mathrm{Hess}(f) \geq C + \tfrac{1}{2}\Delta f - \alpha \quad \text{and} \quad \tfrac{1}{2}\Delta f - \alpha \geq C \tag{5.8}$$

are satisfied in some open subset $W \subset \mathcal{M}$. Observe that, taking traces in (5.7), one obtains immediately the following inequalities for the Laplacian Δf:

$$\tfrac{1}{2}nA \le \frac{1}{2}\Delta f \le \tfrac{1}{2}nB, \tag{5.9}$$

where $n = \dim(\mathcal{M})$.

Using (5.7) and (5.9), we see that (5.8) are satisfied, provided that A and B satisfy:

$$A - \tfrac{1}{2}nB \ge C - \alpha, \quad \text{and} \quad \tfrac{1}{2}nA \ge C + \alpha.$$

Equalities in the above expressions are obtained by setting:

$$A = \tfrac{2}{n}(C+\alpha), \quad B = \tfrac{2}{n}\big[\tfrac{2-n}{n}C + \tfrac{2+n}{n}\alpha\big] = \tfrac{4}{n^2}(C+\alpha) - \frac{2}{n}(C-\alpha).$$

The condition $B > A$ gives:

$$C + \alpha < \tfrac{2-n}{n}C + \tfrac{2+n}{n}\alpha,$$

i.e.,

$$\alpha > (n-1)C. \tag{5.10}$$

5.2.2 Warped Products

The class of warped products offers many examples of Riemannian manifolds (M, g) that admit smooth functions f satisfying the pinching condition (5.7). It is not hard to show that a complete n-dimensional Riemannian manifold admitting a smooth function with Hessian bounded from below by a positive constant and having a (necessarily unique and minimal) critical point, must be diffeomorphic to $\mathbb{R}^n$. However, if one requires the existence of smooth strongly convex functions, without the assumption of admitting a minimum, then one has no topological obstruction. Note that for our purposes, it is not necessary to assume the existence of critical points for functions satisfying the pinching conditions (5.7) on the Hessian, as we will explain below.

A basic examples of manifolds admitting a globally defined function satisfying the pinching condition (5.7) is given as follows: Let $(\widetilde{M}, \widetilde{g})$, and consider the product manifold $M = \mathbb{R} \times \widetilde{M}$. Let M be endowed with the warped product metric:

$$g = \mathrm{d}t^2 + w(t)^2\,\widetilde{g}, \tag{5.11}$$

where t is the coordinate in $\mathbb{R}$. One could consider, more generally, metrics on $\mathbb{R} \times \widetilde{M}$ of the form $g = \alpha(t)^2\,\mathrm{d}t^2 + w(t)^2\,\widetilde{g}$. However, by a standard change of coordinates, one could rewrite such metric as $\mathrm{d}\rho^2 + w\big(t(\rho)\big)^2\,\widetilde{g}$, where $r = \int \alpha(t)$.

Given the metric (5.11), consider the radial function $f : M \to \mathbb{R}$ defined by $f(t, x) = \int_0^t w(s)\,\mathrm{d}s$. An easy computation shows that $\mathrm{Hess}(f)$ at any point p is a multiple of the metric g:

$$\mathrm{Hess}(f) = w'(t)\, g.$$

In particular, if $w : \mathbb{R} \to \mathbb{R}$ is a smooth diffeomorphism satisfying:

$$A \leq w'(t) \leq B, \qquad \forall\, t \in \mathbb{R},$$

then the corresponding function f satisfies (5.7).

It is also interesting to observe that, by a characterization of Cheeger and Colding in [14], Riemannian manifolds admitting functions whose Hessian is a multiple of the metric are basically warped products of the type described here.

5.2.3 The Hessian Comparison Theorem

We will now change our point of view, and try to determine regions of a Riemannian manifold that are domains of smooth functions that satisfy the pinching condition on the Hessian (5.7) using curvature conditions.

