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H–J–B Equations of Optimal Consumption-Investment and Verification Theorems

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Abstract We consider a consumption-investment problem on infinite time horizon maximizing discounted expected HARA utility for a general incomplete market model. Based on dynamic programming approach we derive the relevant H–J–B equation and study the existence and uniqueness of the solution to the nonlinear partial differential equation. By using the smooth solution we construct the optimal consumption rate and portfolio strategy and then prove the verification theorems under certain general settings.

Keywords Utility maximization · Risk-sensitive stochastic control · Factor models · H–J–B equation · Infinite time horizon

Mathematics Subject Classification 35J60 · 49L20 · 60F10 · 91B28 · 93E20

1 Introduction

In this paper we consider an optimal consumption-investment problem on infinite time horizon for a general incomplete market model. The market model considered here consists of $m+1$ securities, one of which is a risk-less asset and the other m assets are risky ones. The price of risk-less asset is governed by an ordinary differential equation, while the prices of risky ones are defined by the stochastic differential equations. We suppose that all coefficients appearing in those dynamics of the asset prices are affected

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by the economic factors. The dynamics of the economic factors are also formulated by the stochastic differential equations (cf. (2.1)–(2.3)). In such a general market model, the investor divides his (her) wealth among those $m+1$ securities and decides the rate for consumption. The goal is to select consumption and investment strategies which maximize the total expected (discounted) power utility of consumption on a long run (cf. (2.5)).

Our approach is based on the dynamic programming principle, in which the H–J–B (Hamilton–Jacobi–Bellman) equation (cf. (2.10) and (2.11)) are derived relevant to the consumption-investment problem. One can construct an optimal consumption-investment strategy from the smooth solution to the equation. Indeed, the optimal consumption rate and optimal portfolio strategy can be explicitly expressed in terms of the smooth solution. In this approach, a pioneering work by Merton for the market with the risky asset price having constant volatility and return has been done in [14] and recent progress of the further studies in this direction is seen in [4–8] etc.. In [4–6] one dimensional H–J–B equations are considered, where the ordinary differential equations concern. On the other hand, in [7,8], they prove existence of the smooth solution to the H–J–B equation in general dimension by employing a modification of the Leray-Schauder fixed point theorem. Then, constructing the optimal consumption-investment strategy by using the solution, the verification theorems are shown under certain incomplete market settings. In their proofs they have also the best possible discount factor. The current paper is motivated by these works [7,8]. Following their methods we prove existence of the solution to the H–J–B equation (cf. (2.11)) by giving sub- and super- solutions under the general settings, while we obtain newly the uniqueness theorems on the solution, which is one of our main concern. In the current paper the definition of the set of admissible strategies is given by using the unique solution of the H–J–B equation and thus our uniqueness theorems have a crucial meaning. As a result our set of admissible strategies generalizes the one defined in [8] and the proofs of the verification theorems have become quite natural and simple in the current paper. Indeed, by introducing the new measure $\overline{P}^{\hat{h}}$ defined by (5.3) from the unique solution $z(x)$ of H–J–B equation (2.11), the relevant criterion function to the optimal strategy $(\hat{c}_s, \hat{h}_s)$ turns out to be described as

$$e^{z(x)}\overline{E}^{\hat{h}}\left[-\int_0^\infty d\varphi_t\right]$$

by using a multiplicative functional $\varphi_t = e^{-\int_0^t e^{-\frac{z(X_s)}{1-\gamma}}ds} = e^{-\int_0^t \hat{c}_s^\gamma e^{-z(X_s)}ds}$ as you see in (5.2) and (5.5). Thus, identification of the value function with the solution to the H–J–B equation can be done by showing $\varphi_\infty = 0$, a.s. Comparing the value function with the criterion for strategy (c_s, h_s) is also seen by looking at $e^{z(x)}\overline{E}^h[-\int_0^\infty d\tilde{\varphi}_t]$ with $\tilde{\varphi}_t = e^{-\int_0^t c_s^\gamma e^{-z(X_s)}ds}$ (cf. Proofs of Theorem 5.1 and 5.2).

There are slight difference between the market models discussed in [7,8] and ours although the both are factor models. They treat a linear Gaussian model and another factor model with the boundedness assumptions, where all coefficients appearing in the dynamics for the asset prices and the factor process are bounded. In the current paper, the general factor model including linear Gaussian models is discussed without boundedness assumptions on the returns of the price process and the drift coefficient of the factor process. Such difference may cause certain technical difference to treat.

We mention some other works with different approach from ours. Approach using the martingale methods for a complete market model appears in [2,11,19]. Analytical solutions are given in [3,12,20]. More attentive introduction to the historical works can be seen in [7].

The paper is organized as follows. Derivation of the H–J–B equation and our assumptions are described in Sect. 2 under the setting of the factor model. We devote Sect. 3 to construction of sub- and super- solutions to the H–J–B equation. In Sect. 4, we present the existence and uniqueness theorems for the H–J–B equation and the proofs of uniqueness are given. The proofs of existence of the solution following the methods due to Hata and Sheu [7] is completed in Appendix 1. We give the proofs of the verification theorems in Sect. 5. Some notes on the useful gradient estimates for the proofs are given in Appendix 2.

2 Derivation of H–J–B Equations and Assumptions

Consider a market model with $m+1$ securities and n factors, where the bond price is governed by ordinary differential equation

$$dS^0(t) = r(X_t)S^0(t)dt, \quad S^0(0) = s^0. \tag{2.1}$$

The other security prices and factors are assumed to satisfy stochastic differential equations

$$\begin{aligned} dS^i(t) &= S^i(t)\{\alpha^i(X_t)dt + \sum_{k=1}^{n+m} \sigma_k^i(X_t)dW_t^k\}, \\ S^i(0) &= s^i, \quad i = 1, \ldots, m, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} dX_t &= \beta(X_t)dt + \lambda(X_t)dW_t, \\ X(0) &= x, \end{aligned} \tag{2.3}$$

where $W_t = (W_t^k)_{k=1,\ldots,(n+m)}$ is an $m+n$-dimensional standard Brownian motion process on a probability space $(\Omega, \mathcal{F}, P)$. Let N_t^i be the number of the shares of ith security. Then the total wealth that the investor possesses is defined as

$$V_t = \sum_{i=0}^{m} N_t^i S_t^i$$

the portfolio proportion invested to ith security as

$$h_t^i = \frac{N_t^i S_t^i}{V_t}, \quad i = 0, 1, 2, \dots, m.$$

We assume that self-financing condition holds for a consumption investment strategy (h_t, C_t):

$$dV_t = \sum_{i=0}^{m} N_t^i dS_t^i - C_t dt = \sum_{i=0}^{m} \frac{V_t h_t^i}{S_t^i} dS_t^i - C_t dt,$$

where C_t is the instantaneous nominal consumption. Setting $C_t = c_t V_t$, we have

$$\begin{aligned}\frac{dV_t}{V_t} &= h_t^0 r(X_t)dt + \sum_{i=1}^{m} h_t^i \{\alpha^i(X_t)dt + \sum_{j=1}^{n+m} \sigma_j^i(X_t)dW_t^j\} - c_t dt \\ &= \{r(X_t) - c_t\}dt + \sum_{i=1}^{m} h_t^i \{(\alpha^i(X_t) - r(X_t))dt + \sum_{j=1}^{n+m} \sigma_j^i(X_t)dW_t^j\}\end{aligned}$$

Thus the equation describing the dynamics of wealth $V_t = V(c, h)_t$ is given by

$$\frac{dV_t}{V_t} = \{r(X_t) + h_t^*(\alpha(X_t) - r(X_t)\mathbf{1}) - c_t\}dt + h_t^* \sigma(X_t)dW_t. \tag{2.4}$$

Here h^* stands for the transposed vector of h and $\mathbf{1} = (1, 1, \dots, 1)^*$. As for the filtration to be satisfied by admissible investment strategies,

$$\mathcal{G}_t = \sigma(S(u), X(u), \quad u \le t)$$

is relevant in the present problem and we introduce the following definition.

Definition 2.1 $h(t)_{0 \le t \le T}$ is said an investment strategy if $h(t)$ is an R^m valued $\mathcal{G}_t$-progressively measurable stochastic process such that

$$P\left(\int_0^T |h(s)|^2 ds < \infty, \quad \forall T \right) = 1.$$

The set of all investment strategies is denoted by $\mathcal{H}(T)$. For a given $h \in \mathcal{H}(T)$, the process $V_t = V_t(h)$ representing the total wealth of the investor at time t is determined by the stochastic differential equation as was seen above. For $\rho \ge 0$, let us consider the following problem

$$v(x) := \sup_{h_\cdot, c_\cdot} E\left[\int_0^\infty \frac{1}{\gamma} e^{-\rho t} \{c_t V_t(c, h)\}^\gamma dt \right], \quad \gamma < 1, \ \gamma \ne 0. \tag{2.5}$$

The following equations are equivalent to (2.5)

$$v^*(x) := \sup_{h_\cdot, c_\cdot} E\left[\int_0^\infty e^{-\rho t}\{c_t V_t(c,h)\}^\gamma dt\right], \quad 0 < \gamma < 1, \tag{2.6}$$

$$v_*(x) := \inf_{h_\cdot, c_\cdot} E\left[\int_0^\infty e^{-\rho t}\{c_t V_t(c,h)\}^\gamma dt\right], \quad \gamma < 0, \tag{2.7}$$

in each case of $0 < \gamma < 1$ and $\gamma < 0$, respectively. It is easy to see that

$$V_t^\gamma = v_0^\gamma e^{\gamma\int_0^t\{\eta(X_s,h_s)-c_s\}ds+\gamma\int_0^t h_s^*\sigma(X_s)dW_s-\frac{\gamma^2}{2}\int_0^t h_s^*\sigma\sigma^*(X_s)h_s ds},$$

where, v_0 is the initial wealth and

$$\eta(x,h) = -\frac{1-\gamma}{2}h^*\sigma\sigma^*(x)h + h^*\hat{\alpha}(x) + r(x), \quad \hat{\alpha}(x) = \alpha(x) - r(x)\mathbf{1}.$$

Therefore

$$E\left[\int_0^T e^{-\rho t}\{c_t V_t(c,h)\}^\gamma dt\right]$$
$$= v_0^\gamma \int_0^T e^{-\rho t} E[c_t^\gamma e^{\gamma\int_0^t\{\eta(X_s,h_s)-c_s\}ds+\gamma\int_0^t h_s^*\sigma(X_s)dW_s-\frac{\gamma^2}{2}\int_0^t h_s^*\sigma\sigma^*(X_s)h_s ds}]dt.$$

