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Decay Rate of Solutions to Timoshenko System with Past History in Unbounded Domains

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Abstract We consider the Cauchy problem for the one-dimensional Timoshenko system coupled with the heat conduction, wherein the latter is described by the Gurtin–Pipkin thermal law. We study the decay properties of the system using the energy method in the Fourier space (to build an appropriate Lyapunov functional) accompanied with some integral estimates. We show that the number α_g (depending on the parameters of the system) found in (Dell'Oro and Pata J Differ Equ 257(2):523–548, 2014), which rules the evolution in bounded domains, also plays a role in an unbounded domain and controls the behavior of the solution. In fact, we prove that if $\alpha_g = 0$, then the L^2-norm of the solution decays with the rate $(1 + t)^{-1/12}$. The same decay rate has been obtained for $\alpha_g \neq 0$, but under some higher regularity assumption. This high regularity requirement is known as regularity loss, which means that in order to get the estimate for the H^s-norm of the solution, we need our initial data to be in the space H^{s+s_0}, $s_0 > 1$.

1 Introduction

The purpose of this paper is to study the Cauchy problem of the Timoshenko system with Gurtin–Pipkin heat conduction for the heat flux [3]:

$$\varphi_{tt} - (\varphi_x - \psi)_x = 0,$$

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$$\begin{aligned} &\psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + \delta\theta_x = 0, \\ &\theta_t - \frac{1}{\beta}\int_0^\infty g(s)\theta_{xx}(t-s)\,ds + \delta\psi_{tx} = 0, \end{aligned} \tag{1.1}$$

where the time variable $t \in (0, \infty)$, the space variable $x \in \mathbb{R}$ and a and δ are strictly positive fixed constants. The constant $\beta > 0$ is equal to $1/\kappa$, where κ is the thermal conductivity as defined below. The memory kernel $g(s)$ is a convex summable function on $[0, \infty)$ with a total mass of:

$$1 = \int_0^\infty g(s)\,ds.$$

The function $\varphi(x,t)$ is the transverse displacement of the beam from an equilibrium state, $\psi(x,t)$ is the rotation angle of the filament of the beam and $\theta(x,t)$ is the temperature difference.

System (1.1) is supplied with the following initial conditions:

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \qquad \varphi_t(x,0) = \varphi_1(x), \qquad \psi(x,0) = \psi_0(x), \\ \psi_t(x,0) = \psi_1(x), \theta_t(x,0) = \theta_1(x). \end{cases} \tag{1.2}$$

Originally the Timoshenko system (without heat conduction) consists of two wave equations describing the transverse vibration of a beam, which can be represented as:

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0, \\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) = 0. \end{cases} \tag{1.3}$$

System (1.3) is an undamped system and its energy

$$\mathscr{E}_k(t) := \frac{1}{2}\int_{\mathbb{R}} \left\{ (\partial_x^k\varphi_t)^2 + (\partial_x^k\psi_t)^2 + (\partial_x^k(\varphi_x - \psi))^2 + a^2(\partial_x^k\psi_x)^2 \right\}(x,t)dx, \quad k \geq 0, \tag{1.4}$$

remains constant when the time t evolves. To stabilize system (1.3), many damping terms have been considered by several authors. By considering a frictional damping term of the form $\gamma_1\psi_t(x,t)$ acting on the left-hand side of the second equation in (1.3), the system (1.3), as well as its nonlinear version, have been recently studied. Results concerning global existence and decay estimates of the solution have been established. Here, we mention some previous work only related to our study, such as Ide et al. [4], Ide and Kawashima [5] and Racke and Said-Houari [7]. In these papers, the authors showed that the assumptions

$$a = 1, \qquad \text{or} \qquad a \neq 1$$

play decisive roles in showing whether or not the decay estimates of the solution are of regularity-loss type. Indeed, in Ide et al. [4], it has been shown that the solution $V = (\varphi_t, \psi_t, a\psi_x, \varphi_x - \psi)^T$ of (1.3) satisfies the following decay estimates:

- When $a = 1$,

$$\left\| \partial_x^k V(t) \right\|_{L^2} \leq C(1+t)^{-1/4-k/2} \|V_0\|_{L^1} + Ce^{-ct} \left\| \partial_x^k V_0 \right\|_{L^2}. \tag{1.5}$$

- When $a \neq 1$,

$$\left\| \partial_x^k V(t) \right\|_{L^2} \leq C(1+t)^{-1/4-k/2} \|V_0\|_{L^1} + C(1+t)^{-\ell/2} \left\| \partial_x^{k+\ell} V_0 \right\|_{L^2}, \tag{1.6}$$

where

$$||\partial_x^k V(t)||_{L^2}^2 = \frac{1}{2}\mathscr{E}_k(t).$$

It has been proved recently in [11], that by considering an additional damping term of the form $\gamma_0 \varphi_t(x,t)$ acting on the left-hand side of the first equation in (1.3), then, surprisingly, the decay rate becomes slower than the ones obtained in (1.5) and (1.6). In fact, the authors proved, without any assumptions on a, the following estimate:

$$\left\| \partial_x^k V(t) \right\|_{L^2} \leq C(1+t)^{-1/8-k/4} \|V_0\|_{L^1} + Ce^{-ct} \left\| \partial_x^k V_0 \right\|_{L^2}. \tag{1.7}$$

On the other hand, they showed that by considering the damping term $\gamma_0 \varphi_t(x,t)$ alone and under the same assumption on a, similar estimates as in (1.5) and (1.6) hold with the decay rate $(1+t)^{-1/8-k/4}$ instead of $(1+t)^{-1/4-k/2}$.

For the Cauchy problem of Timoshenko system in thermoelasticity, where the heat conduction is described by the classical Fourier law in which the heat flux $q(x,t)$ is proportional to the gradient of the temperature $\theta_x(x,t)$; i.e.,

$$q(x,t) = -\kappa \theta_x(x,t), \tag{1.8}$$

where $\kappa > 0$ is the thermal conductivity, the following system (after normalizing the constants) is obtained:

$$\begin{aligned} &\varphi_{tt} - (\varphi_x - \psi)_x = 0, \\ &\psi_{tt} - a^2 \psi_{xx} - (\varphi_x - \psi) + \delta\theta_x = 0, \\ &\theta_t - \theta_{xx} + \delta\psi_{tx} = 0. \end{aligned} \tag{1.9}$$

The initial value problem (1.9) was first studied by Said-Houari and Kasimov in [8] and [9], where the authors proved in [9] that the solution $W = (\varphi_t, \psi_t, a\psi_x, \varphi_x - \psi, \theta)^T$ decays with the rate:

$$\|\partial_x^k W(t)\|_{L^2} \leq C(1+t)^{-1/12-k/6} \|W_0\|_{L^1} + Ce^{-ct} \|\partial_x^k W_0\|_{L^2}, \tag{1.10}$$

for $a = 1$ and $k = 1, 2, ..., s$, and

$$\|\partial_x^k W(t)\|_{L^2} \leq C(1+t)^{-1/12-k/6}\|W_0\|_{L^1} + C(1+t)^{-\ell/2}\|\partial_x^{k+\ell} W_0\|_{L^2}, \tag{1.11}$$

for $a \neq 1$ and $k = 1, 2, ..., s - \ell$. While, in [8], they showed that the same decay estimates can be obtained with the optimal decay rate $(1+t)^{-1/4-k/2}$ instead of $(1+t)^{-1/12-k/6}$, provided that an additional frictional damping term of the form $\lambda\psi_t(x,t)$ is considered in the second equation of (1.9).

It is well known that the use of the Cattaneo law

$$\tau q_t(x,t) + q(x,t) = -\kappa\theta_x(x,t), \tag{1.12}$$

instead of the Fourier law (1.8) of heat conduction removes the paradox of the infinite speed of propagation in the Fourier law. The coupling between the Cattaneo law and the Timoshenko system leads to the following system:

$$\begin{aligned} &\varphi_{tt} - (\varphi_x - \psi)_x = 0,\\ &\psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + \delta\theta_x = 0,\\ &\theta_t + q_x + \delta\psi_{tx} = 0,\\ &\tau q_t + q + \theta_x = 0. \end{aligned} \tag{1.13}$$

It was shown in [9] that the same decay rates as (1.10) and (1.11) hold for the solution of (1.13), but the decay rate is controlled by a new number (found first in [10])

$$\alpha = (\tau - 1)(1 - a^2) - \tau\delta^2 \tag{1.14}$$

rather than by a. In fact, they proved that the estimate (1.10) is obtained under the assumption $\alpha = 0$ which is exactly the assumption $a = 1$ for $\tau = 0$ (Timoshenko–Fourier system).

In (1.12), one can directly express $q(x,t)$ in terms of $\theta(x,t)$ from this equation, but it becomes a *nonlocal* (in time) relationship

$$q(x,t) = -\frac{\kappa}{\tau}\int_{-\infty}^{t} \theta_x(s)\, e^{(s-t)/\tau} ds. \tag{1.15}$$

A more general form of Eq. (1.15) was given by Gurtin and Pipkin [3]:

$$q(x,t) = -\kappa\int_{-\infty}^{t} g(t-s)\,\theta_x(s)\,ds, \tag{1.16}$$

where $g(s)$ is the heat flux relaxation kernel.