Let us recall that, given a Riemannian manifold $(\mathcal{M}, \mathbf{g})$ and a point $p \in \mathcal{M}$, the *injectivity radius* at p, denoted by $\mathrm{inj}(p)$ is the supremum of the set

$$\big\{ r > 0 : \exp_p \big|_{B_p(0,r)} \text{ is a diffeomorphism} \big\}.$$

Here, $B_p(0, r)$ is the open ball of radius r and center 0 in $T_p\mathcal{M}$. Equivalently, $\mathrm{inj}(p)$ is the distance between p and its cut locus. A point $p \in \mathcal{M}$ is said to be a *pole* if $\mathrm{inj}(p) = +\infty$. Given $p \in \mathcal{M}$ and $v, w \in T_p\mathcal{M}$ linearly independent, we will denote by $\sec_{\mathbf{g}}(v \wedge w)$ the sectional curvature of the plane spanned by v and w.

Let us recall the following result on the Hessian of the distance function.

Theorem 5.2 *Let $(\mathcal{M}_i^n, \mathbf{g}_i)$, $i = 1, 2$, be complete Riemannian manifolds, and let $\gamma_i : [0, L] \to \mathcal{M}_i$ be geodesics parameterized by arc-length, such that γ_i does not intersect the cut locus of $\gamma_i(0)$. Denote by $r_i = \mathrm{dist}\big(\cdot, \gamma_i(0)\big)$, $i = 1, 2$. Assume that for all $t \in [0, L]$, one has:*

$$\sec_{g_1}\big(v_1 \wedge \dot{\gamma}_1(t)\big) \geq \sec_{g_2}\big(v_2 \wedge \dot{\gamma}_2(t)\big)$$

for all $v_i \in \dot{\gamma}_i(t)^{\perp}$. Then:

$$\mathrm{Hess}(r_1)(v_1, v_1) \leq \mathrm{Hess}(r_2)(v_2, v_2),$$

for all $v_i \in \dot{\gamma}_i(t)^{\perp}$ and all $t \in [0, L]$.

Remark 6 In the above statement, $r_i\big(\gamma_i(t)\big) = t$ for all $t \in [0, L]$ (because γ_i is parameterized by arc-length, hence $\mathrm{Hess}(r_i)\big(\dot{\gamma}_i(t), \dot{\gamma}_i(t)\big) = 0$ for all t. In order to have a function with control on the Hessian in all directions, one should consider the squared distance function. An easy computation shows that, for all smooth function $f : \mathcal{M} \to \mathbb{R}$:

$$\mathrm{Hess}(\tfrac{1}{2} f^2)(v, v) = \mathbf{g}(\nabla f, v)^2 + f \cdot \mathrm{Hess}(f)(v, v).$$

In particular, since $\nabla r_i\big(\gamma_i(t)\big) = \dot\gamma_i(t)$, then:

$$\mathrm{Hess}(\tfrac{1}{2}r_i^2)(\dot\gamma_i, \dot\gamma_i) = 1.$$

More generally, if $F : \mathbb{R} \to \mathbb{R}$ and $r : \mathcal{M} \to \mathbb{R}$ are smooth functions, then:

$$\mathrm{Hess}(F \circ r)(X, X) = F''(r)\mathbf{g}(\nabla r, X)^2 + F'(r)\,\mathrm{Hess}(r)(X, X). \tag{5.12}$$

Thus, in computing the Hessian of a distance function, one has to distinguish *radial directions*, i.e., parallel to the tangent field $\dot\gamma(t)$, and *spherical directions*, i.e., orthogonal to $\dot\gamma(t)$.

5.2.4 Space Forms

Let $g_{\mathbb{S}^n}$ be the standard round metric on the unit sphere $\mathbb{S}^n$. For $k \in \mathbb{R}$, consider the manifold $\mathcal{M}^n = \mathbb{R}^+ \times \mathbb{S}^{n-1}$ endowed with the metric

$$g_k = \mathrm{d}r^2 + S_k(r)\, g_{\mathbb{S}^{n-1}},$$

where:

$$S_k(r) = \begin{cases} \frac{1}{\sqrt{k}} \sin\big(\sqrt{k}\, r\big), & \text{if } k > 0, \\ r, & \text{if } k = 0, \\ \frac{1}{\sqrt{|k|}} \sinh\big(\sqrt{|k|}\, r\big), & \text{if } k < 0. \end{cases}$$

The (completion of the) Riemannian manifold $(\mathcal{M}^n, g_k)$ is the[3] *n-dimensional space form of curvature k*. An explicit calculation of the Hessian of the distance function for spaceforms yields the following immediate corollary of the Hessian comparison theorem.