Now let us assume that $(\Omega, \mathcal{F})$ be a standard measurable space (cf. [18]). Then, if M_t^h defined by $M_t^h = \gamma\int_0^t h_s^*\sigma(X_s)dW_s$ satisfies

$$E[e^{M_T^h - \frac{1}{2}\langle M^h\rangle_T}] = 1, \quad \forall T, \tag{2.8}$$

then there is a probability measure P^h satisfying

$$\left.\frac{dP^h}{dP}\right|_{\mathcal{F}_T} = e^{M_T^h - \frac{1}{2}\langle M^h\rangle_T}, \quad \forall T.$$

Then we have

$$E\left[\int_0^T e^{-\rho t}\{c_t V_t(c,h)\}^\gamma dt\right]$$
$$= v_0^\gamma E^h\left[\int_0^T e^{-\rho t} c_t^\gamma e^{\gamma\int_0^t\{\eta(X_s,h_s)-c_s\}ds} dt\right],$$

and thus

$$v^*(x) = \sup_{h_\cdot, c_\cdot} v_0^\gamma E^h \left[\int_0^\infty e^{-\rho t} c_t^\gamma e^{\gamma \int_0^t \{\eta(X_s, h_s) - c_s\} ds} dt \right], \quad 0 < \gamma < 1 \tag{2.6'}$$

and

$$v_*(x) = \inf_{h_\cdot, c_\cdot} v_0^\gamma E^h \left[\int_0^\infty e^{-\rho t} c_t^\gamma e^{\gamma \int_0^t \{\eta(X_s, h_s) - c_s\} ds} dt \right], \quad \gamma < 0. \tag{2.7'}$$

Note that, under the probability measure P^h,

$$W_t^h = W_t - \int_0^t \gamma \sigma^*(X_s) h_s ds$$

is a Brownian motion process and the stochastic differential equation for the economic factor X_t is written as

$$dX_t = \{\beta(X_t) + \gamma \lambda \sigma^*(X_t) h_t\} dt + \lambda(X_t) dW_t^h, \quad X_0 = x. \tag{2.9}$$

When setting

$$v(x) := v_0^{-\gamma} v_*(x), \quad \gamma < 0,$$

the H–J–B equation for $v(x)$ turns out to be

$$\rho v = \frac{1}{2} \mathrm{tr}[\lambda \lambda^* D^2 v] + \beta^* Dv + \inf_{c \geq 0, h \in R^m} \{[\gamma \lambda \sigma^*(x) h]^* Dv + \gamma(\eta(x, h) - c) v + c^\gamma\}.$$

By taking a transformation $z(x) = \log v(x)$, we obtain

$$\begin{aligned} \rho = {} & \frac{1}{2} \mathrm{tr}[\lambda \lambda^* D^2 z] + \frac{1}{2} (Dz)^* \lambda \lambda^* Dz + \beta^* Dz \\ & + \inf_{c \geq 0, h \in R^m} \{[\gamma \lambda \sigma^*(x) h]^* Dz + \gamma(\eta(x, h) - c) + c^\gamma e^{-z}\}, \end{aligned} \tag{2.10}$$

which can be written as

$$\rho = \frac{1}{2} \mathrm{tr}[\lambda \lambda^* D^2 z] + \beta_\gamma^* Dz + \frac{1}{2} (Dz)^* \lambda N_\gamma^{-1} \lambda^* Dz + U_\gamma + (1 - \gamma) e^{-\frac{z}{1-\gamma}}, \tag{2.11}$$

where

$$\begin{aligned} \beta_\gamma &= \beta + \frac{\gamma}{1 - \gamma} \lambda \sigma^* (\sigma \sigma^*)^{-1} \hat{\alpha}, \quad N_\gamma^{-1} = I + \frac{\gamma}{1 - \gamma} \sigma^* (\sigma \sigma^*)^{-1} \sigma, \\ U_\gamma &= \frac{\gamma}{2(1 - \gamma)} \hat{\alpha}^* (\sigma \sigma^*)^{-1} \hat{\alpha} + \gamma r. \end{aligned}$$

On the other hand, for $1 > \gamma > 0$, we set

$$\tilde{v}(x) := v_0^{-\gamma} v^*(x).$$

Then, the H–J–B equation of $\tilde{v}(x)$ is seen to be

$$\rho\tilde{v} = \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2\tilde{v}] + \beta^* D\tilde{v} + \sup_{c\geq 0, h\in R^m}\{[\gamma\lambda\sigma^*(x)h]^* D\tilde{v} + \gamma(\eta(x,h) - c)v + c^\gamma\}$$

and, by taking a transformation $\tilde{z}(x) = \log\tilde{v}(x)$, we obtain

$$\begin{aligned}\rho = &\frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2\tilde{z}] + \frac{1}{2}(D\tilde{z})^*\lambda\lambda^* D\tilde{z} + \beta^* D\tilde{z}\\ &+ \sup_{c\geq 0, h\in R^m}\{[\gamma\lambda\sigma^*(x)h]^* D\tilde{z} + \gamma(\eta(x,h) - c) + c^\gamma e^{-\tilde{z}}\},\end{aligned} \tag{2.12}$$

which turns out to be the same equation as (2.11).

When $\gamma < 0$, we assume that

$$\lambda,\ \beta,\ \sigma,\ \alpha \text{ and } r \text{ are globally Lipschitz, smooth,} \tag{2.13}$$

$$\begin{cases} c_1|\xi|^2 \leq \xi^*\lambda\lambda^*(x)\xi \leq c_2|\xi|^2, & c_1,\ c_2 > 0, \quad \xi \in R^n,\\ c_1|\zeta|^2 \leq \zeta^*\sigma\sigma^*(x)\zeta \leq c_2|\zeta|^2, & \zeta \in R^m,\end{cases} \tag{2.14}$$

$$r \text{ is nonnegative },\tag{2.15}$$

and that, in the case of $\rho = 0$,

$$k_1 \leq \hat{\alpha}^*(\sigma\sigma^*)^{-1}\hat{\alpha}(x) + r(x), \quad k_1 > 0. \tag{2.16}$$

On the other hand, when $0 < \gamma < 1$, we assume (2.13)–(2.15),

$$\hat{\alpha}^*(\sigma\sigma^*)^{-1}\hat{\alpha} + r \to \infty, \quad |x| \to \infty, \tag{2.16'}$$

and the following condition

$$\begin{cases} \text{there exists a function } z_0 \text{ bounded below such that}\\ \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 z_0] + \beta_\gamma^* Dz_0 + \frac{1}{2}(Dz_0)^*\lambda N_\gamma^{-1}\lambda^* Dz_0 + U_\gamma \to -\infty,\\ \text{as } |x| \to \infty, \text{ and that } |Dz_0(x)| \leq C_0(|x|+1), \quad C_0 > 0.\end{cases} \tag{2.17}$$

Note that

$$\begin{aligned}&\frac{1}{1-\gamma}I \leq N_\gamma^{-1} \leq I, \quad \gamma < 0,\\ &I \leq N_\gamma^{-1} \leq \frac{1}{1-\gamma}I, \quad 0 < \gamma < 1\end{aligned} \tag{2.18}$$

hold under these assumptions.

In considering (2.6) and (2.7) we formulate the set of strategies defined by

$\mathcal{A}_0 := \{(c_s, h_s); c_s \geq 0$ and h_s are progressively measurable and h_s satisfies (2.9)$\}$.

Then, we confine the sets of admissible strategies in each case of $0 < \gamma < 1$ and $\gamma < 0$ as follows. Set

$$\tilde{M}_t = \int_0^t (Dz)^* \lambda(X_s) dW_s^h,$$

where $z(x)$ is the unique solution to H–J–B equation (2.11) (see Theorem 4.1 and Theorem 4.2), and consider strategies satisfying

$$E^h[e^{\tilde{M}_T - \frac{1}{2}\langle \tilde{M}\rangle_T}] = 1, \quad \forall T. \tag{2.19}$$

For such h_s we have a probability measure $\overline{P}^h$ on $(\Omega, \mathcal{F})$ such that

$$\left.\frac{d\overline{P}^h}{dP^h}\right|_{\mathcal{F}_T} = e^{\tilde{M}_T - \frac{1}{2}\langle \tilde{M}\rangle_T}, \quad \forall T.$$

The set $\mathcal{A}_1$ of admissible strategies is defined by

$$\mathcal{A}_1 = \{(c_s, h_s) \in \mathcal{A}_0; h_s \text{ satisfies (2.20) }\}.$$

3 Sub- and Super-Solution

3.1 Risk-Averse Case ($\gamma < 0$)

We first note that there exists a positive constants c_0 and c such that

$$\frac{1}{2(1-\gamma)} \hat{\alpha}^* (\sigma\sigma^*)^{-1} \hat{\alpha} + r - c \geq c_0 > 0 \tag{3.1}$$

under assumption (2.16). We consider the following stochastic differential equation

$$d\bar{X}_t = \lambda(\bar{X}_t) dW_t + \beta_\gamma(\bar{X}_t) dt, \quad \bar{X}_0 = x$$

and set

$$\underline{z}(x) = (1-\gamma) \log E_x \left[\int_0^\infty e^{\frac{1}{1-\gamma} \int_0^t U_\gamma(\bar{X}_s) ds} dt \right] \tag{3.2}$$

and

$$\overline{z}(x) = \log E_x \left[\int_0^\infty c^\gamma e^{\int_0^t \{U_\gamma(\bar{X}_s) - \gamma c\} ds} dt \right]. \tag{3.3}$$

Then, we have the following lemma.

Lemma 3.1 *Assume assumptions* (2.13)–(2.16)*. Then,* $\underline{z}(x)$ *(respectively* $\overline{z}(x)$*) is a sub-solution (resp. super-solution) to* (2.11) *for* $\rho = 0$*. Further*

$$\underline{z}(x) \le \overline{z}(x)$$

Proof Set

$$\varphi_1(x) = E_x \left[\int_0^\infty e^{\frac{1}{1-\gamma} \int_0^t U_\gamma(\bar{X}_s) ds} dt \right].$$

Then, it satisfies

$$\frac{1}{2} \mathrm{tr}[\lambda\lambda^* D^2 \varphi_1] + \beta_\gamma^* D\varphi_1 + \frac{1}{1-\gamma} U_\gamma \varphi_1 + 1 = 0$$

and thus $\underline{z}$ does

$$\frac{1}{2} \mathrm{tr}[\lambda\lambda^* D^2 \underline{z}] + \beta_\gamma^* D\underline{z} + \frac{1}{2(1-\gamma)} (D\underline{z})^* \lambda\lambda^* D\underline{z} + U_\gamma + (1-\gamma) e^{-\frac{z}{1-\gamma}} = 0.$$

Since $\frac{1}{1-\gamma} I \le N_\gamma^{-1}$, $\underline{z}$ turns out to be a sub-solution to (2.11).

