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Finite Difference Schemes for Stochastic Partial Differential Equations in Sobolev Spaces

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Abstract We discuss L_p-estimates for finite difference schemes approximating parabolic, possibly degenerate, SPDEs, with initial conditions from W_p^m and free terms taking values in W_p^m. Consequences of these estimates include an asymptotic expansion of the error, allowing the acceleration of the approximation by Richardson's method.

Keywords Cauchy problem · Stochastic PDEs · Finite differences · Extrapolation to the limit · Richardson's method

Mathematics Subject Classification 65M06 · 60H15 · 65B05

1 Introduction

In this paper spatial finite difference schemes for parabolic stochastic partial differential equations (SPDEs) are considered. In the literature finite difference approximations for deterministic partial differential equations are well studied, we refer the reader to [1], to recent results in [3], and the references therein. There is a growing number of publications on finite difference schemes also for SPDEs, see e.g. [2,10,22], and their references. In recent papers, see e.g. [6,7,9,11], L_2-theory is used to estimate in W_2^m-norms the error of finite difference approximations for the solutions of parabolic SPDEs. Hence error estimates in supremum norms are proved via Sobolev's

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embedding if $2m$ is larger than the dimension d of the state space $\mathbb{R}^d$. Therefore to get estimates in supremum norm, in these papers unnecessary spatial smoothness of the coefficients of the equation are required. Moreover, the smoothness conditions in these papers depend on the dimension of the state space. Our aim is to overcome this problem and generalize the results of [7] by giving W_p^m-norm estimates, assuming that the initial condition is in W_p^m and the free terms are W_p^m-valued processes. This forces us to give up part of generality, but important examples, like the Zakai equation in case of uncorrelated noises, are included. Since bounded functions, or more generally, functions with polynomial growth, can be seen as elements of suitable weighted Sobolev spaces with arbitrarily large integrability exponent p, for equations with such data we get dimension-invariant conditions on the smoothness of the coefficients.

It should be noted that the L_p- and $L_q(L_p)$-theory of SPDEs are well developed, see e.g. [12,15,16]. Their results, however, will not be used, as these theories deal with uniformly parabolic SPDEs, while the equations in this paper may degenerate and become first order SPDEs.

Following the idea seen in [14], to estimate the solutions of finite difference schemes we consider them in the whole space rather than on a grid. Through the estimates obtained for their Sobolev norms on the whole space, this allows us to estimate their supremum norm on a grid. For the finite difference approximations not only their convergence is proved, but also power series expansion in the mesh size is obtained. As in [9], this allows us to accelerate the rate of convergence, using the well known Richardson extrapolation, introduced in [20].

Finally, let us introduce some notation used throughout the paper. We consider a complete probability space $(\Omega, \mathcal{F}, P)$, which is equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and carries a sequence of independent $\mathcal{F}_t$-Wiener martingales $(w^r)_{r=1}^{\infty}$. We use the notation $\mathcal{P}$ for the σ-algebra of the predictable subsets of $\Omega \times [0, T]$. It is assumed that $\mathcal{F}_0$ contains every P-zero set. For $p \geq 2$ and $m \geq 0$, W_p^m denotes the Sobolev space with exponent p and order m. For integer m, this is the space of functions whose generalized partial derivatives up to order m are in L_p, for non integer real m, W_p^m is a fractional Sobolev space, or, as often cited in the literature, Bessel potential space, for the definition we refer to [21]. The Sobolev spaces of l_2-valued functions will be denoted by $W_p^m(l_2)$. We use the notation

$$D_i = \frac{\partial}{\partial x^i}, \quad \partial_v = \sum_{i=1}^{d} v^i D_i$$

for $v \in (v^1, \ldots, v^d) \in \mathbb{R}^d$, and

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_d^{\alpha_d}$$

for multi-indices $\alpha = (\alpha_1, \ldots, \alpha_d) \in \{0, 1, \ldots\}^d$ of length

$$|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_d.$$

Derivatives are understood in the generalized sense unless otherwise noted. The summation convention with respect to repeated indices is used thorough the paper, where it is not indicated otherwise.

The paper is organized as follows. Formulation of the problem and the statements of the main results are collected in Sect. 2. The appropriate estimate for the finite difference scheme is derived in Sect. 3, and it is used in the proof of the main results in Sect. 4.

2 Formulation of the Results

We consider the SPDE

$$\begin{aligned} du_t(x) = \{&D_i(a_t^{ij}(x)D_ju_t(x)) + b_t^i(x)D_iu_t(x) + c_t(x)u_t(x) + f_t(x)\}\,dt \\ &+(\mu_t^{ir}D_iu_t(x) + \nu_t^r(x)u_t(x) + g^r(x))\,dw_t^r \end{aligned} \tag{2.1}$$

for $(t,x) \in [0,T] \times \mathbb{R}^d =: H_T$, with the initial condition

$$u_0(x) = \psi(x) \quad x \in \mathbb{R}^d, \tag{2.2}$$

with the summation convention here and in the rest of the paper is used with respect to the repeated indices i, j and r.

The initial value ψ is an $\mathcal{F}_0$-measurable random variable with values in W_p^1 for a fixed $p \geq 2$. For all $i, j = 1, 2, \ldots, d$ the coefficients $a^{ij} = a^{ji}$, b^i and c are real-valued $\mathcal{P}\times\mathcal{B}(\mathbb{R}^d)$-measurable bounded functions, and $\mu^i = (\mu^{ir})_{r=1}^\infty$ and $\nu = (\nu^r)_{r=1}^\infty$ are l_2-valued $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable bounded functions on $\Omega \times H_T$. The free terms $f = (f_t)_{t\geq 0}$ and $g = (g_t)_{t\geq 0}$ are W_p^1-valued and $W_p^1(l_2)$ -valued adapted processes.

Let $m \in [1, \infty)$ Set

$$F_{m,p}(t) = \left(\int_0^t |f_t|_{W_p^m}^p dt\right)^{1/p}, \quad G_{m,p}(t) = \left(\int_0^t |g_t|_{W_p^m(l_2)}^p dt\right)^{1/p},$$

and let $K > 0$ be a constant. We make the following assumptions.

Assumption 2.1 The derivatives of the coefficients b^i and c in $x \in \mathbb{R}^d$ up to order $\lceil m \rceil$, and the derivatives of a^{ij} in x up to order $\lceil m \rceil + 1$ are functions, bounded by K. The l_2-valued functions μ^i and ν satisfy either of the following:

(i) their derivatives in x up to order $\lceil m \rceil + 1$ are functions, in magnitude bounded by K.
(ii) $\mu = (\mu^i)_{i=1}^d = 0$ and the derivatives of ν in x up to order $\lceil m \rceil$ are functions, in magnitude bounded by K.

Assumption 2.2 Almost surely $\psi \in W_p^m$, and either

(i) $F_{m,p}(T) + G_{m+1,p}(T) < \infty$ (a.s.), or
(ii) $\mu = 0$ and $F_{m,p}(T) + G_{m,p}(T) < \infty$ (a.s.).

Assumption 2.3 Almost surely the matrix valued function

$$\tilde{a}_t^{ij}(x) := a_t^{ij}(x) - \tfrac{1}{2}\mu_t^{ir}(x)\mu_t^{jr}(x), \quad i, j = 1, \dots, d$$

is positive semidefinite for each $(t, x) \in H_T$.

The notion of (generalised) solution is defined as follows.

Definition 2.1 A W_p^1-valued adapted weakly continuous process $(u_t)_{t\in[0,T]}$ is a solution of (2.1)–(2.2) on the interval $[0, \tau]$ for a stopping time $\tau \le T$, if almost surely

$$(u_t, \varphi) = (\psi, \varphi) + \int_0^t \{-(a_s^{ij} D_j u_s, D_i\varphi) + (b_s^i D_i u_s + c_s u_s + f_s, \varphi)\}\, ds$$
$$+ \int_0^t (\mu_s^{ir} D_i u_s + \nu_s^r u_s + g_s^r, \varphi)\, dw_s^r, \tag{2.3}$$

for all $t \in [0, \tau]$, $\varphi \in C_0^\infty(\mathbb{R}^d)$, where (v, φ) denotes the integral

$$\int_{\mathbb{R}^d} v(x)\varphi(x)\, dx$$

for functions φ and v on $\mathbb{R}^d$, when $v\varphi \in L_1(\mathbb{R}^d)$.

Existence and uniqueness theorems for degenerate SPDEs are established in [19] and [4]. We will need a slight generalization of these results, which will be proven at the end of Sect. 3.

Theorem 2.1 *Let Assumptions 2.1, 2.2 and 2.3 hold. Then (2.1)–(2.2) has a unique solution $u = (u_t)_{t\in[0,T]}$ on $[0, T]$. Moreover, u is a W_p^m-valued weakly continuous process, it is strongly continuous with values in W_p^{m-1}, and for all $l \in [0, m]$ and $q > 0$*

$$E \sup_{t\in[0,T]} |u(t)|_{W_p^l}^q \le N(E|\psi|_{W_p^l}^q + EF_{l,p}^q(T) + EG_{l+\kappa,p}^q). \tag{2.4}$$

where $\kappa = 0$ if $(\mu^i) = 0$ and $\kappa = 1$ otherwise, and N is a constant depending only on T, d, K, p, and m.

While Theorem 2.1 is stated for a general equation of the form (2.1)–(2.2), all of the subsequent results will only be proven under the restriction $\mu = 0$.