Many different constitutive models arise from different choices of $g(s)$. Equation (1.15) can easily be recovered from (1.16) by assuming that

$$g\,(s) = \frac{1}{\tau}e^{-s/\tau}. \tag{1.17}$$

Also, the Fourier law can be seen as the singular limit when $\varepsilon \to 0$ of the Gurtin–Pipkin law (1.16) with the kernel

$$g_\varepsilon(s) = \frac{1}{\varepsilon}g\left(\frac{s}{\varepsilon}\right), \qquad \varepsilon > 0.$$

Furthermore, the heat flux law of Jeffrey's type

$$q\,(x,t) = -\kappa_1\,\theta_x\,(t) - \frac{\kappa_2}{\tau}\int_{-\infty}^{t}\theta_x\,(s)\,e^{(s-t)/\tau}ds,$$

can be obtained by letting

$$g\,(s) = \kappa_1\delta\,(s) + \frac{\kappa_2}{\tau}e^{-s/\tau}$$

in (1.16), where κ_1 and κ_2 are two positive constants and δ is the Dirac function. See [6] for more details.

The main goal of this paper is to investigate the decay rate of problem (1.1). In fact, we prove that the same number α_g

$$\alpha_g := \left(\frac{\beta}{g(0)} - 1\right)(1-a^2) - \delta^2\frac{\beta}{g(0)}, \tag{1.18}$$

which controls the behavior of the solution in bounded domains ([2]), also plays a role in unbounded situation and affects the decay rate of the solution (see Theorem 4.1). More precisely, we prove that the energy

$$E_k(t) = \mathscr{E}_k(t) + \frac{1}{\beta}\int_0^\infty \mu(s)\int_{\mathbb{R}}|\partial_x^k\eta(x,t,s)|^2\,dxds \tag{1.19}$$

satisfies the estimates (4.1) and (4.2) below.

It is obvious that if we take $g(s)$ as in (1.17) (Timoshenko–Cattaneo), then α_g reduces to the same number α in (1.14).

By taking the kernel $g(s) = g_\varepsilon(s)$ and letting $\varepsilon \to 0$, the assumption $\alpha_g = 0$ reduces to $a = 1$ in the Fourier model. See [2] for the details.

This paper is organized as follows: In Sect. 2, we state the problem. Section 3 is devoted to the energy method in the Fourier space and to the construction of the Lyapunov functionals. In Sect. 4, we prove the main estimates of the solution in the energy space.

2 Statement of the Problem

First, we recall system (1.1) and write

$$\begin{aligned}
&\varphi_{tt} - (\varphi_x - \psi)_x = 0,\\
&\psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + \delta\theta_x = 0,\\
&\theta_t - \frac{1}{\beta}\int_0^\infty g(s)\theta_{xx}(t-s)\,ds + \delta\psi_{tx} = 0.
\end{aligned} \tag{2.1}$$

Following [1], we introduce the new variable

$$\eta(x,t,s) = \int_0^s \theta(x,t-\sigma)d\sigma = \int_{t-s}^t \theta(x,\sigma)d\sigma \qquad s \geq 0,\ t \geq 0. \tag{2.2}$$

Differentiating (2.2) with respect to t yields that η satisfies the supplementary equation

$$\eta_t(s) = -\eta_s(s) + \theta(t), \qquad \eta(0) = 0, \quad \forall\, t \geq 0, \tag{2.3}$$

which has to be added to system (2.1). Then, we define the operator $T\eta = -\eta'$. From (2.3), we get the following equation:

$$\eta_t = T\eta + \theta. \tag{2.4}$$

Also, we define $\mu(s) = -g'(s)$ and assume that μ satisfies the following two assumptions:

(M1) μ is a nonnegative nonincreasing and absolutely continuous function on $\mathbb{R}^+$ such that

$$\mu(0) = \lim_{s\to 0}\mu(s) \in (0,\infty).$$

(M2) There exists $\nu > 0$ such that the differential inequality

$$\mu'(s) + \nu\mu(s) \leq 0$$

holds for almost every $s > 0$.

With all these new variables, we rewrite system (2.1) as:

$$\begin{aligned}
&\varphi_{tt} - (\varphi_x - \psi)_x = 0,\\
&\psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + \delta\theta_x = 0,\\
&\theta_t - \frac{1}{\beta}\int_0^\infty \mu(s)\eta_{xx}(s)\,ds + \delta\psi_{tx} = 0,\\
&\eta_t = T\eta + \theta.
\end{aligned} \tag{2.5}$$

To rewrite the system as a first-order (with respect to t) differential system, we define new variables, as follows:

$$v = \varphi_x - \psi, \qquad u = \varphi_t, \qquad z = a\psi_x, \qquad y = \psi_t.$$

Hence, system (1.1) takes the form

$$
\begin{aligned}
&v_t - u_x + y = 0,\\
&u_t - v_x = 0,\\
&z_t - a y_x = 0,\\
&y_t - a z_x - v + \delta\theta_x = 0,\\
&\theta_t - \frac{1}{\beta}\int_0^\infty \mu(s)\eta_{xx}(s)\,ds + \delta y_x = 0,\\
&\eta_t = T\eta + \theta.
\end{aligned} \tag{2.6}
$$

Now, we define the solution

$$
U(x,t) = (v, u, z, y, \theta, \eta)^T. \tag{2.7}
$$

Hence, the initial conditions can be written as

$$
U_0(x) = U(x,0) = U(v_0, u_0, z_0, y_0, \theta_0, \eta_0)^T. \tag{2.8}
$$

Before closing this section, we introduce the following lemma, which will be used later in the proof of our main result.

Lemma 2.1 *For all $k \geq 0$, $c \geq 0$, there exists a constant $C > 0$ such that for all $t \geq 0$ the following estimate holds:*

$$
\int_{|\xi|\leq 1} |\xi|^k e^{-c|\xi|^6 t} d\xi \leq C(1+t)^{-(k+1)/6}, \qquad \xi \in \mathbb{R}. \tag{2.9}
$$

Proof First, observe that

$$
\int_{|\xi|\leq 1} |\xi|^k e^{-c|\xi|^6 t} d\xi = 2\int_0^1 r^k e^{-cr^6 t} dr.
$$

Thus, it is enough to prove that for given $c > 0$ and $k \geq 0$, we have

$$
\int_0^1 r^k e^{-cr^6 t} dr \leq C(1+t)^{-(k+1)/6}, \tag{2.10}
$$

for all $t \geq 0$, where C is a positive constant independent of t. To see this, observe first that, for $0 \leq t \leq 1$, the estimate (2.10) is obvious. On the other hand, for $t \geq 1$, we have

$$
(1+t) \leq 2t. \tag{2.11}
$$

Now, using (2.11) and the change of variables $z = cr^6 t$, we get

$$
\begin{aligned}
2^{-(k+1)/6} c^{(k+1)/6} (1+t)^{(k+1)/6} \int_0^1 r^k e^{-cr^6 t} dr &\le c^{(k+1)/6} t^{(k+1)/6} \int_0^1 r^k e^{-cr^6 t} dr \\
&= \int_0^1 (cr^6 t)^{k/6} e^{-cr^6 t} (ct)^{1/6} dr \\
&= \frac{1}{6} \int_0^{ct} (z)^{k/6} e^{-z} z^{-5/6} dz \\
&\le \frac{1}{6} \int_0^{\infty} z^{(k+1)/6-1} e^{-z} dz \\
&= \frac{1}{6} \Gamma\left(\frac{k+1}{6}\right) < \infty,
\end{aligned}
$$

where Γ is the Gamma function. This yields (2.9). This completes the proof of Lemma 2.1. □

3 The Energy Method in the Fourier Space

Our goal in this section is to obtain some decay estimates of the Fourier image of the energy of (2.6). To achieve this, we use the energy method in the Fourier space and build some appropriate Lyapunov functionals, which lead eventually to our desired estimates.

Applying the Fourier transform to (2.6), we get

$$\hat{v}_t - i\xi \hat{u} + \hat{y} = 0, \tag{3.1}$$

$$\hat{u}_t - i\xi \hat{v} = 0, \tag{3.2}$$

$$\hat{z}_t - ai\xi \hat{y} = 0, \tag{3.3}$$

$$\hat{y}_t - ai\xi \hat{z} - \hat{v} + \delta i\xi \hat{\theta} = 0, \tag{3.4}$$

$$\hat{\theta}_t + \frac{\xi^2}{\beta} \int_0^{\infty} \mu(s)\hat{\eta}(s) ds + \delta i\xi \hat{y} = 0, \tag{3.5}$$

$$\hat{\eta}_t = T\hat{\eta} + \hat{\theta}. \tag{3.6}$$

Together with the initial data, written in terms of the solution vector $\hat{U}(\xi, t) = (\hat{v}, \hat{u}, \hat{z}, \hat{y}, \hat{\theta}, \hat{\eta})^T(\xi, t)$, as

$$\hat{U}(\xi, 0) = \hat{U}_0(\xi). \tag{3.7}$$

The energy functional $\hat{E}(\xi, t)$ associated to the system (3.1)–(3.7) is defined as follows:

$$\hat{E}(\xi, t) = \frac{1}{2}\left\{ |\hat{v}|^2 + |\hat{u}|^2 + |\hat{z}|^2 + |\hat{y}|^2 + |\hat{\theta}|^2 + \frac{\xi^2}{\beta} \int_0^{\infty} \mu(s) |\hat{\eta}(t,s)|^2 \, ds \right\}. \tag{3.8}$$