Proposition 5.1 (Special case of the Hessian comparison theorem) *Given a Riemannian manifold* $(\mathcal{M}^n, g)$ *and a point* $p \in \mathcal{M}$*, set* $r = \mathrm{dist}(\cdot, p)$*. If* $\sec_g \geq k$*, then, in the spherical directions,* $\mathrm{Hess}(r) \leq \frac{1}{n-1} H_k(r)$*, where:*

$$H_k(r) = \begin{cases} (n-1)\sqrt{k} \cdot \cot\big(\sqrt{k} \cdot r\big), & \text{if } k > 0; \\ \dfrac{n-1}{r}, & \text{if } k = 0; \\ (n-1)\sqrt{-k} \cdot \coth\big(\sqrt{-k} \cdot r\big), & \text{if } k < 0. \end{cases}$$

Similarly, $\sec_g \leq k$*, then* $\mathrm{Hess}(r) \geq \frac{1}{n-1} H_k(f)$ *in the spherical directions.*

[3] Actually, for $k > 0$, $M^n = \left]0, \pi\right[\times \mathbb{S}^{n-1}$.

Remark 7 One can consider more generally a metric on $\mathcal{M}^n$ of the form:

$$\mathbf{g} = \mathrm{d}r^2 + \phi(r)^2\, g_{\mathbb{S}^{n-1}}, \tag{5.13}$$

where $\phi : \left]0, +\infty\right[\to \mathbb{R}^+$ is a smooth function with $\lim_{r\to 0} \phi(r) = 0$ and $\lim_{r\to 0} \phi'(r) = 1$. The following formulas for the sectional curvature of these metrics hold:

$$\sec_{\mathrm{rad}} = \frac{\phi''}{\phi}, \quad \sec_{\mathrm{sph}} = \frac{1-(\phi')^2}{\phi^2}, \tag{5.14}$$

where $\sec_{\mathrm{rad}}$ is the sectional curvature of planes containing the radial vector, and $\sec_{\mathrm{sph}}$ is the curvature of planes perpendicular to the radial vector.

5.2.5 Conclusion

Let us consider a simply connected complete Riemannian manifold $(\mathcal{M}^n, \mathbf{g})$ satisfying:

$$k_1 \le \sec_g \le k_2 < 0,$$

for some (negative) constants k_1, k_2. It is well known that every point p of such a manifold, which is necessarily diffeomorphic to $\mathbb{R}^n$, is a pole. Let us fix positive constants $0 < A < B$; for computational convenience, let us choose[4] $B > n-1$ and $A < n-1$. We will exhibit a smooth function $f : \mathcal{M} \to \left[0, +\infty\right[$ and a non empty open subset $W \subset \mathcal{M}$ such that (5.7) holds in W.

Let us fix $p_0 \in \mathcal{M}$, set $r = \mathrm{dist}(\cdot, p_0)$, choose $\gamma \in \left]A, B\right[$, and let $f = \frac{\gamma}{2} r^2$. Using (5.12), the Hessian of f is given by:

$$\mathrm{Hess}(f)(X, X) = \gamma \cdot g(\nabla r, X)^2 + \gamma r\, \mathrm{Hess}(r)(X, X).$$

If X is a radial vector, then $\mathrm{Hess}(r)(X, X) = 0$, and since $g(\nabla r, \nabla r) = 1$, then $\mathrm{Hess}(f)(X, X) = \gamma g(X, X)$, i.e., inequality (5.7) is always satisfied for radial vectors. As to spherical directions X, since $g(\nabla r, X) = 0$, then

$$\mathrm{Hess}(f)(X, X) = \gamma r\, \mathrm{Hess}(r)(X, X),$$

and by the Hessian comparison theorem we then get:

$$\gamma r\sqrt{-k_2}\coth\big(r\sqrt{-k_2}\big) \le \mathrm{Hess}(f)(X, X) \le \gamma r\sqrt{-k_1}\coth\big(r\sqrt{-k_1}\big).$$

Thus, inequality (5.7) is satisfied in the region:

$$r_1 \le r \le r_2,$$

[4] Note that, when imposing the pinching condition (5.7), the relevant quantity is simply the quotient B/A. Namely, a function f satisfies (5.7) if and only if the function $\widetilde{f} = \frac{1}{A} f$ satisfies $1 \le \mathrm{Hess}(\widetilde{f}) \le B/A$. This implies, in particular, that there is no loss of generality in assuming $A < \alpha < B$ for any $\alpha > 0$.

