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On the Rate of Convergence of the 2-D Stochastic Leray-α Model to the 2-D Stochastic Navier–Stokes Equations with Multiplicative Noise

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Abstract In the present paper we study the convergence of the solution of the two dimensional (2-D) stochastic Leray-α model to the solution of the 2-D stochastic Navier–Stokes equations. We are mainly interested in the rate, as $\alpha \to 0$, of the following error function

$$\varepsilon_\alpha(t) = \sup_{s\in[0,t]} |\mathbf{u}^\alpha(s) - \mathbf{u}(s)| + \left(\int_0^t |\mathrm{A}^{\frac{1}{2}}[\mathbf{u}^\alpha(s) - \mathbf{u}(s)]|^2 ds\right)^{\frac{1}{2}},$$

where $\mathbf{u}^\alpha$ and $\mathbf{u}$ are the solution of stochastic Leray-α model and the stochastic Navier–Stokes equations, respectively. We show that when properly localized the error function ε_α converges in mean square as $\alpha \to 0$ and the convergence is of order $O(\alpha)$. We also prove that ε_α converges in probability to zero with order at most $O(\alpha)$.

Keywords Navier–Stokes equations · Leray-α model · Rate of convergence in mean square · Rate of convergence in probability · Turbulence models · Navier–Stokes-α

Mathematics Subject Classification 60H15 · 76D05 · 60H35

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1 Introduction

The Navier–Stokes system is the most used model in turbulence theory. In recent years, various regularization models were introduced as an efficient subgrid model scale of the Navier–Stokes equations (NSE), see for eg. [9,11–13,16,26,27,29,30,32]. Moreover, numerical analyses in [14,28,29,31,34,35,37,38] seem to confirm that these models can capture remarkably well the physical phenomenon of turbulence in fluid flows at a lower computational cost. Among them are the Navier–Stokes-α, Leray-α, modified Leray-α, Clark-α to name just a few.

Another tool used to tackle the closure problem in turbulent flows is to introduce a stochastic forcing that will mimic all the terms that can't be handled. This approach is basically motivated by Reynolds' work which stipulates that hydrodynamic turbulence is composed of slow (deterministic) and fast (stochastic) components. This approach was used in [36] to derive a stochastic Navier–Stokes equations with gradient and nonlinear diffusion coefficient.

It is worth emphasizing that the presence of the stochastic term (noise) in the model often leads to qualitatively new types of behavior, which are very helpful in understanding real processes and is also often more realistic. In particular, for the 2d Navier–Stokes equations, some ergodic properties are proved when adding a random perturbation,

There is an extensive literature about the convergence of α-models to the Navier–Stokes equations, see for eg. [1,7,8,17,21–24,27]. However, only a few papers deal with the rate of convergence, see [10,15]. In [10] the rates of convergence of four α-models (NS-α model, Leray-α model, modified Leray-α model, and simplified Bardina model) in the two-dimensional (2D) case, subject to periodic boundary conditions on the periodic box $[0, L]^2$ are studied. The authors of [10] mainly showed that all the four α-models have the same order of convergence and error estimates; that is, the convergences in the $\mathbb{L}^2$-norms are all of the order $O(\frac{\alpha}{L}(\log(\frac{L}{\alpha}))^{\frac{1}{2}})$ as $\frac{\alpha}{L}$ tends to zero, while in [15] the rate of convergence of order $O(\alpha)$ is obtained in a mixed $L^1 - L^2$ time-space norm with small initial data in Besov-type function spaces.

Despite the numerous papers, there are only very few addressing the convergence of stochastic α-models to the stochastic Navier–Stokes. It is proved in [17] that the stochastic Leray-α model has a unique invariant measure which converges to the stationary solution (unique invariant measure) of 3-D (resp. 2-D) stochastic Navier–Stokes equations. In [21,22] Deugoué and Sango proved that one can find a sequence of weak martingales of the 3-D stochastic Navier–Stokes-α and Leray-α model respectively which converges in distribution to the weak martingale solution of the 3-D stochastic Navier–Stokes equations.

Here in this paper, we are interested in the analysis of the rate of convergence of the two-dimensional stochastic Leray-α model to the stochastic Navier–Stokes equations. More precisely we consider the Leray-α model with multiplicative stochastic perturbation on a periodic domain $\mathcal{O} = [0, L]^2$, $L > 0$, given by the following system

$$d\mathbf{v}^\alpha(t) + [\nu \mathrm{A}\mathbf{v}^\alpha(t) + B(\mathbf{u}^\alpha(t), \mathbf{v}^\alpha(t))]dt = Q(\mathbf{u}^\alpha(t))dW(t), \ t \in (0, T] \tag{1.1a}$$

$$\mathbf{u}^\alpha + \alpha^2 \mathrm{A}\mathbf{u}^\alpha = \mathbf{v}^\alpha, \tag{1.1b}$$

$$\mathbf{u}^{\alpha}(0) = \mathbf{u}_0, \tag{1.1c}$$

where W is a cylindrical Wiener process on a separable Hilbert space K, A is the Stokes operator and B is the well-known bilinear map in the mathematical theory of the Navier–Stokes equations. We refer to Sect. 2 for the functional setting.

Our main goal in the present paper is to study the convergenve of the solution $\mathbf{u}^{\alpha}$ to (1.1) to the solution of the stochastic Navier–Stokes equations given by

$$d\mathbf{u}(t) = [-\mathrm{A}\mathbf{u}(t) - B(\mathbf{u}(t), \mathbf{u}(t))]dt + Q(\mathbf{u}(t))dW(t), \; t \in (0, T], \tag{1.2a}$$

$$\mathbf{u}(0) = \mathbf{u}_0. \tag{1.2b}$$

To the best of our knowledge it seems that the investigation of the rate of convergence of the stochastic α-model to the stochastic Navier–Stokes has never been done before. In this paper we initiate this direction of research by studying the rate of convergence of the error function for $t \in [0, T]$

$$\varepsilon_{\alpha}(t) = \sup_{s\in[0,t]} |\mathbf{u}^{\alpha}(s) - \mathbf{u}(s)| + \left(\int_0^t |\mathrm{A}^{\frac{1}{2}}[\mathbf{u}^{\alpha}(s) - \mathbf{u}(s)]|^2 ds\right)^{\frac{1}{2}},$$

as α tends to zero. Here $|\cdot|$ denotes the $\mathbb{L}^2(\mathcal{O})$-norm. By deriving several important uniform estimates for the sequence of stochastic processes $\mathbf{u}^{\alpha}$ we can prove that for an appropriate family of stopping times $\{\tau_R; R > 0\}$ the stopped error function $\varepsilon_{\alpha}(t \wedge \tau_R)$ converges to 0 in mean square as α goes to zero and the convergence is of order $O(\alpha)$. In particular, this shows that when the error function ε_{α} is properly localized then the order of convergence in the stochastic case is better than the one in the deterministic case. In this paper, we also prove that the convergence in probability (see for example [39] for the definition) of ε_{α} is also of order $O(\alpha)$. These results can be found in Theorems 4.1 and 4.2. We mainly combine the approaches used in [4,10].

In Sect. 2 we introduce the notations and some frequently used lemmata. In Sect. 3, we introduce the main assumptions on the diffusion coefficient. Moreover, several important uniform estimates which are the backbone of our analysis will be derived. In Sect. 4, we state and prove our main results; we mainly show that when properly localized the error function ε_{α} converges in mean square to zero as α tend to zero. Owing to the uniform estimates obtained in Sect. 3, we also show in Sect. 4 that it converges in probability with order $O(\alpha)$ as α tends to zero.

Throughout the paper C, c denote some unessential constants which do not depend on α and may change from one place to the next one.

2 Notations

In this section we introduce some notations that are frequently used in this paper. We will mainly follow the presentation of Cao and Titi [10].

Let $\mathcal{O}$ be a bounded subset of $\mathbb{R}^2$. For any $p \in [1, \infty)$ and $k \in \mathbb{N}$, $\mathbb{L}^p(\mathcal{O})$ and $\mathbb{W}^{k,p}(\mathcal{O})$ are the well-known Lebesgue and Sobolev spaces, respectively, of $\mathbb{R}^2$-valued functions. The corresponding spaces of scalar functions we will denote by standard letter, e.g. $W^{k,p}(\mathcal{O})$. The usual scalar product on $\mathbb{L}^2(\mathcal{O})$ is denoted by $\langle u, v\rangle$ for $u, v \in \mathbb{L}^2(\mathcal{O})$. Its associated norm is $|u|$, $u \in \mathbb{L}^2(\mathcal{O})$.

Let $L > 0$ and $\mathcal{P}$ be the set of (periodic) trigonometric polynomials of two variables defined on the periodic domain $\mathcal{O} = [0, L]^2$ and with zero spatial average; that is, for every $\phi \in \mathcal{P}$, $\int_{\mathcal{O}} \phi(x)dx = 0$. We also set

$$\begin{aligned} \mathcal{V} &= \left\{\mathbf{u} \in [\mathcal{P}]^2 \text{ such that } \nabla\cdot\mathbf{u} = 0\right\} \\ \mathbf{V} &= \text{ closure of } \mathcal{V} \text{in } \mathbb{H}^1(\mathcal{O}) \\ \mathbf{H} &= \text{ closure of } \mathcal{V} \text{in } \mathbb{L}^2(\mathcal{O}). \end{aligned}$$

We endow the spaces $\mathbf{H}$ with the scalar product and norm of $\mathbb{L}^2$. We equip the space $\mathbf{V}$ with the scalar product $((\mathbf{u}, \mathbf{v})) := \int_{\mathcal{O}} \nabla\mathbf{u}(x)\cdot\nabla\mathbf{v}(x)dx$ which is equivalent to the $\mathbb{H}^1(\mathcal{O})$-scalar product on $\mathbf{V}$. The norm corresponding to the scalar product $((\cdot,\cdot))$ is denoted by $\|\cdot\|$.

Let $\Pi : \mathbb{L}^2(\mathcal{O}) \to \mathbf{H}$ be the projection from $\mathbb{L}^2(\mathcal{O})$ onto $\mathbf{H}$. We denote by A the Stokes operator defined by

$$\begin{cases} D(\mathrm{A}) = \{u \in \mathbf{H},\ \Delta u \in \mathbf{H}\}, \\ \mathrm{A}u = -\Pi\Delta u,\ u \in D(\mathrm{A}). \end{cases} \tag{2.1}$$

Note that in the space-periodic case

$$\mathrm{A}\mathbf{u} = -\Pi\Delta\mathbf{u} = -\Delta\mathbf{u}, \text{ for all } \mathbf{u} \in D(\mathrm{A}).$$

The operator A is a self-adjoint, positive definite, and a compact operator on $\mathbf{H}$ (see, for instance, [20,41]). We will denote by $\lambda_1 \le \lambda_2 \le \cdots$ the eigenvalues of A; the correspoding eigenfunctions $\{\Psi_i : i = 1, 2, \ldots\}$ form an orthonormal basis of $\mathbf{H}$ and an orthogonal basis of $\mathbf{V}$. For any positive integer $n \in \mathbb{N}$ we set

$$\mathbf{H}_n = \text{linspan}\{\Psi_i : i = 1, \ldots, n\}$$

and we denote by P_n the orthogonal projection onto $\mathbf{H}_n$ defined by

$$P_n\mathbf{u} = \sum_{i=1}^{n} \langle \mathbf{u}, \Psi_i\rangle\Psi_i, \text{ for all } \mathbf{u} \in \mathbf{H}.$$

We also recall that in the periodic case we have $D(\mathrm{A}^{\frac{n}{2}}) = \mathbb{H}^n(\mathcal{O}) \cap \mathbf{H}$, for $n > 0$ (see, for instance, [20,41]). In particular we have $\mathbf{V} = D(\mathrm{A}^{\frac{1}{2}})$.