To introduce the finite difference schemes approximating (2.1) first let $\Lambda_0, \Lambda_1 \subset \mathbb{R}^d$ be two finite sets, the latter being symmetric to the origin, and $0 \in \Lambda_1 \setminus \Lambda_0$. Denote

$$\Lambda = \Lambda_0 \cup -\Lambda_0 \cup \Lambda_1$$

and $|\Lambda| = \sum_{\lambda\in\Lambda} |\lambda|$. On Λ we make the following assumption.

Assumption 2.4 If any subset $\Lambda' \subset \Lambda$ is linearly dependent, then Λ' is linearly dependent over the rationals.

Let $\mathbb{G}_h$ denote the grid

$$\mathbb{G}_h = \{h(\lambda_1 + \cdots + \lambda_n) : \lambda_i \in \Lambda, n = 1, 2, \ldots\},$$

for $h > 0$, and define the finite difference operators

$$\delta_{h,\lambda}\varphi(x) = (1/h)(\varphi(x + h\lambda) - \varphi(x))$$

and the shift operators

$$T_{h,\lambda}\varphi(x) = \varphi(x + h\lambda)$$

for $\lambda \in \Lambda$ and $h \neq 0$. Notice that $\delta_{h,0}\varphi = 0$ and $T_{h,0}\varphi = \varphi$. For a fixed $h > 0$ consider the finite difference equation

$$du_t^h(x) = (L_t^h(x)u_t^h(x) + f_t(x))\,dt + (\nu_t^r(x)u_t^h(x) + g_t^r(x))\,dw_t^r, \tag{2.5}$$

for $(t, x) \in [0, T] \times \mathbb{G}_h$, with the initial condition

$$u_0^h(x) = \psi(x) \tag{2.6}$$

for $x \in \mathbb{G}_h$, where

$$L_t^h\varphi = \sum_{\lambda\in\Lambda_0} \delta_{-h,\lambda}(\mathfrak{a}_h^\lambda \delta_{h,\lambda}\varphi) + \sum_{\gamma\in\Lambda_1} \mathfrak{p}_h^\gamma \delta_{h,\gamma}\varphi + \sum_{\gamma\in\Lambda_1} \mathfrak{c}_h^\gamma T_{h,\gamma}\varphi$$

for functions φ on $\mathbb{G}_h$. The coefficients $\mathfrak{a}_h^\lambda$, $\mathfrak{p}_h^\gamma$, and $\mathfrak{c}_h^\gamma$ are $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable bounded functions on $\Omega \times [0, T] \times \mathbb{R}^d$, with values in $\mathbb{R}$, and $\mathfrak{p}_h^0 = 0$ is assumed. All of them are supposed to be defined for $h = 0$ as well, and to depend continuously on h.

Note that Assumption 2.4 ensures that $\mathbb{G}_h \cap B$ is finite for any bounded set $B \subset \mathbb{R}^d$. This condition is necessary for (2.5) to be useful from a practical point of view.

One can look for solutions of the above scheme in the space of adapted stochastic processes with values in $l_{p,h}$, the space of real functions ϕ on $\mathbb{G}_h$ such that

$$|\phi|_{l_{p,h}}^p = \sum_{x\in\mathbb{G}_h} |\phi(x)|^p h^d < \infty.$$

The similar space is defined for l_2-valued functions and will be denoted by $l_{p,h}(l_2)$. For a fixed h Eq. (2.5) is an SDE in $l_{p,h}$, with Lipschitz coefficients, by the boundedness of $\mathfrak{a}_h^\lambda$, $\mathfrak{p}_h^\gamma$, $\mathfrak{c}_h^\gamma$, and ν^r. Hence if almost surely

$$|\psi|_{l_{p,h}}^p + \int_0^T |f_t|_{l_{p,h}}^p + |g_t|_{l_{p,h}(l_2)}^p\,dt < \infty,$$

then (2.5)–(2.6) admits a unique $l_{p,h}$-valued solution $(u_t^h)_{t\in[0,T]}$.

Remark 2.1 By well-known results on Sobolev embeddings, if $m > k + d/p$, there exists a bounded operator J from W_p^m to the space of functions with bounded and continuous derivatives up to order k such that $Jv = v$ almost everywhere. In the rest of the paper we will always identify functions with their continuous modifications if they have one, without introducing new notation for them. It is also known, and can be easily seen, that if Assumption 2.4 holds and $m > d/p$, then the for $v \in W_p^m$ the restriction of Jv onto the grid $\mathbb{G}_h$ is in $l_{p,h}$, moreover,

$$|Jv|_{l_{p,h}} \le C|v|_{W_p^m}, \tag{2.7}$$

where C is independent of v and h.

Remark 2.2 The h-dependency of the coefficients may seem artificial and in fact does not mean any additional difficulty in the proof of Theorems 2.2–2.4 below. However, we will make use of this generality to extend our results to the case when the data in the problem (2.1)–(2.2) are in some weighted Sobolev spaces.

Clearly

$$\delta_{h,\lambda}\varphi(x) \to \partial_\lambda\varphi(x)$$

as $h \to 0$ for smooth functions φ, so in order to get that our finite difference operators approximate the corresponding differential operators, we make the following assumption.

Assumption 2.5 We have, for every $i, j = 1, \dots, d$

$$a^{ij} = \sum_{\lambda\in\Lambda_0} \mathfrak{a}_0^\lambda \lambda^i \lambda^j, \tag{2.8}$$

$$b^i = \sum_{\gamma\in\Lambda_1} \mathfrak{p}_0^\gamma \gamma^i, \quad c = \sum_{\gamma\in\Lambda_1} \mathfrak{c}_0^\gamma, \tag{2.9}$$

and for $P \times dt \times dx$-almost all (ω, t, x) we have

$$\mathfrak{a}_h^\lambda \ge 0, \quad \mathfrak{p}_h^\gamma \ge 0 \quad \text{for every } \lambda \in \Lambda_0, \gamma \in \Lambda_1, h \ge 0. \tag{2.10}$$

Remark 2.3 The restriction (2.8) together with $\mathfrak{a}_0^\lambda \ge 0$ is not too severe, we refer the reader to [17] for a detailed discussion about matrix-valued functions which possess this property.

Example 2.1 Suppose that the matrix (a^{ij}) is diagonal. Then taking $\Lambda_0 = \{e_i : i = 1 \dots d\}$ and $\Lambda_1 = \{0\} \cup \{\pm e_i : i = 1 \dots d\}$, where (e_i) is the standard basis in $\mathbb{R}^d$, one can set

$$\mathfrak{a}_h^{e_i} = a^{ii}, \ \mathfrak{p}_h^{e_i} = b^i + \theta^i, \ \mathfrak{p}_h^{-e_i} = \theta_i, \ \mathfrak{c}_h^0 = c, \ \mathfrak{p}_h^0 = \mathfrak{c}_h^{\pm e_i} = 0,$$

with any $\theta^i \geq \max(0, -b^i)$, $i = 1 \dots d$.

Example 2.2 Suppose that (a^{ij}) is a $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable function of (ω, t, x), with values in a closed bounded polyhedron in the set of symmetric non-negative $d \times d$ matrices, such that its first and second order derivatives in $x \in \mathbb{R}^d$ are continuous in x and are bounded by a constant K. Then it is shown in [17] that one can obtain a finite set $\Lambda_0 \subset \mathbb{R}^d$ and $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable, bounded, nonnegative functions $\mathfrak{a}_0^\lambda$, $\lambda \in \Lambda_0$ such that (2.8) holds, and the first order derivatives of $(\mathfrak{a}_0^\lambda)^{1/2}$ in x are bounded by a constant N depending only on K, d and the polyhedron. Such situation arises in applications when, for example, $(a_t^{ij}(x))$ is a diagonally dominant symmetric non-negative definite matrix for each (ω, t, x), which by definition means that

$$2a_t^{ii}(x) \geq \sum_{j=1}^{d} |a_t^{ij}(x)|, \quad \text{for all } i = 1, 2, \dots, d, \text{ and } (\omega, t, x),$$

and hence it clearly follows that (a^{ij}) takes values in a closed polyhedron in the set of symmetric non-negative $d \times d$ matrices. Clearly, this polyhedron can be chosen to be bounded if (a^{ij}) is a bounded function. Moreover, in the case $d = 2$ explicit formulas are given in [18] to represent diagonally dominant symmetric non-negative definite matrices (a^{ij}) in the form (2.8).

The coefficients of the first and zero order terms, i.e., $\mathfrak{p}_h^\gamma$ and $\mathfrak{c}_h^\gamma$ can be chosen as in Example 2.1.

If (a^{ij}) does not depend on x, and it is a bounded $\mathcal{P}$-measurable function of (ω, t) with values in the set of diagonally dominant symmetric non-negative definite matrices, then we can take

$$\Lambda_0 := \{e_i, e_i + e_j, e_i - e_j : i, j = 1, 2, \dots, d\}, \quad \Lambda_1 = \{0\} \cup \Lambda_0 \cup -\Lambda_0,$$

where $(e_i)_{i=1}^d$ is the standard basis in $\mathbb{R}^d$, and set

$$\mathfrak{a}_h^\lambda = \begin{cases} a^{ij} - \sum_{j\neq i} |a^{ij}| & \text{if } \lambda = e_i \\ \frac{1}{2}\sum_{j\neq i}(a^{ij})^+ & \text{if} \lambda = e_i + e_j \\ \frac{1}{2}\sum_{j\neq i}(a^{ij})^- & \text{if} \lambda = e_i - e_j \end{cases},$$

$$\mathfrak{p}_h^\gamma = \begin{cases} \pm\frac{1}{2}b^i + \theta^i & \text{if} \gamma = \pm e_i \\ \theta^{ij} & \text{if} \gamma = \pm(e_i + e_j) \\ \theta^{ij} & \text{if} \gamma = \pm(e_i - e_j) \end{cases},$$

$$\mathfrak{c}_h^0 = c, \quad \mathfrak{p}_h^0 = \mathfrak{c}_h^\gamma = 0 \quad \text{for } \gamma \in \Lambda_1 \setminus \{0\},$$

with any constants $\theta^{ij} \geq \kappa$ and $\theta^i \geq \kappa - \frac{1}{2}|b^i|$, for $i, j = 1, \dots, d$, where κ is any nonnegative constant, and $a^\pm := (|a| \pm a)/2$ for $a \in \mathbb{R}$. Then clearly, Λ_0, Λ_1, $\mathfrak{a}_h^\lambda$, $\mathfrak{p}_h^\gamma$ and $\mathfrak{c}_h$ satisfy Assumptions 2.4, 2.5 above, and Assumption 2.6 below.