Lemma 3.1 *Let* $(\hat{v}, \hat{u}, \hat{z}, \hat{y}, \hat{\theta}, \hat{\eta})$ *be the solution of* (3.1)–(3.7)*, then the energy* $\hat{E}(\xi, t)$ *given by (3.8) is a nonincreasing function and satisfies, for all* $t \geq 0$,

$$\frac{d}{dt}\hat{E}(\xi,t) = \frac{\xi^2}{2\beta}\int_0^\infty \mu'(s)|\hat{\eta}(t,s)|^2 ds. \tag{3.9}$$

Proof Multiplying Eq. (3.1) by $\bar{\hat{v}}$, Eq. (3.1) by $\bar{\hat{u}}$, Eq. (3.2) by $\bar{\hat{z}}$, Eq. (3.3) by $\bar{\hat{y}}$ and Eq. (3.4) by $\bar{\hat{\theta}}$ adding the results and taking the real part, we get

$$\frac{1}{2}\frac{d}{dt}\left(|\hat{v}|^2 + |\hat{u}|^2 + |\hat{z}|^2 + |\hat{y}|^2 + |\hat{\theta}|^2\right) = -\operatorname{Re}\left\{\frac{\xi^2}{\beta}\bar{\hat{\theta}}(t,\xi)\int_0^\infty \mu(s)\hat{\eta}(t,s)ds\right\}. \tag{3.10}$$

Taking the conjugate of Eq. (3.5), then multiplying the resulting equation by $\mu(s)\hat{\eta}(\xi, t, s)$ and taking the integration with respect to s, we obtain

$$\begin{aligned}&\int_0^\infty \mu(s)\hat{\eta}(\xi,t,s)\bar{\hat{\eta}}_t(\xi,t,s)ds\\ &= -\int_0^\infty \mu(s)\hat{\eta}(\xi,t,s)\bar{\hat{\eta}}_s(\xi,t,s)\,ds + \int_0^\infty \mu(s)\hat{\eta}(\xi,t,s)\bar{\hat{\theta}}(\xi,t)\,ds.\end{aligned}$$

Hence, we have

$$\begin{aligned}-\operatorname{Re}&\left\{\frac{\xi^2}{\beta}\int_0^\infty \mu(s)\hat{\eta}(\xi,t,s)\bar{\hat{\theta}}(\xi,t)\,ds\right\}\\ &= -\operatorname{Re}\left\{\frac{\xi^2}{\beta}\int_0^\infty \mu(s)\hat{\eta}(\xi,t,s)\bar{\hat{\eta}}_t(\xi,t,s)ds\right\}\\ &\quad -\operatorname{Re}\left\{\frac{\xi^2}{\beta}\int_0^\infty \mu(s)\hat{\eta}(\xi,t,s)\bar{\hat{\eta}}_s(\xi,t,s)\,ds\right\}\\ &= -\frac{1}{2}\frac{d}{dt}\left\{\frac{\xi^2}{\beta}\int_0^\infty \mu(s)|\hat{\eta}(\xi,t,s)|^2\,ds\right\}\\ &\quad -\operatorname{Re}\left\{\frac{\xi^2}{\beta}\int_0^\infty \mu(s)\hat{\eta}(\xi,t,s)\bar{\hat{\eta}}_s(\xi,t,s)\,ds\right\}.\end{aligned} \tag{3.11}$$

Integrating the second term in the right-hand side of (3.11) by parts and using the assumption (M1) and (2.2), we have

$$-\frac{\xi^2}{\beta}\operatorname{Re}\left\{\int_0^\infty \mu(s)\bar{\hat{\eta}}_s(\xi,t,s)\hat{\eta}(\xi,t,s)ds\right\} = \frac{\xi^2}{2\beta}\int_0^\infty \mu'(s)|\hat{\eta}(\xi,t,s)|^2 ds.$$

Hence, collecting (3.10) and (3.11), then (3.9) holds. □

Now, let

$$g(0) = \int_0^\infty \mu(s)\, ds. \tag{3.12}$$

Following [9], we define the functional

$$\begin{aligned} G(\xi,t) = \mathrm{Re}\left\{ -\frac{\beta}{g(0)}\hat{v}\bar{\hat{y}} - \frac{\beta}{g(0)}a\hat{u}\bar{\hat{z}} + \left(\frac{1}{\delta^2} - \frac{a^2}{\delta^2} + \frac{\beta}{g(0)}\right)\delta\bar{\hat{\theta}}\hat{u}\right\} \\ +\frac{1-a^2}{\delta g(0)}\,\mathrm{Re}\left(i\xi\int_0^\infty \mu(s)\hat{\eta}(s)\bar{\hat{v}}\, ds\right). \end{aligned} \tag{3.13}$$

Then, we have the following lemma.

Lemma 3.2 *The functional $G(\xi,t)$ satisfies*

$$\begin{aligned} &\frac{d}{dt}G(\xi,t) - \frac{\beta}{g(0)}|\hat{y}|^2 + \frac{\beta}{g(0)}|\hat{v}|^2 \\ &= \frac{\alpha_g}{\delta\beta}\,\mathrm{Re}\left(\xi^2\bar{\hat{u}}\int_0^\infty \mu(s)\hat{\eta}(s)ds\right) + \alpha_g\,\mathrm{Re}(i\xi\bar{\hat{u}}\hat{y}) \\ &\quad + \frac{1-a^2}{\delta g(0)}\,\mathrm{Re}\left(i\xi\int_0^\infty \mu(s)\hat{y}\bar{\hat{\eta}}(s)\, ds\right) + \frac{1-a^2}{\delta g(0)}\,\mathrm{Re}\left(i\xi\int_0^\infty \mu'(s)\hat{\eta}(s)\bar{\hat{v}}\, ds\right), \end{aligned} \tag{3.14}$$

where

$$\alpha_g := \left(\frac{\beta}{g(0)} - 1\right)(1-a^2) - \delta^2\frac{\beta}{g(0)}. \tag{3.15}$$

Proof Multiplying Eq. (3.1) by $-\bar{\hat{y}}$ and Eq. (3.3) by $-\bar{\hat{v}}$, adding the results and taking the real part, we have

$$-\frac{d}{dt}\mathrm{Re}(\hat{v}\bar{\hat{y}}) - |\hat{y}|^2 + |\hat{v}|^2 = -\mathrm{Re}(i\xi\hat{u}\bar{\hat{y}}) - \mathrm{Re}(ai\xi\bar{\hat{v}}\hat{z}) + \mathrm{Re}(\delta i\xi\hat{\theta}\bar{\hat{v}}). \tag{3.16}$$

Multiplying Eq. (3.1) by $-a\bar{\hat{z}}$ and Eq. (3.2) by $-a\bar{\hat{u}}$, adding the results and taking the real part, we find

$$-\frac{d}{dt}\mathrm{Re}(a\hat{u}\bar{\hat{z}}) = -\mathrm{Re}(ai\xi\hat{v}\bar{\hat{z}}) - \mathrm{Re}(a^2 i\xi\bar{\hat{u}}\hat{y}). \tag{3.17}$$

Multiplying Eq. (3.1) by $\delta\bar{\hat{\theta}}$ and Eq. (3.4) by $\delta\bar{\hat{u}}$, adding the results and taking the real part, we obtain

$$\frac{d}{dt}\operatorname{Re}(\delta\bar{\hat{\theta}}\hat{u}) = \operatorname{Re}(\delta i\xi\bar{\hat{\theta}}\hat{v}) - \operatorname{Re}\left(\frac{\delta\xi^2}{\beta}\bar{\hat{u}}\int_0^\infty \mu(s)\hat{\eta}(s)ds\right) - \operatorname{Re}(\delta^2 i\xi\bar{\hat{u}}\hat{y}). \tag{3.18}$$

Now, computing $\frac{\beta}{g(0)}$(3.16)+$\frac{\beta}{g(0)}$(3.17)+$(\frac{1}{\delta^2} - \frac{a^2}{\delta^2} + \frac{\beta}{g(0)})$(3.18), we obtain (after collecting the similar terms)

$$\begin{aligned}
&\frac{d}{dt}F(\xi,t) - \frac{\beta}{g(0)}|\hat{y}|^2 + \frac{\beta}{g(0)}|\hat{v}|^2 \\
&= -\left(\frac{1}{\delta^2} - \frac{a^2}{\delta^2} + \frac{\beta}{g(0)}\right)\operatorname{Re}\left(\frac{\delta\xi^2}{\beta}\bar{\hat{u}}\int_0^\infty \mu(s)\hat{\eta}(s)ds\right) \\
&\quad + \alpha_g \operatorname{Re}(i\xi\bar{\hat{u}}\hat{y}) + \frac{1-a^2}{\delta}\operatorname{Re}(i\xi\bar{\hat{\theta}}\hat{v}),
\end{aligned} \tag{3.19}$$

where

$$F(\xi,t) = \operatorname{Re}\left\{-\frac{\beta}{g(0)}\hat{v}\bar{\hat{y}} - \frac{\beta}{g(0)}a\hat{u}\bar{\hat{z}} + \left(\frac{1}{\delta^2} - \frac{a^2}{\delta^2} + \frac{\beta}{g(0)}\right)\delta\bar{\hat{\theta}}\hat{u}\right\}$$

and

$$\begin{aligned}
\alpha_g &= \left\{\frac{\beta}{g(0)} - \frac{\beta}{g(0)}a^2 - \delta^2\left(\frac{1}{\delta^2} - \frac{a^2}{\delta^2} + \frac{\beta}{g(0)}\right)\right\} \\
&= \left(\frac{\beta}{g(0)} - 1\right)(1-a^2) - \delta^2\frac{\beta}{g(0)}.
\end{aligned}$$