For every $\mathbf{w} \in \mathbf{V}$, we have the following Poincaré inequality

$$\lambda_1|\mathbf{w}|^2 \le \|\mathbf{w}\|^2, \text{ for all } \mathbf{w} \in \mathbf{V}. \tag{2.2}$$

Also, there exists $c > 0$ such that

$$c|\mathrm{A}\mathbf{w}| \leq \|\mathbf{w}\|_2 \leq c^{-1}|\mathrm{A}\mathbf{w}| \text{ for every } \mathbf{w} \in D(\mathrm{A}), \tag{2.3}$$

$$c|\mathrm{A}^{\frac{1}{2}}\mathbf{w}| \leq \|\mathbf{w}\|_1 \leq c^{-1}|\mathrm{A}^{\frac{1}{2}}\mathbf{w}| \text{ for every } \mathbf{w} \in \mathbf{V}. \tag{2.4}$$

Thanks to (2.4) the norm $\|\cdot\|$ of $\mathbf{V}$ is equivalent to the usual $\mathbb{H}^1(\mathcal{O})$-norm. Recall that the following estimate, valid for all $\mathbf{w} \in \mathbb{H}^1$ (or $\mathbf{w} \in H^1$), is a special case of Gagliardo-Nirenberg's inequalities:

$$\|\mathbf{w}\|_{\mathbb{L}^4} \leq c|\mathbf{w}|^{\frac{1}{2}}|\nabla\mathbf{w}|^{\frac{1}{2}}. \tag{2.5}$$

The inequality (2.5) can be written in the spirit of the continuous embedding

$$\mathbb{H}^1 \subset \mathbb{L}^4. \tag{2.6}$$

Next, for avery $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{V}$ we define the bilinear operator

$$B(\mathbf{w}_1, \mathbf{w}_2) = \Pi\big[(\mathbf{w}_1 \cdot \nabla)\mathbf{w}_2\big]. \tag{2.7}$$

In the following lemma we recall some properties of the bilinear operator B.

Lemma 2.1 *The bilinear operator B defined in (2.7) satisfies the following*

(i) *B can be extended as a continuous bilinear map $B : \mathbf{V} \times \mathbf{V} \to \mathbf{V}^*$, where $\mathbf{V}^*$ is the dual space of $\mathbf{V}$. In particular, the following properties hold for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}$:*

$$|\langle B(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle| \leq c|\mathbf{u}|^{\frac{1}{2}}\|\mathbf{u}\|^{\frac{1}{2}}\|\mathbf{v}\||\mathbf{w}|^{\frac{1}{2}}\|\mathbf{w}\|^{\frac{1}{2}}, \tag{2.8}$$

$$\langle B(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle = -\langle B(\mathbf{u}, \mathbf{w}), \mathbf{v}\rangle. \tag{2.9}$$

As consequence of (2.9) we have

$$\langle B(\mathbf{u}, \mathbf{v}), \mathbf{v}\rangle = 0 \tag{2.10}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbf{V}$.

(ii) *In the 2D periodic boundary condition case, we have*

$$\langle B(\mathbf{u}, \mathbf{u}), \mathrm{A}\mathbf{u}\rangle = 0, \tag{2.11}$$

for every $\mathbf{u} \in D(\mathrm{A})$.

Proof Part (i) is very classical and can be found in any reference related to Navier–Stokes equations, for instance in [20,41]. Part (ii) can be found in [40, Lemma 3.1]. □

We also recall the following lemma.

Lemma 2.2 *For every* $\mathbf{u} \in D(\mathrm{A})$ *and* $\mathbf{v} \in \mathbf{V}$, *we have*

$$|\langle B(\mathbf{v}, \mathbf{u}), \mathrm{A}\mathbf{u}\rangle| \le c\|\mathbf{v}\|\|\mathbf{u}\|\,|\mathrm{A}\mathbf{u}|. \tag{2.12}$$

Proof For the proof we refer to [10, Lemma 2.2]. □

3 A Priori Estimates for the Stochastic Navier–Stokes Equations and the Stochastic Leray-α Model

The stochastic Leray-α model (1.1) and the stochastic Navier–Stokes equations (1.2) have been extensively studied. Their well posedness are established in several mathematical papers. In this section we just recall the most recent results which are very close to our purpose. Most of these results were obtained from Galerkin approximation and energy estimates. However, the estimates derived in previous papers are not sufficient for our analysis. Therefore, we will also devote this section to derive several important estimates which are the backbone of our analysis.

We consider a prescribed complete probability system $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} := \{\mathcal{F}_t; t \ge 0\}$. We assume that the filtration satisfies the usual condition, that is, the family $\mathbb{F}$ is increasing, right-continuous and $\mathcal{F}_0$ contains all null sets of $\mathcal{F}$. Let K be a separable Hilbert space. On the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ we suppose that we are given a cylindrical Wiener process W on K.

For two Banach spaces X and Y, we denote by $\mathcal{L}(X, Y)$ the space of all bounded linear maps $L : X \to Y$. The space of all Hilbert–Schmidt operators $L : X \to Y$ is denoted by $\mathcal{L}_2(X, Y)$. The Hilbert-Schmidt norm of $L \in \mathcal{L}_2(X, Y)$ is denoted by $\|L\|_{\mathcal{L}_2(X,Y)}$. When $X = Y$ we just write $\mathcal{L}_2(X) := \mathcal{L}_2(X, X)$.

Now, we can introduce the standing assumptions of the paper.

Assumption 3.1 Throughout this paper we assume that $Q : D(\mathrm{A}^{\frac{1}{2}}) \to \mathcal{L}(\mathrm{K}, D(\mathrm{A}^{\frac{1}{2}}))$ satisfies:

(i) there exists $\ell_0 > 0$ such that for any $\mathbf{u}_1, \mathbf{u}_2 \in D(\mathrm{A}^{\frac{1}{2}})$ we have

$$\|Q(\mathbf{u}_1) - Q(\mathbf{u}_2)\|_{\mathcal{L}_2(\mathrm{K},\mathbf{H})} \le \ell_0|\mathbf{u}_1 - \mathbf{u}_2|,$$

(ii) there exists $\ell_1 > 0$ such that for any $\mathbf{u}_1, \mathbf{u}_2 \in D(\mathrm{A}^{\frac{1}{2}})$ we have

$$\left\|\mathrm{A}^{\frac{1}{2}}\big[Q(\mathbf{u}_1) - Q(\mathbf{u}_2)\big]\right\|_{\mathcal{L}_2(\mathrm{K},\mathbf{H})} \le \ell_1 \left|\mathrm{A}^{\frac{1}{2}}\mathbf{u}_1 - \mathrm{A}^{\frac{1}{2}}\mathbf{u}_2\right|.$$

Remark 3.1 Assumption 3.1 implies in particular that

(1) there exists $\ell_2 > 0$ such that for any $\mathbf{u} \in D(\mathrm{A}^{\frac{1}{2}})$ we have

$$\|Q(\mathbf{u})\|_{\mathcal{L}_2(\mathrm{K},\mathbf{H})} \le \ell_2(1 + |\mathbf{u}|),$$

(2) there exists $\ell_3 > 0$ such that for any $\mathbf{u} \in D(\mathrm{A}^{\frac{1}{2}})$ we have

$$\left\| \mathrm{A}^{\frac{1}{2}} Q(\mathbf{u}) \right\|_{\mathcal{L}_2(\mathrm{K},\mathbf{H})} \leq \ell_3 \left(1 + |\mathrm{A}^{\frac{1}{2}}\mathbf{u}|\right),$$

3.1 A Priori Estimates for the Navier–Stokes Equations

The study of stochastic Navier–Stokes equations was pioneered by Bensoussan and Temam in [3]. Since then, an intense investigation about the qualitative and quantitative properties of this model has generated an extensive literature, see e.g. [2,5,6,18,25,36].

The following definition of solution is mainly taken from [18] (see also [2,25]).

Definition 3.1 A weak solution to (1.2) is a stochastic process $\mathbf{u}$ such that

(1) $\mathbf{u}$ is progressively measurable,
(2) $\mathbf{u}$ belongs to $C([0,T];\mathbf{H}) \cap L^2(0,T,\mathbf{V})$ almost surely,
(3) for all $t \in [0,T]$, almost surely

$$(\mathbf{u}(t),\phi) + \nu \int \langle \mathrm{A}^{\frac{1}{2}}\mathbf{u}(s), \mathrm{A}^{\frac{1}{2}}\phi\rangle ds + \int_0^t \langle B(\mathbf{u}(s),\mathbf{u}(s)),\phi\rangle ds$$
$$= \mathbf{u}_0 + \int_0^t \langle \phi, Q(\mathbf{u}(s))dW(s)\rangle,$$

for any $\phi \in \mathbf{V}$.

We state the following theorem which was proved in [18, Theorem 2.4], see also [2,25].

Theorem 3.2 *Let $\mathbf{u}_0$ be a $\mathbf{H}$-valued $\mathcal{F}_0$-measurable such that $\mathbb{E}|\mathbf{u}_0|^4 < \infty$. Assume that (3.1) holds. Then (1.2) has a unique solution $\mathbf{u}$ in the sense of the above definition. Moreover, for any $p \in \{2,4\}$ and $T > 0$ there exists $C > 0$ such that*

$$\mathbf{E}\left(\sup_{t\in[0,T]} |\mathbf{u}(t)|^p + \int_0^T |\mathbf{u}(s)|^{p-2}|\mathrm{A}^{\frac{1}{2}}\mathbf{u}(s)|^2 ds \right) < C(1+|\mathbf{u}_0|^4) \qquad (3.1)$$

3.2 A Priori Estimates for the Leray-α Model

The Leray-α model was introduced and analyzed in [16]. Since, then it has been extensively studied; we refer to [10] and references therein for a brief historical description and review of results. It is worth noticing that the Leray-α model is a particular example of a more general regularization used by Leray in his seminal work, [33], in the context of establishing the existence of solutions for the 2D and 3D NSE.

The stochastic Leray-α model was studied in [18,19,22]. For the inviscid case, we refer to the recent work [1] where the uniqueness of solutions were investigated. The following definition of solutions to (1.1) is taken from [22] (see also [18]).

Definition 3.3 A weak solution to (1.1) is a stochastic process $\mathbf{u}^\alpha$ such that

(1) $\mathbf{u}^\alpha$ is progressively measurable,
(2) $\mathbf{v}^\alpha$, with $\mathbf{v}^\alpha := (I+\alpha^2 \mathrm{A})\mathbf{u}^\alpha$, belongs to $C([0,T];\mathbf{H})\cap L^2(0,T,\mathbf{V})$ almost surely,
(3) for all $t\in[0,T]$, almost surely

$$(\mathbf{v}^\alpha(t),\phi)+\nu\int \langle \mathrm{A}^{\frac{1}{2}}\mathbf{v}^\alpha(s),\mathrm{A}^{\frac{1}{2}}\phi\rangle ds+\int_0^t\langle B(\mathbf{u}^\alpha(s),\mathbf{v}^\alpha(s)),\phi\rangle ds$$
$$=\mathbf{v}_0^\alpha+\int_0^t\langle\phi, Q(\mathbf{u}^\alpha(s))dW(s)\rangle,$$

for any $\phi\in\mathbf{V}$.

Again, we refer to [18, Theorem 2.4] for the statement of the result below. See also [22].

Theorem 3.4 *Let $\mathbf{u}_0$ be a $D(\mathrm{A})$-valued $\mathcal{F}_0$-measurable such that $\mathbb{E}|\mathrm{A}\mathbf{u}_0|^4<\infty$. Assume that (3.1) holds. Then for any $\alpha>0$ the system (1.1) has a unique solution $\mathbf{u}^\alpha$ in the sense of the above definition. Moreover, for any $p\in\{2,4\}$ and $T>0$ there exists $C>0$ such that*

$$\mathbf{E}\left(\sup_{t\in[0,T]}|\mathbf{v}^\alpha(t)|^p+\int_0^T|\mathbf{v}^\alpha(s)|^{p-2}|\mathrm{A}^{\frac{1}{2}}\mathbf{v}^\alpha(s)|^2ds\right)<C(1+|(I+\alpha^2\mathrm{A})\mathbf{u}_0|^4) \tag{3.2}$$

As stated in [22], the constant C above depends on α and may explode as α tends to zero. The uniform estimates, with respect to α, obtained in [22] are not helpful for our analysis. Our aim in this subsection is to derive several a priori estimates for the stochastic Leray-α model (1.1). These estimates summarized in the following two propositions, will be used in Sect. 4 to derive a rate of convergence of the stochastic Leray-α model to the stochastic Navier–Stokes equations.