Since the compatibility condition (2.8)–(2.9) will always be assumed, any subsequent conditions will be formulated for the coefficients in (2.5), which then automatically imply the corresponding properties for the coefficients in (2.1).

Assumption 2.6 The coefficients $\mathfrak{a}_h^\lambda$ (resp., $\mathfrak{p}_h^\gamma$, $\mathfrak{c}_h^\gamma$, ν), and their partial derivatives in the variable (h, x) up to order $\lceil m \rceil + 1$ (resp., $\lceil m \rceil$) are functions bounded by K.

Assumption 2.7 The initial value ψ is in W_p^m, and the free terms f and g are W_p^m-valued and $W_p^m(l_2)$-valued processes, respectively, such that almost surely $F_{m,p}(T) + G_{m,p}(T) < \infty$.

We are now about to present the main results. The first three theorems correspond to similar results in the L_2 setting from [7]. The key role in their proof is played by Theorem 3.1 below, which presents an upper bound for the W_p^m norms of the solutions to (2.5)–(2.6). After obtaining this estimate, Theorems 2.2 through 2.4 can be proved in the same fashion as their counterparts in the L_2 setting. Therefore, in Sect. 4 only a sketch of the proof will be provided in which we highlight the main differences; for the complete argument we refer to [7].

Theorem 2.2 *Let $k \geq 0$ be an integer and let Assumptions 2.4 through 2.7 hold with $m > 2k + 3 + d/p$. Then there are continuous random fields $u^{(1)}, \dots u^{(k)}$ on $[0, T] \times \mathbb{R}^d$, independent of h, such that almost surely*

$$u_t^h(x) = \sum_{j=0}^{k} \frac{h^j}{j!} u_t^{(j)}(x) + h^{k+1} r_t^h(x) \tag{2.11}$$

for $t \in [0, T]$ and $x \in \mathbb{G}_h$, where $u^{(0)} = u$, r^h is a continuous random field on $[0, T] \times \mathbb{R}^d$, and for any $q > 0$

$$\mathbb{E} \sup_{t \in [0,T]} \sup_{x \in \mathbb{G}_h} |r_t^h(x)|^q + \mathbb{E} \sup_{t \in [0,T]} |r_t^h|_{l_{p,h}}^q \leq N(\mathbb{E}|\psi|_{W_p^m}^q + \mathbb{E} F_{m,p}^q(T) + \mathbb{E} G_{m,p}^q(T))$$

with $N = N(K, T, m, p, q, d, |\Lambda|)$.

Once we have the expansion above, we can use Richardson extrapolation to improve the rate of convergence. For a given k set

$$(c_0, c_1, \dots, c_k) = (1, 0, 0, \dots, 0) V^{-1}, \tag{2.12}$$

where V denotes the $(k + 1) \times (k + 1)$ Vandermonde matrix $V = (V^{ij}) = (2^{-(i-1)(j-1)})$, and define

$$v^h = \sum_{i=0}^{k} c_i u^{h_i},$$

where $h_i = h/2^i$.

Theorem 2.3 *Let $k \geq 0$ be an integer and let Assumptions 2.4 through 2.7 hold with $m > 2k + 3 + d/p$. Then for every $q > 0$ we have*

$$\mathbb{E} \sup_{t\in[0,T]} \sup_{x\in\mathbb{G}_h} |u_t(x) - v_t^h(x)|^q + \mathbb{E} \sup_{t\in[0,T]} |u_t - v_t^h|^q_{l_{p,h}}$$
$$\leq h^{q(k+1)} N(\mathbb{E}|\psi|^q_{W^m_p} + \mathbb{E}F^q_{m,p}(T) + \mathbb{E}G^q_{m,p}(T))$$

with $N = N(K, T, m, k, p, q, d, |\Lambda|)$.

Theorem 2.4 *Let $(h_n)_{n=1}^\infty \in l_q$ be a nonnegative sequence for some $q \geq 1$. Let $k \geq 0$ be an integer and let Assumptions 2.4 through 2.7 hold with $m > 2k + 3 + d/p$. Then for every $\varepsilon > 0$ there exists a random variable ξ_ε such that almost surely*

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{G}_h} |u_t(x) - v_t^h(x)| \leq \xi_\varepsilon h^{k+1-\varepsilon}$$

for $h = h_n$.

Remark 2.4 We can use $h_i = h/n_i$, $i = 1 \ldots k$, with any set of different integers n_i, with $n_1 = 1$. Then changing the matrix V to $\tilde{V} = (\tilde{V}^{ij}) = (n_i^{-j+1})$ in (2.12), Theorems 2.3–2.4 remain valid. The choice $n_i = i$, for example, yields a more coarse grid, and can reduce computation time.

Choosing p large enough, in some cases one can get rid of the term d/p in the conditions of the theorems above, thus obtaining dimension-invariant conditions. To this end, first denote the function $\rho_s(x) = 1/(1 + |x|^2)^{s/2}$ defined on $\mathbb{R}^d$ for all $s \geq 0$. We say that a function F on $\mathbb{R}^d$ has polynomial growth of order s if the L_∞ norm of $F\rho_s$ is finite. For any integer $m \geq 0$, the set of functions on $\mathbb{R}^d$ which have polynomial growth of order s and whose derivatives up to order m are functions and have polynomial growth of order s is denoted by P_s^m, and its equipped with the norm

$$\|F\|_{P_s^m} = |F\rho_s|_{W^m_\infty} < \infty.$$

The similar space is defined for l_2-valued functions and is denoted by $P_s^m(l_2)$. Note that for any integers $m > k \geq 0$, if $F \in P_s^m$, then its partial derivatives up to order k exist in the classical sense and along with F are continuous functions. The polynomial growth property of order s for functions on $\mathbb{G}_h$ can also be defined analogously, the set of such functions is denoted by $P_{h,s}$.

Let $s \geq 0$ and m be a nonnegative integer. Consider again the equation

$$du_t(x) = (D_i(a_t^{ij}(x)D_j u_t(x)) + b_t^i(x)D_j u_t(x) + c_t(x)u_t(x) + f_t(x))\,dt$$
$$+(\nu_t^r(x)u_t(x) + g^r(x))\,dw_t^r \tag{2.13}$$

for $(t, x) \in [0, T] \times \mathbb{R}^d$, with the initial condition

$$u_0(x) = \psi(x) \quad x \in \mathbb{R}^d, \tag{2.14}$$

where we keep all our measurability conditions from (2.1)–(2.2). However, instead of the integrability conditions on ψ, f_t, g_t, we now assume the following.

Assumption 2.8 The initial data ψ is an $\mathcal{F}_0 \times \mathcal{B}(\mathbb{R}^d)$-measurable mapping from $\Omega \times \mathbb{R}^d$ to $\mathbb{R}$, such that $\psi \in P_s^m$ (a.s.). The free data f and g are $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable mappings from $\Omega \times [0, T] \times \mathbb{R}^d$ to $\mathbb{R}$ and l_2, respectively. Moreover, almost surely (f_t) is a P_s^m-valued process and (g_t) is a $P_s^m(l_2)$-valued process, such that

$$\left| \| f_t \|_{P_s^m} + \| g_t \|_{P_s^m(l_2)} \right|_{L_\infty[0,T]} < \infty.$$

Definition 2.2 A $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable mapping u from $\Omega \times [0, T] \times \mathbb{R}^d$ to $\mathbb{R}$ such that $(u_t)_{t\in[0,T]}$ is almost surely a P_s^1-valued bounded process, is called a classical solution of (2.13)–(2.14) on $[0, T]$, if almost surely u and its first and second order partial derivatives in x are continuous functions of $(t, x) \in [0, T] \times \mathbb{R}^d$, and almost surely

$$\begin{aligned} u_t(x) = \psi(x) &+ \int_0^t [D_i(a_s^{ij}(x) D_j u_s(x)) + b_s^i(x) D_j u_s(x) + c_s(x) u_s(x) + f_s(x)]\, ds \\ &+ \int_0^t [\nu_s^r(x) u_s(x) + g_s^r(x)]\, dw_s^r \end{aligned}$$

for all $(t, x) \in [0, T] \times \mathbb{R}^d$ for a suitable modification of the stochastic integral in the right-hand side of the equation.

If $m \geq 1$, then as noted above the initial condition and free terms are continuous in space. This makes it reasonable to consider the finite difference scheme (2.5)–(2.6) as an approximation for the problem (2.13)–(2.14).