Multiplying Eq. (3.1) by $-i\xi\mu(s)\bar{\hat{\eta}}(s)$ and Eq. (3.5) by $i\xi\mu(s)\bar{\hat{v}}$, adding the results, taking the integration with respect to s and taking the real part, we have

$$\begin{aligned}
&\frac{d}{dt}\operatorname{Re}\left(i\xi\int_0^\infty \mu(s)\hat{\eta}(s)\bar{\hat{v}}\,ds\right) \\
&= \operatorname{Re}\left(\xi^2\int_0^\infty \mu(s)\hat{u}\bar{\hat{\eta}}(s)\,ds\right) + \operatorname{Re}\left(i\xi\int_0^\infty \mu(s)\hat{y}\bar{\hat{\eta}}(s)\,ds\right) \\
&\quad + \operatorname{Re}\left(i\xi\int_0^\infty \mu(s)T\hat{\eta}\bar{\hat{v}}\,ds\right) + \operatorname{Re}\left(i\xi\int_0^\infty \mu(s)\hat{\theta}\bar{\hat{v}}\,ds\right).
\end{aligned} \tag{3.20}$$

Computing (3.19)+$\frac{1-a^2}{\delta g(0)}$ (3.20), we have

$$\begin{aligned}
&\frac{d}{dt}G(\xi,t)-\frac{\beta}{g(0)}|\hat{y}|^2+\frac{\beta}{g(0)}|\hat{v}|^2\\
&=-\frac{\delta}{\beta}\left(\frac{1}{\delta^2}-\frac{a^2}{\delta^2}+\frac{\beta}{g(0)}\right)\mathrm{Re}\left(\xi^2\bar{\hat{u}}\int_0^\infty \mu(s)\hat{\eta}(s)ds\right)+\frac{1-a^2}{\delta}\mathrm{Re}(i\xi\bar{\hat{\theta}}\hat{v})\\
&\quad+\alpha_g\,\mathrm{Re}(i\xi\bar{\hat{u}}\hat{y})+\frac{1-a^2}{\delta g(0)}\mathrm{Re}\left(\xi^2\bar{\hat{u}}\int_0^\infty \mu(s)\hat{\eta}(s)\,ds\right)\\
&\quad+\frac{1-a^2}{\delta g(0)}\mathrm{Re}\left(i\xi\int_0^\infty \mu(s)\hat{y}\bar{\hat{\eta}}(s)\,ds\right)+\frac{1-a^2}{\delta g(0)}\mathrm{Re}\left(i\xi\int_0^\infty \mu(s)T\hat{\eta}\bar{\hat{v}}\,ds\right)\\
&\quad-\frac{1-a^2}{\delta g(0)}\mathrm{Re}\left(i\xi\int_0^\infty \mu(s)\bar{\hat{\theta}}\hat{v}\,ds\right).
\end{aligned}\tag{3.21}$$

Using (3.12), the last term of (3.21) can be written as

$$\begin{aligned}
-\frac{1-a^2}{\delta g(0)}\mathrm{Re}\left(i\xi\int_0^\infty \mu(s)\bar{\hat{\theta}}\hat{v}\,ds\right)&=-\frac{1-a^2}{\delta g(0)}\mathrm{Re}\left(i\xi\bar{\hat{\theta}}\hat{v}\int_0^\infty \mu(s)\,ds\right)\\
&=-\frac{1-a^2}{\delta}\mathrm{Re}(i\xi\bar{\hat{\theta}}\hat{v}).
\end{aligned}$$

While, integration by parts leads to

$$\frac{1-a^2}{\delta g(0)}\mathrm{Re}\left(i\xi\int_0^\infty \mu(s)T\hat{\eta}\bar{\hat{v}}\,ds\right)=\frac{1-a^2}{\delta g(0)}\mathrm{Re}\left(i\xi\int_0^\infty \mu'(s)\hat{\eta}(s)\bar{\hat{v}}\,ds\right).$$

Plugging the above estimates into (3.21), hence, Eq. (3.14) holds and this ends the proof of Lemma 3.2. □

We define the functional:

$$K(\xi,t)=\mathrm{Re}\left\{i\xi\bar{\hat{v}}\hat{u}+ai\xi\hat{y}\bar{\hat{z}}\right\}.\tag{3.22}$$

Thus, we have the following estimate.

Lemma 3.3 *For any* $\epsilon_1,\ \epsilon_2>0$, *the estimate*

$$\begin{aligned}
&\frac{d}{dt}K(\xi,t)+\xi^2(1-\epsilon_1)|\hat{u}|^2+\xi^2(a^2-\epsilon_2)|\hat{z}|^2\\
&\le C(\epsilon_1)(1+\xi^2)|\hat{y}|^2+C(\epsilon_2)(1+\xi^2)|\hat{v}|^2+C(\epsilon_2)\xi^2|\hat{\theta}|^2,
\end{aligned}\tag{3.23}$$

is satisfied.

Proof Multiplying Eq. (3.1) by $i\xi\bar{\hat{u}}$ and Eq. (3.1) by $-i\xi\bar{\hat{v}}$, adding the results and taking the real part, we obtain

$$\frac{d}{dt}\,\mathrm{Re}(i\xi\bar{\hat{v}}\hat{u})+\xi^2|\hat{u}|^2-\xi^2|\hat{v}|^2=-\,\mathrm{Re}(i\xi\bar{\hat{u}}\hat{y}).$$

Applying Young's inequality, we have, for any $\epsilon_1>0$,

$$|\,\mathrm{Re}(i\xi\bar{\hat{u}}\hat{y})|\le\epsilon_1\xi^2|\hat{u}|^2+C(\epsilon_1)|\hat{y}|^2.$$

Hence,

$$\frac{d}{dt}\,\mathrm{Re}(i\xi\bar{\hat{v}}\hat{u})+(1-\epsilon_1)\xi^2|\hat{u}|^2\le\xi^2|\hat{v}|^2+C(\epsilon_1)|\hat{y}|^2. \tag{3.24}$$

Multiplying Eq. (3.2) by $-ai\xi\bar{\hat{y}}$ and Eq. (3.3) by $ai\xi\bar{\hat{z}}$, adding the results and taking the real part, we obtain

$$\frac{d}{dt}\,\mathrm{Re}(ai\xi\hat{y}\bar{\hat{z}})-a^2\xi^2|\hat{y}|^2+a^2\xi^2|\hat{z}|^2=\mathrm{Re}(ai\xi\hat{v}\bar{\hat{z}})+\mathrm{Re}(a\xi^2\delta\hat{\theta}\bar{\hat{z}}).$$

Applying Young's inequality, we have, for any $\epsilon_2>0$,

$$|\,\mathrm{Re}(ai\xi\hat{v}\bar{\hat{z}})|\le\frac{\epsilon_2}{2}\xi^2|\hat{z}|^2+C(\epsilon_2)|\hat{v}|^2$$

and

$$|\,\mathrm{Re}(a\xi^2\delta\hat{\theta}\bar{\hat{z}})|\le\frac{\epsilon_2}{2}\xi^2|\hat{z}|^2+C(\epsilon_2)\xi^2|\hat{\theta}|^2.$$

Consequently, we have from above

$$\frac{d}{dt}\,\mathrm{Re}(ai\xi\hat{y}\bar{\hat{z}})+(a^2-\epsilon_2)\xi^2|\hat{z}|^2\le a^2\xi^2|\hat{y}|^2+C(\epsilon_2)|\hat{v}|^2+C(\epsilon_2)\xi^2|\hat{\theta}|^2. \tag{3.25}$$

Summing up Eqs. (3.24) and (3.25), then (3.23) holds. This ends the proof of Lemma 3.3.

Lemma 3.4 *The following inequality holds true:*

$$\begin{aligned}\frac{d}{dt}\,\mathrm{Re}(i\xi\hat{y}\bar{\hat{\theta}})+(\delta-\epsilon_4)\xi^2|\hat{y}|^2&\le C(\epsilon_4)\frac{\xi^4g(0)}{\beta^2}\int_0^\infty\mu(s)|\hat{\eta}(s,t)|^2ds\\&\quad+C(\epsilon_3)(1+\xi^2)|\hat{\theta}|^2+\epsilon_3\frac{\xi^4}{1+\xi^2}|\hat{z}|^2+\epsilon_3\xi^2|\hat{v}|^2,\end{aligned} \tag{3.26}$$

where ϵ_3 and ϵ_4 are arbitrary positive constants.