We start with some estimates in the weak norms of the solution $\mathbf{u}^\alpha$, these are refinements of estimates obtained in [22].

Proposition 3.5 *Let $\mathbf{u}_0$ be a $\mathcal{F}_0$-measurable random variable such that $\mathbb{E}|\mathbf{u}_0+\mathrm{A}\mathbf{u}_0|^4<\infty$. Assume that the set of hypotheses stated in Assumption 3.1 holds. Then, there exists a constant $C>0$ such that for any $\alpha\in(0,1)$ we have*

$$\mathbb{E}\sup_{t\in[0,T]}\left[|\mathbf{u}^\alpha(t)|^2+2\alpha^2|\mathrm{A}^{\frac{1}{2}}\mathbf{u}^\alpha(t)|^2+\alpha^4|\mathrm{A}\mathbf{u}^\alpha(t)|^2\right]^2\le\mathfrak{K}_0, \tag{3.3}$$

$$\mathbb{E}\int_0^T|\mathrm{A}^{\frac{1}{2}}\mathbf{u}^\alpha(s)|^2|\mathbf{u}^\alpha(s)|^2ds\le\mathfrak{K}_0, \tag{3.4}$$

$$2\alpha^2\mathbb{E}\int_0^T|\mathbf{u}^\alpha(s)|^2\Big(|\mathrm{A}\mathbf{u}^\alpha(s)|^2+\frac{\alpha^2}{2}|\mathrm{A}^{\frac{3}{2}}\mathbf{u}^\alpha(s)|^2\Big)ds\le\mathfrak{K}_0, \tag{3.5}$$

$$2\alpha^2\mathbb{E}\int_0^T |A^{\frac{1}{2}}\mathbf{u}^\alpha(s)|^2\Big(|A^{\frac{1}{2}}\mathbf{u}^\alpha(s)|^2+2\alpha^2|A\mathbf{u}^\alpha(s)|^2+\alpha^4|A^{\frac{3}{2}}\mathbf{u}^\alpha(s)|^2\Big)ds \le \mathfrak{K}_0, \tag{3.6}$$

$$\alpha^4\mathbb{E}\int_0^T |A\mathbf{u}^\alpha(s)|^2\Big(|A^{\frac{1}{2}}\mathbf{u}^\alpha(s)|^2+2\alpha^2|A\mathbf{u}^\alpha(s)|^2+\alpha^4|A^{\frac{3}{2}}\mathbf{u}^\alpha(s)|^2\Big)ds \le \mathfrak{K}_0, \tag{3.7}$$

where

$$\mathfrak{K}_0 := \Big(2\mathbb{E}|\mathbf{u}_0 + A\mathbf{u}_0|^4 + CT\Big)\Big(1+Ce^{CT}\Big).$$

Proof For any positive integer $n \in \mathbb{N}$, we will consider the Galerkin approximation of (1.1) which is a system of SDEs in $\mathbf{H}_n$

$$d\mathbf{v}_n^\alpha(t) + [\nu A\mathbf{v}_n^\alpha(t) + B(\mathbf{u}_n^\alpha(t), \mathbf{v}_n^\alpha(t))]dt = P_n Q(\mathbf{u}_n^\alpha(t))dW(t),\ t \in (0, T] \tag{3.8a}$$
$$\mathbf{u}_n^\alpha + \alpha^2 A\mathbf{u}_n^\alpha = \mathbf{v}_n^\alpha, \tag{3.8b}$$
$$\mathbf{u}_n^\alpha(0) = \mathbf{u}_{0n}, \tag{3.8c}$$

where $\mathbf{u}_{0n} = P_n\mathbf{u}_0$. Let $\Psi(\cdot)$ be a mapping defined on $\mathbf{H}_n$ defined by $\Psi(\cdot) := |\cdot|^4$. The mapping $\Psi(\cdot)$ is twice Fréchet differentiable with first and second derivative defined by

$$\Psi'(\mathbf{u})[\mathbf{f}] = 4|\mathbf{u}|^2\langle\mathbf{u}, \mathbf{f}\rangle,$$
$$\Psi''(\mathbf{u})[\mathbf{f}, \mathbf{g}] = 4|\mathbf{u}|^2\langle\mathbf{g}, \mathbf{f}\rangle + 8\langle\mathbf{u}, \mathbf{g}\rangle\langle\mathbf{u}, \mathbf{f}\rangle,$$

for any $\mathbf{u}, \mathbf{f}, \mathbf{g} \in \mathbf{H}_n$. In particular, the last identity implies that

$$\Psi''(\mathbf{u})[\mathbf{f}, \mathbf{f}] \le 12|\mathbf{u}|^2|\mathbf{f}|^2,$$

for any $\mathbf{u}, \mathbf{f} \in \mathbf{H}_n$. Therefore by Itô's formula to $\Psi(\mathbf{v}_n^\alpha) := |\mathbf{v}_n^\alpha(t)|^4$ we obtain

$$d|\mathbf{v}_n^\alpha(t)|^4 + 4|\mathbf{v}_n^\alpha(t)|^2\left[\nu\langle A\mathbf{v}_n^\alpha(t) + B(\mathbf{u}_n^\alpha(t), \mathbf{v}_n^\alpha(t)), \mathbf{v}_n^\alpha(t)\rangle\right]dt$$
$$\le C|\mathbf{v}_n^\alpha(t)|^2\|Q(\mathbf{u}_n^\alpha(t))\|^2_{\mathcal{L}_2(K,\mathbf{H})}dt + 4|\mathbf{v}_n^\alpha(t)|^2\langle\mathbf{v}_n^\alpha(t), P_nQ(\mathbf{u}_n^\alpha(t))dW(t)\rangle.$$

By using the identity (2.10), the Cauchy's inequality and Assumption 3.1-(i) along with Remark 3.1-(1) we infer the existence of a constant $c > 0$ such that

$$d|\mathbf{v}_n^\alpha(t)|^4 + 4\nu|\mathbf{v}_n^\alpha(t)|^2\langle A\mathbf{v}_n^\alpha(t), \mathbf{v}_n^\alpha(t)\rangle dt \le c|\mathbf{v}_n^\alpha(t)|^4dt + c\,dt + 4|\mathbf{v}_n^\alpha(t)|^2\langle\mathbf{v}_n^\alpha(t), P_nQ(\mathbf{u}_n^\alpha(t))dW(t)\rangle. \tag{3.9}$$

Since, by definition of $\mathbf{v}^\alpha$, $|\mathbf{u}^\alpha| \le c|\mathbf{v}^\alpha|$ we deduce from Assumption 3.1-(i) that

$$\|Q(\mathbf{u}_n^\alpha(s))\|^2_{\mathcal{L}_2(K,\mathbf{H})} \le c(1 + |\mathbf{v}^\alpha|)^2.$$

Now, using Burkholder–Davis–Gundy and Cauchy–Schwarz inequalities, we deduce that

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s |\mathbf{v}_n^\alpha(s)|^2\langle \mathbf{v}_n^\alpha(s),\, P_n Q(\mathbf{u}_n^\alpha(s))dW(s)\rangle\right|\\
&\le c\mathbb{E}\biggl(\int_0^t |\mathbf{v}_n^\alpha(s)|^4|\mathbf{v}_n^\alpha(s)|^2\|Q(\mathbf{u}_n^\alpha(s))\|^2_{\mathcal{L}_2(\mathrm{K},\mathbf{H})}ds\biggr)^{\frac{1}{2}},\\
&\le c\mathbb{E}\left[\sup_{s\in[0,t]}|\mathbf{v}_n^\alpha(s)|^2\left(\int_0^t c(1+|\mathbf{v}_n^\alpha(s)|)^4ds\right)^{\frac{1}{2}}\right]\\
&\le \frac{1}{2}\mathbb{E}\sup_{s\in[0,t]}|\mathbf{v}_n^\alpha(s)|^4 + cT + c\mathbb{E}\int_0^t |\mathbf{v}_n^\alpha(s)|^4ds.
\end{aligned}$$

From this last estimate and (3.9) we derive that there exists $C>0$ such that

$$\begin{aligned}
\mathbb{E}\sup_{s\in[0,t]}|\mathbf{v}_n^\alpha(s)|^4 + 8\nu\mathbb{E}\int_0^t |\mathbf{v}_n^\alpha(s)|^2\langle \mathrm{A}\mathbf{v}_n^\alpha(s),\, \mathbf{v}_n^\alpha(s)\rangle ds \le 2\mathbb{E}|\mathbf{v}^\alpha(0)|^4 + CT\\
+ C\mathbb{E}\int_0^t |\mathbf{v}_n^\alpha(s)|^4 ds.
\end{aligned}$$

Since $\langle \mathrm{A}\mathbf{v}_n^\alpha(s), \mathbf{v}_n^\alpha(s)\rangle = |\mathrm{A}^{\frac{1}{2}}\mathbf{v}_n^\alpha(s)|^2$ is nonnegative, and using Gronwall's lemma, we deduce that

$$\mathbb{E}\sup_{s\in[0,t]}|\mathbf{v}_n^\alpha(s)|^4+8\nu\mathbb{E}\int_0^t |\mathbf{v}_n^\alpha(s)|^2\langle \mathrm{A}\mathbf{v}_n^\alpha(s),\, \mathbf{v}_n^\alpha(s)\rangle ds \le (2\mathbb{E}|\mathbf{v}^\alpha(0)|^4+CT)(1+Ce^{CT})$$

Since α tend to 0, we can assume that $\alpha\in(0,1)$. Therefore, by lower semicontinuity of the norm and the fact that $|\mathbf{v}^\alpha(0)|^4 \le |\mathbf{u}_0 + \mathrm{A}\mathbf{u}_0|^4$, we infer that as $n\to\infty$

$$\mathbb{E}\sup_{s\in[0,t]}|\mathbf{v}^\alpha(s)|^4 + 8\nu\mathbb{E}\int_0^t |\mathbf{v}^\alpha(s)|^2\langle \mathrm{A}\mathbf{v}^\alpha(s),\, \mathbf{v}^\alpha(s)\rangle ds \le \mathfrak{K}_0 \tag{3.10}$$

where $\mathfrak{K}_0 := (2\mathbb{E}|\mathbf{u}_0 + \mathrm{A}\mathbf{u}_0|^4 + CT)(1+Ce^{CT})$. Since

$$\begin{aligned}
|\mathbf{v}^\alpha|^2 &= |\mathbf{u}^\alpha|^2 + 2\alpha^2|\mathrm{A}^{\frac{1}{2}}\mathbf{u}^\alpha|^2 + \alpha^4|\mathrm{A}\mathbf{u}^\alpha|^2,\\
\langle \mathrm{A}\mathbf{v}^\alpha, \mathbf{v}^\alpha\rangle &= |\mathrm{A}^{\frac{1}{2}}\mathbf{u}^\alpha|^2 + 2\alpha^2|\mathrm{A}\mathbf{u}^\alpha|^2 + \alpha^4|\mathrm{A}^{\frac{3}{2}}\mathbf{u}^\alpha|^2,
\end{aligned}$$

we deduce from (3.10) that the five estimates (3.3)–(3.7) hold. □

As a consequence of the estimate (3.4) and the Gagliardo–Nirenberg's inequality

$$\|\mathbf{v}^\alpha\|_{\mathbb{L}^4} \le c|\mathbf{v}^\alpha|^{\frac{1}{2}}|\mathrm{A}^{\frac{1}{2}}\mathbf{v}^\alpha|^{\frac{1}{2}},$$

we state the following corollary.