On the other hand, set

$$\varphi_2(x) = E_x \left[\int_0^\infty c^\gamma e^{\int_0^t \{U_\gamma(\bar{X}_s) - \gamma c\} ds} dt \right].$$

Then, it satisfies

$$\frac{1}{2} \mathrm{tr}[\lambda\lambda^* D^2 \varphi_2] + \beta_\gamma^* D\varphi_2 + (U_\gamma - \gamma c)\varphi + c^\gamma = 0.$$

Therefore, $\overline{z}$ satisfies

$$\frac{1}{2} \mathrm{tr}[\lambda\lambda^* D^2 \overline{z}] + \beta_\gamma^* D\overline{z} + \frac{1}{2} (D\overline{z})^* \lambda\lambda^* D\overline{z} + (U_\gamma - \gamma c) + c^\gamma e^{-\overline{z}} = 0.$$

The left hand side is obtained when taking in the right hand of (2.12) $h = \frac{1}{1-\gamma}(\sigma\sigma^*)^{-1}\hat{\alpha}(x)$ and the constant $c > 0$ satisfying (3.1), and thus we see that $\overline{z}$ is a super-solution to (2.11) with $\rho = 0$.

It is a direct consequence from the following lemma that $\underline{z}(x) \leq \overline{z}(x)$. □

Lemma 3.2 *The following inequality holds*

$$E\left[\int_0^T e^{\frac{1}{1-\gamma}\int_0^t U_\gamma(\bar{X}_s)ds}dt\right]^{1-\gamma} \leq E\left[\int_0^T c^\gamma e^{\int_0^t \{U_\gamma(\bar{X}_s)-\gamma c\}ds}dt\right](1-e^{-cT})^{-\gamma}.$$

Proof Set

$$f_t = e^{\frac{1}{1-\gamma}\int_0^t U_\gamma(\bar{X}_s)ds} c^{\frac{\gamma}{1-\gamma}} e^{-\frac{\gamma}{1-\gamma}ct},$$
$$g_t = c^{-\frac{\gamma}{1-\gamma}} e^{\frac{\gamma}{1-\gamma}ct}.$$

Then, we have

$$E\left[\int_0^T f_t g_t dt\right] \leq E\left[\int_0^T f_t^{1-\gamma} dt\right]^{\frac{1}{1-\gamma}} E\left[\int_0^T g_t^{-\frac{1-\gamma}{\gamma}} dt\right]^{-\frac{\gamma}{1-\gamma}}$$
$$= E\left[\int_0^T c^\gamma e^{\int_0^t \{U_\gamma(\bar{X}_s)-\gamma c\}ds}dt\right]^{\frac{1}{1-\gamma}}(1-e^{-cT})^{-\frac{\gamma}{1-\gamma}}.$$

Hence, we obtain the present lemma. □

Remark When $\rho > 0$, we do not need to assume (2.16) because there exist positive constants c and c_0 such that

$$\frac{1}{2(1-\gamma)}\hat{\alpha}^*(\sigma\sigma^*)^{-1}\hat{\alpha} + r - \frac{\rho}{\gamma} - c \geq c_0 > 0$$

and thus, considering $\tilde{U}_\gamma := \frac{\gamma}{2(1-\gamma)}\hat{\alpha}^*(\sigma\sigma^*)^{-1}\hat{\alpha} + \gamma r - \rho$ in place of U_γ the parallel arguments to the above apply.

3.2 Risk Seeking Case ($0 < \gamma < 1$)

It is not easy to construct a super-solution to (2.11) in risk seeking case, $0 < \gamma < 1$. Let us start with considering the H–J–B equation of risk-sensitive portfolio optimization without consumption:

$$\chi = \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 z] + \beta_\gamma^* Dz + \frac{1}{2}(Dz)^*\lambda N_\gamma^{-1}\lambda^* Dz + U_\gamma. \tag{3.4}$$

To study the existence and uniqueness of the solution to (3.4), we introduce the discounted type equation:

$$\epsilon z_\epsilon = \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 z_\epsilon] + \beta_\gamma^* D z_\epsilon + \frac{1}{2}(Dz_\epsilon)^*\lambda N_\gamma^{-1}\lambda^* D z_\epsilon + U_\gamma. \tag{3.5}$$

We first note that (3.5) can be written as

$$\epsilon z_\epsilon = L z_\epsilon + Q(x, Dz_\epsilon), \tag{3.6}$$

where

$$Lz := \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 z] + \beta^* Dz$$

and

$$Q(x,p) := \frac{1}{2}(\lambda^* p + \gamma\sigma^*(\sigma\sigma^*)^{-1}\hat{\alpha})^* N_\gamma^{-1}(\lambda^* p + \gamma\sigma^*(\sigma\sigma^*)^{-1}\hat{\alpha}) + \frac{\gamma}{2}\hat{\alpha}^*(\sigma\sigma^*)^{-1}\hat{\alpha} + \gamma r.$$

We have the following lemma.

Lemma 3.3 *Under assumptions* (2.13)–(2.15) *and* (2.17), (3.5) *has a solution* $z_\epsilon \in C^2(R^n)$ *such that* $z_\epsilon - z_0$ *is bounded above.*

Proof In light of assumption (2.17), we can assume that $z_0 \geq 0$ and also take R_0 such that, for $R \geq R_0$,

$$Lz_0 + Q(x, Dz_0) < 0, \quad x \in B_R^c.$$

Set

$$\Phi_\epsilon(x) = \frac{M}{\epsilon} + z_0, \quad M := \sup_{x \in B_{R_0}} |Lz_0 + Q(x, Dz_0)|.$$

Then, $\Phi_\epsilon(x)$ turns out to be a super-solution to (3.5). In proving the existence of the solution to (3.5), we first consider the Dirichlet problem for $R > R_0$:

$$\begin{cases} \epsilon z_{\epsilon,R} = L z_{\epsilon,R} + Q(x, Dz_{\epsilon,R}), & x \in B_R \\ z_{\epsilon,R} = \Phi_\epsilon, & x \in \partial B_R. \end{cases} \tag{3.7}$$

Owing to Theorem 8.3 [13, Chapter 4], Dirichlet problem (3.7) has a solution $z_{\epsilon,R} \in C^{2,\mu}(R^n)$. We extend $z_{\epsilon,R}$ to the whole Euclidean space as

$$\tilde{z}_{\epsilon,R}(x) = \begin{cases} z_{\epsilon,R}(x) & x \in B_R, \\ \Phi_\epsilon(x) & x \in B_R^c. \end{cases}$$

Then, we can see that $\tilde{z}_{\epsilon,R}(x)$ is non-increasing with respect to R. Indeed, for $R < R'$,

$$\epsilon(\tilde{z}_{\epsilon,R} - \tilde{z}_{\epsilon,R'}) = L(\tilde{z}_{\epsilon,R} - \tilde{z}_{\epsilon,R'})$$
$$+\frac{1}{2}\{\lambda^*(D\tilde{z}_{\epsilon,R} + D\tilde{z}_{\epsilon,R'} + 2\gamma\sigma^*(\sigma\sigma^*)^{-1}\hat{\alpha}\}^* N_\gamma^{-1}\lambda^*(\tilde{z}_{\epsilon,R} - \tilde{z}_{\epsilon,R'}).$$

Therefore, from the maximum principle, we see that

$$\tilde{z}_{\epsilon,R}(x) - \tilde{z}_{\epsilon,R'}(x) \geq 0 \quad \text{on} \quad B_{R'}$$

since $\tilde{z}_{\epsilon,R}(x) = \tilde{z}_{\epsilon,R'}(x),\ \ x \in \partial B_{R'}$. We further note that $\tilde{z}_{\epsilon,R}(x) \geq 0$ for each R because $z_1(x) \equiv 0$ is a sub-solution to (3.5) for $\gamma > 0$ and the maximum principle again applies.

Similarly to the proof of Proposition 3.2 in [17], we have the following gradient estimate: for each R, $r < \frac{R}{2}$, and $x \in B_r$,

$$|\nabla\tilde{z}_{\epsilon,R}(x)|^2 \leq C(|\nabla\Psi|_{2r}^2 + \frac{1}{r^2}|\Psi|_{2r}^2 + |\beta_\gamma|_{2r}^2 + |\nabla\beta_\gamma|_{2r}$$
$$+\frac{|\beta_\gamma|_{2r}}{2r} + |U_\gamma|_{2r} + |\nabla U_\gamma|_{2r} + 1)), \tag{3.8}$$

where C is a positive constant independent of R and ϵ, $|f|_r = |f|_{L^\infty(B_r(x))}$, and $\Psi = \lambda N_\gamma^{-1}\lambda^*$. Thus, by similar arguments to the proof of Lemma 2.8 in [9], we can see that $\tilde{z}_{\epsilon,R}(x)$ converges H^1_{loc} strongly and uniformly on each compact set to the solution $z_\epsilon \in C^2(R^n)$. Since $\tilde{z}_{\epsilon,R}(x) \leq \Phi_\epsilon(x)$ for each $R > R_0$ we see that $z_\epsilon \leq \Phi_\epsilon$, and hence, $z_\epsilon - z_0$ is bounded above. □

Lemma 3.4 *Assume assumptions* (2.13)–(2.15) *and* (2.17). *Then, the solution z_ϵ to* (3.5) *such that $z_\epsilon - z_0$ is bounded above satisfies*

$$z_\epsilon(x) - z_0(x) \to -\infty, \quad \text{as} \ \ |x| \to \infty.$$

Proof Let z_ϵ be a solution to (3.5) such that $z_\epsilon - z_0$ is bounded above and set

$$V := -\left\{\frac{1}{2}\text{tr}[\lambda\lambda^* D^2 z_0] + \beta_\gamma^* D z_0 + \frac{1}{2}(Dz_0)^*\lambda N_\gamma^{-1}\lambda^* D z_0 + U_\gamma\right\}.$$

Then, we have

$$\epsilon z_\epsilon + V = \frac{1}{2}\text{tr}[\lambda\lambda^* D^2(z_\epsilon - z_0)] + \beta_\gamma^* D(z_\epsilon - z_0) + \frac{1}{2}(Dz_\epsilon)^*\lambda N_\gamma^{-1}\lambda^* D z_\epsilon$$
$$-\frac{1}{2}(Dz_0)^*\lambda N_\gamma^{-1}\lambda^* D z_0 = \frac{1}{2}\text{tr}[\lambda\lambda^* D^2(z_\epsilon - z_0)]$$
$$+\tilde{\beta}_\gamma^* D(z_\epsilon - z_0) + \frac{1}{2}D(z_\epsilon - z_0)^*\lambda N_\gamma^{-1}\lambda^* D(z_\epsilon - z_0), \tag{3.9}$$

where

$$\tilde{\beta}_\gamma = \beta_\gamma + \lambda N_\gamma^{-1}\lambda^* Dz_0.$$

Then, similarly to the proof of Lemma 2.1 in [16], we can see that $z_\epsilon - z_0 \to -\infty$ as $|x| \to \infty$ since $V(x) + \epsilon z_\epsilon \to \infty, \quad |x| \to \infty$. □