Proof Multiplying Eq. (3.3) by $i\xi\bar{\hat{\theta}}$ and Eq. (3.4) by $-i\xi\bar{\hat{y}}$, adding the results and taking the real parts, we have

$$\begin{aligned}&\frac{d}{dt}\operatorname{Re}(i\xi\hat{y}\bar{\hat{\theta}})+\delta\xi^2(|\hat{y}|^2-|\hat{\theta}|^2)-\frac{1}{\beta}\operatorname{Re}\left(i\xi^3\bar{\hat{y}}\int_0^\infty\mu(s)\hat{\eta}(s)ds\right)\\&=-\operatorname{Re}(a\xi^2\bar{\hat{\theta}}\hat{z})+\operatorname{Re}(i\xi\bar{\hat{\theta}}\hat{v}).\end{aligned}\tag{3.27}$$

Applying Young's inequality, we have, for any $\epsilon_3,\epsilon_4>0$,

$$|\operatorname{Re}(a\xi^2\bar{\hat{\theta}}\hat{z})|\le\epsilon_3\frac{\xi^4}{1+\xi^2}|\hat{z}|^2+C(\epsilon_3)(1+\xi^2)|\hat{\theta}|^2,\tag{3.28}$$

$$|\operatorname{Re}(i\xi\bar{\hat{\theta}}\hat{v})|\le\epsilon_3\xi^2|\hat{v}|^2+C(\epsilon_3)|\hat{\theta}|^2\tag{3.29}$$

and

$$\left|\operatorname{Re}\left(i\frac{\xi^3}{\beta}\bar{\hat{y}}\int_0^\infty\mu(s)\hat{\eta}(s)ds\right)\right|\le\epsilon_4\xi^2|\hat{y}|^2+C(\epsilon_4)\frac{\xi^4}{\beta^2}\left|\int_0^\infty\mu(s)\hat{\eta}(s)ds\right|^2.$$

Now, using (3.12) together with the following inequality

$$\begin{aligned}\left|\int_0^\infty\mu(s)\hat{\eta}(s)ds\right|^2&=\left|\int_0^\infty(\mu(s))^{\frac12}(\mu(s))^{\frac12}\hat{\eta}(s)\,ds\right|^2\\&\le\left|\left(\int_0^\infty\mu(s)ds\right)^{\frac12}\left(\int_0^\infty\mu(s)(\hat{\eta}(s))^2ds\right)^{\frac12}\right|^2\\&=\left(\int_0^\infty\mu(s)ds\right)\int_0^\infty\mu(s)|\hat{\eta}(s,t)|^2ds\\&=g(0)\int_0^\infty\mu(s)|\hat{\eta}(s,t)|^2ds,\end{aligned}$$

we obtain

$$\left|\operatorname{Re}\left(i\frac{\xi^3}{\beta}\bar{\hat{y}}\int_0^\infty\mu(s)\hat{\eta}(s)ds\right)\right|\le\epsilon_4\xi^2|\hat{y}|^2+C(\epsilon_4)\frac{\xi^4g(0)}{\beta^2}\int_0^\infty\mu(s)|\hat{\eta}(s,t)|^2ds.$$

Hence, plugging this last estimate together with (3.28) and (3.29) into (3.27), we obtain (3.26). This completes the proof of Lemma 3.4.

Lemma 3.5 *The following inequality holds true:*

$$
\begin{aligned}
&-\frac{d}{dt}\operatorname{Re}\left(\xi^2\int_0^\infty \mu(s)\bar{\hat{\eta}}(s)\hat{\theta}\,ds\right)+(g(0)-\epsilon_6)\xi^2|\hat{\theta}|^2\\
&\le (C(\epsilon_5)+\frac{1}{\beta})g(0)\xi^2(1+\xi^2)\int_0^\infty \mu(s)|\hat{\eta}(s,t)|^2\,ds\\
&+\epsilon_5\tfrac{\xi^4}{1+\xi^2}|\hat{y}|^2+C(\epsilon_6)g'(0)\xi^2\int_0^\infty \mu'(s)|\hat{\eta}(s,t)|^2\,ds,
\end{aligned}\tag{3.30}
$$

where ϵ_5 and ϵ_6 are arbitrary positive constants.

Proof Multiplying equation (3.5) by $-\xi^2\mu(s)\bar{\hat{\theta}}$ and Eq. (3.4) by $-\xi^2\mu(s)\bar{\hat{\eta}}(s)$, adding the results and taking the integration with respect to s for the real parts, we have

$$
\begin{aligned}
&-\frac{d}{dt}\operatorname{Re}\left(\xi^2\int_0^\infty \mu(s)\bar{\hat{\eta}}(s)\hat{\theta}\,ds\right)+\xi^2\int_0^\infty \mu(s)|\hat{\theta}|^2\,ds\\
&\quad=\operatorname{Re}\left(\frac{\xi^4}{\beta}\int_0^\infty \mu(s)\bar{\hat{\eta}}(s)\int_0^\infty \mu(s)\hat{\eta}(s)\,ds\,ds\right)-\operatorname{Re}\left(\xi^2\int_0^\infty \mu(s)\bar{\hat{\theta}}T\hat{\eta}\,ds\right)\\
&\quad+\operatorname{Re}\left(\delta i\xi^3\int_0^\infty \mu(s)\bar{\hat{\eta}}(s)\hat{y}\,ds\right).
\end{aligned}
$$

Now, integrating by parts the second term on the right-hand side of the above equality, we get

$$
\begin{aligned}
&-\frac{d}{dt}\operatorname{Re}\left(\xi^2\int_0^\infty \mu(s)\bar{\hat{\eta}}(s)\hat{\theta}\,ds\right)+\xi^2 g(0)|\hat{\theta}|^2\\
&\quad=\operatorname{Re}\left(\frac{\xi^4}{\beta}\left(\int_0^\infty \mu(s)\hat{\eta}(s)\,ds\right)^2\right)-\operatorname{Re}\left(\xi^2\int_0^\infty \mu'(s)\hat{\eta}(s)\bar{\hat{\theta}}\,ds\right)\\
&\quad+\operatorname{Re}\left(\delta i\xi^3\int_0^\infty \mu(s)\bar{\hat{\eta}}(s)\hat{y}\,ds\right).
\end{aligned}\tag{3.31}
$$

Applying Young's inequality, we have, for any $\epsilon_5,\ \epsilon_6>0$,

$$
\begin{aligned}
&\left|\operatorname{Re}\left(\delta i\xi^3\int_0^\infty \mu(s)\bar{\hat{\eta}}(s)\hat{y}\,ds\right)\right|\le C(\epsilon_5)\xi^2(1+\xi^2)g(0)\\
&\qquad\int_0^\infty \mu(s)|\hat{\eta}(s,t)|^2\,ds+\epsilon_5\frac{\xi^4}{1+\xi^2}|\hat{y}|^2\\
&\qquad\left|\operatorname{Re}\left(\xi^2\int_0^\infty \mu'(s)\hat{\eta}(s)\bar{\hat{\theta}}\,ds\right)\right|\le \epsilon_6\xi^2|\hat{\theta}|^2+C(\epsilon_6)\xi^2 g'(0)\\
&\qquad\int_0^\infty \mu'(s)|\hat{\eta}(s,t)|^2\,ds
\end{aligned}
$$

and

$$\left|\operatorname{Re}\left(\frac{\xi^4}{\beta}\left(\int_0^\infty \mu(s)\hat{\eta}(s)\,ds\right)^2\right)\right| \le \frac{\xi^4 g(0)}{\beta}\int_0^\infty \mu(s)|\hat{\eta}(s,t)|^2 ds.$$

Hence, inserting the above estimates into (3.31), then (3.30) is fulfilled. Thus, the proof of Lemma 3.5 is finished. □

Proposition 3.6 *Let $\hat{U}(\xi,t) = (\hat{v},\hat{u},\hat{z},\hat{y},\hat{\theta},\hat{\eta})$ be the solution of (3.1)–(3.6) and*

$$\alpha_g = \left(\frac{\beta}{g(0)} - 1\right)(1-a^2) - \frac{\beta}{g(0)}\delta^2. \tag{3.32}$$

Then, there exist two positive constants, C and c, such that for all $t \ge 0$:

$$\hat{E}(\xi,t) \le C\hat{E}(\xi,0)e^{-c\rho(\xi)t}, \tag{3.33}$$

where

$$\rho(\xi) = \begin{cases} \dfrac{\xi^6}{(1+\xi^2)^3}, & \text{if} \quad \alpha_g = 0, \\ \dfrac{\xi^6}{(1+\xi^2)^6}, & \text{if} \quad \alpha_g \neq 0. \end{cases} \tag{3.34}$$

Proof **Case one:** $\alpha_g = 0$. Substituting $\alpha_g = 0$ into Eq. (3.14), we obtain

$$\begin{aligned} &\frac{d}{dt}G(\xi,t) - \frac{\beta}{g(0)}|\hat{y}|^2 + \frac{\beta}{g(0)}|\hat{v}|^2 \\ &= \frac{1-a^2}{\delta g(0)}\operatorname{Re}\left(i\xi\int_0^\infty \mu(s)\bar{\hat{\eta}}(s)\hat{y}\,ds\right) + \frac{1-a^2}{\delta g(0)}\operatorname{Re}\left(i\xi\int_0^\infty \mu'(s)\hat{\eta}(s)\bar{\hat{v}}\,ds\right). \end{aligned} \tag{3.35}$$

Now, applying Young's inequality, we find, for any $\epsilon_0 > 0$,

$$\begin{aligned} \left|\operatorname{Re}\left(i\xi\int_0^\infty \mu(s)\bar{\hat{\eta}}(s)\hat{y}\,ds\right)\right| &\le \epsilon_0|\hat{y}|^2 + C(\epsilon_0)\xi^2\left|\int_0^\infty \mu(s)\hat{\eta}(s)\,ds\right|^2 \\ &\le \epsilon_0|\hat{y}|^2 + C(\epsilon_0)\xi^2 g(0)\int_0^\infty \mu(s)|\hat{\eta}(s,t)|^2\,ds \end{aligned}$$

and

$$\begin{aligned} \left|\operatorname{Re}\left(i\xi\int_0^\infty \mu'(s)\hat{\eta}(s)\bar{\hat{v}}\,ds\right)\right| &\le \epsilon_0|\hat{v}|^2 + C(\epsilon_0)\xi^2\left|\int_0^\infty \mu'(s)\hat{\eta}(s)\,ds\right|^2 \\ &\le \epsilon_0|\hat{v}|^2 + C(\epsilon_0)\xi^2 g'(0)\int_0^\infty \mu'(s)|\hat{\eta}(s,t)|^2\,ds. \end{aligned}$$