where r_1 and r_2 are defined by:

$$r_1\sqrt{-k_2}\coth\big(r_1\sqrt{-k_2}\big) = \frac{A}{\gamma}, \quad r_2\sqrt{-k_1}\coth\big(r_2\sqrt{-k_1}\big) = \frac{B}{\gamma}.$$

By elementary arguments, one can determine a large class of examples of smooth functions satisfying pinching conditions on the Hessian, using wisely formulas (5.12) for the Hessian of composite functions and formulas (5.14) for the radial curvature of warped metrics of the type (5.13) in the Hessian comparison theorem.

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6 Appendix

We start this section by considering a uniqueness result due to Bur and Gérard [10] (see p. 80, Sect. 6) or Lasiecka et. al. [26]. Let us consider the wave equation posed in a compact Riemannian manifold (M, g) with boundary. If (ω, T_0) controls M, then the following observability inequality holds for ultra weak solutions to problem

$$\begin{cases} v_{tt} - \Delta_g v = 0 \ \text{ in } M \times (0, T), \\ v = 0 \ \text{ on } \partial M \times (0, T), \\ v(0) = v_0 \in L^2(M);\, v_t(0) = v_1 \in H^{-1}(M), \end{cases} \tag{6.1}$$

holds:

$$||v_0||^2_{L^2(M)} + ||v_1||_{H^{-1}(M)} \leq C \int_0^T \int_\omega |v(x,t)|^2 \, d\mathcal{M}dt, \tag{6.2}$$

for a certain constant C and for all $T \geq T_0$ and for all $\{v_0, v_1\} \in L^2(M) \times H^{-1}(M)$.

Following we will show that if the Riemannian manifold M admits a function with positive defined Hessian in some subset $U \subset \text{int}M$, then any geodesic on U hits ∂U.

We consider a Riemannian manifold M Riemannian metric $G = \langle \cdot, \cdot \rangle$ and Riemannian connection $\tilde{\nabla}$. We denote its Laplace–Beltrami operator by Δ. Fix a coordinate system $(x_1, \ldots, x_n)$ on M, denote $G_{ij} = \big\langle \partial/\partial x_i, \partial/\partial x_j \big\rangle$, let G^{ij} be the inverse matrix of G_{ij}. The Laplace–Beltrami operator in this coordinate system is given by

$$\Delta u = \frac{1}{\sqrt{\det G_{ij}}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{\det G_{ij}}\, G^{ij} \frac{\partial u}{\partial x_j} \right),$$

where ∇ is the usual gradient correspondent to the Euclidean metric on the domain $(x_1, \ldots, x_n)$.

The Hessian of a smooth function $\phi : M \to \mathbb{R}$ is a symmetric 2-form on M defined as

$$\nabla^2\phi(X, Y) = XY(\phi) - \tilde{\nabla}_X Y(\phi), \tag{6.3}$$

where X and Y are vector fields on M. Here $X(\phi)$ denotes the directional derivative of ϕ in the direction of the vector field X. It is well known that the value of $\nabla^2(X, Y)(p)$ depends only on the values of X and Y on p. It means that the right-hand side of (6.3) does not depend on the smooth extension we take for $X(p)$ and $Y(p)$.

A curve $\gamma : (-\varepsilon, \varepsilon) \to M$ is a geodesic if $\tilde{\nabla}_{\gamma'(t)}\gamma'(t) \equiv 0$.

Lemma 6.1 *Let M be a complete Riemannian manifold, eventually with boundary, and let $\phi : intM \to \mathbb{R}$ be a smooth function. Suppose that ϕ is bounded and $\nabla^2\phi(v, v) \geq c\|v\|^2$ on an open subset $U \subset intM$, where $c > 0$ is a constant. Then any geodesic on U hits ∂U.*

Proof Let γ be a geodesic on U. Then

$$\begin{aligned}\frac{d^2}{dt^2}\phi(\gamma(t)) &= \gamma'(t)\gamma'(t)\phi = \nabla^2\phi(\gamma'(t), \gamma'(t)) + (\nabla_{\gamma'(t)}\gamma'(t))(\phi)\\ &= \tilde{\nabla}^2\phi(\gamma'(t), \gamma'(t)) \geq c\|\gamma'(t)\|^2\end{aligned}$$

where the last equality holds because γ is a geodesic. Observe that the last term is a positive real constant because $\|\gamma'(t)\|$ does not depend on t.