Lemma 3.5 *Under assumptions* (2.13)–(2.15) *and*(2.17), (3.4) *has a solution* $(\hat{\chi}, \hat{z})$ *such that* $\hat{z} - z_0$ *is bounded above and* $\hat{z} \in C^2(R^n)$*. Moreover, the solution* (χ, z) *such that* $z - z_0$ *is bounded above is unique, when admitting ambiguity of additive constants with respect to* z*.*

Proof Let us first note that the same estimate as (3.8) holds for z_ϵ for each $\epsilon > 0$. Owing to assumptions (2.13) - (2.15), estimate (3.8) implies that

$$|\nabla z_\epsilon(x)|^2 \le C(|x|^2 + 1), \tag{3.10}$$

where C is a positive constant independent of ϵ. According to Lemma 3.4, $z_\epsilon(x) - z_0(x) \to -\infty$ as $|x| \to \infty$. Therefore, we can take x_ϵ such that

$$z_\epsilon(x_\epsilon) - z_0(x_\epsilon) = \sup_x\{z_\epsilon(x) - z_0(x)\}.$$

Then, at x_ϵ

$$D(z_\epsilon - z_0) = 0, \quad \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2(z_\epsilon - z_0)] \le 0.$$

Therefore, from (3.9), we have

$$V(x_\epsilon) + \epsilon z_\epsilon(x_\epsilon) \le 0,$$

and thus, $V(x_\epsilon) \le 0$. Since $V(x) \to \infty$ as $|x| \to \infty$, there exists a compact set K such that $x_\epsilon \in K$ for each $\epsilon > 0$. Therefore, we can take a subsequence $\{x_{\epsilon_n}\} \subset \{x_\epsilon\}$ and $\hat{x}$ such that $x_{\epsilon_n} \to \hat{x}, n \to \infty$. Once more again from (3.9) we see that

$$0 \le \overline{\lim_{n\to\infty}}\, \epsilon_n z_{\epsilon_n}(x_{\epsilon_n}) \le -V(\hat{x}).$$

Thus, by taking a subsequence if necessary, $\epsilon_n z_{\epsilon_n}(x_{\epsilon_n}) \to \hat{\chi}, n \to \infty$. On the other hand, by using (3.10) we can see that $\{z_{\epsilon_n}(x) - z_{\epsilon_n}(\hat{x})\}$ forms a sequence of uniformly bounded and equicontinuous functions on each compact set K' including K. Thus, similarly to the proof of Theorem 3.1 in [9], we can see that it converges to $\hat{z}(x) \in C^2(R^n)$ in H^1_{loc} strongly and uniformly on each compact set, by using estimate (3.8) for z_ϵ, and that $(\hat{\chi}, \hat{z}(x))$ satisfies (3.4). Further, we can see that $\epsilon_n z_{\epsilon_n}(\hat{x}) \to \hat{\chi}$. Note that

$$z_{\epsilon_n}(x) - z_{\epsilon_n}(\hat{x}) - z_0(x) \le z_{\epsilon_n}(x_{\epsilon_n}) - z_{\epsilon_n}(\hat{x}) - z_0(x_{\epsilon_n})$$

and the left hand side converges to $\hat{z}(x) - z_0(x)$. Therefore we see that $\hat{z}(x) - z_0(x)$ is bounded above by the constant $-z_0(\hat{x})$ which is the limit of the right hand side.

Further, we can see that $\hat{z}(x) - z_0(x) \to -\infty$ as $|x| \to \infty$ similarly to the proof of Lemma 3.4. Therefore, similarly to the proof of Lemma 3.2 in [16], we see that the solution to (3.4) such that $\hat{z}(x) - z_0(x)$ is bounded above is unique when admitting additive constant with respect to z. □

Let us define the operator by

$$\hat{L}\varphi := \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2\varphi] + \{\beta_\gamma^* + (\nabla\hat{z})^*\lambda N_\gamma^{-1}\lambda^*\}D\varphi. \tag{3.11}$$

Then,we have the following lemma.

Lemma 3.6 *Under the assumptions of Lemma 3.3, the diffusion process with the generator $\hat{L}$ is ergodic.*

Proof We have shown that $z_0(x) - \hat{z}(x) \to \infty$ as $|x| \to \infty$ in the proof of Lemma 3.5. We moreover see that

$$\begin{aligned}
\hat{L}(z_0 - \hat{z}) &= \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2(z_0 - \hat{z})] + \beta_\gamma^* D(z_0 - \hat{z}) + (D\hat{z})^*\lambda N_\gamma^{-1}\lambda^* D(z_0 - \hat{z}) \\
&= \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 z_0] + \beta_\gamma^* D z_0 + \frac{1}{2}(Dz_0)^*\lambda N_\gamma^{-1}\lambda^* D z_0 + U_\gamma \\
&\quad - \left\{\frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2\hat{z}] + \beta_\gamma^* D\hat{z} + \frac{1}{2}(D\hat{z})^*\lambda N_\gamma^{-1}\lambda^* D\hat{z} + U_\gamma\right\} \\
&\quad - \frac{1}{2}D(z_0 - \hat{z})^*\lambda N_\gamma^{-1}\lambda^* D(z_0 - \hat{z}) \\
&\le -V - \hat{\chi}.
\end{aligned}$$

Since $-V - \hat{\chi} \to -\infty$ as $|x| \to \infty$, we see that the Has'minskii's conditions are satisfied and that $\hat{L}$ is ergodic. □

Lemma 3.7 *Besides the assumptions of Lemma 3.3, we assume (2.16'). Then, $\hat{z}(x)$ is bounded below.*

Proof Note that

$$Q(z, p) \ge \frac{\gamma}{2}\hat{\alpha}^*(\sigma\sigma^*)^{-1}\hat{\alpha} + \gamma r \equiv f_\gamma(x).$$

Then, from (3.6), it follows that

$$(\epsilon - L)z_\epsilon \ge f_\gamma(x).$$

Set

$$\bar{z}_\epsilon := z_\epsilon(\hat{x}), \quad \tilde{z}_\epsilon := z_\epsilon - \bar{z}_\epsilon.$$

Then, $|\epsilon \bar{z}_\epsilon| < M$ for some positive constant M and we can take $R > 0$ such that

$$f_\gamma(x) > M, \quad x \in B_R^c$$

under assumption (2.16'). Therefore, by using Itô's formula, we have

$$\begin{aligned}
\tilde{z}_\epsilon(x) &\geq E_x\left[\int_0^{\sigma\wedge T} e^{-\epsilon t}(f_\gamma - \epsilon\bar{z}_\epsilon)(Y_t)dt + e^{-\epsilon\sigma\wedge T}\tilde{z}_\epsilon(Y_\sigma)\right] \\
&\geq E_x[e^{-\epsilon\sigma\wedge T}\tilde{z}_\epsilon(Y_{\sigma\wedge T})] \\
&\geq E_x[e^{-\epsilon\sigma}\tilde{z}_\epsilon(Y_\sigma); \sigma < T] - \bar{z}_\epsilon e^{-\epsilon T} P_x(T \leq \sigma)
\end{aligned}$$

since $z_\epsilon(x) \geq 0$, where (Y_t, P_x) is the diffusion process with the generator L and

$$\sigma = \begin{cases} \inf\{t; Y_t \in B_R\}, & \text{if } \{t; Y_t \in B_R\} \neq \phi \\ \infty, & \text{if } \{t; Y_t \in B_R\} = \phi. \end{cases}$$

Therefore, as $T \to \infty$ we see that

$$\tilde{z}_\epsilon(x) \geq \inf_{|x|=R} \tilde{z}_\epsilon(x), \quad x \in B_R^c.$$

Since $\tilde{z}_\epsilon$ converges to $\hat{z}$ uniformly on each compact set we have $\inf_{|x|=R} \tilde{z}_\epsilon(x) \geq -K, \quad K > 0$ and hence obtain $\hat{z}(x) \geq -K$. □

Now, we consider H–J–B equation (2.11) for $0 < \gamma < 1$.

Proposition 3.1 *Assume assumptions* (2.13)–(2.15)*,* (2.16') *and* (2.17)*. Then, when taking C to be sufficiently large, $\bar{z}(x) = \hat{z}(x) + C$ is a super-solution to* (2.11) *with $\rho > \hat{\chi}$. Moreover, $\underline{z}(x) \equiv -C'$ is a sub-solution to* (2.11) *if $C' > 0$ is sufficiently large.*

Proof Take $\epsilon > 0$ such that $\rho - \hat{\chi} > \epsilon$. Then, we can see that

$$\frac{1}{1-\gamma} e^{-\frac{\hat{z}(x)+C}{1-\gamma}} \leq \frac{1}{1-\gamma} e^{-\frac{C+\inf \hat{z}}{1-\gamma}} < \epsilon,$$

by taking C to be sufficiently large because of Lemma 3.7. Since $(\hat{\chi}, \hat{z})$ is a solution to (3.4), $\bar{z} = \hat{z}(x) + C$ turns out to be a super-solution to (2.11).

It is easy to see that $\underline{z}(x)$ is a sub-solution to (2.11). □

4 Existence and Uniqueness

We first prepare the following lemma.

Lemma 4.1 *Assume assumptions* (2.13)–(2.16) *and $\gamma < 0$. Then, the bounded above solution to H–J–B equation* (2.11) *is unique.*

Proof Note that each smooth solution z to (2.11) satisfies the estimate

$$|\nabla z(x)| \le C(1+|x|)$$

for a positive constant $C > 0$ under our assumptions (cf. Appendix 2). Let z and z_1 be bounded above solutions to (2.11) and set

$$\phi(x) = e^{\frac{1}{1-\gamma}(z-z_1)(x)}.$$

Then, we have

$$\begin{aligned}
&\frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2\phi] + (\beta_\gamma + \lambda N_\gamma^{-1}\lambda^* Dz_1)^* D\phi \\
&= \frac{1}{1-\gamma}\Bigg[\frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 z] + (\beta_\gamma + \lambda N_\gamma^{-1}\lambda^* Dz_1)^* Dz \\
&\quad -\left\{\frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 z_1] + (\beta_\gamma + \lambda N_\gamma^{-1}\lambda^* Dz_1)^* Dz_1\right\} \\
&\quad +\frac{1}{2(1-\gamma)} D(z-z_1)^*\lambda\lambda^* D(z-z_1)\Bigg]\phi \\
&= \frac{1}{1-\gamma}\Bigg\{-(1-\gamma)e^{-\frac{z}{1-\gamma}} + (1-\gamma)e^{-\frac{z_1}{1-\gamma}} \\
&\quad -\frac{1}{2}D(z-z_1)^*\lambda(N_\gamma^{-1} - \frac{1}{1-\gamma}I)\lambda^* D(z-z_1)\Bigg\}\phi \\
&\le e^{-\frac{z}{1-\gamma}}(e^{\frac{z-z_1}{1-\gamma}} - 1)\phi = e^{-\frac{z}{1-\gamma}}(\phi-1)\phi
\end{aligned} \tag{4.1}$$

since $\frac{1}{1-\gamma}I \le N_\gamma^{-1}$ for $\gamma < 0$.