Corollary 3.6 *Under the assumptions of Proposition 3.5 there exists* $C > 0$ *such that for any* $\alpha \in (0, 1)$ *we have*

$$\mathbb{E}\int_0^T \|\mathbf{u}^\alpha(s)\|^4 ds \leq (2\mathbb{E}|\mathbf{u}_0 + \mathrm{A}\mathbf{u}_0|^4 + CT)(1 + Ce^{CT}). \tag{3.11}$$

Now we state several important estimates for the norm of $\mathbf{u}^\alpha$ in stronger norms.

Proposition 3.7 *Let Assumption 3.1 holds and let* $\mathbf{u}_0$ *be a* $\mathcal{F}_0$*-measurable random variable such that* $\mathbb{E}(|\mathrm{A}^{\frac{1}{2}}\mathbf{u}_0|^2 + |\mathrm{A}\mathbf{u}_0|^2)^2 < \infty$. *Then, there exists a generic constant* $K_0 > 0$ *such that for any* $\alpha \in (0, 1)$ *we have*

$$\mathbb{E}\sup_{t\in[0,T]}\Big[|\mathrm{A}^{\frac{1}{2}}\mathbf{u}^\alpha(t)|^4 + \alpha^4|\mathrm{A}\mathbf{u}^\alpha(t)|^4 + 2\alpha^2|\mathrm{A}^{\frac{1}{2}}\mathbf{u}^\alpha(t)|^2|\mathrm{A}\mathbf{u}(t)|^2\Big] \leq K_0, \tag{3.12}$$

$$\mathbb{E}\int_0^T |\mathrm{A}^{\frac{1}{2}}\mathbf{u}^\alpha(s)|^2|\mathrm{A}\mathbf{u}^\alpha(s)|^2 ds \leq K_0, \tag{3.13}$$

$$\alpha^2\mathbb{E}\int_0^T |\mathrm{A}\mathbf{u}^\alpha(s)|^4 ds \leq K_0, \tag{3.14}$$

$$\alpha^2\mathbb{E}\int_0^T |\mathrm{A}^{\frac{1}{2}}\mathbf{u}^\alpha(s)|^2|\mathrm{A}^{\frac{3}{2}}\mathbf{u}^\alpha(s)|^2 ds \leq K_0, \tag{3.15}$$

$$\alpha^4\mathbb{E}\int_0^T |\mathrm{A}\mathbf{u}^\alpha(s)|^2|\mathrm{A}^{\frac{3}{2}}\mathbf{u}^\alpha(s)|^2 ds \leq K_0, \tag{3.16}$$

where

$$K_0 := \left[2\mathbb{E}\left(|\mathrm{A}^{\frac{1}{2}}\mathbf{u}_0|^2 + |\mathrm{A}\mathbf{u}_0|^2\right)^2 + CT\right]\cdot\left[1 + Ce^{CT}\right].$$

Proof As in the proof of Proposition 3.5 we still consider the solution $\mathbf{v}_n^\alpha$ (or $\mathbf{u}_n^\alpha$) of the n-th Galerkin approximation of (1.1) defined by the system of SDEs (3.8). Let N_α be the self-adjoint and positive definite operator defined by $\mathrm{N}_\alpha\mathbf{v} = (\mathrm{I} + \alpha^2\mathrm{A})^{-1}\mathbf{v}$ for any $\mathbf{v} \in \mathbf{H}$. It is well-known that N_α^{-1} with domain $D(\mathrm{A})$ is also positive definite and self-adjoint on $\mathbf{H}$. Thus, the fractional powers $\mathrm{N}_\alpha^{\frac{1}{2}}$ and $\mathrm{N}_\alpha^{-\frac{1}{2}}$ are also self-adjoint. Since $\mathbf{v}^\alpha = \mathrm{N}_\alpha^{-1}\mathbf{u}^\alpha$ it follows from (3.8a) that

$$d\mathbf{u}_n^\alpha(t) + [\nu\mathrm{A}\mathbf{u}_n^\alpha(t) + \mathrm{N}_\alpha B(\mathbf{u}_n^\alpha(t), \mathbf{v}_n^\alpha(t))]dt = \mathrm{N}_\alpha P_n Q(\mathbf{u}_n^\alpha(t))dW(t), \tag{3.17}$$

Let $\Phi : D(\mathrm{A}) \to [0, \infty)$ be the mapping defined by $\Phi(\mathbf{v}) = \langle \mathrm{A}\mathbf{v}, \mathrm{N}_\alpha^{-1}\mathbf{v}\rangle$ for any $\mathbf{v} \in D(\mathrm{A})$. It is not difficut to show that $\Phi(\cdot)$ is twice Fréchet differentiable and its first and second derivatives satisfy

$$\begin{aligned}\Phi'(\mathbf{u}^\alpha)[\mathbf{f}] &= \langle \mathrm{A}\mathbf{u}^\alpha, \mathrm{N}_\alpha^{-1}\mathbf{f}\rangle + \langle \mathrm{N}_\alpha^{-1}\mathbf{u}^\alpha, \mathrm{A}\mathbf{f}\rangle\\ &= \langle A\mathbf{v}^\alpha, \mathbf{f}\rangle + \langle \mathbf{v}^\alpha, \mathrm{A}\mathbf{f}\rangle,\\ \Phi''(\mathbf{u}^\alpha)[\mathbf{f}, \mathbf{g}] &= \langle \mathrm{A}\mathbf{g}, \mathrm{N}_\alpha^{-1}\mathbf{f}\rangle + \langle \mathrm{A}\mathbf{f}, \mathrm{N}_\alpha^{-1}\mathbf{g}\rangle,\end{aligned}$$

for any $\mathbf{f}, \mathbf{g} \in D(A)$. In particular, the last identity and $\mathrm{A}^{\frac{1}{2}}$ and $\mathrm{N}_\alpha^{-\frac{1}{2}}$ being self-adjoint imply that

$$\Phi''(\mathbf{u}^\alpha)[\mathbf{f}, \mathbf{f}] = 2\left|\mathrm{A}^{\frac{1}{2}}\mathrm{N}_\alpha^{-\frac{1}{2}}\mathbf{f}\right|^2.$$

Therefore, using the Itô formula for $\Phi(\mathbf{u}_n^\alpha)$ and (3.17), we derive that there exists $c > 0$

$$\begin{aligned} d\Phi(\mathbf{u}_n^\alpha(t)) \leq\; & \Phi'(\mathbf{u}_n^\alpha(t))[-\nu \mathrm{A}\mathbf{u}_n^\alpha(t) - \mathrm{N}_\alpha B(\mathbf{u}_n^\alpha(t), \mathbf{v}_n^\alpha(t))]dt \\ & + \Phi'(\mathbf{u}_n^\alpha(t))[\mathrm{N}_\alpha P_n Q(\mathbf{u}_n^\alpha(t))]dW(t) \\ & + c\|\mathrm{A}^{\frac{1}{2}}\mathrm{N}_\alpha^{-\frac{1}{2}}\mathrm{N}_\alpha P_n Q(\mathbf{u}_n^\alpha(t))\|^2_{\mathcal{L}_2(\mathrm{K},\mathbf{H})}dt. \end{aligned}$$

Referring to the equation for $\Phi'(\mathbf{u}^\alpha)[\cdot]$ we see that

$$\begin{aligned} & \Phi'(\mathbf{u}_n^\alpha(t))\left[-\nu \mathrm{A}\mathbf{u}_n^\alpha(t) - \mathrm{N}_\alpha B(\mathbf{u}_n^\alpha(t), \mathbf{v}_n^\alpha(t))\right] \\ & = \langle \mathrm{A}\mathbf{v}_n^\alpha(t), -\nu \mathrm{A}\mathbf{u}_n^\alpha(t) - \mathrm{N}_\alpha B(\mathbf{u}_n^\alpha(t), \mathbf{v}_n^\alpha(t)) \\ & \quad + \left\langle \mathbf{v}_n^\alpha(t), \mathrm{A}\left[-\nu \mathrm{A}\mathbf{u}_n^\alpha(t) - \mathrm{N}_\alpha B(\mathbf{u}_n^\alpha(t), \mathbf{v}_n^\alpha(t))\right]\right\rangle, \end{aligned}$$

and

$$\begin{aligned} \Phi'(\mathbf{u}_n^\alpha(t))[\mathrm{N}_\alpha P_n Q(\mathbf{u}_n^\alpha(t))]dW(t) = & \left\langle \mathrm{A}\mathbf{v}_n^\alpha(t), \mathrm{N}_\alpha P_n Q(\mathbf{u}_n^\alpha(t))dW(t)\right\rangle \\ & + \left\langle \mathbf{v}_n^\alpha(t), \mathrm{A}[\mathrm{N}_\alpha P_n Q(\mathbf{u}_n^\alpha(t))dW(t)]\right\rangle. \end{aligned}$$

Hence

$$\begin{aligned} d\langle \mathrm{A}\mathbf{u}_n^\alpha(t), \mathbf{v}_n^\alpha(t)\rangle \leq & \left\langle \mathrm{A}\mathbf{u}_n^\alpha(t), d\mathbf{v}_n^\alpha(t)\right\rangle + \left\langle \mathbf{v}_n^\alpha(t), d\mathrm{A}\mathbf{u}_n^\alpha(t)\right\rangle \\ & + c\|\mathrm{A}^{\frac{1}{2}}\mathrm{N}_\alpha^{-\frac{1}{2}}\mathrm{N}_\alpha P_n Q(\mathbf{u}_n^\alpha(t))\|^2_{\mathcal{L}_2(\mathrm{K},\mathbf{H})}dt. \end{aligned} \tag{3.18}$$

First, we estimate the term $\langle \mathrm{A}\mathbf{u}_n^\alpha(t), d\mathbf{v}_n^\alpha(t)\rangle$. We derive from (3.8a) that

$$\begin{aligned} \left\langle \mathrm{A}\mathbf{u}_n^\alpha(t), d\mathbf{v}_n^\alpha(t)\right\rangle = & \left[-\langle \mathrm{A}\mathbf{u}_n^\alpha(t), \mathrm{A}\mathbf{v}_n^\alpha(t)\rangle - \langle B(\mathbf{u}_n^\alpha(t), \mathbf{v}_n^\alpha(t)), \mathrm{A}\mathbf{u}_n^\alpha(t)\rangle\right]dt \\ & + \left\langle \mathrm{A}\mathbf{u}_n^\alpha(t), P_n Q dW(t)\right\rangle. \end{aligned} \tag{3.19}$$

Recalling the definition of $\mathbf{v}_n^\alpha$ we derive that

$$\left\langle \mathrm{A}\mathbf{u}_n^\alpha(t), \mathrm{A}\mathbf{v}_n^\alpha(t)\right\rangle = \left|\mathrm{A}\mathbf{u}_n^\alpha(t)\right|^2 + \alpha^2\left|\mathrm{A}^{\frac{3}{2}}\mathbf{u}_n^\alpha(t)\right|^2.$$

Owing to the definition of $\mathbf{v}_n^\alpha$, (2.10) and (2.11) we have

$$\begin{aligned} \left\langle B(\mathbf{u}_n^\alpha(t), \mathbf{v}_n^\alpha(t)), \mathrm{A}\mathbf{u}_n^\alpha(t)\right\rangle & = \alpha^2\langle B(\mathbf{u}_n^\alpha(t), \mathrm{A}\mathbf{u}_n^\alpha(t), \mathrm{A}\mathbf{u}_n^\alpha(t)\rangle \\ & \quad + \left\langle B(\mathbf{u}_n^\alpha(t), \mathbf{u}_n^\alpha(t), \mathrm{A}\mathbf{u}_n^\alpha(t)\right\rangle \\ & = 0. \end{aligned} \tag{3.20}$$

Therefore we derive from (3.19)–(3.20) that

$$\left\langle \mathrm{A}\mathbf{u}_n^\alpha(t), d\mathbf{v}_n^\alpha(t)\right\rangle = -\left|\mathrm{A}\mathbf{u}_n^\alpha(t)\right|^2 - \alpha^2\left|\mathrm{A}^{\frac{3}{2}}\mathbf{u}_n^\alpha(t)\right|^2 + \left\langle \mathrm{A}\mathbf{u}_n^\alpha(t), P_n Q dW(t)\right\rangle. \quad (3.21)$$