Then $\phi(\gamma(t))$ is a smooth real valued function which second derivative is bounded below by a strictly positive constant. Then it is not difficult to prove that if γ is defined on a interval (a, ∞), then $\lim_{t\to\infty}\phi(\gamma(t)) = \infty$. Analogously $\lim_{t\to-\infty}\phi(\gamma(t)) = \infty$ whenever γ is defined on a interval $(-\infty, a)$. But neither of the cases are possible if γ remain forever in U because ϕ is bounded there. Therefore γ must hit ∂U. □

According to the construction of the function f give in [13] and mentioned in Remark 4 we have that $\nabla^2 f(v, v) \geq C\|v\|_g^2$ in V, where $V = \bigcup_{i=1}^k V_i$. Thus, we ensure that the geodesics find the effective dissipation region, namely $(M\backslash\overline{\Omega}) \cup M_*$, That is, there are no geodesic "trapped" within the free sets of dissipative effects V_i, for all $i = 1, \ldots, k$. It is worth remembering that $\Omega \subset\subset \Omega^*$.

Now we can show that, in fact, the function v given in (4.64) is null in all Ω^*. For this we have to show that $v = 0$ in V_i, for all $i = 1, \ldots, k$, since $(\overline{\Omega^*}\backslash V) \subset \mathcal{M}_*$ and $v = 0$ in $\mathcal{M}_*$.

By (4.64), we have

$$\begin{cases} v'' - \Delta v + v = 0 & in\ \ \Omega^* \times (0, T)\\ v = 0 & in\ \ \partial\Omega^* \times (0, T)\\ v(0) = v_0 \in L^2(\Omega^*);\ v'(0) = v_1 \in H^{-1}(\Omega^*), \end{cases} \tag{6.4}$$

with $v = 0$ in $(\Omega^*\backslash\Omega) \cup \mathcal{M}_*$.

Now, consider the problems

$$\begin{cases} \varphi'' - \Delta\varphi = 0 & in \ \ \Omega^* \times (0,T) \\ \varphi = 0 & in \ \ \partial\Omega^* \times (0,T) \\ \varphi(0) = v_0 \in L^2(\Omega^*); \ \varphi'(0) = v_1 \in H^{-1}(\Omega^*), \end{cases} \tag{6.5}$$

and,

$$\begin{cases} z'' - \Delta z = -v & in \ \ \Omega^* \times (0,T) \\ z = 0 & in \ \ \partial\Omega^* \times (0,T) \\ z(0) = z'(0) = 0. \end{cases} \tag{6.6}$$

Defining $w = \varphi + z$, we have that w is solution of

$$\begin{cases} w'' - \Delta w = -v & in \ \ \Omega^* \times (0,T) \\ w = 0 & in \ \ \partial\Omega^* \times (0,T) \\ w(0) = v_0 \in L^2(\Omega^*); \ w'(0) = v_1 \in H^{-1}(\Omega^*), \end{cases} \tag{6.7}$$

and, if $y = v - w$ then y is solution of

$$\begin{cases} y'' - \Delta y = 0 & in \ \ \Omega^* \times (0,T) \\ y = 0 & in \ \ \partial\Omega^* \times (0,T) \\ y(0) = y'(0) = 0. \end{cases} \tag{6.8}$$

By uniqueness of solution, we conclude that $y = 0$, that is, $v = w = \varphi + z$. Note that $z'' - \Delta v = -v = 0$ is $(\overline{\Omega^*}\backslash\Omega) \cup \mathcal{M}_*$ with $z(0) = 0 = z'(0)$, where it follows that $z = 0$ in $(\overline{\Omega^*}\backslash\Omega) \cup \mathcal{M}_*$, and consequently, $\varphi = 0$ in $(\overline{\Omega^*}\backslash\Omega) \cup \mathcal{M}_*$.

As $\varphi = 0$ in $\mathcal{M}_* \supset\supset \overline{\Omega^*}\backslash V$ we have that exists an open neighborhood ω_i of ∂V_i, with $\omega_i \subset V_i$, such that $\varphi = 0$ in ω_i.