Theorem 2.5 *Let $k \geq 0$ be integer, and let $\bar{s} > s \geq 0$ be real numbers. Suppose that Assumptions 2.4 2.5, 2.6, and 2.8 hold with $m > 2k + 3$.*

(i) *Equation (2.13)–(2.14) admits a unique $P_{\bar{s}}^{m-1}$-valued classical solution $(u_t)_{t\in[0,T]}$.*
(ii) *For fixed h the corresponding finite difference Eqs. (2.5)–(2.6) admits a unique $P_{h,\bar{s}}$-valued solution $(u_t^h)_{t\in[0,T]}$.*
(iii) *Suppose furthermore $\mathfrak{p}_h^\gamma \geq \kappa$ for $\gamma \in \Lambda_1$, for some constant $\kappa > 0$, and*

$$\Lambda_0 \cup -\Lambda_0 \subset \Lambda_1.$$

Then there are continuous random fields $u^{(1)}, \ldots u^{(k)}$ on $[0, T] \times \mathbb{R}^d$, independent of h, such that almost surely

$$u_t^h = \sum_{j=0}^k \frac{h^j}{j!} u_t^{(j)}(x) + h^{k+1} r_t^h(x)$$

for $t \in [0,T]$ and $x \in \mathbb{G}_h$, where $u^{(0)} = u$, r^h is a continuous random field on $[0,T] \times \mathbb{R}^d$, and for any $q > 0$

$$\mathbb{E} \sup_{t\in[0,T]} \sup_{x\in\mathbb{G}_h} |r_t^h(x)\rho_{\bar{s}}(x)|^q + \mathbb{E} \sup_{t\in[0,T]} |r_t^h \rho_{\bar{s}}|^q_{l_{p,h}}$$
$$\leq N \left(\mathbb{E}\|\psi\|^q_{P_s^m} + \mathbb{E}\big|\|f_t\|_{P_s^m} + \|g_t\|_{P_s^m(l_2)}\big|^q_{L_\infty[0,T]} \right)$$

with some $N = N(K, T, m, s, \bar{s}, q, d, |\Lambda|, \kappa)$.

(iv) Let $(h_n)_{n=1}^\infty \in l_q$ be a nonnegative sequence for some $q \geq 1$. Then for every $\varepsilon, M > 0$ there exists a random variable $\xi_{\varepsilon,M}$ such that almost surely

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{G}_h, |x|\leq M} |u_t(x) - v_t^h(x)| \leq \xi_{\varepsilon,M} h^{k+1-\varepsilon}$$

for $h = h_n$.

This theorem will be proved in Sect. 4.

Remark 2.5 Condition $\mathfrak{p}_h^\gamma \geq \kappa$ in assertion (iii) of the above theorem is harmless, similarly to the second part of (2.10). As seen in Examples 2.1 and 2.2, we can always satisfy this additional requirement by adding a sufficiently large constant to $\mathfrak{p}_h^\gamma$.

3 Estimate on the Finite Difference Scheme

First let us collect some properties of the finite difference operators. Throughout this section we consider a fixed $h > 0$ and use the notation $u_\alpha = D^\alpha u$. It is easy to see that, analogously to the integration by parts,

$$\int_{\mathbb{R}^d} v(\delta_{h,\lambda} u)\, dx = \int_{\mathbb{R}^d} (\delta_{h,-\lambda} v) u\, dx = -\int_{\mathbb{R}^d} (\delta_{-h,\lambda} v) u\, dx, \tag{3.1}$$

when $v \in L_{q/q-1}$ and $u \in L_q$ for some $1 \leq q \leq \infty$, with the convention $1/0 = \infty$ and $\infty/(\infty - 1) = 1$. The discrete analogue of the Leibniz rule can be written as

$$\delta_{h,\lambda}(uv) = u(\delta_{h,\lambda} v) + (\delta_{h,\lambda} u)(T_{h,\lambda} v). \tag{3.2}$$

Finally, we will also make use of the simple identities

$$T_{h,\alpha}\delta_{h,\beta} u = \delta_{h,\alpha+\beta} u - \delta_{h,\alpha} u, \tag{3.3}$$
$$v v_\lambda = (1/2)(\delta_\lambda(v^2) - h(\delta_\lambda v)^2), \tag{3.4}$$

and the estimate

$$|\delta_{h,\lambda} v|_{L_p} \leq |\int_0^1 \partial_\lambda v(\cdot + \theta h\lambda)\, d\theta|_{L_p} \leq |\lambda||v|_{W_p^1}, \tag{3.5}$$

valid for $p \in [1, \infty]$ and $v \in W_p^1$, $h \neq 0$ and $\lambda \in \mathbb{R}^d$.

Lemma 3.1 *For any $p \geq 2$, $\lambda \in \mathbb{R}^d$, $h \neq 0$ and real function v on $\mathbb{R}^d$ we can write*

$$\delta_{h,\lambda}(|v|^{p-2}v) = F_p^{h,\lambda}(v)\delta_{h,\lambda}v,$$

where $F_p^{h,\lambda}(v) \geq 0$, and for $p > 2$, $q = p/(p-2)$ and for all $v \in L_p(\mathbb{R}^d)$

$$|F_p^{h,\lambda}(v)|_{L_q}^q \leq (p-1)^q |v|_{L_p}^p. \tag{3.6}$$

Proof The derivative of the function $G(r) = |r|^{p-2}r$ is

$$G'(r) = (p-1)|r|^{p-2} \geq 0,$$

so we have

$$\begin{aligned}\delta_{h,\lambda}(|v|^{p-2}v)(x) &= (1/h)G((1-\theta)v(x) + \theta v(x+\lambda h))|_{\theta=0}^1 \\ &= \int_0^1 G'((1-\theta)v(x) + \theta v(x+\lambda h))\delta_{h,\lambda}v(x)\,d\theta = F_p^{h,\lambda}(v)\delta_{h,\lambda}v(x).\end{aligned}$$

By Jensen's inequality and the convexity of the function $|r|^p$,

$$|F_p^{h,\lambda}(v)|_{L_q}^q \leq (p-1)^q \int_{\mathbb{R}^d}\int_0^1 \theta|v(x+\lambda h)|^p + (1-\theta)|v(x+\lambda h)|^p\,d\theta\,dx.$$

Hence (3.6) follows by Fubini's theorem and the shift invariance of the Lebesgue measure. □

Lemma 3.2 *Let m be a nonnegative integer, and let α be a multi-index of length m. Then the following statements hold.*

(i) *Let $\mathfrak{a}$ be a nonnegative function on $\mathbb{R}^d$ such that its generalised derivatives up to order $m+1$ are functions, in magnitude bounded by a constant K. If $m \geq 1$ then let the first order generalised derivatives of $\sigma := \sqrt{\mathfrak{a}}$ be also functions, bounded by K. Then for $u \in W_p^m$, $p \in [2,\infty)$, $\lambda \in \mathbb{R}^d$ and $h \neq 0$*

$$\int_{\mathbb{R}^d} |D^\alpha u|^{p-2} D^\alpha u D^\alpha \delta_{-h,\lambda}(\mathfrak{a}\delta_{h,\lambda}u)\,dx \leq N|u|_{W_p^m}^p. \tag{3.7}$$

(ii) *Let $\mathfrak{p}$ be a nonnegative function on $\mathbb{R}^d$ such that its generalised derivatives up to order $m \vee 1$ are functions bounded by K. Let $p = 2^k$ for an integer $k \geq 1$. Then for $u \in W_p^m$, $\lambda \in \mathbb{R}^d$ and $h > 0$*

$$\int_{\mathbb{R}^d} |D^\alpha u|^{p-2} D^\alpha u D^\alpha (\mathfrak{p}\delta_{h,\lambda}u)\,dx \leq N|u|_{W_p^m}^p. \tag{3.8}$$

The constant N in the above estimates depend only on m, p, d, K and $|\lambda|$.

Proof Recall the notation $u_\alpha = D^\alpha u$. For real functions v and w defined on $\mathbb{R}^d$ we write $v \sim w$ if their integrals over $\mathbb{R}^d$ are the same. We use the notation $v \preceq w$ if $v = w + F$ with a function F whose integral over $\mathbb{R}^d$ can be estimated by $N|u|^p_{W^m_p}$, where N is a constant depending only on m, K, p, d and $|\lambda|$. To prove (3.7) we consider first the case $m = 0$. By (3.1) and Lemma 3.1

$$\begin{aligned} |u|^{p-2}u\delta_{-h\lambda}(\mathfrak{a}\delta_{h,\lambda}u) &\sim -\delta_{h,\lambda}(|u|^{p-2}u)\mathfrak{a}\delta_{h,\lambda}u \\ &= -F^{h,\lambda}_p(u)\mathfrak{a}(\delta_{h,\lambda}u)^2 \le 0, \end{aligned} \tag{3.9}$$

where F is the functional obtained from Lemma 3.1. Consequently, (3.7) holds for $m = 0$. Assume now $m \ge 1$. Then it is easy to see that

$$|D^\alpha u|^{p-2}D^\alpha u D^\alpha \delta_{-h,\lambda}(\mathfrak{a}\delta_{h,\lambda}u) \preceq I_1 + I_2, \tag{3.10}$$

with

$$\begin{aligned} I_1 &:= |u_\alpha|^{p-2}u_\alpha \sum_{(\alpha',\alpha'')\in A} \delta_{-h,\lambda}D^{\alpha'}\mathfrak{a}D^{\alpha''}\delta_{h,\lambda}u \\ I_2 &:= |u_\alpha|^{p-2}u_\alpha\delta_{-h,\lambda}(\mathfrak{a}\delta_{h,\lambda}u_\alpha), \end{aligned}$$

where A is the set of ordered pairs of multi-indices (α', α'') such that $|\alpha'| = 1$ and $\alpha' + \alpha'' = \alpha$. By (3.1) and Lemma 3.1

$$\begin{aligned} I_1 &\sim -2F^{h,\lambda}_p(u_\alpha)\sigma\delta_{h,\lambda}u_\alpha \sum_{(\alpha',\alpha'')\in A} D^{\alpha'}\sigma\delta_{h,\lambda}u_{\alpha''} \\ &\le \varepsilon F^{h,\lambda}_p(u_\alpha)\mathfrak{a}(\delta_{h,\lambda}u_\alpha)^2 + \varepsilon^{-1}dK^2F^{h,\lambda}_p(u_\alpha)(\delta_{h,\lambda}u_{\alpha''})^2 \end{aligned} \tag{3.11}$$

for every $\varepsilon > 0$, where the simple inequality $2yz \le \varepsilon y^2 + \varepsilon^{-1}z^2$ is used with $y = \sigma\delta_{h,\lambda}u_\alpha$ and $z = \sum_{(\alpha',\alpha'')\in A} D^{\alpha'}\sigma\delta_{h,\lambda}u_{\alpha''}$. Using (3.9) with u_α in place of u we get

$$I_2 \preceq -F^{h,\lambda}_p(u_\alpha)\mathfrak{a}(\delta_{h,\lambda}u_\alpha)^2.$$