Hence, taking the above estimates into account, then (3.35) can be written as:

$$\frac{d}{dt}G(\xi,t)+\left(\frac{\beta}{g(0)}-\epsilon_0\right)|\hat{v}|^2 \leq C(\epsilon_0)\left(|\hat{y}|^2-\int_0^\infty \xi^2\mu'(s)|\hat{\eta}(s,t)|^2\,ds+\int_0^\infty \xi^2\mu(s)|\hat{\eta}(s,t)|^2\,ds\right). \tag{3.36}$$

Define the functional

$$L_1(\xi,t)=\frac{\xi^2}{1+\xi^2}\left[\gamma_1\xi^2 G(\xi,t)+\gamma_2\frac{\xi^2}{1+\xi^2}K(\xi,t)+\gamma_3\,\mathrm{Re}(i\xi\hat{y}\bar{\hat{\theta}})\right] -\gamma_4\,\mathrm{Re}\left(\int_0^\infty \xi^2\mu(s)\bar{\hat{\eta}}(s)\hat{\theta}\,ds\right), \tag{3.37}$$

where $\gamma_1,\ \gamma_2,\ \gamma_3$ and γ_4 are positive constants to be fixed later. Taking the derivative of $L_1(\xi,t)$ with respect to t and using the estimates (3.9), (3.23), (3.26), (3.30) and (3.36), we have

$$\begin{aligned}
&\frac{d}{dt}L_1(\xi,t)+\left[(\delta-\epsilon_4)\gamma_3-C(\epsilon_0)\gamma_1-C(\epsilon_1)\gamma_2-\epsilon_5\gamma_4\right]\frac{\xi^4}{(1+\xi^2)}|\hat{y}|^2\\
&-\left[C(\epsilon_0)\gamma_1+\gamma_3\frac{g(0)}{\beta^2}C(\epsilon_4)+(C(\epsilon_5)+\frac{1}{\beta})g(0)\gamma_4\right](1+\xi^2)\int_0^\infty \xi^2\mu(s)|\hat{\eta}(s,t)|^2\,ds\\
&+\left[\left(\frac{\beta}{g(0)}-\epsilon_0\right)\gamma_1-C(\epsilon_2)\gamma_2-\epsilon_3\gamma_3\right]\frac{\xi^4}{(1+\xi^2)}|\hat{v}|^2\\
&+\left[(g(0)-\epsilon_6)\gamma_4-C(\epsilon_2)\gamma_2-C(\epsilon_3)\gamma_3\right]\xi^2|\hat{\theta}|^2\\
&+\gamma_2(1-\epsilon_1)\frac{\xi^6}{(1+\xi^2)^2}|\hat{u}|^2+\left[(a^2-\epsilon_2)\gamma_2-\epsilon_3\gamma_3\right]\frac{\xi^6}{(1+\xi^2)^2}|\hat{z}|^2\\
&-\left[C(\epsilon_0)\gamma_1\frac{\xi^4}{1+\xi^2}+C(\epsilon_6)\gamma_4 g'(0)\right]\int_0^\infty(-\mu'(s))\xi^2|\hat{\eta}(s,t)|^2\,ds\\
&\leq\ 0.
\end{aligned} \tag{3.38}$$

In the above estimate, we have used some trivial inequalities, such as $\xi^4/(1+\xi^2)\leq \xi^2,\ \xi^2/(1+\xi^2)\leq 1$ and so on.

Now, using the assumption (M2), we may write

$$\int_0^\infty \xi^2\mu(s)|\hat{\eta}(s,t)|^2\,ds\leq\frac{1}{\nu}\int_0^\infty(-\mu'(s))\xi^2|\hat{\eta}(s,t)|^2\,ds. \tag{3.39}$$

Consequently, taking into account the above estimate, we may rewrite (3.38) as:

$$
\begin{aligned}
&\frac{d}{dt}L_1(\xi,t)+\left[(\delta-\epsilon_4)\gamma_3-C(\epsilon_0)\gamma_1-C(\epsilon_1)\gamma_2-\epsilon_5\gamma_4\right]\frac{\xi^4}{(1+\xi^2)}|\hat{y}|^2\\
&+\left[\left(\frac{\beta}{g(0)}-\epsilon_0\right)\gamma_1-C(\epsilon_2)\gamma_2-\epsilon_3\gamma_3\right]\frac{\xi^4}{(1+\xi^2)}|\hat{v}|^2\\
&+\left[(g(0)-\epsilon_6)\gamma_4-C(\epsilon_2)\gamma_2-C(\epsilon_3)\gamma_3\right]\xi^2|\hat{\theta}|^2\\
&+\gamma_2(1-\epsilon_1)\frac{\xi^6}{(1+\xi^2)^2}|\hat{u}|^2+\left[(a^2-\epsilon_2)\gamma_2-\epsilon_3\gamma_3\right]\frac{\xi^6}{(1+\xi^2)^2}|\hat{z}|^2\\
&-C_1(1+\xi^2)\int_0^\infty(-\mu'(s))\xi^2|\hat{\eta}(s,t)|^2\,ds\\
&\le 0, \qquad (3.40)
\end{aligned}
$$

where C_1 is a generic positive constant that depends on $\epsilon_i,\ \gamma_j$ and ν, yet is independent on t and ξ.

Now, we choose the constants in (3.40) very carefully in order to make all the coefficients (except the last one) in (3.40) positive. Let us fix $\epsilon_0,\ \epsilon_1,\ \epsilon_2,\ \epsilon_4$, and ϵ_6 small enough such that

$$
\epsilon_0<\frac{\beta}{g(0)},\qquad \epsilon_1<1,\qquad \epsilon_2<a^2,\qquad \epsilon_4<\delta,\qquad \epsilon_6<g(0).
$$

We take $\gamma_2=1$ and choose γ_1 large enough such that

$$
\gamma_1\left(\frac{\beta}{g(0)}-\epsilon_0\right)-C(\epsilon_2)>0.
$$

Next, we choose γ_3 large enough such that

$$
-\gamma_1C(\epsilon_0)-C(\epsilon_1)+\gamma_3(\delta-\epsilon_4)>0.
$$

Then, after fixing $\gamma_1,\ \gamma_3,\ \epsilon_0$ and ϵ_2, we choose ϵ_3 small enough such that

$$
\epsilon_3<\min\left\{\frac{\gamma_1(\frac{\beta}{g(0)}-\epsilon_0)-C(\epsilon_2)}{\gamma_3},\frac{a^2-\epsilon_2}{\gamma_3}\right\}.
$$

Now, we choose γ_4 large enough such that

$$
(g(0)-\epsilon_6)\gamma_4-C(\epsilon_2)-\gamma_3C(\epsilon_3)>0.
$$

Finally, we fix ϵ_5 small enough such that

$$
\epsilon_5<\frac{-\gamma_1C(\epsilon_0)-C(\epsilon_1)+\gamma_3(\delta-\epsilon_4)}{\gamma_4}.
$$

Consequently, we deduce that there exists a positive constant $\eta_1 > 0$ such that

$$\frac{d}{dt}L_1(\xi,t) + \eta_1 Q_1(\xi,t) \leq C_1(1+\xi^2)\int_0^\infty (-\mu'(s))\xi^2|\hat{\eta}(s,t)|^2\,ds, \quad (3.41)$$

where

$$Q_1(\xi,t) = \tfrac{\xi^6}{(1+\xi^2)^2}\left(|\hat{u}|^2 + |\hat{z}|^2\right) + \xi^2|\hat{\theta}|^2 + \tfrac{\xi^4}{(1+\xi^2)}\left(|\hat{v}|^2 + |\hat{y}|^2\right). \quad (3.42)$$

It is straightforward to see that

$$Q_1(\xi,t) \geq \tfrac{\xi^6}{(1+\xi^2)^2}\left(|\hat{u}|^2 + |\hat{z}|^2 + |\hat{v}|^2 + |\hat{y}|^2 + |\hat{\theta}|^2\right). \quad (3.43)$$

Now, we define the Lyapunov functional

$$\mathscr{L}_1(\xi,t) = N(1+\xi^2)\hat{E}(\xi,t) + L_1(\xi,t), \quad (3.44)$$

where N is a large positive number that will be chosen later on. Using (3.9) and (3.41), the functional $\mathscr{L}_1(\xi,t)$ satisfies the estimate

$$\frac{d}{dt}\mathscr{L}_1(\xi,t) + \eta_1 Q_1(\xi,t) + \Big(\frac{N}{2\beta} - C_1\Big)(1+\xi^2)\int_0^\infty (-\mu'(s))\xi^2|\hat{\eta}(t,s)|^2\,ds \leq 0. \quad (3.45)$$

By choosing N large enough such that

$$N > 2\beta C_1,$$

and exploiting the estimate (3.39), we deduce from (3.8) and (3.43) that there exists a positive constant η_2 such that

$$\frac{d}{dt}\mathscr{L}_1(\xi,t) + \eta_2\frac{\xi^6}{(1+\xi^2)^2}\hat{E}(\xi,t) \leq 0, \qquad \forall t \geq 0. \quad (3.46)$$

Now, using (3.44) and (3.37), together with the definitions of all functionals involved in (3.37), we deduce that there exist two positive constants β_1 and β_2, such that, for all $t \geq 0$,

$$\beta_1(1+\xi^2)\hat{E}(\xi,t) \leq \mathscr{L}_1(\xi,t) \leq \beta_2(1+\xi^2)\hat{E}(\xi,t). \quad (3.47)$$

Combining (3.46) and (3.47), we find that for all $t \geq 0$, we have

$$\frac{d}{dt}\mathscr{L}_1(\xi,t) \leq -\frac{\eta_2}{\beta_2}\frac{\xi^6}{(1+\xi^2)^3}\mathscr{L}_1(\xi,t), \qquad \forall t \geq 0. \quad (3.48)$$

Applying Gronwall's lemma and using (3.47) once again, then (3.33) holds.