Second, we treat the term $\langle \mathbf{v}_n^\alpha(t), d\mathrm{A}\mathbf{u}_n^\alpha(t)\rangle$, but before proceeding further we observe that

$$\mathrm{AN}_\alpha = \frac{1}{\alpha^2}[\mathrm{I} - \mathrm{N}_\alpha]$$

from which it follows that

$$\begin{aligned}\left\langle \mathrm{AN}_\alpha B(\mathbf{u}_n^\alpha(t), \mathbf{v}_n^\alpha(t)), \mathbf{v}_n^\alpha(t)\right\rangle &= \frac{1}{\alpha^2}\left\langle B(\mathbf{u}_n^\alpha(t), \mathbf{v}_n^\alpha(t), \mathbf{v}_n^\alpha(t)\right\rangle \\ &\quad - \frac{1}{\alpha^2}\left\langle \mathrm{N}_\alpha B(\mathbf{u}_n^\alpha(t), \mathbf{v}_n^\alpha(t)), \mathbf{v}_n^\alpha(t)\right\rangle \\ &= -\frac{1}{\alpha^2}\left\langle \mathrm{N}_\alpha B(\mathbf{u}_n^\alpha(t), \mathbf{v}_n^\alpha(t)), \mathbf{v}_n^\alpha(t)\right\rangle,\end{aligned}$$

where (2.9) was used to derive the last line. Since $\mathbf{v}_n^\alpha = \mathrm{N}_\alpha^{-1}\mathbf{u}_n^\alpha$, we obtain that

$$\begin{aligned}\left\langle \mathrm{N}_\alpha B(\mathbf{u}_n^\alpha(t), \mathbf{v}_n^\alpha(t)), \mathbf{v}_n^\alpha(t)\right\rangle &= \left\langle B(\mathbf{u}_n^\alpha(t), \mathbf{v}_n^\alpha(t)), \mathbf{u}_n^\alpha(t)\right\rangle \\ &= \left\langle B(\mathbf{u}_n^\alpha(t), \mathbf{u}_n^\alpha(t)), \mathbf{u}_n^\alpha(t)\right\rangle \\ &\quad + \alpha^2\left\langle B(\mathbf{u}_n^\alpha(t), \mathrm{A}\mathbf{u}_n^\alpha(t)), \mathbf{u}_n^\alpha(t)\right\rangle,\end{aligned}$$

Owing to this last identity, (2.9)–(2.11) we infer that

$$\left\langle \mathrm{AN}_\alpha B(\mathbf{u}_n^\alpha(t), \mathbf{v}_n^\alpha(t)), \mathbf{v}_n^\alpha(t)\right\rangle = 0. \quad (3.22)$$

Since

$$d\mathrm{A}\mathbf{u}_n^\alpha(t) = \left[-\mathrm{A}^2\mathbf{u}_n^\alpha(t) - \mathrm{AN}_\alpha B(\mathbf{u}_n^\alpha(t), \mathbf{v}_n^\alpha(t))\right]dt + \mathrm{AN}_\alpha P_n Q(\mathbf{u}_n^\alpha(t))dW(t),$$

it follows by invoking (3.22) and using the definition of $\mathbf{v}_n^\alpha$ that

$$\begin{aligned}\left\langle \mathbf{v}_n^\alpha(t), d\mathrm{A}\mathbf{u}_n^\alpha(t)\right\rangle &= -\left\langle \mathbf{u}_n^\alpha(t), \mathrm{A}^2\mathbf{u}_n^\alpha(t)\right\rangle - \alpha^2\left\langle \mathrm{A}\mathbf{u}_n^\alpha(t), \mathrm{A}^2\mathbf{u}_n^\alpha(t)\right\rangle \\ &\quad + \left\langle \mathbf{v}_n^\alpha(t), \mathrm{AN}_\alpha P_n Q(\mathbf{u}_n^\alpha(t))dW(t)\right\rangle.\end{aligned}$$

From this latter identity we easily derive that

$$\left\langle \mathbf{v}_n^\alpha(t), d\mathrm{A}\mathbf{u}_n^\alpha(t)\right\rangle = -\left|\mathrm{A}\mathbf{u}_n^\alpha(t)\right|^2 - \alpha^2\left|\mathrm{A}^{\frac{3}{2}}\mathbf{u}_n^\alpha(t)\right|^2 + \left\langle \mathbf{v}_n^\alpha(t), \mathrm{AN}_\alpha P_n Q(\mathbf{u}_n^\alpha(t))dW(t)\right\rangle. \quad (3.23)$$

Plugging (3.21) and (3.23) in (3.18) implies that

$$\left|\mathrm{A}^{\frac{1}{2}}\mathbf{u}_n^\alpha(t)\right|^2+\alpha^2\left|\mathrm{A}\mathbf{u}_n^\alpha(t)\right|^2-\left|\mathrm{A}^{\frac{1}{2}}\mathbf{u}_{0n}\right|^2-\alpha^2\left|\mathrm{A}\mathbf{u}_{0n}\right|^2$$
$$+2\int_0^t\left(|\mathrm{A}\mathbf{u}_n^\alpha(s)|^2+\alpha^2|\mathrm{A}^{\frac{3}{2}}\mathbf{u}_n^\alpha(t)|^2\right)ds$$
$$\leq c\int_0^t\|\mathrm{A}^{\frac{1}{2}}\mathrm{N}_\alpha^{\frac{1}{2}}P_nQ(\mathbf{u}_n^\alpha(t))\|^2_{\mathcal{L}_2(H)}ds+2\int_0^t\langle\mathrm{A}^{\frac{1}{2}}\mathbf{u}_n^\alpha(s),\mathrm{A}^{\frac{1}{2}}P_nQ(\mathbf{u}_n^\alpha(t))dW(s)\rangle, \tag{3.24}$$

where we have used the fact that

$$\begin{aligned}\left\langle\mathbf{v}_n^\alpha(s),\mathrm{A}\mathrm{N}_\alpha P_nQ(\mathbf{u}_n^\alpha(t))dW(s)\right\rangle&=\left\langle A\mathbf{u}_n^\alpha(s),P_nQ(\mathbf{u}_n^\alpha(t))dW(s)\right\rangle\\&=\left\langle\mathrm{A}^{\frac{1}{2}}\mathbf{u}_n^\alpha(s),\mathrm{A}^{\frac{1}{2}}P_nQ(\mathbf{u}_n^\alpha(t))dW(s)\right\rangle.\end{aligned}$$

By the Burkholder–Davis–Gundy, Cauchy–Schwarz, Cauchy inequalities and Assumption 3.1-(i) along with Remark 3.1-(2) we derive that

$$\mathbb{E}\sup_{r\in[0,t]}\left|\int_0^r\langle\mathrm{A}\mathbf{u}_n^\alpha(s),P_nQ(\mathbf{u}_n^\alpha(t))dW(s)\rangle\right|$$
$$\leq c\mathbb{E}\left[\sup_{s\in[0,t]}|\mathrm{A}^{\frac{1}{2}}\mathbf{u}_n^\alpha(s)|\times\left(\int_0^t\|\mathrm{A}^{\frac{1}{2}}Q(\mathbf{u}_n^\alpha(s))\|_{\mathcal{L}_2(\mathrm{K},\mathbf{H})}ds\right)^{\frac{1}{2}}\right]$$
$$\leq\frac{1}{4}\mathbb{E}\left(\sup_{s\in[0,t]}\left[|\mathrm{A}^{\frac{1}{2}}\mathbf{u}_n^\alpha(s)|^2+\alpha^2|\mathrm{A}\mathbf{u}_n^\alpha(s)|^2\right]\right)+c^2\mathbb{E}\int_0^t\|\mathrm{A}^{\frac{1}{2}}Q(\mathbf{u}_n^\alpha(s))\|^2_{\mathcal{L}_2(\mathrm{K},\mathbf{H})}ds$$
$$\leq\frac{1}{4}\mathbb{E}\left(\sup_{s\in[0,t]}\left[|\mathrm{A}^{\frac{1}{2}}\mathbf{u}_n^\alpha(s)|^2+\alpha^2|\mathrm{A}\mathbf{u}_n^\alpha(s)|^2\right]\right)+c^2\ell_3^2\mathbb{E}\int_0^t(1+|\mathrm{A}^{\frac{1}{2}}\mathbf{u}_n^\alpha(s)|)^2ds$$
$$\leq\frac{1}{4}\mathbb{E}\left(\sup_{s\in[0,t]}\left[|\mathrm{A}^{\frac{1}{2}}\mathbf{u}_n^\alpha(s)|^2+\alpha^2|\mathrm{A}\mathbf{u}_n^\alpha(s)|^2\right]\right)+cT$$
$$+c\mathbb{E}\int_0^t\left[|\mathrm{A}^{\frac{1}{2}}\mathbf{u}_n^\alpha(s)|^2+\alpha^2|\mathrm{A}\mathbf{u}_n^\alpha(s)|^2\right]ds. \tag{3.25}$$

Since $\mathrm{N}_\alpha^{\frac{1}{2}}$ is self-adjoint and $\|\mathrm{N}_\alpha\|_{\mathcal{L}(H)}\leq 1$ we infer that

$$\|\mathrm{N}_\alpha^{\frac{1}{2}}\|_{\mathcal{L}(\mathbf{H})}\leq 1. \tag{3.26}$$

Thus, it follows from Assumption 3.1-(ii) along with Remark 3.1-(2)

$$\int_0^t\|\mathrm{A}^{\frac{1}{2}}\mathrm{N}_\alpha^{\frac{1}{2}}P_nQ(\mathbf{u}_n^\alpha(t))\|^2_{\mathcal{L}_2(\mathrm{K},\mathbf{H})}ds\leq cT+c\mathbb{E}\int_0^t\left[|\mathrm{A}^{\frac{1}{2}}\mathbf{u}_n^\alpha(s)|^2+\alpha^2|\mathrm{A}\mathbf{u}_n^\alpha(s)|^2\right]ds. \tag{3.27}$$

Hence, the calculations between (3.24) and (3.27) yield

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left[|\mathrm{A}^{\frac{1}{2}}\mathbf{u}_n^\alpha(t)|^2+\alpha^2|\mathrm{A}\mathbf{u}_n^\alpha(t)|^2\right]\right)+4\mathbb{E}\int_0^t\left(|\mathrm{A}\mathbf{u}_n^\alpha(s)|^2+\alpha^2|\mathrm{A}^{\frac{3}{2}}\mathbf{u}_n^\alpha(t)|^2\right)ds$$
$$\le cT+c\mathbb{E}\int_0^t\left[|\mathrm{A}^{\frac{1}{2}}\mathbf{u}_n^\alpha(s)|^2+\alpha^2|\mathrm{A}\mathbf{u}_n^\alpha(s)|^2\right]ds+2|\mathrm{A}^{\frac{1}{2}}\mathbf{u}_{0n}|^2+2\alpha^2|\mathrm{A}\mathbf{u}_{0n}|^2.$$

Since $\alpha\in(0,1)$ we derive from the last estimate that for any $\alpha\in(0,1)$ and $n\in\mathbb{N}$

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left[|\mathrm{A}^{\frac{1}{2}}\mathbf{u}_n^\alpha(t)|^2+\alpha^2|\mathrm{A}\mathbf{u}_n^\alpha(t)|^2\right]\right)+2\mathbb{E}\int_0^t\left(|\mathrm{A}\mathbf{u}_n^\alpha(s)|^2+\alpha^2|\mathrm{A}^{\frac{3}{2}}\mathbf{u}_n^\alpha(t)|^2\right)ds$$
$$\le cT+c\mathbb{E}\int_0^t\left[|\mathrm{A}^{\frac{1}{2}}\mathbf{u}_n^\alpha(s)|^2+\alpha^2|\mathrm{A}\mathbf{u}_n^\alpha(s)|^2\right]ds+2|\mathrm{A}^{\frac{1}{2}}\mathbf{u}_0|^2+2|\mathrm{A}\mathbf{u}_0|^2.$$