Combining this with (3.11) with sufficiently small ε, from (3.10) we obtain

$$\begin{aligned} I &\preceq NF^{h,\lambda}_p(u_\alpha) \sum_{(\alpha',\alpha'')\in A} (\delta^h_\lambda u_{\alpha''})^2 \\ &\le N|F^{h,\lambda}_p(u_\alpha)|^q + N|\sum_{(\alpha',\alpha'')\in A} (\delta^h_\lambda u_{\alpha''})^2|^{p/2}, \end{aligned}$$

with $q = p/(p-2)$, which gives (3.7), due to the estimates (3.6) and (3.5).To prove (3.8) notice that for $p = 2^k$

$$\begin{aligned} J := |D^\alpha u|^{p-2}D^\alpha \ \ uD^\alpha(\mathfrak{p}\delta_{h,\lambda}u) &= (D^\alpha u)^{p-1}D^\alpha(\mathfrak{p}\delta_{h,\lambda}u) \\ &\preceq (D^\alpha u)^{p-1}\mathfrak{p}\delta_{h,\lambda}u_\alpha. \end{aligned}$$

Hence we can repeatedly use (3.4) and the nonnegativity of $\mathfrak{p}_h^\lambda$ to get

$$\begin{aligned} u_\alpha^{p-1}\mathfrak{p}\delta_{h,\lambda}u_\alpha &\le (1/2)u_\alpha^{p-2}\mathfrak{p}\delta_{h,\lambda}u_\alpha^2 \\ &\le (1/4)u_\alpha^{p-4}\mathfrak{p}\delta_{h,\lambda}u_\alpha^4 \le \cdots \le (1/2^k)\mathfrak{p}\delta_{h,\lambda}u_\alpha^{2^k}. \end{aligned}$$

By (3.1), $\mathfrak{p}\delta_{h,\lambda}u_\alpha^p$ has the same integral over $\mathbb{R}^d$ as $\delta_{h,-\lambda}\mathfrak{p}u_\alpha^p$, and hence (3.8) follows, since $|\delta_{h,-\lambda}\mathfrak{p}| \le K|\lambda|$ by (3.5). □

Corollary 3.1 *Let $m \ge 1$ be an integer and $p = 2^k$ for some integer $k \ge 1$, and let Assumptions 2.6 and 2.7, along with the condition (2.10) be satisfied. Then for $u \in W_p^m$, $f \in W_p^m$, $g \in W_p^m(l_2)$ and for all multi-indices α of length $|\alpha| \le m$ we have*

$$\begin{aligned} &\int_{\mathbb{R}^d} (p-1)|u_\alpha|^{p-2}u_\alpha(x)D^\alpha(L_t^h u(x) + f(x)) \\ &\quad +(1/2)(p-1)(p-2)|u_\alpha|^{p-2}(x)|D^\alpha(\nu^r(x)u(x) + g^r(x))|^2dx \\ &\le N(|u|_{W_p^m}^p + |f|_{W_p^m}^p + |g|_{W_p^m(l_2)}^p) \end{aligned} \tag{3.12}$$

for $P \times dt$-almost all $(\omega, t) \in \Omega \times [0, T]$, where N is a constant depending only on $d, p, m, |\Lambda|$, and K.

Proof Using the notation of the preceding proof, by Hölder's inequality

$$u_\alpha^{p-1}D^\alpha(\mathfrak{c}_h^\gamma T_{h,\gamma}u) + u_\alpha^{p-1}f_\alpha + u_\alpha^{p-2}|D^\alpha(\nu^r u + g^r)|^2 \preceq N(|f|_{W_p^m}^p + |g|_{W_p^m(l_2)}^p).$$

The remaining two terms are estimated in Lemma 3.2. □

The following is a stochastic version of Gronwall's lemma, for its proof we refer to [5].

Lemma 3.3 *Let $(y_t)_{t\in[0,T]}$, $(F_t)_{t\in[0,T]}$, and $(G_t)_{t\in[0,T]}$ be two nonnegative adapted processes, and let $(m_t)_{t\in[0,T]}$ be a continuous local martingale such that for a constant N almost surely*

$$dy_t \le N(y_t + F_t)dt + dm_t$$

for all $t \in [0, T]$. Assume furthermore that for some $p \ge 2$ almost surely

$$d\langle m\rangle_t \le N(y_t^2 + G_t y_t^{2-(2/p)})dt.$$

Then for every $q \ge 0$ there exists a constant C, depending only on N, q, p, and T, such that

$$\mathbb{E}\sup_{t\le T} y_t^q \le C\mathbb{E}y_0^q + C\mathbb{E}\left(\int_0^T F_t dt\right)^q + C\mathbb{E}\left(\int_0^T G_t^{p/2}dt\right)^q.$$

Consider (2.5) without restricting it to the grid $\mathbb{G}_h$, that is,

$$du_t^h = (L_t^h u_t^h + f_t)dt + \nu_t^r u_t^h + g_t^r)dw_t^r, \quad t \in (0, T],\ x \in \mathbb{R}^d \tag{3.13}$$

with the initial condition

$$u_0^h(x) = \psi(x), \quad x \in \mathbb{R}^d. \tag{3.14}$$

The solution of (3.13)–(3.14) is understood in the spirit of Definition 2.1.

Definition 3.1 An $L_p(\mathbb{R}^d)$-valued continuous adapted process $(u_t^h)_{t\in[0,T]}$ is a solution to (3.13)–(3.14) on $[0, \tau]$ for a stopping time $\tau \le T$ if almost surely

$$(u_t^h, \varphi) = (\psi, \varphi) + \int_0^t (L_s^h u_s^h + f_s, \varphi)\, ds + \int_0^t (\nu_s^r u_s^h + g_s, \varphi)\, dw_s^r \tag{3.15}$$

for all $t \le \tau$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$.

Assumption 2.6 implies that the operators $u \to L_t^h u$ and $u \to (b_t^r u)_{r=1}^\infty$ are bounded linear operators from W_p^m to W_p^m and to $W_p^m(l_2)$, respectively, with operator norm uniformly bounded in (t, ω). Therefore if Assumption 2.7 is also satisfied, (3.13) is a SDE in the space W_p^m with Lipschitz continuous coefficients. As such, it admits a unique continuous solution.

Theorem 3.1 *Let Assumptions 2.6 and 2.7 hold with $m \ge 1$, and let condition (2.10) be satisfied. Then (3.13)–(3.14) has a unique continuous W_p^m-valued solution $(u_t^h)_{t\in[0,T]}$, and for each $q > 0$ there exists a constant $N = N(d, q, p, m, K, T, |\Lambda|)$ such that*

$$\mathbb{E} \sup_{t\le T} |u_t^h|_{W_p^m}^q \le N(\mathbb{E}|\psi|_{W_p^m}^q + \mathbb{E}F_{m,p}^q(T) + \mathbb{E}G_{m,p}^q(T)) \tag{3.16}$$

for all $h > 0$.

Proof By the preceding argument, we need only prove estimate (3.16). First let m and p be as in the conditions of Corollary 3.1, and fix a $q > 1$. Let α be a multi-index such that $|\alpha| \le m$. If we apply Itô's formula to $|D^\alpha u^h|_{L_p}^p$ by Lemma 5.1 in [13], one can notice that the term appearing in the drift is the left-hand side of (3.12), with u^h in place of u. Using Corollary 3.1 and summing over $|\alpha| \le m$ we get

$$d|u_t^h|_{W_p^m}^p \le N(|u|_{W_p^m}^p + |f|_{W_p^m}^p + |g|_{W_p^m}^p)\, dt + dm_t^h$$

with some N depending only on $p, m, d, |\Lambda|$, and K, where

$$dm_t^h = (p-1)\int_{\mathbb{R}^d} |\partial_\alpha u_t^h|^{p-1} \partial_\alpha(\nu_t^r u_t^h + g_t^r)\, dx\, dw_t^r$$

with α used as a repeated index. It is clear that

$$\begin{aligned} d\langle m^h\rangle_t &= (p-1)^2 \sum_{r=1}^{\infty} \left(\int_{\mathbb{R}^d} |\delta_\alpha u_t^h|^{p-1} \partial_\alpha (v_t^r u_t^h + g_t^r) dx \right)^2 dt \\ &\le N((|u_t^h|_{W_p^m}^p)^2 + |g|_{W_p^m(l_2)}^2 |u_t^h|_{W_p^m}^{2p-2})\, dt, \end{aligned}$$

so Lemma 3.3 can be applied to the function $|u_t^h|_{W_p^m}^p$ and the power q/p, which proves (3.16) for integer m, $p = 2^k$.