Let Y_t be a solution of the stochastic differential equation

$$dY_t = \lambda(Y_t)dW_t + \{\beta_\gamma(Y_t) + \lambda N_\gamma^{-1}\lambda^* Dz_1(Y_t)\}dt, \quad Y_0 = x$$

and set

$$f(x) = -e^{-\frac{z}{1-\gamma}}\phi = -e^{-\frac{z_1}{1-\gamma}}.$$

Then, from (4.1) it follows that

$$\begin{aligned}
d\{(\phi(Y_t)-1)e^{\int_0^t f(Y_s)ds}\} &= f(Y_t)(\phi(Y_t)-1)e^{\int_0^t f(Y_s)ds}dt + e^{\int_0^t f(Y_s)ds}d\phi(Y_t) \\
&\le e^{\int_0^t f(Y_s)ds} D\phi(Y_t)^*\lambda(Y_t)dW_t.
\end{aligned}$$

Setting $t = \tau_G \wedge T$, where

$$\tau_G = \inf\{t; z(Y_t) \ge z_1(Y_t)\},$$

we have

$$\begin{aligned}\phi(x)-1 &\geq E[(\phi(Y_{\tau_G\wedge T})-1)e^{\int_0^{\tau_G\wedge T} f(Y_s)ds}]\\ &= E[(\phi(Y_{\tau_G})-1)e^{\int_0^{\tau_G} f(Y_s)ds};\tau_G\leq T]\\ &\quad +E[(\phi(Y_T)-1)e^{\int_0^T f(Y_s)ds};T<\tau_G]\\ &\geq E[(\phi(Y_T)-1)e^{\int_0^T f(Y_s)ds};T<\tau_G].\end{aligned}$$

Note that

$$(\phi(Y_T)-1)e^{\int_0^T f(Y_s)ds}\geq -e^{-\int_0^T e^{-\frac{1}{1-\gamma}z_1(Y_s)}ds}\geq -e^{-\frac{T}{K}},,$$

where K is a positive constant such that $e^{\frac{1}{1-\gamma}z_1(x)}\leq K$. Thus, we see that

$$\phi(x)-1\geq -e^{-\frac{T}{K}}.$$

Sending T to ∞ we have $\phi(x)-1\geq 0$ and so $z(x)\geq z_1(x)$.

Exchanging a role of z_1 by z, we obtain converse inequality $z_1(x)\geq z(x)$. □

Let $\underline{z}(x)$ and $\overline{z}(x)$ be, respectively sub- and super- solution to (2.11) with $\rho=0$ obtained in Lemma 3.1. Then, we have the following theorem.

Theorem 4.1 *For $\gamma<0$, we assume assumptions* (2.13)–(2.16)*. Then, for $\rho=0$,* (2.11) *has a solution z such that $\underline{z}(x)\leq z(x)\leq \overline{z}(x)$. Moreover, the bounded above solution to* (2.11) *is unique.*

Proof Since we have a sub- and a super- solution as was seen in Lemma 3.1, the existence of a solution can be shown in a similar manner to the proof of Theorem 3.5 in [7]. We complete the proof in Appendix 1.

Let us prove uniqueness. Let $z(x)$ be the solutions to (2.11) with $\rho=0$ such that $\underline{z}(x)\leq z(x)\leq \overline{z}(x)$. Under our assumptions we can see that

$$\overline{z}(x)\leq \gamma\log c+\log\left(\frac{-1}{\gamma c_0}\right)$$

holds and so $z(x)$ is bounded above. Further, owing to Lemma 4.1 we have the uniqueness of the bounded above solution to (2.11). □

Remark It is to be noted that even in the case of $\rho=0$, H–J–B equation (2.11) has the unique solution without ambiguity of additive constants with respect to $z(x)$. Moreover, considering (2.11) with $\rho>0$ and without assumption (2.16) can be reduced to the case of $\rho=0$ with assumption (2.16) (cf. Remark in Sect. 3).

Theorem 4.2 *For $0<\gamma<1$, we assume assumptions* (2.13)–(2.15), (2.16') *and* (2.17)*, and let $\underline{z}(x)$ and $\overline{z}(x)$ be, respectively sub- and super- solutions to* (2.11) *appeared in Proposition* 3.1*. Then, for each $\rho>\hat{\chi}$,* (2.11) *has a solution z such that $\underline{z}(x)\leq z(x)\leq \overline{z}(x)$ and that it satisfies*

$$z(x) - z_0(x) \to -\infty, \quad as \ \ |x| \to \infty. \tag{4.2}$$

Moreover, the solution satisfying (4.2) *is unique.*

Proof As in the proof of the previous theorem, the existence of a solution is given in a similar manner to the proof of Theorem 3.5 in [7] (cf. Appendix 1). We give the proof of unique existence of the solution satisfying (4.2). First note that the solution $z(x)$ to (2.11) necessarily satisfies (4.2) under our assumptions. Indeed, we have seen that $\hat{z}(x) - z_0(x) \to -\infty$ as $|x| \to \infty$ in the proof of Lemma 3.5, and thus, (4.2) follows from $z(x) \le \hat{z}(x) + C$. Let us prove uniqueness. Set

$$\psi = z - z_0$$

for a solution z to (2.11). Then,

$$\begin{aligned}\rho = &\frac{1}{2}\text{tr}[\lambda\lambda^* D^2\psi] + \frac{1}{2}\text{tr}[\lambda\lambda^* D^2 z_0] + \beta_\gamma^* D\psi + \beta_\gamma^* D z_0 + U_\gamma \\ &+\frac{1}{2}D(\psi + z_0)^*\lambda N_\gamma^{-1}\lambda^* D(\psi + z_0) + (1-\gamma)e^{-\frac{\psi+z_0}{1-\gamma}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\rho = &\frac{1}{2}\text{tr}[\lambda\lambda^* D^2\psi] + \tilde{\beta}_\gamma^* D\psi + \frac{1}{2}(D\psi)^*\lambda N_\gamma^{-1}\lambda^* D\psi \\ &-V + (1-\gamma)e^{-\frac{\psi+z_0}{1-\gamma}},\end{aligned} \tag{4.3}$$

where

$$V = -\left\{\frac{1}{2}\text{tr}[\lambda\lambda^* D^2 z_0] + \beta_\gamma^* D z_0 + \frac{1}{2}(Dz_0)^*\lambda N_\gamma^{-1}\lambda^* D z_0 + U_\gamma\right\} \tag{4.4}$$

and

$$\tilde{\beta}_\gamma = \beta_\gamma + \lambda N_\gamma^{-1}\lambda^* D z_0.$$

Let z_1 and z_2 be solutions to (2.11) satisfying (4.2) and set $\psi_i = z_i - z_0$, $i = 1, 2$. Assume that there exists x_0 such that $z_2(x_0) > z_1(x_0)$. Then, $\psi_2(x_0) > \psi_1(x_0)$ and $\psi_i(x) \to -\infty$, as $|x| \to \infty$. Therefore, for each $\epsilon > 0$ there exists x_ϵ such that

$$\psi_\epsilon(x) := \sup_x\{e^{\epsilon\psi_2(x)} - e^{\epsilon\psi_1(x)}\} = e^{\epsilon\psi_2(x_\epsilon)} - e^{\epsilon\psi_1(x_\epsilon)} \ge e^{\epsilon\psi_2(x_0)} - e^{\epsilon\psi_1(x_0)} > 0. \tag{4.5}$$

At x_ϵ we have

$$\begin{aligned}0 &\le -\frac{1}{2}\text{tr}[\lambda\lambda^* D^2\psi_\epsilon] - \tilde{\beta}_\gamma^* D\psi_\epsilon \\ &= -\frac{\epsilon}{2}e^{\epsilon\psi_2}\text{tr}[\lambda\lambda^* D^2\psi_2] - \frac{\epsilon^2}{2}e^{\epsilon\psi_2}(D\psi_2)^*\lambda\lambda^* DZ_2\end{aligned}$$

$$
\begin{aligned}
&+\frac{\epsilon}{2}e^{\epsilon\psi_1}\mathrm{tr}[\lambda\lambda^* D^2\psi_1] + \frac{\epsilon^2}{2}e^{\epsilon\psi_1}(D\psi_1)^*\lambda\lambda^* D\psi_1 \\
&-\epsilon e^{\epsilon\psi_2}\tilde{\beta}_\gamma^* D\psi_2 + \epsilon e^{\epsilon\psi_1}\tilde{\beta}_\gamma^* D\psi_1 \\
= {}& \epsilon e^{\epsilon\psi_2}\left\{\frac{1}{2}(D\psi_2)^*\lambda N_\gamma^{-1}\lambda^* D\psi_2 - V + (1-\gamma)e^{-\frac{\psi_2+z_0}{1-\gamma}} - \rho\right\} \\
&-\epsilon e^{\epsilon\psi_1}\left\{\frac{1}{2}(D\psi_1)^*\lambda N_\gamma^{-1}\lambda^* D\psi_1 - V + (1-\gamma)e^{-\frac{\psi_1+z_0}{1-\gamma}} - \rho\right\} \\
&-\frac{\epsilon^2}{2}e^{\epsilon\psi_2}(D\psi_2)^*\lambda\lambda^* D\psi_2 + \frac{\epsilon^2}{2}e^{\epsilon\psi_1}(D\psi_1)^*\lambda\lambda^* D\psi_1 \\
= {}& \epsilon e^{\epsilon\psi_2}\left\{\frac{1}{2}(D\psi_2)^*\lambda(N_\gamma^{-1} - \epsilon I)\lambda^* D\psi_2 - V + (1-\gamma)e^{-\frac{\psi_2+z_0}{1-\gamma}} - \rho\right\} \\
&-\epsilon e^{\epsilon\psi_1}\left\{\frac{1}{2}(D\psi_1)^*\lambda(N_\gamma^{-1} - \epsilon I)\lambda^* D\psi_1 - V + (1-\gamma)e^{-\frac{\psi_1+z_0}{1-\gamma}} - \rho\right\}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{1}{2}e^{\epsilon\psi_2}(D\psi_2)^*\lambda(N_\gamma^{-1} - \epsilon I)\lambda^* D\psi_2 - \frac{1}{2}e^{\epsilon\psi_1}(D\psi_1)^*\lambda(N_\gamma^{-1} - \epsilon I)\lambda^* D\psi_1 \\
&\quad \geq e^{\epsilon\psi_2}(V + \rho - (1-\gamma)e^{-\frac{\psi_2+z_0}{1-\gamma}}) - e^{\epsilon\psi_1}(V + \rho - (1-\gamma)e^{-\frac{\psi_1+z_0}{1-\gamma}}) \qquad (4.6)
\end{aligned}
$$