Case two: $\alpha_g \neq 0$. From Eq. (3.14), we estimate the following terms by applying Young's inequality, for any $\epsilon_0' > 0$,

$$|\alpha_g \operatorname{Re}(i\xi \bar{\hat{u}} \hat{y})| \leq \frac{\epsilon_0'}{2} \frac{\xi^2}{1+\xi^2} |\hat{u}|^2 + C(\epsilon_0')(1+\xi^2)|\hat{y}|^2,$$
$$\left| \frac{\alpha_g}{\delta\beta} \operatorname{Re}\left(\xi^2 \bar{\hat{u}} \int_0^\infty \mu(s)\hat{\eta}(s)\, ds \right) \right| \leq \frac{\epsilon_0'}{2} \frac{\xi^2}{1+\xi^2} |\hat{u}|^2 + C(\epsilon_0') g(0)(1+\xi^2) \int_0^\infty \xi^2 \mu(s) |\hat{\eta}(s,t)|^2\, ds.$$

Hence, taking the above estimates into account, (3.14) can be written as:

$$\frac{d}{dt} G(\xi,t) + \left(\frac{\beta}{g(0)} - \epsilon_0 \right) |\hat{v}|^2 \leq \epsilon_0' \frac{\xi^2}{1+\xi^2} |\hat{u}|^2 + C(\epsilon_0, \epsilon_0')(1+\xi^2)|\hat{y}|^2 + C(\epsilon_0, \epsilon_0')(1+\xi^2) g(0) \int_0^\infty \xi^2 \mu(s) |\hat{\eta}(s,t)|^2\, ds + C(\epsilon_0) g'(0) \int_0^\infty \xi^2 \mu'(s) |\hat{\eta}(s,t)|^2\, ds. \tag{3.49}$$

Also, we estimate Eq. (3.27) as follows

$$\frac{d}{dt} \operatorname{Re}(i\xi \hat{y} \bar{\hat{\theta}}) + (\delta - \epsilon_4)\xi^2 |\hat{y}|^2 \leq C(\epsilon_4) \frac{\xi^4 g(0)}{\beta^2} \int_0^\infty \mu(s) |\hat{\eta}(s,t)|^2\, ds + C(\epsilon_3)(1+\xi^2)^2 |\hat{\theta}|^2 + \epsilon_3 \frac{\xi^4}{(1+\xi^2)^2} |\hat{z}|^2 + \epsilon_3 \frac{\xi^2}{1+\xi^2} |\hat{v}|^2, \tag{3.50}$$

where we used the inequalities

$$|\operatorname{Re}(a\xi^2 \bar{\hat{\theta}} \hat{z})| \leq \epsilon_3 \frac{\xi^4}{(1+\xi^2)^2} |\hat{z}|^2 + C(\epsilon_3)(1+\xi^2)^2 |\hat{\theta}|^2,$$
$$|\operatorname{Re}(i\xi \bar{\hat{\theta}} \hat{v})| \leq \epsilon_3 \frac{\xi^2}{1+\xi^2} |\hat{v}|^2 + C(\epsilon_3)(1+\xi^2) |\hat{\theta}|^2.$$

On the other hand, we can estimate (3.31) as follows:

$$-\frac{d}{dt} \operatorname{Re}\left(\xi^2 \int_0^\infty \mu(s) \bar{\hat{\eta}}(s) \hat{\theta}\, ds \right) + (g(0) - \epsilon_6)\xi^2 |\hat{\theta}|^2 \leq \epsilon_5 \frac{\xi^4}{(1+\xi^2)^2} |\hat{y}|^2 + (C(\epsilon_5) + \frac{1}{\beta}) g(0)(1+\xi^2)^2 \int_0^\infty \xi^2 \mu(s) |\hat{\eta}(s,t)|^2\, ds + C(\epsilon_6) g'(0) \int_0^\infty \xi^2 \mu'(s) |\hat{\eta}(s,t)|^2\, ds, \tag{3.51}$$

where we made use of the estimate:

$$\left|\mathrm{Re}\left(\delta i\xi^3\int_0^\infty \mu(s)\bar{\hat{\eta}}(s)\hat{y}\right)ds\right| \le C(\epsilon_5)\xi^2(1+\xi^2)^2 g(0)\int_0^\infty \mu(s)|\hat{\eta}(s,t)|^2\,ds + \epsilon_5\frac{\xi^4}{(1+\xi^2)^2}|\hat{y}|^2.$$

Now, we define the functional

$$L_2(\xi,t) = \frac{\xi^2}{(1+\xi^2)^2}\left[\lambda_1\frac{\xi^2}{1+\xi^2}G(\xi,t) + \lambda_2\frac{\xi^2}{(1+\xi^2)^2}K(\xi,t) + \lambda_3\,\mathrm{Re}(i\xi\hat{y}\bar{\hat{\theta}})\right] - \lambda_4\,\mathrm{Re}\left(\xi^2\int_0^\infty \mu(s)\bar{\hat{\eta}}(s)\hat{\theta}\,ds\right), \tag{3.52}$$

where $\lambda_1,\ \lambda_2,\ \lambda_3$ and λ_4 are positive constants to be fixed later. Taking the derivative of $L_2(\xi,t)$ with respect to t and using the inequalities (3.23), (3.49), (3.50) and (3.51), we have

$$\begin{aligned}
&\frac{d}{dt}L_2(\xi,t) + \left[-C(\epsilon_0,\epsilon_0')\lambda_1 - C(\epsilon_1)\lambda_2 + (\delta-\epsilon_4)\lambda_3 - \epsilon_5\lambda_4\right]\frac{\xi^4}{(1+\xi^2)^2}|\hat{y}|^2\\
&-\left[C(\epsilon_0,\epsilon_0')\lambda_1 + C(\epsilon_4)\frac{\lambda_3}{\beta^2} + (C(\epsilon_5)+\frac{1}{\beta})\lambda_4\right]g(0)(1+\xi^2)^2\int_0^\infty \xi^2\mu(s)|\hat{\eta}(s,t)|^2\,ds\\
&+\left[-\epsilon_0'\lambda_1 + \lambda_2(1-\epsilon_1)\right]\frac{\xi^6}{(1+\xi^2)^4}|\hat{u}|^2 + \left[\lambda_2(a^2-\epsilon_2) - \lambda_3\epsilon_3\right]\frac{\xi^6}{(1+\xi^2)^4}|\hat{z}|^2\\
&+\left[\lambda_1\left(\frac{\beta}{g(0)}-\epsilon_0\right) - C(\epsilon_2)\lambda_2 - \lambda_3\epsilon_3\right]\frac{\xi^4}{(1+\xi^2)^3}|\hat{v}|^2\\
&+\left[-\lambda_2 C(\epsilon_2) - C(\epsilon_3)\lambda_3 + (g(0)-\epsilon_6)\lambda_4\right]\xi^2|\hat{\theta}|^2\\
&+\left[\lambda_1 C(\epsilon_0) + \lambda_4 C(\epsilon_6)\right]g'(0)\int_0^\infty \xi^2(-\mu'(s))|\hat{\eta}(s,t)|^2\,ds\\
&\le 0.
\end{aligned} \tag{3.53}$$

Now, using (3.39), we may write (3.53) as:

$$\begin{aligned}
&\frac{d}{dt}L_2(\xi,t) + \left[-C(\epsilon_0,\epsilon_0')\lambda_1 - C(\epsilon_1)\lambda_2 + (\delta-\epsilon_4)\lambda_3 - \epsilon_5\lambda_4\right]\frac{\xi^4}{(1+\xi^2)^2}|\hat{y}|^2\\
&+\left[-\epsilon_0'\lambda_1 + \lambda_2(1-\epsilon_1)\right]\frac{\xi^6}{(1+\xi^2)^4}|\hat{u}|^2 + \left[\lambda_2(a^2-\epsilon_2) - \lambda_3\epsilon_3\right]\frac{\xi^6}{(1+\xi^2)^4}|\hat{z}|^2\\
&+\left[\lambda_1\left(\frac{\beta}{g(0)}-\epsilon_0\right) - C(\epsilon_2)\lambda_2 - \lambda_3\epsilon_3\right]\frac{\xi^4}{(1+\xi^2)^3}|\hat{v}|^2\\
&+\left[-\lambda_2 C(\epsilon_2) - C(\epsilon_3)\lambda_3 + (g(0)-\epsilon_6)\lambda_4\right]\xi^2|\hat{\theta}|^2
\end{aligned}$$

$$- C(1+\xi^2)^2 \int_0^\infty \xi^2(-\mu'(s))|\hat{\eta}(s,t)|^2\, ds$$
$$\leq 0, \tag{3.54}$$

where C is a generic positive constant that depends on $\epsilon_i,\ \gamma_j$ and ν, yet is independent on t and ξ.