Note that (3.16) is equivalent to

$$[\mathbb{E} \sup_{t \le T} |u_t^h|_{W_p^m}^q]^{\frac{1}{q}} \le N([\mathbb{E}|\psi|_{W_p^m}^q]^{\frac{1}{q}} + [\mathbb{E}F_{m,p}^q]^{\frac{1}{q}} + [\mathbb{E}G_{m,p}^q]^{\frac{1}{q}}),$$

which implies

$$[\mathbb{E} \left(\int_0^T |u_t^h|_{W_p^m}^r \right)^{\frac{q}{r}}]^{\frac{1}{q}} \le N([\mathbb{E}|\psi|_{W_p^m}^q]^{\frac{1}{q}} + [\mathbb{E}F_{m,p}^q]^{\frac{1}{q}} + [\mathbb{E}G_{m,p}^q]^{\frac{1}{q}}), \tag{3.17}$$

for any $r > 1$, with another constant N, independent from r. In other words, this means that for the special case of m and p considered so far the solution operator

$$(\psi, f, g) \to u^h$$

continuously maps $\Psi_p^m \times \mathcal{F}_p^m \times \mathcal{G}_p^m$ to $\mathcal{U}_p^m$, where

$$\begin{aligned} \Psi_p^m &= L_q(\Omega, W_p^m), \\ \mathcal{F}_p^m &= L_q(\Omega, L_p([0,T], W_p^m)), \\ \mathcal{G}_p^m &= L_q(\Omega, L_p([0,T], W_p^m(l_2))), \\ \mathcal{U}_p^m &= L_q(\Omega, L_r([0,T], W_p^m)). \end{aligned}$$

Let us denote the complex interpolation space between any Banach spaces A_0 and A_1 with parameter θ by $[A_0, A_1]_\theta$. Recall the following interpolation properties (see 1.9.3, 1.18.4, and 2.4.2 from [21])

(i) If a linear operator T is continuous from A_0 to B_0 and from A_1 to B_1, then it is also continuous from $[A_0, A_1]_\theta$ to $[B_0, B_1]_\theta$, moreover, its norm between the interpolated spaces depends only on θ and its norm between the original spaces.

(ii) For a measure space M and $1 < p_0, p_1 < \infty$,

$$[L_{p_0}(M, A_0), L_{p_1}(M, A_1)]_\theta = L_{p_\theta}(M, [A_0, A_1]_\theta),$$

where $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$.

(iii) For $m_0, m_1 \in \mathbb{R}$, $1 < p_0, p_1 < \infty$,

$$[W^{m_0}_{p_0}, W^{m_1}_{p_1}]_\theta = W^{m_\theta}_{p_\theta},$$

where $m_\theta = (1-\theta)m_0 + \theta m_1$, and $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$.

Now take any $p \geq 2$, and take $p_0 \leq p \leq p_1$ such that $p_0 = 2^k$ and $p = 2^{k+1}$ for $k \in \mathbb{N}$, and set $\theta \in [0, 1]$ such that $1/p = (1-\theta)/p_0 + \theta/p_1$. By property (ii) we have

$$\begin{aligned} \Psi^m_p &= [\Psi^m_{p_0}, \Psi^m_{p_1}]_\theta,\ \mathcal{F}^m_p = [\mathcal{F}^m_{p_0}, \mathcal{F}^m_{p_1}]_\theta, \\ \mathcal{G}^m_p &= [\mathcal{G}^m_{p_0}, \mathcal{G}^m_{p_1}]_\theta,\ \mathcal{U}^m_p = [\mathcal{U}^m_{p_0}, \mathcal{U}^m_{p_1}]_\theta, \end{aligned}$$

and therefore by (i) the solution operator is continuous for any $p \geq 2$ and integer $m \geq 1$.

For arbitrary $m \geq 1$, $p \geq 2$, set $\theta = \lfloor m \rfloor / \lceil m \rceil$. Then properties (ii) and (iii) imply that

$$\begin{aligned} \Psi^m_p &= [\Psi^{\lfloor m \rfloor}_p, \Psi^{\lceil m \rceil}_p]_\theta,\ \mathcal{F}^m_p = [\mathcal{F}^{\lfloor m \rfloor}_p, \mathcal{F}^{\lceil m \rceil}_p]_\theta, \\ \mathcal{G}^m_p &= [\mathcal{G}^{\lfloor m \rfloor}_p, \mathcal{G}^{\lceil m \rceil}_p]_\theta,\ \mathcal{U}^m_p = [\mathcal{U}^{\lfloor m \rfloor}, \mathcal{U}^{\lceil m \rceil}_p]_\theta. \end{aligned}$$

If Assumptions 2.6 and 2.7 hold, then the solution operator is continuous from $\Psi^{\lceil m \rceil}_p \times \mathcal{F}^{\lceil m \rceil}_p \times \mathcal{G}^{\lceil m \rceil}_p$ to $\mathcal{U}^{\lceil m \rceil}_p$, and from $\Psi^{\lfloor m \rfloor}_p \times \mathcal{F}^{\lfloor m \rfloor}_p \times \mathcal{G}^{\lfloor m \rfloor}_p$ to $\mathcal{U}^{\lfloor m \rfloor}_p$. Applying property (i) again therefore yields (3.17) for m, p. Letting $r \to \infty$ and keeping in mind that u^h is a continuous in W^m_p-valued process, using Fatou's lemma we get (3.16) when $q > 1$. Hence for $q > 1$ we obtain

$$\begin{aligned} E(1_A \sup_{t \leq \tau \wedge \tau_n} |u^h_t|^q_l) \quad &\leq \quad NE|1_A 1_{\tau_n > 0} \psi|^q_{W^m_p} + N\mathbb{E}(1_A F^q_{m,p}(\tau \wedge \tau_n)) \\ &\quad + N\mathbb{E}(1_A G^q_{m,p}(\tau \wedge \tau_n)) \end{aligned}$$

for every stopping time $\tau \leq T$, integer $n \geq 1$, and $A \in \mathcal{F}_0$, where

$$\begin{aligned} \tau_n &:= \inf\{t \in [0, T] : \mathcal{R}_{m,p}(t) \geq n\}, \\ \mathcal{R}^p_{m,p}(t) &:= |\psi|^p_{W^m_p} + \int_0^t |f_s|^p_{W^m_p} + |g_s|^p_{W^m_p}\, ds, \end{aligned}$$

and N is a constant depending only on K, T, m, q, d and $|\Lambda|$. By virtue of Lemma 3.2 from [8] this implies

$$\mathbb{E}(\sup_{t \leq \tau \wedge \tau_n} |u^h_t|^q_{W^m_p}) \leq N(\mathbb{E}|\psi|^q_{W^m_p} + \mathbb{E}F^q_{m,p}(\tau \wedge \tau_n) + \mathbb{E}G^q_{m,p}(\tau \wedge \tau_n))$$

for any $q > 0$ with a constant $N = N(K, T, p, d, m, |\Lambda|)$. We finish the proof by letting here $n \to \infty$. □

Proof of Theorem 2.1 By rewriting the equation in the non-divergence form, the theorem for integer m follows from Theorem 2.1 in [4]. Thus we need only prove it when m is non integer. From [4], we know that under the conditions of the theorem, (2.1)–(2.2) admits a unique solution u. Moreover, it is $W_p^{\lfloor m \rfloor}$-valued, and

$$\mathbb{E} \sup_{t\in[0,T]} |u|^q_{W^l_p} \le N(\mathbb{E}|\psi|^q_{W^l_p} + \mathbb{E}F^q_{l,p}(T) + \mathbb{E}G_{l+\kappa}, p^q(T)) \tag{3.18}$$

holds for $l = 0, 1, \ldots, \lceil m \rceil$ and $q > 0$, where $\kappa = 0$ when $(\mu^i) = 0$ and $\kappa = 1$ when (μ^i) is not identically zero. Assume first that $q > 1$. Then following the same interpolation arguments as above, we find that (3.18) holds for all $l \in [0, \lceil m \rceil]$, with ess sup in place of sup on the left-hand side. Then by substituting $(1-\Delta)^{(m-1)/2}\phi$ in place of ϕ in (2.3), we obtain that for any $\phi \in C_0^\infty$, almost surely for all $t \in [0, T]$,

$$\begin{aligned}((1-\Delta)^{(m-1)/2}u_t, \phi) &= ((1-\Delta)^{(m-1)/2}\psi, \phi)\\ &+ \int_0^t \sum_{i=1}^d (\mathbf{F}^i_s, D_i\phi) + (\mathbf{F}^0_s, \phi)\, ds + \int_0^t (\mathbf{G}^k_s, \phi)\, dw^k_s,\end{aligned}$$

where, due to estimates such as

$$|(1-\Delta)^{(m-1)/2}(a^{ij}D_j v)|_{L_p} \le K|v|_{W^m_p},$$

$\mathbf{F}^i$ and $\mathbf{G} = (\mathbf{G}^k)_{k=1}^\infty$ are predictable processes with values in L_p, such that

$$\int_0^T \left(\sum_{i=0}^d |\mathbf{F}^i_t|_{L_p} + |\mathbf{G}_t|_{L_p(l_2)} \right) dt < \infty.$$