Restricting the problem (6.5) to V_i, follows by (6.2) that

$$||v_0||^2_{L^2(V_i)} + ||v_1||_{H^{-1}(V_i)} \leq C \int_0^T \int_{\omega_i} |\varphi(x,t)|^2 \, d\mathcal{M} dt, \tag{6.9}$$

where it follows that $v_0 = v_1 = 0$ em V_i, for all $i = 1, \ldots, k$, and therefore, by the problem (6.8), we conclude that $v = 0$ in V_i, for all $i = 1, \ldots, k$, as we wanted to show. □

References

1. Adams, R.A.: Sobolev Spaces. Academic Press, New York (1975)
2. Alabau-Boussouira, F.: Convexity and weighted inequalities for energy decay rates of nonlinear dissipative hyperbolic systems. Appl. Math. Optim. **51**(1), 61–105 (2005)
3. Bessa, G.P., Montenegro, F., Piccione, P.: Riemannian submersions with discrete spectrum. J. Geom. Anal. (to appear). arXiv:1001.0853
4. Barbu, V.: Nonlinear semigroups and differential equations in Banach spaces. Editura Academiei, Noordhoff International Publishing, Bucuresti Romania, Leyden (1976)
5. Bardos, C., Lebeau, G., Rauch, J.: Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. SIAM J. Control Optim. **30**(5), 1024–1065 (1992)

6. Brézis, H.: Análisis Funcional. Teoría y Aplicaciones. Alianza Editorial, S.A., Madrid (1984)
7. Brézis, H.: Operateurs Maximaux Monotones et Semigroups de Contractions dans les Spaces de Hilbert. North Holland Publishing Co., Amsterdam (1973)
8. Brézis, H.: Autumn Course On Semigoups, Theory and Applications. Lecture Notes taken by L. COHEN. International Center for Theoretical Physics, Triest (1984)
9. Brézis, H.: Cazenave T. Nonlinear Evolution Equations. Preliminary Version of Chapters 1,2 and 3 and the Appendix (1994)
10. Burq, N., Gérard, P.: Contrôle Optimal des équations aux dérivées partielles. Url:http://www.math.u-psud.fr/~burq/articles/coursX (2001)
11. Cavalcanti, M.M., Domingos Cavalcanti, V.N., Lasiecka, I.: Wellposedness and optimal decay rates for wave equation with nonlinear boundary damping-source interaction. J. Differ. Equ. **236**, 407–459 (2007)
12. Cavalcanti, M.M., Domingos Cavalcanti, V.N., Fukuoka, R., Soriano, J.A.: Uniform Stabilization of the wave equation on compact surfaces and locally distributed damping. Trans. AMS **361**(9), 4561–4580 (2009)
13. Cavalcanti, M.M., Domingos Cavalcanti, V.N., Fukuoka, R., Soriano, J.A.: Asymptotic stability of the wave equation on compact manifolds and locally distributed damping: a sharp result, -. Arch. Ration. Mech. Anal. **197**, 925–964 (2010)
14. Cheeger, J., Colding, T.H.: Lower bounds on Ricci curvature and the almost rigidity of warped products. Ann. Math. **144**(1), 189–237 (1996)
15. Christianson, H.: Semiclassical non-concetration near hyperbolic orbits. J. Funct. Anal. **246**(2), 145–195 (2007)
16. Dautray, R., Lions, J.L.: Mathematical Analysis and Numerical Methods for Science and Technology, vol. II. Springer, Berlin (1990)
17. Dehman, B., Lebeau, G., Zuazua, E.: Stabilization and control for the subcritical semilinear wave equation. Ann. Sci. École Norm. Sup. **36**, 525–551 (2003)
18. Do Carmo, M.P.: Riemannian geometry Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston (1992)
19. Donnelly, H., Li, P.: Pure point spectrum and negative curvature for noncompact manifolds. Duke Math. J. **46**, 497–503 (1979)
20. Evans, L.C. Partial Differential Equations, Graduate Studies in Mathematics, vol. 19, A.M.S
21. Fukuoka, R.: Mollifier smoothing of tensor fields on differentiable manifolds and applications to Riemannian Geometry. arXiv:math.DG/0608230
22. Greene, R.E., Wu, H.: C^{∞}-approximations of convex, subharmonic, and plurisubharmonic functions. Ann. Sci. École Norm. Sup. **12**(1), 47–84 (1979)
23. Hebey, E.: Sobolev Space on Riemannian Manifolds. Springer, Berlin (1996)
24. Hitrik, M.: Expansions and eigenfrequencies for damped wave equations. Journèes"Équations aux Dérivées Patielles" (Plestin-les-Gréves, 2001), Exp. No. VI, 10 pp., Univ. Nantes, Nantes (2001)
25. Jorge, L.P., Xavier, F.: An inequality between the exterior diameter and the mean curvature of bounded immersions. Math. Z. **178**, 77–82 (1981)
26. Lasiecka, I., Triggiani, R., Yao, P.F.: An observability estimate in $L^2 \times H^{-1}$ for send order hyperbolic equations with variable coefficients. In: Control of Distributed Parameter and Stochastic systems. Kluwer (1999)
27. Lasiecka, I., Tataru, D.: Uniform Boundary Stabilization of Semilinear Wave Equations with Nonlinear Boundary Damping. Lecture Notes in Pure and Applied Maths, vol. 142. Dekker, New York (1993)
28. Lasiecka, I., Triggiani, R.: Uniform stabilization of the wave equation with Dirichlet or Neumann feedback control without geometric conditions. Appl. Math. Optim. **25**, 189–224 (1992)
29. Lions, J.L.: Contrôlabilité Exacte Pertubations et Stabilisation de Systèmes Distribués. Masson, Paris (1988)
30. Lions, J.L., Magenes, E.: Non-homogeneous Boudary Value Problems and Applications. Springer, Berlin (1972)
31. Milnor, J.: Lectures on the h-Cobordism Theorem. Princeton University Press, Princeton. Notes by L. Siebenmann and J. Sondow (1965)
32. Pigola, S., Rigoli, M., Setti, A.: Maximum principle on Riemannian manifolds and applications. Mem. Am. Math. Soc. **174**(822) (2005)
33. Rauch, J., Taylor, M.: Decay of solutions to n ondissipative hyperbolic systems on compact manifolds. Commun. Pure Appl. Math. **28**(4), 501–523 (1975)