As we did in case one, we fix $\epsilon_0,\ \epsilon_1,\ \epsilon_2,\ \epsilon_4$ and ϵ_6 as follows:

$$\epsilon_0 < \frac{\beta}{g(0)}, \qquad \epsilon_1 < 1, \qquad \epsilon_2 < a^2, \qquad \epsilon_4 < \delta, \qquad \epsilon_6 < g(0).$$

Also, we choose λ_i as we did for γ_i. That is we fix $\lambda_2 = 1$ and choose λ_1 large enough such that

$$\lambda_1\left(\frac{\beta}{g(0)} - \epsilon_0\right) - C(\epsilon_2) > 0.$$

Furthermore, we choose ϵ_0' small enough such that

$$\epsilon_0' < \frac{1-\epsilon_1}{\lambda_1}.$$

Next, we choose λ_3 large enough such that

$$-C(\epsilon_0, \epsilon_0')\lambda_1 - C(\epsilon_1) + (\delta - \epsilon_4)\lambda_3 > 0.$$

Then, we fix ϵ_3 small enough such that

$$\epsilon_3 < \min\left\{\frac{\lambda_1(\frac{\beta}{g(0)} - \epsilon_0) - C(\epsilon_2)}{\lambda_3}, \frac{a^2 - \epsilon_2}{\lambda_3}\right\}.$$

Now, we choose λ_4 large enough such that

$$(g(0) - \epsilon_6)\lambda_4 - C(\epsilon_2) - \lambda_3 C(\epsilon_3) > 0.$$

Finally, we pick ϵ_5 small enough such that

$$\epsilon_5 < \frac{-\lambda_1 C(\epsilon_0, \epsilon_0') - C(\epsilon_1) + \lambda_3(\delta - \epsilon_4)}{\lambda_4}.$$

Consequently, we deduce that there exists a positive constant $\eta_3 > 0$, such that

$$\frac{d}{dt}L_2(\xi,t) + \eta_3 Q_2(\xi,t) \leq C(1+\xi^2)^2 \int_0^\infty \xi^2(-\mu'(s))|\hat{\eta}(s,t)|^2\, ds, \tag{3.55}$$

where

$$Q_2(\xi,t) = \frac{\xi^6}{(1+\xi^2)^4}\left(|\hat{u}|^2 + |\hat{z}|^2\right) + \frac{\xi^4}{(1+\xi^2)^3}|\hat{v}|^2 + \frac{\xi^4}{(1+\xi^2)^2}|\hat{y}|^2 + \xi^2|\hat{\theta}|^2. \tag{3.56}$$

It is straightforward to see that

$$Q_2(\xi,t) \geq \frac{\xi^6}{(1+\xi^2)^4}\left(|\hat{u}|^2 + |\hat{z}|^2 + |\hat{v}|^2 + |\hat{y}|^2 + |\hat{\theta}|^2\right). \tag{3.57}$$

Now, we define the functional

$$\mathscr{L}_2(\xi,t) = M(1+\xi^2)^2\hat{E}(\xi,t) + L_2(\xi,t), \tag{3.58}$$

where M is a large positive number that will be chosen later. Using (3.9) and (3.55), the functional $\mathscr{L}_2(\xi,t)$ satisfies the estimate

$$\frac{d}{dt}\mathscr{L}_2(\xi,t) + \eta_3 Q_2(\xi,t) + \left(\frac{M}{2\beta} - C\right)(1+\xi^2)^2\int_0^\infty \xi^2(-\mu'(s))|\hat{\eta}(t,s)|^2\,ds \leq 0. \tag{3.59}$$

By choosing M large enough such that

$$M > 2C\beta,$$

we deduce from (3.8) and (3.57) that for M large enough, there exists a positive constant η_4 such that

$$\frac{d}{dt}\mathscr{L}_2(\xi,t) + \eta_4\frac{\xi^6}{(1+\xi^2)^4}\hat{E}(\xi,t) \leq 0, \qquad \forall t \geq 0. \tag{3.60}$$

Now, using (3.58) and (3.52), together with the definitions of all functionals involved in (3.52) for all $\xi \in \mathbb{R}$, there exist two positive constants β_3 and β_4, such that, for all $t \geq 0$,

$$\beta_3(1+\xi^2)^2\hat{E}(\xi,t) \leq \mathscr{L}_2(\xi,t) \leq \beta_4(1+\xi^2)^2\hat{E}(\xi,t). \tag{3.61}$$

Combining (3.60) and (3.61), we have

$$\frac{d}{dt}\mathscr{L}_2(\xi,t) \leq -\frac{\eta_4}{\beta_4}\frac{\xi^6}{(1+\xi^2)^6}\mathscr{L}_2(\xi,t), \qquad \forall t \geq 0. \tag{3.62}$$

Applying Gronwall's lemma and again using (3.59) give (3.33). □

4 The Decay Estimate

In this section, we derive the decay rates of the energy of (2.5) together with (1.2).

Theorem 4.1 *Let s be a nonnegative integer, $\alpha_g = \left(\frac{\beta}{g(0)} - 1\right)(1-a^2) - \delta^2 \frac{\beta}{g(0)}$ as in (3.15), and assume that $E_s(0)$ and $\sup_{|\xi|\leq 1}\{\hat{E}(\xi,0)\}$ are bounded. Then, the energy $E_k(t)$, defined in (1.19), satisfies the following decay estimates:*

- *if $\alpha_g = 0$*

$$E_k(t) \leq C \sup_{|\xi|\leq 1}\{\hat{E}(\xi,0)\}(1+t)^{-1/6-k/3} + Ce^{-ct}E_k(0), \tag{4.1}$$

- *if $\alpha_g \neq 0$*

$$E_k(t) \leq C \sup_{|\xi|\leq 1}\{\hat{E}(\xi,0)\}(1+t)^{-1/6-k/3} + C(1+t)^{-\ell/3}E_{k+\ell}(0), \tag{4.2}$$

where k and ℓ are nonnegative integers satisfying $k+\ell \leq s$ and C and c are positive constants.

Remark 4.2 In many situation like the Fourier model of the Cattaneo one, if we assume that the initial data to belong to $L^1(\mathbb{R})$, then we can easily see that (using the Hausdroff–Young inequality) that $\sup_{|\xi|\leq 1}\{\hat{E}(\xi,0)\}$ can be estimate by the L^1-norm of the initial data.

Proof of Theorem 4.1 **Case One:** $\alpha_g = 0$. In this case, using (3.34), we have

$$\rho(\xi) \geq \begin{cases} c\xi^6 & \text{for } |\xi| \leq 1, \\ c & \text{for } |\xi| \geq 1. \end{cases} \tag{4.3}$$

Applying the Plancherel theorem together with inequality (3.33), we have:

$$\begin{aligned} E_k(t) &= \int_{\mathbb{R}} |\xi|^{2k}\hat{E}(\xi,t)\,d\xi \\ &\leq C\int_{\mathbb{R}} |\xi|^{2k}e^{-c\rho(\xi)t}\hat{E}(\xi,0)\,d\xi \\ &= C\int_{|\xi|\leq 1} |\xi|^{2k}e^{-c\rho(\xi)t}\hat{E}(\xi,0)\,d\xi + C\int_{|\xi|\geq 1} |\xi|^{2k}e^{-c\rho(\xi)t}\hat{E}(\xi,0)\,d\xi \\ &= I_1(t) + I_2(t). \end{aligned}$$

Here, we split the integral into two parts, so that $I_1(t)$ is the low-frequency part where $|\xi| \leq 1$ and $I_2(t)$ is the high-frequency part where $|\xi| \geq 1$. Using the first inequality in (4.3), we can estimate $I_1(t)$ as:

$$
\begin{aligned}
I_1(t) &= C \int_{|\xi|\leq 1} |\xi|^{2k} e^{-c\rho(\xi)t} \hat{E}(\xi,0)\, d\xi \\
&\leq C \int_{|\xi|\leq 1} |\xi|^{2k} e^{-c\xi^6 t} \hat{E}(\xi,0)\, d\xi \\
&\leq C \sup_{|\xi|\leq 1} \{\hat{E}(\xi,0)\} \int_{|\xi|\leq 1} |\xi|^{2k} e^{-c\xi^6 t}\, d\xi. \qquad (4.4)
\end{aligned}
$$

Finally, using Lemma 2.1, we obtain

$$
\int_0^1 |\xi|^{2k} e^{-c\xi^6 t}\, d\xi \leq C(1+t)^{-1/6-k/3}. \qquad (4.5)
$$

Hence, we have:

$$
I_1(t) \leq C \sup_{|\xi|\leq 1} \{\hat{E}(\xi,0)\}(1+t)^{-1/6-k/3}. \qquad (4.6)
$$

Using the second inequality of (4.3), we can find the estimate for $I_2(t)$ as follows:

$$
\begin{aligned}
I_2(t) &= C \int_{|\xi|\geq 1} |\xi|^{2k} e^{-c\rho(\xi)t} \hat{E}(\xi,0)\, d\xi \\
&= Ce^{-ct} \int_{|\xi|\geq 1} |\xi|^{2k} \hat{E}(\xi,0)\, d\xi \\
&\leq Ce^{-ct} E_k(0). \qquad (4.7)
\end{aligned}
$$

Now, adding estimates (4.6) and (4.7) shows that estimate (4.1) holds.

The second case $\alpha_g \neq 0$ can be proved as in [9] with slight modifications. We omit the details. □

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