□

Using Itô's formula for the L_p norm from [13], we find that $(1-\Delta)^{(m-1)/2}u$ is a strongly L_p-valued process, and thus u is a strongly continuous W_p^{m-1}-valued process. Hence almost surely

$$((1-\Delta)^{m/2}u_t, \varphi) = ((1-\Delta)^{(m-1)/2}u_t, (1-\Delta)^{1/2}\varphi) \tag{3.19}$$

is continuous in $t \in [0, T]$ for all $\varphi \in C_0^\infty$. Let Φ denote the set of those C_0^∞ functions which belong to the unit ball of L_{p^*}, where $p^* = p/(p-1)$. Then

$$\begin{aligned}\sup_{t\in[0,T]} |u_t|_{W^m_p} &= \sup_{t\in[0,T]} \sup_{\varphi\in\Phi} ((1-\Delta)^{m/2}u_t, \varphi) = \sup_{\varphi\in\Phi} \sup_{t\in[0,T]} ((1-\Delta)^{m/2}u_t, \varphi)\\ &= \sup_{\varphi\in\Phi} ess\ sup_{t\in[0,T]}((1-\Delta)^{m/2}u_t, \varphi_j) \le ess\ sup_{t\in[0,T]}|u_t|_{W^m_p} < \infty \quad (a.s.).\end{aligned}$$

This, the continuity in $t \in [0, T]$ of the expression in (3.19) and the denseness of C_0^∞ in $W_{p^*}^{-m}$ imply that almost surely u is a W_p^m-valued weakly continuous process.

Consequently, (3.18) holds for all $l \in [0, m]$ and $q > 1$. Hence using Lemma 3.2 from [8] in the same way as at the end of the proof of Theorem 3.1, we obtain (3.18) for all $l \in [0, m]$ and $q > 0$.

4 Proof of the Main Results

Proof of Theorems 2.2,2.3,2.4 To prove Theorem 2.2, first consider the functions

$$F(h) = \delta_{h,\lambda}\phi(x) = \int_0^1 \partial_\lambda \phi(x + h\theta\lambda)\, d\theta,$$
$$G(h) = \delta_{h,\lambda}\delta_{h,\lambda}\psi(x) = \int_0^1 \int_0^1 \partial_\lambda \partial_\lambda \psi(x + h\lambda(\theta_1 + \theta_2)) d\theta_1 d\theta_2$$

for fixed $\phi \in W_p^{n+l+2}$, $\psi \in W_p^{n+l+3}$, $n, l \geq 0$. Applying Taylor's formula at $h = 0$ up to $n + 1$ terms we get that

$$|\delta_{h,\lambda}\phi - \sum_{i=0}^{n} h^i A_i \partial_\lambda^{i+1}\phi|_{W_p^l} \leq N|h|^{n+1}|\phi|_{W_p^{n+l+2}},$$
$$|\delta_{-h,\lambda}\delta_{h,\lambda}\psi - \sum_{i=0}^{n} h^i B_i \partial_\lambda^{i+2}\psi|_{W_p^l} \leq N|h|^{n+1}|\psi|_{W_p^{l+n+3}}$$

with constants $A_i = 1/(i + 1)!$ and

$$B_i = \begin{cases} 0 & \text{if } i \text{ is odd} \\ \frac{2}{(i+2)!} & \text{if } i \text{ is even} \end{cases},$$

where $N = N(|\Lambda|, d, l, n)$ is a constant. Similarly, or in fact equivalently to the first inequality, we have

$$|T_{h,\lambda}\varphi - \sum_{i=0}^{n} \frac{h^i}{i!} \partial_\lambda^i \varphi|_{W_p^l} \leq N|h|^{n+1}|\varphi|_{W_p^{n+l+1}}$$

for $\varphi \in W_p^{n+l+1}$, where ∂_λ^0 denotes the identity operator. Without going into details, it is clear that, due to Assumption 2.6, from these expansions one can obtain operators $\mathfrak{L}_t^{(i)}$ for integers $i \in [0, \lceil m \rceil]$ such that $\mathfrak{L}_t^0\phi = \partial_i a^{ij}\partial_j\phi + b^i\partial_i\phi + c\phi$,

$$|\mathfrak{L}_t^{(i)}\phi|_{W_p^l} \leq N|\phi|_{W_p^{l+i+1}} \quad \text{for } i \text{ odd}, i + l \leq \lceil m \rceil, \tag{4.1}$$
$$|\mathfrak{L}_t^{(i)}\phi|_{W_p^l} \leq N|\phi|_{W_p^{l+i+2}} \quad \text{for } i \text{ even}, i + l \leq \lceil m \rceil, \tag{4.2}$$

and

$$|\left(L_t^h - \sum_{i=0}^{n} \frac{h^i}{i!}\mathfrak{L}_t^{(i)}\right)\phi|_{W_p^l} \le N|h|^{n+1}|\phi|_{W_p^{n+l+3}} \quad \text{for} n+l < \lceil m \rceil \tag{4.3}$$

with $N = N(|\Lambda|, K, d, p, m)$. The random fields $u^{(j)}$ in expansion (2.11) can then be obtained from the system of SPDEs

$$du_t^{(j)} = \left(\mathfrak{L}_t^{(0)} u_t^{(j)} + \sum_{l=1}^{j} \binom{j}{l} \mathfrak{L}_t^{(l)} u_t^{(j-l)}\right) dt + v_t^r(x) u_t^{(j)}(x)\, dw_t^r \tag{4.4}$$

$$u_0^{(j)} = 0, \quad j = 1, \ldots, k, \tag{4.5}$$

where $v^{(0)} = u$, the solution of (2.1)–(2.2) when $\mu = 0$. The following theorem holds, being the exact analogue of Theorem 5.1 from [7]. It can be proven inductively on j, by a straightforward application of Theorem 2.1 and (4.1)–(4.2). □

Theorem 4.1 *Let $k \ge 1$ be an integer, and let Assumptions 2.5, 2.6 and 2.7 hold with $m \ge 2k+1$. Then there is a unique solution $u^{(1)}, \ldots, u^{(k)}$ of (4.4)–(4.5). Moreover, $u^{(j)}$ is a W_p^{m-2j}-valued weakly continuous process, it is strongly continuous as a W_p^{m-2j-1}-valued process, and*

$$\mathbb{E} \sup_{t\in[0,T]} |u_t^{(j)}|^q_{W_p^{m-2j}} \le N(\mathbb{E}|\psi|^q_{W_p^m} + \mathbb{E}F^q_{m,p}(T) + \mathbb{E}G^q_{m,p}(T))$$

for $j = 1, \ldots, k$, for any $q > 0$, with a constant $N = N(K, m, p, q, T, |\Lambda|)$.

Set

$$\bar{r}_t^h(x) = u_t^h(x) - \sum_{j=0}^{k} \frac{h^j}{j!} u_t^{(j)}(x),$$

for $t \in [0, T]$ and $x \in \mathbb{R}^d$, where u^h is the solution of (3.13)–(3.14). Then it is not difficult to verify that $\bar{r}^h$ is the solution, in the sense of Definition 3.1, of the finite difference equation

$$\bar{r}_t^h(x) = (L_t^h \bar{r}_t^h(x) + F_t^h(x))\, dt + v_t^r(x)\bar{r}_t^h(x)\, dw_t^r, \quad t \in (0, T],\ x \in \mathbb{R}^d$$

with initial condition $\bar{r}_0^h(x) = 0$ for $x \in \mathbb{R}^d$, where

$$F_t^h = \sum_{j=0}^{k} \frac{h^j}{j!} \left(L_t^h - \sum_{i=0}^{k-j} \frac{h^i}{i!}\mathfrak{L}_t^{(i)}\right) u_t^{(j)}.$$

Hence by applying Theorem 3.1 we get

$$\mathbb{E}\sup_{t\in[0,T]}|\bar{r}_t^h|^q_{W_p^{m-2k-3}} \le N\mathbb{E}\left(\int_0^t |F_t|^p_{W_p^{m-2k-3}}dt\right)^{q/p}.$$

Now using $m-2k-3>d/p$, for the left-hand side we can write

$$\mathbb{E}\sup_{t\in[0,T]}\sup_{x\in\mathbb{G}_h}|\bar{r}_t^h(x)|^q+\mathbb{E}\sup_{t\in[0,T]}|\bar{r}_t^h|^q_{l_{p,h}} \le N\mathbb{E}\sup_{t\in[0,T]}|\bar{r}_t^h|^q_{W_p^{m-2k-3}},$$

while (4.3) and the theorem above yield

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}|F_t^h|^q_{W_p^{m-2k-3}} &\le Nh^{q(k+1)}\sum_{j=0}^k\mathbb{E}\sup_{t\in[0,T]}|u_t^{(j)}|^q_{W_p^{m-2j}}\\ &\le Nh^{q(k+1)}(\mathbb{E}|\psi|^q_{W_p^m}+\mathbb{E}F^q_{m,p}(T)+\mathbb{E}G^q_{m,p}(T)),\end{aligned}$$

where N denotes some constants which depend only on K, m, d, q, p, T and $|\Lambda|$. Putting these inequalities together we obtain the estimate

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,T]}\sup_{x\in\mathbb{G}_h}|\bar{r}_t^h(x)|^q+\mathbb{E}\sup_{t\in[0,T]}|\bar{r}_t^h|^q_{l_{p,h}}\\ &\le Nh^{q(k+1)}(\mathbb{E}|\psi|^q_{W_p^m}+\mathbb{E}F^q_{m,p}(T)+\mathbb{E}G^q_{m,p}(T)),\end{aligned}\tag{4.6}$$

for all $h>0$ with a constant $N=N(K,m,d,q,p,T,|\Lambda|)$. Thus we have the following theorem.