On the other hand, $D\psi_2 = D\psi_1 e^{\epsilon\psi_1 - \epsilon\psi_2}$ because $D\psi_\epsilon = 0$ at x_ϵ, and thus,

$$
\begin{aligned}
&\frac{1}{2}e^{\epsilon\psi_2}(D\psi_2)^*\lambda(N_\gamma^{-1} - \epsilon I)\lambda^* D\psi_2 - \frac{1}{2}e^{\epsilon\psi_1}(D\psi_1)^*\lambda(N_\gamma^{-1} - \epsilon I)\lambda^* D\psi_1 \\
&= \frac{1}{2}e^{2\epsilon\psi_1 - \epsilon\psi_2}(D\psi_1)^*\lambda(N_\gamma^{-1} - \epsilon I)\lambda^* D\psi_1 - \frac{1}{2}e^{\epsilon\psi_1}(D\psi_1)^*\lambda(N_\gamma^{-1} - \epsilon I)\lambda^* D\psi_1 \\
&= \frac{1}{2}e^{\epsilon\psi_1 - \epsilon\psi_2}(e^{\epsilon\psi_1} - e^{\epsilon\psi_2})(D\psi_1)^*\lambda(N_\gamma^{-1} - \epsilon I)\lambda^* D\psi_1 \\
&\leq \frac{1-\epsilon}{2}e^{\epsilon\psi_1 - \epsilon\psi_2}(e^{\epsilon\psi_1} - e^{\epsilon\psi_2})(D\psi_1)^*\lambda\lambda^* D\psi_1 \leq 0.
\end{aligned}
$$

Then, from (4.6), it follows that

$$
(e^{\epsilon\psi_2} - e^{\epsilon\psi_1})(V + \rho) - (1-\gamma)e^{-\frac{z_0}{1-\gamma}}(e^{(\epsilon - \frac{1}{1-\gamma})\psi_2} - e^{(\epsilon - \frac{1}{1-\gamma})\psi_1}) \leq 0,
$$

and so,

$$
(e^{\epsilon\psi_2} - e^{\epsilon\psi_1})(V + \rho) \leq (1-\gamma)e^{-\frac{z_0}{1-\gamma}}(e^{(\epsilon - \frac{1}{1-\gamma})\psi_2} - e^{(\epsilon - \frac{1}{1-\gamma})\psi_1}) < 0 \qquad (4.7)
$$

by taking ϵ such that $\epsilon < \frac{1}{1-\gamma}$. Thus, we obtain

$$
V(x_\epsilon) < -\rho
$$

from (4.7). Since $V(x) \to \infty$ as $|x| \to \infty$, there exists $R > 0$ independent of ϵ such that $x_\epsilon \in B_R$ and ϵ_n such that $x_{\epsilon_n} \to \hat{x} \in \overline{B}_R$. From (4.5), we have

$$\frac{1}{\epsilon_n}(e^{\epsilon_n \psi_2(x_{\epsilon_n})} - e^{\epsilon_n \psi_1(x_{\epsilon_n})}) \geq \frac{1}{\epsilon_n}(e^{\epsilon_n \psi_2(x_0)} - e^{\epsilon_n \psi_1(x_0)}) > 0$$

and, by letting $\epsilon_n \to 0$, we obtain

$$\psi_2(\hat{x}) - \psi_1(\hat{x}) \geq \psi_2(x_0) - \psi_1(x_0) > 0.$$

From (4.7), at x_ϵ, we have

$$-K_0 \leq V + \rho \leq \frac{(1-\gamma)e^{-\frac{z_0}{1-\gamma}}(e^{(\epsilon - \frac{1}{1-\gamma})\psi_2} - e^{(\epsilon - \frac{1}{1-\gamma})\psi_1})}{e^{\epsilon\psi_2} - e^{\epsilon\psi_1}}$$

and letting $\epsilon_n \to 0$, the right-hand side tends to $-\infty$, which is a contradiction. Therefore $\psi_2(x) \leq \psi_1(x)$ for each x. In the same way, we have the converse inequality, and hence, we proved uniqueness of the solution to (2.11). □

Let us set the operator $\overline{L}$ by

$$\overline{L}g := \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 g] + \beta_\gamma^* Dg + (Dz)^* \lambda N_\gamma^{-1} \lambda^* Dg. \tag{4.8}$$

Then, inspired by Lemma 3.6 we have the following proposition useful in the proof of the verification theorem.

Proposition 4.1 *Under the assumptions of Theorem 4.2, the diffusion process with the generator $\overline{L}$ is ergodic.*

Proof Let us set

$$\bar{\psi}(x) := z_0(x) - z(x).$$

Then, as was seen in the proof of Theorem 4.2, $\bar{\psi}(x) \to \infty$, as $|x| \to \infty$. Further, from (4.3) it follows that

$$\begin{aligned}-\rho = &\frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2\bar{\psi}] + \beta_\gamma^* D\bar{\psi} + (Dz_0)^*\lambda N_\gamma^{-1}\lambda^* D\bar{\psi} \\ &-\frac{1}{2}(D\bar{\psi})^*\lambda N_\gamma^{-1}\lambda^* D\bar{\psi} + V - (1-\gamma)e^{-\frac{z}{1-\gamma}}.\end{aligned}$$

Since $z_0 = z + \bar{\psi}$ we have

$$\begin{aligned}&\frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2\bar{\psi}] + \beta_\gamma^* D\bar{\psi} + (Dz)^*\lambda N_\gamma^{-1}\lambda^* D\bar{\psi} \\ &\quad = -\frac{1}{2}(D\bar{\psi})^*\lambda N_\gamma^{-1}\lambda^* D\bar{\psi} - V + (1-\gamma)e^{-\frac{z}{1-\gamma}} - \rho\end{aligned}$$

Thus, we can see that $\overline{L}\bar{\psi}(x) \to -\infty$ as $|x| \to \infty$ and hence the proof is complete. □

5 Verification Theorems

Let us set the value functions

$$\hat{v}(x) = \inf_{(h_\cdot, c_\cdot) \in \mathcal{A}_1} E^h \left[\int_0^\infty e^{-\rho t} c_t^\gamma e^{\gamma \int_0^t \{\eta(X_s, h_s) - c_s\} ds} dt \right],$$

for $\gamma < 0$, and

$$\check{v}(x) = \sup_{(h_\cdot, c_\cdot) \in \mathcal{A}_1} E^h \left[\int_0^\infty e^{-\rho t} c_t^\gamma e^{\gamma \int_0^t \{\eta(X_s, h_s) - c_s\} ds} dt \right],$$

for $0 < \gamma < 1$, where $\mathcal{A}_1$ is the sets of admissible strategies defined in the end of Sect. 2. For a solution $z(x)$ to (2.11), we shall prove that $e^{z(x)} = \hat{v}(x)$, for $\gamma < 0$, under assumptions (2.13)-(2.15) (resp. (2.13)-(2.16)) for $\rho > 0$ (resp. $\rho = 0$), and that $e^{z(x)} = \check{v}(x)$, for $0 < \gamma < 1$, under the assumptions of Theorem 4.2. For the solution $z(x)$ to (2.11), define a function $\hat{h}(x)$ by

$$\hat{h}(x) = \frac{1}{1-\gamma} (\sigma\sigma^*)^{-1} (\sigma\lambda^* Dz + \hat{\alpha})(x)$$

and

$$\hat{c}(x) = e^{-\frac{z(x)}{1-\gamma}}.$$

We define also

$$\hat{h}_s = \hat{h}(X_s), \quad \hat{c}_s = \hat{c}(X_s).$$

Let us prepare the following lemma for the proof of the verification theorem.

Lemma 5.1 *Let Y_t be a solution to the stochastic differential equation*

$$dY_t = \sigma(t, Y_t) dW_t + \mu(t, Y_t) dt, \quad Y_0 = x,$$

where σ and μ are locally Lipschitz continuous with respect to x and continuous in t. We moreover assume that $|\mu(t, x)| \le C(1 + |x|)$ and σ is bounded. For a given continuous function $H(t, x)$ satisfying $|H(t, x)| \le C(1 + |x|)$, define ρ_t by

$$\rho_t := e^{\int_0^t H(s, Y_s)^* dW_s - \frac{1}{2} \int_0^t |H(s, X_s)|^2 ds}.$$

Then, we have

$$E[\rho_t] = 1, \quad \forall t.$$

The proof of this lemma is similar to that of Lemma 4.1.1 in [1] and we omit the proof.

We have the following theorem.

Theorem 5.1 *For $\rho = 0$ (resp. $\rho > 0$), assume assumptions (2.13)–(2.16) (resp. (2.13)–(2.15)). Then, for a solution $z(x)$ to (2.11), we have*

$$e^{z(x)} = \inf_{h_\cdot, c_\cdot \in \mathcal{A}_1} E^h \left[\int_0^\infty e^{-\rho t} c_t^\gamma e^{\gamma \int_0^t \{\eta(X_s, h_s) - c_s\} ds} dt \right]$$
$$= E^{\hat{h}} \left[\int_0^\infty e^{-\rho t} \hat{c}_t^\gamma e^{\gamma \int_0^t \{\eta(X_s, \hat{h}_s) - \hat{c}_s\} ds} dt \right] \quad (5.1)$$

with $\gamma < 0$.