34. Rauch, J., Taylor, M.: Exponential decay of solutions to hyperbolic equations in bounded domains. Indiana Univ. Math. J. **24**, 79–86 (1974)
35. Simon, J.: Compact Sets in the Space $\mathbf{L}^p(0, T; B)$. Annali di Matematica pura ed applicata **146**(1), 65–96 (1987)
36. Taylor Michael, E.: Partial Differential Equations I. Basic Theory, 2nd edn. Springer, Berlin (2010)
37. Triggiani, R., Yao, P.E.: Calerman estimates with no lower-Oder terms for general Riemannian wave equations. Global uniquenees and observability in one shot. Appl. Math. Optim. **46**, 331–375 (2002)
38. Warner, F.W.: Foundations of Differentiable Manifolds and Lie Groups. Scott, Foresman and Company, Glenview (1971)
39. Yao, P., Lasiecka, I., Triggiani, R.: Inverse/observability estimates for second-order hiperbolic equations with variable coefficients. J. Math. Anal. Apl. **235**, 13–57 (1999)
40. Yao, P.: On the observability inequalities for exact controllability of wave equations with variable coefficients. SIAM J. Control Optim. **37**(5), 1568–1599 (1999)
41. Yao, P.: Modeling and control in vibrational and structural dynamics. A differential geometric approach. Chapman e Hall/CRC Applied Mathematics and Nonlinear Science Series. CRC Press, Boca Raton (2011). xiv+405 pp. ISBN: 978-1-4398-3455-8. (o livro dele)
42. Yao, P., Liu, Y., Li, J.: Decay rates of the hyperbolic equation in an exterior domain with half-linear and nonlinear boundary dissipations. J. Syst. Sci. Complex. **29**(3), 657–680 (2016)
43. Zeidler, E.: Nonlinear Functional Analysis and its Aplications. Vol 2A: Linear Monotone Operators. Springer, Berlin (1990)
44. Zuazua, E.: Exponential decay for the semilinear wave equation whith localized damping in unbounded domains. J. Math. Pures Appl. **70**, 513–529 (1992)