Now it follows from Gronwall's lemma that there exists $C>0$ such that for any $\alpha\in(0,1)$ and $n\in\mathbb{N}$ we have

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left[|\mathrm{A}^{\frac{1}{2}}\mathbf{u}_n^\alpha(t)|^2+\alpha^2|\mathrm{A}\mathbf{u}_n^\alpha(t)|^2\right]\right)+2\mathbb{E}\int_0^t\left(|\mathrm{A}\mathbf{u}_n^\alpha(s)|^2+\alpha^2|\mathrm{A}^{\frac{3}{2}}\mathbf{u}_n^\alpha(t)|^2\right)ds$$
$$\le\left[CT+2|\mathrm{A}^{\frac{1}{2}}\mathbf{u}_0|^2+2|\mathrm{A}\mathbf{u}_0|^2\right](1+Ce^{CT}).$$

As $n\to\infty$, by lower semicontinuity we deduce that

$$\mathbb{E}\left(\sup_{t\in[0,T]}[|\mathrm{A}^{\frac{1}{2}}\mathbf{u}^\alpha(t)|^2+\alpha^2|\mathrm{A}\mathbf{u}^\alpha(t)|^2]\right)$$
$$+2\mathbb{E}\int_0^t\left(|\mathrm{A}\mathbf{u}^\alpha(s)|^2+\alpha^2|\mathrm{A}^{\frac{3}{2}}\mathbf{u}^\alpha(t)|^2\right)ds\le K. \tag{3.28}$$

where

$$K:=\left[CT+2|\mathrm{A}^{\frac{1}{2}}\mathbf{u}_0|^2+2|\mathrm{A}\mathbf{u}_0|^2\right](1+Ce^{CT}).$$

Now, let $y_n^\alpha(t)=|\mathrm{A}^{\frac{1}{2}}\mathbf{u}_n^\alpha(t)|^2+\alpha^2|\mathrm{A}\mathbf{u}_n^\alpha(t)|^2$. Observing that

$$\langle\mathbf{v}_n^\alpha(t),\mathrm{A}\mathbf{u}_n^\alpha(t)\rangle=|\mathrm{A}^{\frac{1}{2}}\mathbf{u}_n^\alpha(t)|^2+\alpha^2|\mathrm{A}\mathbf{u}_n^\alpha(t)|^2,$$

we see that $[y_n^\alpha(t)]^2=[\Phi(\mathbf{u}_n^\alpha(t))]^2$. Therefore, from Itô's formula and (3.26) we deduce that

$$d([y_n^\alpha]^2(t)) \le [-4y_n^\alpha(t)(|\mathrm{A}\mathbf{u}_n^\alpha(t)|^2+\alpha^2|\mathrm{A}^{\frac{3}{2}}\mathbf{u}_n^\alpha(t)|^2)+2cy_n^\alpha(t)\|\mathrm{A}^{\frac{1}{2}}Q(\mathbf{u}_n^\alpha(t))\|^2_{\mathcal{L}_2(\mathrm{K},\mathbf{H})}$$
$$+c|\mathrm{A}^{\frac{1}{2}}\mathbf{u}_n^\alpha(t)|^2\|\mathrm{A}^{\frac{1}{2}}Q(\mathbf{u}_n^\alpha(t))\|^2_{\mathcal{L}_2(\mathrm{K},\mathbf{H})}]dt$$
$$+4y_n^\alpha(t)\langle \mathrm{A}^{\frac{1}{2}}\mathbf{u}_n^\alpha(t), \mathrm{A}^{\frac{1}{2}}Q(\mathbf{u}_n^\alpha(t))dW(t)\rangle.$$

From this last inequality we infer that

$$\mathbb{E}\sup_{s\in[0,t]}[y_n^\alpha]^2(s)-[y_n^\alpha(0)]^2+4\int_0^t y_n^\alpha(s)\left[|\mathrm{A}\mathbf{u}_n^\alpha(t)|^2+\alpha^2|\mathrm{A}^{\frac{3}{2}}\mathbf{u}_n^\alpha(t)|^2\right]ds$$
$$\le 4\mathbb{E}\sup_{r\in[0,t]}\left|\int_0^r y_n^\alpha(s)\left\langle \mathrm{A}^{\frac{1}{2}}\mathbf{u}_n^\alpha(t), \mathrm{A}^{\frac{1}{2}}Q(\mathbf{u}_n^\alpha(s))dW(s)\right\rangle\right|$$
$$+c\mathbb{E}\int_0^t y_n^\alpha(s)\|\mathrm{A}^{\frac{1}{2}}Q(\mathbf{u}_n^\alpha(s))\|^2_{\mathcal{L}_2(\mathrm{K},\mathbf{H})}ds. \tag{3.29}$$

Thanks to Remark 3.1-(2) we easily derive that

$$y_n^\alpha(s)\|\mathrm{A}^{\frac{1}{2}}Q(\mathbf{u}_n^\alpha(s))\|^2_{\mathcal{L}_2(\mathrm{K},\mathbf{H})}\le C(1+[y_n^\alpha(s)]^2) \tag{3.30}$$

Now, arguing as in the proof of (3.25) and using this last inequality we obtain the following estimates

$$4\mathbb{E}\sup_{r\in[0,t]}\left|\int_0^r y_n^\alpha(s)\left\langle \mathrm{A}^{\frac{1}{2}}\mathbf{u}_n^\alpha(t), \mathrm{A}^{\frac{1}{2}}Q(\mathbf{u}_n^\alpha(s))dW(s)\right\rangle\right|$$
$$\le \frac{1}{2}\mathbb{E}\sup_{s\in[0,t]}[y_n^\alpha(s)|\mathrm{A}^{\frac{1}{2}}\mathbf{u}_n^\alpha(s)|^2]+c\mathbb{E}\int_0^t y_n^\alpha(s)\|\mathrm{A}^{\frac{1}{2}}Q(\mathbf{u}_n^\alpha(s))\|^2_{\mathcal{L}_4(\mathbf{H})}ds$$
$$\le \frac{1}{2}\mathbb{E}\sup_{s\in[0,t]}[y_n^\alpha(s)\times\left(|\mathrm{A}^{\frac{1}{2}}\mathbf{u}_n^\alpha(s)|^2(s)+\alpha^2|\mathrm{A}\mathbf{u}_n^\alpha(s)|^2\right)]$$
$$+c\mathbb{E}\int_0^t y_n^\alpha(s)(1+|\mathrm{A}^{\frac{1}{2}}\mathbf{u}_n^\alpha(s)|^2)ds$$
$$\le \frac{1}{2}\mathbb{E}\sup_{s\in[0,t]}[y_n^\alpha(s)]^2+cT+c\mathbb{E}\int_0^t[y_n^\alpha(s)]^2ds.$$

Taking the latter estimate and (3.30) into (3.29)

$$\mathbb{E}\sup_{s\in[0,t]}[y_n^\alpha]^2(s)+8\mathbb{E}\int_0^t y_n^\alpha(s)\left[|\mathrm{A}\mathbf{u}_n^\alpha(t)|^2+\alpha^2|\mathrm{A}^{\frac{3}{2}}\mathbf{u}_n^\alpha(t)|^2\right]ds$$
$$\le 2[y_n^\alpha(0)]^2+cT+c\mathbb{E}\int_0^t[y_n^\alpha(s)]^2ds.$$

Applying Gronwall's lemma and (3.28) imply that there exists $C>0$ such that

$$\mathbb{E}\sup_{s\in[0,t]}[y_n^\alpha]^2(s)+8\mathbb{E}\int_0^t y_n^\alpha(s)[|\mathrm{A}\mathbf{u}_n^\alpha(t)|^2+\alpha^2|\mathrm{A}^{\frac{3}{2}}\mathbf{u}_n^\alpha(t)|^2]ds\le K_0,$$

where

$$K_0 := \left[2(|A^{\frac{1}{2}}\mathbf{u}_0|^2 + |A\mathbf{u}_0|^2)^2 + CT\right] \cdot \left[1 + Ce^{KT}\right].$$

Recalling the definition of y_n^α and by lower semicontinuity we infer from the last estimate that as $n \to \infty$

$$\mathbb{E} \sup_{s\in[0,T]} [|A^{\frac{1}{2}}\mathbf{u}^\alpha(t)|^2 + \alpha^2|A\mathbf{u}^\alpha(t)|^2]^2 \le K_0, \tag{3.31}$$

$$8\mathbb{E}\int_0^T \left(|A^{\frac{1}{2}}\mathbf{u}^\alpha(t)|^2 + \alpha^2|A\mathbf{u}^\alpha(t)|^2\right)\left(|A\mathbf{u}^\alpha(t)|^2 + \alpha^2|A^{\frac{3}{2}}\mathbf{u}^\alpha(t)|^2\right)ds \le K_0. \tag{3.32}$$

By straightforward calculations we easily derive from (3.31) and (3.32) the set of estimates (3.12)–(3.16) stated in Proposition 3.7. □

4 Rate of Convergence of the Sequence $\mathbf{u}^\alpha$ to u

In this section we consider a sequence $\{\alpha_n; n \in \mathbb{N}\} \subset (0, 1)$ such that $\alpha_n \to 0$ as $n \nearrow \infty$. For each $n \in \mathbb{N}$ let $\mathbf{u}^{\alpha_n}$ be the unique solution to (1.1) and for each $R \in \mathbb{R}$ define a family of stopping times τ_R^n by

$$\tau_R^n := \inf\left\{t \in [0, T]; \int_0^t \|\mathbf{u}^{\alpha_n}(s)\|^2 ds \ge R\right\}. \tag{4.1}$$

Let **u** be the solution of the stochastic Navier–Stokes equations; that is, **u** solves (1.2). In the following theorem we will show that by localization procedure the sequence $\mathbf{u}^\alpha$ converges strongly in $L^2(\Omega, L^\infty(0, T; \mathbf{H}))$ and $L^2(\Omega, L^2(0, T; \mathbf{V}))$ and the strong speed of convergence is of order $O(\alpha)$.

Theorem 4.1 *Let Assumption 3.1 holds and let* $\mathbf{u}_0$ *be a* $\mathcal{F}_0$*-measurable random variable such that* $\mathbb{E}(|A^{\frac{1}{2}}\mathbf{u}_0|^2 + |A\mathbf{u}_0|^2)^2 < \infty$. *Then there exists* $C > 0$, $\kappa_0(T) > 0$ *such that for any* $R > 0$ *and* $n \in \mathbb{N}$ *we have*

$$\mathbb{E} \sup_{s\in[0,t\wedge\tau_R^n]} |\mathbf{u} - \mathbf{u}^{\alpha_n}|^2 + 4\mathbb{E}\int_0^{t\wedge\tau_R^n} |A^{\frac{1}{2}}[\mathbf{u} - \mathbf{u}^{\alpha_n}]|^2 ds \le \alpha_n^2\beta(R)\kappa_0 e^{C(T)\beta(R)T}, \tag{4.2}$$

where $C(T) := c(1 + T)$, $\beta(R) := 1 + CRe^{CR}$ *and*

$$\kappa_0(T) := CT + CT^2 + CK_0 + CTK_1 + C(T).$$

Proof Let us fix $n \in \mathbb{N}$ and let $\mathbf{u}^{\alpha_n}$ be the unique solution to (1.1). Let **u** be the unique solution to (1.2). Let us also fix $R > 0$ and let τ_R^n be the stopping time defined above.