Proof Let us first note that $\hat{h}_s$ satisfies (2.8). Indeed, because of the gradient estimates for the solution z given in (7.1) in Appendix 2, we can see it by using the above lemma. Thus, we have a probability measure $P^{\hat{h}}$ under which X_t satisfies the stochastic differential equation

$$dX_t = \lambda(X_t) dW_t^{\hat{h}} + \{\beta_\gamma(X_t) + \frac{\gamma}{1-\gamma} \lambda \sigma^* (\sigma\sigma^*)^{-1} \lambda^* Dz(X_t)\} dt,$$

where

$$\beta_\gamma(x) = \beta(x) + \frac{\gamma}{1-\gamma} \lambda \sigma^* (\sigma\sigma^*)^{-1} \hat{\alpha}(x).$$

Set

$$L^\gamma f := \frac{1}{2} \mathrm{tr}[\lambda\lambda^* D^2 f] + \beta_\gamma^* Df + \frac{\gamma}{1-\gamma} (Dz)^* \lambda \sigma^* (\sigma\sigma^*)^{-1} \sigma \lambda^* Df.$$

Then, by Itô's formula, we have

$$z(X_t) - z(X_0) = \int_0^t L^\gamma z(x_s) ds + \int_0^t (Dz)^* \lambda(X_s) dW_s^{\hat{h}}$$
$$= \int_0^t \left\{ \frac{1}{2} \mathrm{tr}[\lambda\lambda^* D^2 z] + \beta_\gamma^* Dz + \frac{1}{2} (Dz)^* \lambda N_\gamma^{-1} \lambda^* Dz \right.$$
$$\left. + \frac{\gamma}{2(1-\gamma)} (Dz)^* \lambda \sigma^* (\sigma\sigma^*)^{-1} \sigma \lambda^* Dz \right\} (X_s) ds$$
$$+ \int_0^t (Dz)^* \lambda(X_s) dW_s^{\hat{h}} - \frac{1}{2} \int_0^t (Dz)^* \lambda\lambda^* Dz(X_s) ds$$

$$= \int_0^t \Big\{ - U_\gamma(X_s) - (1-\gamma)e^{-\frac{z(X_s)}{1-\gamma}} + \rho$$
$$+ \frac{\gamma}{2(1-\gamma)}(Dz)^* \lambda\sigma^*(\sigma\sigma^*)^{-1}\sigma\lambda^* Dz \Big\}(X_s)ds$$
$$+ \int_0^t (Dz)^*\lambda(X_s)dW_s^{\hat{h}} - \frac{1}{2}\int_0^t (Dz)^*\lambda\lambda^* Dz(X_s)ds.$$

Note that

$$\eta(x,\hat{h}(x)) = -\frac{1}{2(1-\gamma)}(Dz)^*\lambda\sigma^*(\sigma\sigma^*)^{-1}\sigma\lambda^* Dz + \frac{1}{2(1-\gamma)}\hat{\alpha}^*(\sigma\sigma^*)^{-1}\hat{\alpha} + r,$$

and hence,

$$\gamma\eta(x,\hat{h}(x)) = U_\gamma - \frac{\gamma}{2(1-\gamma)}(Dz)^*\lambda\sigma^*(\sigma\sigma^*)^{-1}\sigma\lambda^* Dz.$$

Therefore,

$$z(X_t) - z(X_0) = -\gamma\int_0^t (\eta(X_s,\hat{h}_s) - \hat{c}_s)ds + \rho t - \int_0^t e^{-\frac{z(X_s)}{1-\gamma}}ds + M_t - \frac{1}{2}\langle M\rangle_t$$

where

$$M_t = \int_0^t (Dz)^*\lambda(X_s)dW_s^{\hat{h}}.$$

Once again, owing to the above lemma, we see that M_t satisfies (2.19) with respect to $P^{\hat{h}}$ and therefore $(\hat{c},\hat{h}) \in \mathcal{A}_1$. Thus, we obtain

$$E^{\hat{h}}\left[\int_0^\infty e^{-\rho t}\hat{c}_t^\gamma e^{\gamma\int_0^t(\eta(X_s,\hat{h}_s)-\hat{c}_s)ds}dt\right]$$
$$= E^{\hat{h}}\left[\int_0^\infty e^{-\frac{\gamma}{1-\gamma}z(X_t)-z(X_t)+z(X_0)-\int_0^t e^{-\frac{z(X_s)}{1-\gamma}}ds+M_t-\frac{1}{2}\langle M\rangle_t}dt\right]$$
$$= e^{z(x)}E^{\hat{h}}\left[\int_0^\infty e^{-\frac{1}{1-\gamma}z(X_t)-\int_0^t e^{-\frac{z(X_s)}{1-\gamma}}ds+M_t-\frac{1}{2}\langle M\rangle_t}dt\right]$$
$$= e^{z(x)}\overline{E}^{\hat{h}}\left[\int_0^\infty e^{-\frac{1}{1-\gamma}z(X_t)-\int_0^t e^{-\frac{z(X_s)}{1-\gamma}}ds}dt\right], \tag{5.2}$$

since

$$\left.\frac{d\overline{P}^{\hat{h}}}{dP^{\hat{h}}}\right|_{\mathcal{F}_t} = e^{M_t - \frac{1}{2}\langle M\rangle_t}. \tag{5.3}$$

When setting $\varphi_t = e^{-\int_0^t \hat{c}_s^\gamma e^{-z(X_s)}ds} = e^{-\int_0^t e^{-\frac{z(X_s)}{1-\gamma}}ds}$, we have

$$-d\varphi_t = e^{-\frac{1}{1-\gamma}z(X_t) - \int_0^t e^{-\frac{z(X_s)}{1-\gamma}}ds}dt.$$

From Theorem 4.1, we have

$$0 \le \varphi_t \le e^{-\int_0^t e^{-\frac{\overline{z}(X_s)}{1-\gamma}}ds} \le e^{-K_\gamma t}$$

with $K_\gamma = c^{-\frac{\gamma}{1-\gamma}}(\frac{-1}{\gamma c_0})^{-\frac{1}{1-\gamma}}$ and thus, $\lim_{T\to\infty}\varphi_T = \varphi_\infty = 0$. Therefore we see that

$$\overline{E}^{\hat{h}}\left[\int_0^\infty e^{-\frac{1}{1-\gamma}z(X_t) - \int_0^t e^{-\frac{z(X_s)}{1-\gamma}}ds}dt\right] = \overline{E}^{\hat{h}}[\varphi_0 - \varphi_\infty] = 1.$$

Hence,

$$E^{\hat{h}}\left[\int_0^\infty e^{-\rho t}\hat{c}_t^\gamma e^{\gamma\int_0^t(\eta(X_s,\hat{h}_s)-\hat{c}_s)ds}dt\right] = e^{z(x)}.$$

Now, we shall prove that

$$e^{z(x)} \le E^h\left[\int_0^\infty e^{-\rho t}c_t^\gamma e^{\gamma\int_0^t(\eta(X_s,h_s)-c_s)ds}dt\right], \quad \forall (c,h) \in \mathcal{A}_1. \tag{5.4}$$

For the controlled process defined by

$$dX_t = \lambda(X_t)dW_t^h + \{\beta(X_t) + \gamma\lambda^*\sigma(X_t)h_t\}dt, \quad X_0 = x, \quad (c,h) \in \mathcal{A}_1,$$

we have

$$z(X_t) - z(X_0) = \int_0^t (Dz)^*\lambda(X_s)dW_s^h + \int_0^t \left\{\frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 z] + (\beta + \gamma\lambda\sigma^* h_s)^* Dz\right\}(X_s)ds$$

$$\geq \int_0^t \{\rho - \gamma(\eta(X_s, h_s) - c_s) - c_s^\gamma e^{-z(X_s)}\} ds$$
$$+ \int_0^t (Dz)^* \lambda(X_s) dW_s^h - \frac{1}{2} \int_0^t (Dz)^* \lambda\lambda^* Dz(X_s) ds,$$

from H–J–B equation (2.11). Therefore,

$$-\rho t + \gamma \int_0^t (\eta(X_s, h_s) - c_s) ds \geq z(X_0) - z(X_t) - \int_0^t c_s^\gamma e^{-z(X_s)} ds + \tilde{M}_t - \frac{1}{2}\langle \tilde{M} \rangle_t,$$

where

$$\tilde{M}_t = \int_0^t (Dz)^* \lambda(X_s) dW_s^h.$$

Thus, we obtain

$$E^h \left[\int_0^\infty e^{-\rho t} c_t^\gamma e^{\gamma \int_0^t (\eta(X_s, h_s) - c_s) ds} dt \right]$$
$$\geq e^{z(x)} E^h \left[\int_0^\infty c_t^\gamma e^{-z(X_t) - \int_0^t c_s^\gamma e^{-z(X_s)} ds} e^{\tilde{M}_t - \frac{1}{2}\langle \tilde{M} \rangle_t} dt \right]$$
$$= e^{z(x)} \overline{E}^h \left[\int_0^\infty c_t^\gamma e^{-z(X_t) - \int_0^t c_s^\gamma e^{-z(X_s)} ds} dt \right],$$

where $\overline{P}^h$ is a probability measure defined by

$$\left. \frac{d\overline{P}^h}{dP^h} \right|_{\mathcal{F}_t} = e^{\tilde{M}_t - \frac{1}{2}\langle \tilde{M} \rangle_t}.$$

Let us first assume that $c_t \leq K, \forall t$, for some positive constant $K > 0$. In this case,

$$\int_0^t c_s^\gamma e^{-z(X_s)} ds \geq K^\gamma e^{-C} t$$

holds since $z(x)$ is bounded above by a constant C. Then we have

$$\tilde{\varphi}_t := e^{-\int_0^t c_s^\gamma e^{-z(X_s)} ds} \leq e^{-K^\gamma e^{-C} t}$$

and thus obtain

$$\overline{E}^h\left[\int_0^\infty c_t^\gamma e^{-z(X_t)-\int_0^t c_s^\gamma e^{-z(X_s)}ds}dt\right] = \overline{E}^h[\tilde{\varphi}_0 - \tilde{\varphi}_\infty] = 1.$$

Hence (5.4) holds.

For general $(c, h) \in \mathcal{A}_1$ we set $c_s^{(n)} := \min\{c_s, n\}$. Then, we have

$$E^h\left[\int_0^\infty e^{-\rho t}(c_t^{(n)})^\gamma e^{\gamma\int_0^t(\eta(X_s,h_s)-c_s^{(n)})ds}dt\right] \le E^h\left[\int_0^\infty e^{-\rho t}(c_t^{(n)})^\gamma e^{\gamma\int_0^t(\eta(X_s,h_s)-c_s)ds}dt\right]$$

and therefore

$$e^{z(x)} \le E^h\left[\int_0^\infty e^{-\rho t}(c_t^{(n)})^\gamma e^{\gamma\int_0^t(\eta(X_s,h_s)-c_s)ds}dt\right].$$

Hence, by monotone convergence theorem we have (5.4). □

Theorem 5.2 *Under the assumptions of Theorem 4.2, for a solution $z(x)$ to (2.11), we have*

$$\begin{aligned} e^{z(x)} &= E^{\hat{h}}\left[\int_0^\infty e^{-\rho t}\hat{c}_t^\gamma e^{\gamma\int_0^t\{\eta(X_s,\hat{h}_s)-\hat{c}_s\}ds}dt\right] \\ &= \sup_{h_\cdot,c_\cdot\in\mathcal{A}_1} E^h\left[\int_0^\infty e^{-\rho t}c_t^\gamma e^{\gamma\int_0^t\{\eta(X_s,h_s)-c_s\}ds}dt\right] \end{aligned}$$