Theorem 4.2 *Let $k\ge 0$ be an integer and let Assumptions 2.5, 2.6 and 2.7 hold with $m>2k+3+d/p$. Then there are continuous random fields $u^{(1)},\dots u^{(k)}$ on $[0,T]\times\mathbb{R}^d$, independent of h, such that almost surely*

$$u_t^h(x)=\sum_{j=0}^k\frac{h^j}{j!}u_t^{(j)}(x)+\bar{r}_t^h(x)\tag{4.7}$$

for all $t\in[0,T]$ and $x\in\mathbb{R}^d$, where $u^{(0)}=u$, u^h is the solution of (3.13)–(3.14), and $\bar{r}^h$ is a continuous random field on $[0,T]\times\mathbb{R}^d$, which for any $q>0$ satisfies the estimate (4.6).

Proof By Theorems 2.1, 3.1 and 4.1 $u^h,u^{(0)},u^{(1)},\dots,u^{(k)}$ are W_p^{m-1}-valued continuous processes. Since due to our assumption $m-1>d/p$, by Sobolev's theorem on embedding $W_p^m(\mathbb{R}^d)$ into $C_b(\mathbb{R}^d)$ we get that these random fields are continuous in $(t,x)\in[0,T]\times\mathbb{R}^d$. (Remember that we always identify the functions with their continuous version when they have one.) Hence (4.7) holds by the definition of $\bar{r}^h$, and estimate (4.6) is proved above. □

To finish the proof of Theorem 2.2 we need only show that if Assumption 2.4 holds then under the conditions of Theorem 4.2 the restriction of the solution u^h of (3.13)–(3.14) onto $[0,T]\times\mathbb{G}_h$ is a continuous l_p-valued process which solves (2.5)–(2.6). To

this end note that under the conditions of Theorem 4.2 u^h is a continuous W_p^{m-1} valued process, and if Assumption 2.4 also holds then by (2.7) its restriction to $[0,T]\times\mathbb{G}_h$ is a continuous l_p-valued process. To see that this process satisfies (2.5)–(2.6) we fix a point $x\in\mathbb{G}_h$ and take a nonnegative smooth function φ with compact support in $\mathbb{R}^d$ such that its integral over $\mathbb{R}^d$ is one. Define for each integer $n\geq 1$ the function $\varphi^{(n)}(z)=n^d\varphi(n(z-x))$, $z\in\mathbb{R}^d$. Then by Definition 3.1 we have for u^h, the solution of (2.5)–(2.6), that almost surely

$$(u_t^h,\varphi^{(n)})=(\psi,\varphi^{(n)})+\int_0^t(L_t^hu_s^h+f_s,\varphi^{(n)})\,ds+\int_0^t(\nu_s^ru_s^h+g_s^r,\varphi^{(n)})\,dw_s^r$$

for all $t\in[0,T]$ and for all $n\geq 1$. Letting here $n\to\infty$, for each $t\in[0,T]$ we get

$$u_t^h(x)=\psi(x)+\int_0^t(L_s^hu_s^h(x)+f_s(x))\,ds+\int_0^t(\nu_t^ru_s^h(x)+g_s^r(x))\,dw_s^r \quad (4.8)$$

almost surely, since u^h, ψ, f, ν, g and the coefficients of L^h are continuous in x, due to Sobolev's theorem on embedding $W_p^m(\mathbb{R}^d)$ into $C_b(\mathbb{R}^d)$ in the case $m>d/p$. Note that both $u_t^h(x)$ and the random field on the the right-hand side of Eq. (4.6) are continuous in $t\in[0,T]$. Therefore we have this equality almost surely for all $t\in[0,T]$ and $x\in\mathbb{G}_h$. The proof of Theorem 2.2 is complete.

The extrapolation result, Theorem 2.3, follows from Theorem 2.2 by standard calculations, and hence Theorem 2.4 on the rate of almost sure convergence follows by a standard application of the Borel–Cantelli Lemma.

Proof of Theorem 2.5 Let $\rho(x)=\rho_{\bar s}(\epsilon x)=1/(1+|\epsilon x|^2)^{\bar s/2}$, where $\epsilon>0$ is to be determined later and choose p large enough so that $1>d/p$—and therefore $m>2k+3+d/p$—and Assumption 2.7 holds for $\psi\rho$, $f\rho$ and $g\rho$ in place of ψ, f and g, respectively. After some calculations one gets that u is the solution of (2.13)–(2.14) if and only if $u\rho$ is the solution of the equation

$$\begin{aligned}dv_t(x)=&(D_i\hat a_t^{ij}(x)D_jv_t(x)+\hat b_t^i(x)D_iv_t(x)+\hat c_t(x)v_t(x)+f_t\rho(x))\,dt\\&+(\nu_t^r(x)v_t(x)+g_t^r\rho(x))\,dw_t^r\end{aligned} \quad (4.9)$$

for $(t,x)\in[0,T]\times\mathbb{R}^d$, with the initial condition

$$v_0(x)=\psi\rho(x), \quad (4.10)$$

for $x\in\mathbb{R}^d$, where the coefficients are given by

$$\begin{aligned}\hat a^{ij}&=a^{ij},\\ \hat b^i&=b_t^i-2\sum_{j=1}^d a^{ij}\frac{D_j\rho}{\rho}\ \text{for } i\neq 0,\end{aligned}$$

$$\hat{c} = c - \sum_{i,j=1}^{d} a^{ij} \frac{D_i D_j \rho}{\rho} - \sum_{i,j=1}^{d} D_i a^{ij} \frac{D_j \rho}{\rho} - \sum_{i=1}^{d} \hat{b}^i \frac{D_i \rho}{\rho}.$$

□

Due to our choice of ρ, these coefficients still satisfy the conditions of Theorem 2.1 from [4]. Applying this theorem, we obtain a W_p^m-valued unique solution v. Using Sobolev embedding, we get that v/ρ—which is a solution of (2.13) - is a P_p^{m-1}-valued process.

One can similarly transform the finite difference equations, using (3.2)–(3.3). It turns out that u^h is a solution of (2.5)–(2.6) if and only if $u^h \rho$ is a solution of the equation

$$v_t^h(x) = \{\hat{L}_t^h(x) v_t^h(x) + f_t \rho(x)\}\, dt + (v_t^r(x) v_t^h(x) + g_t^r \rho(x))\, dw_t^r \tag{4.11}$$

for $(t, x) \in [0, T] \times \mathbb{G}_h$ with initial condition

$$v_0^h(x) = \psi \rho(x), \tag{4.12}$$

for $x \in \mathbb{G}_h$, where

$$\hat{L}_t^h \varphi = \sum_{\lambda \in \Lambda_0} \delta_{-h,\lambda}(\hat{\mathfrak{a}}_h^\lambda \delta_{h,\lambda} \varphi) + \sum_{\gamma \in \Lambda_1} \hat{\mathfrak{p}}_h^\gamma \delta_{h,\gamma} \varphi + \sum_{\gamma \in \Lambda_1} \hat{\mathfrak{c}}_h^\gamma T_{h,\gamma} \varphi,$$

with

$$\begin{aligned}
\hat{\mathfrak{a}}_h^\lambda &= \mathfrak{a}_h^\lambda, \\
\hat{\mathfrak{p}}_h^\gamma &= \mathfrak{p}_h^\gamma + \frac{(T_{h,-\lambda}\mathfrak{a}^\lambda)\delta_{h,-\lambda}\rho - (T_{h,\lambda}\mathfrak{a}^{-\lambda})\delta_{h,\lambda}\rho}{\rho}, \\
\hat{\mathfrak{c}}_h^\lambda &= \mathfrak{c}_h^\lambda \frac{\rho}{T_{h,\lambda}\rho} - \frac{(\delta_{h,-\lambda}\mathfrak{a}^\lambda)\delta_{h,-\lambda}\rho - \mathfrak{a}^\lambda \delta_{h,-\lambda}\delta_{h,\lambda}\rho + \hat{\mathfrak{p}}^\lambda \delta_{h,\lambda}\rho}{T_{h,\lambda}\rho},
\end{aligned}$$

where $\mathfrak{a}^\lambda$ is understood to be 0 when not defined.

As it was mentioned earlier, the restriction to $\mathbb{G}_h$ of the continuous modifications of $\psi\rho$, $f\rho$, $g\rho$ are in $l_{p,h}$, $l_{p,h}$-valued, and $l_{p,h}(l_2)$-valued processes, respectively. The coefficients above are bounded, so as we have already seen, there exists a unique $l_{p,h}$-valued solution v^h, in particular, it is bounded. Therefore v^h/ρ is a solution of (2.5) and has polynomial growth.

By choosing ϵ small enough, $|\delta_{h,\lambda}\rho/\rho|$ can be made arbitrarily small, uniformly in $x \in \mathbb{R}^d$, $\lambda \in \Lambda$, $|h| < 1$. In particular, we can choose it to be small enough such that $\hat{\mathfrak{p}}_h^\gamma \geq 0$. Moreover, all of the smoothness and boundedness properties of the coefficients are preserved. Therefore (4.11)–(4.12) is a finite difference scheme for the Eqs. (4.9)–(4.10) such that it satisfies Assumptions 2.4 through 2.7. Claims (iii) and (iv) then follow from applying Theorems 2.2 and 2.4.

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