For sake of simplicity we set $\tau = t \wedge \tau_R^n$ for any $t \in [0, T]$ and $\alpha := \alpha_n$ for any $n \in \mathbb{N}$. Let $\delta = \mathbf{u} - \mathbf{u}^\alpha$. The stochastic process $\delta(t)$ with initial condition $\delta(0) = 0$ solves

$$\begin{aligned} d\delta(t) = &\big[-\mathrm{A}\delta(t) - B(\mathbf{u}(t), \mathbf{u}(t) + \mathrm{N}_\alpha B(\mathbf{u}^\alpha(t), \mathbf{v}^\alpha(t))\big]\, dt \\ &+ Q(\mathbf{u}(t)) - \mathrm{N}_\alpha Q(\mathbf{u}^\alpha(t)))dW(t). \end{aligned}$$

Equivalently,

$$\begin{aligned} &d\delta(t) + \big[\mathrm{A}\delta(t) + B(\mathbf{u}(t), \mathbf{u}(t) - B(\mathbf{u}^\alpha(t), \mathbf{u}^\alpha(t))\big]\, dt - (Q(\mathbf{u}(t)) \\ &-\mathrm{N}_\alpha Q(\mathbf{u}^\alpha(t)))dW(t) \\ &= \big[\mathrm{N}_\alpha B(\mathbf{u}^\alpha(t), \mathbf{v}^\alpha(t)) - B(\mathbf{u}^\alpha(t), \mathbf{u}^\alpha(t))\big]\, dt. \end{aligned}$$

From Itô's formula we infer that

$$\begin{aligned} &\sup_{s\in[0,\tau]} |\delta(s)|^2 + 2\int_0^\tau \|\delta(s)\|^2 ds \\ &\le 2\int_0^\tau \big\langle B(\mathbf{u}^\alpha(s), \mathbf{u}^\alpha(s) - B(\mathbf{u}(s), \mathbf{u}(s)), \delta(s)\big\rangle\, ds \\ &\quad +2\int_0^\tau \big\langle [\mathrm{N}_\alpha B(\mathbf{u}^\alpha(s), \mathbf{v}^\alpha(s)) - B(\mathbf{u}^\alpha(s), \mathbf{u}^\alpha(s))], \delta(s)]\big\rangle\, ds \\ &\quad + \int_0^\tau \|Q(\mathbf{u}(s)) - \mathrm{N}_\alpha Q(\mathbf{u}(s))\|^2_{\mathcal{L}_2(\mathrm{K},\mathbf{H})} ds \\ &\quad + 2\int_0^\tau \langle \delta(s), (Q(\mathbf{u}(t)) - \mathrm{N}_\alpha Q(\mathbf{u}(t)))dW(s)\rangle \\ &\le |J_1| + |J_2| + |J_3| + J_4 + J_5(t), \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} J_1 &:= 2\int_0^\tau \big\langle B(\mathbf{u}^\alpha(s), \mathbf{u}^\alpha(s) - B(\mathbf{u}(s), \mathbf{u}(s)), \delta(s)\big\rangle\, ds, \\ J_2 &:= 2\int_0^\tau \big\langle \mathrm{N}_\alpha[B(\mathbf{u}^\alpha(s), \mathbf{v}^\alpha(s)) - B(\mathbf{u}^\alpha(s), \mathbf{u}^\alpha(s))], \delta(s)\big\rangle\, ds, \\ J_3 &:= \int_0^\tau \big\langle (\mathrm{N}_\alpha - \mathrm{I})B(\mathbf{u}^\alpha(s), \mathbf{u}^\alpha(s)), \delta(s)\big\rangle\, ds, \\ J_4 &:= \int_0^\tau \|Q(\mathbf{u}(s)) - \mathrm{N}_\alpha Q(\mathbf{u}(s))\|^2_{\mathcal{L}_2(\mathrm{K},\mathbf{H})} ds, \\ J_5(t) &:= 2\int_0^\tau \langle \delta(s), (Q(\mathbf{u}(t)) - \mathrm{N}_\alpha Q(\mathbf{u}(t)))dW(s)\rangle\,. \end{aligned}$$

Using the well-known fact

$$\langle B(\mathbf{u}^\alpha(s), \mathbf{u}^\alpha(s) - B(\mathbf{u}(s), \mathbf{u}(s)), \delta(s)\rangle = -\langle B(\delta(s), \delta(s)), \mathbf{u}^\alpha(s)\rangle,$$

the Cauchy–Schwarz inequality, the Gagliardo–Nirenberg inequality and the Young inequality we obtain the chain of inequalities

$$\begin{aligned} |J_1| &\leq 2c \int_0^\tau \|\delta(s)\|_{\mathbb{L}^4} \|\delta(s)\| \|\mathbf{u}^\alpha(s)\|_{\mathbb{L}^4} ds \\ &\leq 2c \int_0^\tau |\delta(s)|^{\frac{1}{2}} \|\delta(s)\|^{\frac{3}{2}} \|\mathbf{u}^\alpha(s)\|_{\mathbb{L}^4} ds, \\ |J_1| &\leq \frac{1}{2} \int_0^\tau \|\delta(s)\|^2 ds + c \int_0^\tau |\delta(s)|^2 \|\mathbf{u}^\alpha(s)\|_{\mathbb{L}^4}^4 ds. \end{aligned} \tag{4.4}$$

By using the definition $\mathbf{v}^\alpha = \mathbf{u}^\alpha + \alpha^2 \mathrm{A}\mathbf{u}^\alpha$ we see that

$$\begin{aligned} \langle \mathrm{N}_\alpha [B(\mathbf{u}^\alpha(s), \mathbf{v}^\alpha(s)) - B(\mathbf{u}^\alpha(s), \mathbf{u}^\alpha(s))], \delta(s)\rangle &= \alpha^2 \langle \mathrm{N}_\alpha B(\mathbf{u}^\alpha(s), \mathrm{A}\mathbf{u}^\alpha(s)), \delta(s)\rangle \\ &= \alpha^2 \langle B(\mathbf{u}^\alpha(s), \mathrm{A}\mathbf{u}^\alpha(s)), \mathrm{N}_\alpha \delta(s)\rangle \\ &= \alpha^2 \langle B(\mathbf{u}^\alpha(s), \mathrm{N}_\alpha \delta(s)), \mathrm{A}\mathbf{u}^\alpha(s)\rangle. \end{aligned}$$

From the last line along with the Cauchy–Schwarz inequality, the embedding $\mathbb{H}^2 \subset \mathbb{L}^\infty$, (3.26), and (2.3) it follows that

$$\begin{aligned} &|\langle \mathrm{N}_\alpha [B(\mathbf{u}^\alpha(s), \mathbf{v}^\alpha(s)) - B(\mathbf{u}^\alpha(s), \mathbf{u}^\alpha(s))], \delta(s)\rangle| \\ &\quad \leq c\alpha^2 \|\mathbf{u}^\alpha(s)\|_{\mathbb{L}^\infty(\mathcal{O})} |\mathrm{N}_\alpha \mathrm{A}^{\frac{1}{2}} \delta(s)| |\mathrm{A}\mathbf{u}^\alpha(s)|, \\ &\quad \leq c\alpha^2 |\mathrm{A}\mathbf{u}^\alpha(s)|^2 |\mathrm{A}^{\frac{1}{2}} \delta(s)|. \end{aligned}$$

Applying the Cauchy inequality in the last estimate implies that

$$|\langle \mathrm{N}_\alpha [B(\mathbf{u}^\alpha(s), \mathbf{v}^\alpha(s)) - B(\mathbf{u}^\alpha(s), \mathbf{u}^\alpha(s))], \delta(s)\rangle| \leq \frac{1}{2} \|\delta(s)\|^2 + c\alpha^4 |\mathrm{A}\mathbf{u}^\alpha(s)|^4.$$

Thus,

$$|J_2| \leq \frac{1}{2} \int_0^\tau \|\delta(s)\|^2 ds + c\alpha^2 \int_0^\tau \alpha^2 |\mathrm{A}\mathbf{u}^\alpha(s)|^2 ds. \tag{4.5}$$

Invoking [10, Lemma 4.1] we infer that

$$|\langle (\mathrm{N}_\alpha - \mathrm{I}) B(\mathbf{u}^\alpha(s), \mathbf{u}^\alpha(s)), \delta(s)\rangle| \leq c\frac{\alpha}{2} |B(\mathbf{u}^\alpha(s), \mathbf{u}^\alpha(s))| \|\delta(s)\|,$$

from which along Cauchy–Schwarz, the embedding $\mathbb{H}^1(\mathcal{O}) \subset \mathbb{L}^4(\mathcal{O})$, (2.3) and the Cauchy inequality we derive that

$$\begin{aligned} |\langle (\mathrm{N}_\alpha - \mathrm{I}) B(\mathbf{u}^\alpha(s), \mathbf{u}^\alpha(s)), \delta(s)\rangle| &\leq c\frac{\alpha}{2} \|\mathbf{u}^\alpha(s)\|_{\mathbb{L}^4(\mathcal{O})} \|\nabla \mathbf{u}^\alpha(s)\|_{\mathbb{L}^4(\mathcal{O})} \|\delta(s)\| \\ &\leq c\frac{\alpha}{2} \|\mathbf{u}^\alpha(s)\| |\mathrm{A}\mathbf{u}^\alpha(s)| \|\delta(s)\| \\ &\leq \frac{1}{2} \|\delta(s)\| + c\frac{\alpha^2}{4} \|\mathbf{u}^\alpha(s)\|^2 |\mathrm{A}\mathbf{u}^\alpha(s)|^2. \end{aligned}$$

Hence

$$|J_3| \leq \frac{1}{2}\int_0^\tau \|\delta(s)\|^2 ds + c\frac{\alpha^2}{4}\int_0^\tau \|\mathbf{u}^\alpha(s)\|^2 |\mathrm{A}\mathbf{u}^\alpha(s)|^2 ds. \tag{4.6}$$

Since

$$Q(\mathbf{u}) - \mathrm{N}_\alpha Q(\mathbf{u}^\alpha) = [Q(\mathbf{u}) - \mathrm{N}_\alpha Q(\mathbf{u})] + [\mathrm{N}_\alpha Q(\mathbf{u}) - \mathrm{N}_\alpha Q(\mathbf{u}^\alpha)],$$

we infer that

$$\begin{aligned} J_4 &\leq c\int_0^\tau \|Q(\mathbf{u}(s)) - \mathrm{N}_\alpha Q(\mathbf{u}(s))\|^2_{\mathcal{L}_2(\mathrm{K},\mathbf{H})} ds \\ &\quad + c\int_0^\tau \|[\mathrm{N}_\alpha Q(\mathbf{u}(s)) - \mathrm{N}_\alpha Q(\mathbf{u}^\alpha(s))]\|^2_{\mathcal{L}_2(\mathrm{K},\mathbf{H})} \\ &:= J_{4,1} + J_{4,2}. \end{aligned}$$

Since $Q(\mathbf{u}(t)) - \mathrm{N}_\alpha Q(\mathbf{u}(t)) = \alpha^2 \mathrm{A}\mathrm{N}_\alpha Q(\mathbf{u}(t))$ we easily check that

$$\begin{aligned} J_{4,1} &\leq c\alpha^2 \int_0^\tau \|\alpha \mathrm{A}\mathrm{N}_\alpha Q(\mathbf{u}(s))\|^2_{\mathcal{L}_2(\mathrm{K},\mathbf{H})} ds \\ &\leq c\alpha^2 \int_0^\tau \|\alpha \mathrm{A}^{\frac{1}{2}}\mathrm{N}_\alpha \mathrm{A}^{\frac{1}{2}} Q(\mathbf{u}(s))\|^2_{\mathcal{L}_2(\mathrm{K},\mathbf{H})} ds \\ &\leq c\alpha^2 \|\alpha \mathrm{A}^{\frac{1}{2}}\mathrm{N}_\alpha\|^2_{\mathcal{L}(\mathbf{H})} \int_0^\tau \|\mathrm{A}^{\frac{1}{2}} Q(\mathbf{u}(s))\|^2_{\mathcal{L}_2(\mathrm{K},\mathbf{H})} ds. \end{aligned}$$

Owing to Assumption 3.1-(ii) altogether with Remark 3.1-(2) we obtain that

$$\begin{aligned} J_{4,1} &\leq c\alpha^2 \|\alpha \mathrm{A}^{\frac{1}{2}}\mathrm{N}_\alpha\|^2_{\mathcal{L}(\mathbf{H})} \int_0^\tau (1 + |\mathrm{A}^{\frac{1}{2}}\mathbf{u}(s)|)^2 ds \\ &\leq c\alpha^2 \|\alpha \mathrm{A}^{\frac{1}{2}}\mathrm{N}_\alpha\|^2_{\mathcal{L}(\mathbf{H})} [cT + c\int_0^\tau |\mathrm{A}^{\frac{1}{2}}\mathbf{u}(s)|^2 ds] \end{aligned}$$

It follows from the last estimate and [10, ProofofLemma4.1] that

$$J_{4,1} \leq c\frac{\alpha^2}{4}\left(T + \int_0^\tau |\mathrm{A}^{\frac{1}{2}}\mathbf{u}(s)|^2 ds\right). \tag{4.7}$$

Now to estimate $J_{4,2}$ we use the fact that $\|\mathrm{N}_\alpha\|_{\mathcal{L}(\mathbf{H})} \leq 1$ and Assumption 3.1-(i) and derive that

$$\begin{aligned} J_{4,2} &\leq c\int_0^\tau \|\mathrm{N}_\alpha\|^2_{\mathcal{L}(\mathbf{H})} \|Q(\mathbf{u}(s)) - Q(\mathbf{u}^\alpha(s))\|^2_{\mathcal{L}_2(\mathrm{K},\mathbf{H})} ds \\ &\leq c\int_0^\tau |\delta(s)|^2 ds. \end{aligned} \tag{4.8}$$

Thus, the two estimates (4.7) and (4.8) yield that

$$J_4 \le \frac{\alpha^2}{4}\left(cT + c\int_0^\tau |\mathrm{A}^{\frac{1}{2}}\mathbf{u}(s)|^2 ds\right) + c\int_0^\tau |\delta(s)|^2 ds. \tag{4.9}$$

It follows from (4.4), (4.5), (4.6) and (4.9) that

$$\sup_{s\in[0,\tau]} |\delta(s)|^2 + 4\int_0^\tau |\mathrm{A}^{\frac{1}{2}}\delta(s)|^2 ds \le \alpha^2 \int_0^\tau \mathfrak{X}(s)|\delta(s)|^2 ds + \alpha^2\mathfrak{Y}(t) + \mathfrak{Z}(t), \tag{4.10}$$

where

$$\begin{aligned}
\mathfrak{Y}(t) &:= c\int_0^\tau \left(\alpha^2|\mathrm{A}\mathbf{u}^\alpha(s)|^4 + |\mathrm{A}^{\frac{1}{2}}\mathbf{u}^\alpha(s)|^2|\mathrm{A}\mathbf{u}^\alpha(s)|^2 + |\mathrm{A}^{\frac{1}{2}}\mathbf{u}(s)|^2\right) ds + cT,\\
\mathfrak{Z}(t) &:= \sup_{s\in[0,\tau]} |J_5(s)| + c\int_0^\tau |\delta(s)|^2 ds,\\
\mathfrak{X}(t) &:= \|\mathbf{u}^\alpha(t)\|^4_{\mathbb{L}^4}.
\end{aligned}$$

Note that using the definition of τ_R^α it is not difficult to see that

$$\int_0^\tau \mathfrak{X}(s) ds \le R.$$

Note also that it follows from Gronwall's lemma that

$$\sup_{s\in[0,\tau]} |\delta(s)|^2 + 4\int_0^\tau |\mathrm{A}^{\frac{1}{2}}\delta(s)|^2 ds \le (\alpha^2\mathfrak{Y}(t) + \mathfrak{Z}(t))\cdot(1 + CRe^{CR})$$

Hence taking the mathematical expectation in (4.10) we obtain that

$$\mathbb{E}\sup_{s\in[0,\tau]} |\delta(s)|^2 + 4\mathbb{E}\int_0^\tau |\mathrm{A}^{\frac{1}{2}}\delta(s)|^2 ds \le (\alpha^2\mathbb{E}\mathfrak{Y}(t) + \mathbb{E}\mathfrak{Z}(t))\cdot(1 + CRe^{CR}) \tag{4.11}$$

Owing to the definition of $\mathfrak{Y}$ and Proposition 3.7 we derive that

$$\mathbb{E}\mathfrak{Y}(t) \le CT + CK_0. \tag{4.12}$$

Now we deal with the estimation of $\mathbb{E}\mathfrak{Z}(t)$. By the Burkholder–Davis–Gundy inequality we deduce that

$$\begin{aligned}
\mathbb{E}\sup_{s\in[0,t]} |J_5(t)| &\le cT^{\frac{1}{2}}\mathbb{E}\sup_{s\in[0,\tau]} |\delta(s)|\sqrt{J_4}\\
&\le \frac{1}{2}\mathbb{E}\sup_{s\in[0,\tau]} |\delta(s)|^2 + cTJ_4.
\end{aligned}$$

Thus, by the last estimate, the inequality (4.9) and the defintion of $\mathfrak{Z}(t)$ we derive that

$$\mathbb{E}\mathfrak{Z}(t) \leq \frac{1}{2}\mathbb{E}\sup_{s\in[0,\tau]}|\delta(s)|^2+\frac{\alpha^2}{4}\left(cT^2+cT\int_0^\tau |\mathrm{A}^{\frac{1}{2}}\mathbf{u}(s)|^2ds\right)+c(1+T)\int_0^\tau |\delta(s)|^2ds. \tag{4.13}$$

Therefore, we derive from (1.2), (4.11), (4.12) and (4.13) that

$$\mathbb{E}\sup_{s\in[0,\tau]}|\delta(s)|^2+8\mathbb{E}\int_0^\tau \|\delta(s)\|^2ds \leq \alpha^2\kappa_0\beta(R)+\beta(R)C(T)\mathbb{E}\int_0^\tau |\delta(s)|^2ds, \tag{4.14}$$

where $C(T) := c(1+T)$, $\beta(R) := 1 + CRe^{CR}$ and

$$\kappa_0 := CT + CT^2 + CK_0 + CTK_1 + C(T)$$

Applying Gronwall's lemma into (4.14) implies that

$$\mathbb{E}\sup_{s\in[0,\tau]}|\delta(s)|^2+4\mathbb{E}\int_0^\tau \|\delta(s)\|^2ds \leq \alpha_n^2\beta(R)\kappa_0e^{C(T)\beta(R)T}, \tag{4.15}$$

where the positive constant $\beta(R)\kappa_0e^{C(T)\beta(R)T}$ does not depend on n and the sequence α_n. □

For every $R > 0$, $t \in [0, T]$ and any integer $n \geq 1$, let

$$\Omega_R^n(t) := \left\{\omega \in \Omega \ :\ \int_0^t \|\mathbf{u}^{\alpha_n}(s,\omega)\|_{\mathbb{L}^4}^4 ds \leq R\right\}. \tag{4.16}$$

This definition shows that $\Omega_R^n(t) \subset \Omega_R^n(s)$ for $s \leq t$ and that $\Omega_R^n(t) \in \mathcal{F}_t$ for any $t \in [0, T]$. Let τ_R^n be the stopping time defined in (4.1). It is not difficult to show that $\tau_R^n = T$ on the set $\Omega_R^n(T)$.

Owing to the intermediate estimate we obtained in the proof of Theorem 4.1 we derive the following result which tells us about the rate of convergence in probability of $\mathbf{u}^\alpha$ to $\mathbf{u}$.

Theorem 4.2 *Let the assumptions of Theorem 4.1 be satisfied. For any integer $n \geq 1$ let $\varepsilon_n(T)$ denote the error term defined by*

$$\varepsilon_n(T) = \sup_{s\in[0,T]}\left|\mathbf{u}^{\alpha_n}(s)-\mathbf{u}(s)\right|+\left(\int_0^T\left|\mathrm{A}^{\frac{1}{2}}\left[\mathbf{u}^{\alpha_n}(s)-\mathbf{u}(s)\right]\right|^2ds\right)^{1/2}.$$

Then $\varepsilon_n(T)$ converges to 0 in probability and the convergence is of order $O(\alpha_n)$. To be precise, for any sequence $\left(\Gamma_n\right)_{n=1}^\infty$ converging to ∞,

$$\lim_{n\to\infty}\mathbb{P}\left(\varepsilon_n(T)\geq\frac{\Gamma_n}{\alpha_n}\right)=0. \tag{4.17}$$

Therefore, the sequence $\mathbf{u}^\alpha$ *converges to* u *in probability in* $\mathbf{H}$ *and the rate of convergence is of order* $O(\alpha)$.

Proof First from the definition of $\Omega_R^n(T)$ and Corollary 3.6 we can show that for each n

$$\lim_{R\to\infty} \mathbb{P}\left(\Omega\backslash\Omega_R^n(T)\right) = 0. \tag{4.18}$$

In fact, it follows by Markov's inequality and (3.11) that

$$\begin{aligned}\sup_{n\geq 1} \mathbb{P}(\Omega\backslash\Omega_R^n(T)) &\leq \frac{1}{R}\sup_{n\geq 1}\mathbb{E}\int_0^T \|\mathbf{u}^{\alpha_n}(s)\|_{\mathbb{L}^4}^4 ds\\ &\leq \frac{1}{R}(\mathbb{E}|\mathbf{u}_0 + \mathrm{A}\mathbf{u}_0|^4 + CT)(1 + Ce^{CT}) \to 0,\end{aligned}$$

as $R \to \infty$.

Now let $\{\Gamma_n; n \in \mathbb{N}\}$ be a sequence of positive numbers such that $\Gamma_n \to \infty$ as $n \to \infty$. By straightforward calculation we deduce that

$$\begin{aligned}\mathbb{P}\Big(\varepsilon_n(T) \geq \Gamma_n\alpha_n\Big) &\leq \mathbb{P}\left(\Omega\backslash\Omega_R^n(T)\right) + \mathbb{P}\left(\varepsilon_n(T) \geq \Gamma_n\alpha_n, \Omega_R^n(T)\right)\\ &\leq \mathbb{P}\left(\Omega\backslash\Omega_R^n(T)\right) + \mathbb{E}\Big(1_{\Omega_R^n(T)}\varepsilon_n(T) \geq \Gamma_n\alpha_n\Big).\end{aligned}$$

Using Markov's inequality and (4.2) in the last inequality implies that

$$\begin{aligned}\mathbb{P}\Big(\varepsilon_n(T) \geq \Gamma_n\alpha_n\Big) &\leq \mathbb{P}\left(\Omega\backslash\Omega_R^n(T)\right) + \frac{1}{\Gamma_n^2\alpha_n^2}\mathbb{E}\Big(1_{\Omega_R^n(T)}\varepsilon_n^2(T)\Big)\\ &\leq \mathbb{P}\left(\Omega\backslash\Omega_R^n(T)\right) + \frac{1}{\Gamma_n^2}\beta(R)\kappa_0 e^{C(T)\beta(R)T}\end{aligned}$$

where $C(T), \beta(R), \kappa_0$ are previously defined.

Note that $\beta(R) \leq Ce^{2CR}$ for any $R > 0$. Hence there exist positive constants $C > 0$ and $C(T)$ such that for all $n \in \mathbb{N}$ we have

$$\mathbb{P}\Big(\varepsilon_n(T) \geq \Gamma_n\alpha_n\Big) \leq \mathbb{P}\left(\Omega\backslash\Omega_R^n(T)\right) + \frac{1}{\Gamma_n^2}\kappa_0 e^{C(T)e^{CR}}. \tag{4.19}$$

Let $\{\Gamma_n; n \in \mathbb{N}\}$ be a sequence such that $\Gamma_n \to \infty$ as $n \to \infty$ and

$$R(n) = \frac{1}{C}\log\left(\frac{1}{C(T)}\log\left(\log\left(\log(\Gamma_n)\right)\right)\right).$$

As $n \to \infty$ we see that $R(n) \to \infty$. Thus, since there exists $c > 0$ such that $\log(\log(\Gamma_n)) \leq c\Gamma_n$ for n large enough, it follows from (4.18) and (4.19) that

$$\mathbb{P}\Big(\varepsilon_n(T) \geq \Gamma_n\alpha_n\Big) \leq \mathbb{P}\left(\Omega\backslash\Omega^n_{R(n)}(T)\right) + \frac{1}{\Gamma_n^2}\kappa_0 \log(\log(\Gamma_n)) \to 0.$$

as $n \to \infty$; this concludes the proof. □

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