Proof Similarly to the proof of Theorem 5.1, we see that $(\hat{c}_s, \hat{h}_s) \in \mathcal{A}_1$ and have

$$\begin{aligned} E^{\hat{h}}\left[\int_0^\infty e^{-\rho t}\hat{c}_t^\gamma e^{\gamma\int_0^t(\eta(X_s,\hat{h}_s)-\hat{c}_s)ds}dt\right] &= e^{z(x)}\overline{E}^{\hat{h}}\left[\int_0^\infty e^{-\frac{1}{1-\gamma}z(X_t)-\int_0^t e^{-\frac{z(X_s)}{1-\gamma}}ds}dt\right] \\ &= e^{z(x)}\overline{E}^{\hat{h}}\left[-\int_0^\infty d\varphi_t\right], \end{aligned} \tag{5.5}$$

where

$$\varphi_t = e^{-\int_0^t e^{-\frac{z(X_s)}{1-\gamma}}ds}.$$

We first note that $(X_t, \overline{P}^{\hat{h}})$ is the ergodic diffusion process with the generator $\overline{L}$ defined by (4.8) according to Proposition 4.1. Moreover,

$$z(x) \le \hat{z}(x) + C$$

and, for each $R > 0$, there exists a positive constant C_R such that

$$e^{-C_R} \le e^{-\frac{\hat{z}(x)+C}{1-\gamma}}, \quad x \in B_R.$$

Therefore,

$$\frac{1}{T}\int_0^T e^{-\frac{\hat{z}(X_s)+C}{1-\gamma}} ds \ge \frac{1}{T} e^{-C_R} \int_0^T \mathbf{1}_{B_R}(X_s) ds \to e^{-C_R} m(B_R)$$

as $T \to \infty$, where $m(dx)$ is the invariant measure of $(X_t, \overline{P}^{\hat{h}})$. Therefore, we see that

$$\int_0^T e^{-\frac{\hat{z}(X_s)+C}{1-\gamma}} ds \to \infty, \quad \overline{P}^{\hat{h}} \ \text{a.s.}$$

as $T \to \infty$, and thus we have $\varphi_T \to 0, \quad \overline{P}^{\hat{h}}$ a.s. since

$$e^{-\int_0^T e^{-\frac{\hat{z}(X_s)+C}{1-\gamma}} ds} \ge e^{-\int_0^T e^{-\frac{z(X_s)}{1-\gamma}} ds} = \varphi_T = e^{-\int_0^T \hat{c}_t^\gamma e^{-z(X_t)} dt}.$$

Thus, we have the first equality.

To prove

$$e^{z(x)} \ge E^h \left[\int_0^\infty e^{-\rho t} c_t^\gamma e^{\gamma \int_0^t (\eta(X_s, h_s) - c_s) ds} dt \right], \quad \forall (c, h) \in \mathcal{A}_1,$$

we use H–J–B equation (2.12) similarly to the above, and then we arrive at

$$\begin{aligned} E^h \left[\int_0^\infty e^{-\rho t} c_t^\gamma e^{\gamma \int_0^t (\eta(X_s, h_s) - c_s) ds} dt \right] &\le e^{z(x)} \overline{E}^h \left[\int_0^\infty c_t^\gamma e^{-z(X_t) - \int_0^t c_s^\gamma e^{-z(X_s)} ds} dt \right] \\ &= e^{z(x)} \lim_{T \to \infty} \overline{E}^h [\tilde{\varphi}_0 - \tilde{\varphi}_T] \\ &\le e^{z(x)}. \end{aligned}$$

□

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6 Appendix 1

Here, we give the proof of the existence of a solution to (2.11) along the line of Hata and Sheu [7], which complete the proofs of Theorems 4.1 and 4.2. We rewrite the equation (2.11) as follows.

$$\frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 z] + b(x, z, Dz; \gamma) = 0, \quad \text{in} \quad R^n, \tag{6.1}$$

where

$$b(x, z, p; \gamma) = \beta_\gamma(x)^* p + \frac{1}{2} p^* \lambda N_\gamma^{-1} \lambda^* p + U_\gamma(x) + (1-\gamma) e^{-\frac{z}{1-\gamma}} - \rho.$$

To prove the existence of a solution to (6.1), we first consider the Dirichlet problem on B_R:

$$\begin{cases} \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 z] + b(x, z, Dz; \gamma) = 0, & x \in B_R \\ z = \underline{z}, & x \in \partial B_R, \end{cases} \tag{6.2}$$

where $\underline{z}$ is a smooth sub-solution to (2.11). For $\gamma < 0$, $\underline{z}$ is the sub-solution appeared in Lemma 3.1 and for $0 < \gamma < 1$, $\underline{z} = -C$ with sufficiently large C. In proving the existence of a solution to (6.2) we appeal to the following theorem, a modification of the Leray-Schauder fixed point theorem, according to the scheme due to Hata and Sheu ([7]).

Theorem 6.1 *(Hata and Sheu [7]) Let $\mathcal{B}$ be a Banach space with the norm $\| \cdot \|_{\mathcal{B}}$ and T a continuous, compact operator from $\mathcal{B} \times [0, 1]$ to $\mathcal{B}$. Assume that there exists a constant $M > 0$ such that $\| \xi \|_{\mathcal{B}} < M$ for all (ξ, τ) satisfying $\xi = T(\xi, \tau)$, or $\xi = \tau T(\xi, 0)$. Then, there exists a fixed point $\xi \in \mathcal{B}$: $\xi = T(\xi, 1)$.*

In applying this theorem, we consider the following problem for each $\tau \in [0, 1]$:

$$\begin{cases} \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 z] + b(x, z, Dz; \tau\gamma) = 0, & x \in B_R \\ z = \tau\underline{z}, & x \in \partial B_R. \end{cases} \tag{6.3}$$

We note that, under assumptions (2.13)–(2.15), for $x \in B_R,\ |z| \le M$, we have

$$c_1|\xi|^2 \le \frac{1}{2}\xi^*\lambda\lambda^*(x)\xi \le c_2|\xi|^2, \quad \xi \in R^n$$
$$|b(x, z, p; \tau\gamma)| + \left|\frac{\frac{1}{2}\partial(\lambda\lambda^*)^{ij}(x)}{\partial x_j}\right| \le c_3(1 + |p|^2),$$

where c_1, c_2 and c_3 are positive constants. To study (A.3), we consider the linear partial differential equation

$$\begin{cases} \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 z] + b(x, w, Dw; \tau\gamma) = 0, & x \in B_R \\ z = \tau\underline{z}, & x \in \partial B_R. \end{cases} \tag{6.4}$$

for a given function $w \in C^{1,\mu}(\overline{B}_R)$, $1 > \mu > 0$. Under assumptions (2.13)–(2.15), we have a unique solution $z \in C^{2,\mu\mu'}(\overline{B}_R)$ since $\underline{z} \in C^{2,\mu}(\overline{B}_R)$ (cf. [13]). Thus, we can define a continuous, compact mapping $T(w, \tau)$ of $\mathcal{B} \times [0, 1]$ into $\mathcal{B}$ as $T(w, \tau) = z$, where z is the solution to (6.4) for a given function $w \in \mathcal{B} := C^{1,\mu}(\overline{B}_R)$ and $\tau \in [0, 1]$. Indeed, since $b(x, w, Dw; \tau\gamma) \in C^{0,\mu}(\overline{B}_R)$ for $w \in C^{1,\mu}(\overline{B}_R)$, and we assume that $\tau\underline{z} \in C^{2,\mu}(\overline{B}_R)$, $T(w, \tau)$, for every τ, transforms the function $w \in C^{1,\mu}(\overline{B}_R)$ into $z(x; \tau)$ in $C^{2,\mu'\mu}(\overline{B}_R)$. Further, we have

$$\| z \|_{C^{2,\mu'\mu}(\overline{B}_R)} \leq f(\| w \|_{C^{1,\mu}(\overline{B}_R)}),$$

where f is a continuous monotonically increasing function of $t \in [0, \infty)$. Since an arbitrary bounded set in $C^{2,\mu'\mu}(\overline{B}_R)$ is compact in the space $C^{1,\mu}(\overline{B}_R)$, $T(w, \tau)$ maps each bounded set of the pairs (w, τ) in $C^{1,\mu}(\overline{B}_R) \times [0, 1]$ into a compact set in $C^{1,\mu}(\overline{B}_R)$. On the other hand, when $\| w_1 - w_2 \|_{C^{1,\mu}(\overline{B}_R)}$ goes to 0, $\| b(x, w_1, Dw_1; \tau\gamma) - b(x, w_2, Dw_2; \tau\gamma) \|_{C^{0,\mu}(\overline{B}_R)}$ tends to 0 uniformly with respect to τ. Moreover, $T(w, \tau)$ is continuous in τ uniformly with respect to $(x, w, Dw) \in \overline{B}_R \times \{u \in R^n; |u| \leq c\} \times \{p \in R^n; |p| \leq c\}$. Therefore $T(w, \tau)$ is a continuous map of $(w, \tau) \in C^{1,\mu}(\overline{B}_R) \times [0, 1]$ into $C^{1,\mu}(\overline{B}_R)$.

Note that a fixed point $z^{(\tau)}$ of T, $z^{(\tau)} = T(z^{(\tau)}, \tau)$, is a solution to (6.3) and $z^{(1)}$ is a solution to (6.2). On the other hand, if we set $z_0 = T(w, 0)$, then z_0 satisfies

$$\begin{cases} \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 z] + b(x, w, Dw; 0) = 0, & x \in B_R \\ z = 0, & x \in \partial B_R, \end{cases} \tag{6.5}$$

with

$$b(x, w, Dw; 0) = \beta(x)^* Dw + \frac{1}{2}(Dw)^* \lambda\lambda^* Dw + e^{-w} - \rho.$$

Therefore, $z_0^{(\tau)} = \tau T(w; 0)$ turns out to be a solution to the equation

$$\begin{cases} \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 z] + \tau b(x, w, Dw; 0) = 0, & x \in B_R \\ z = 0, & x \in \partial B_R. \end{cases} \tag{6.6}$$

Hence, a fixed point $\hat{z}_0^\tau = \tau T(\hat{z}_0^\tau; 0)$ is a solution to

$$\begin{cases} \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 z] + \tau b(x, z, Dz; 0) = 0, & x \in B_R \\ z = 0, & x \in \partial B_R. \end{cases} \tag{6.7}$$

Now, let us give the proof of the existence of a solution to (6.1).

Step 1. Proof of the existence of a solution to (6.2).

Owing to Theorem 3.8 in Hata and Sheu [7], we have the estimate for $z^{(\tau)}$:

$$e^{z^{(\tau)}(x)} \leq \tau e^{\overline{z}(x)} + (1 - \tau) f_0(x), \tag{6.8}$$

where $f_0(x)$ is the solution to

$$\begin{cases} \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 f_0] + \beta(x)Df_0 - \rho + 1 = 0, & x \in B_R \\ f_0 = 1, & x \in \partial B_R. \end{cases} \tag{6.9}$$

Moreover, we see that

$$z^{(\tau)}(x) \geq -C \tag{6.10}$$

for sufficiently large $C > 0$. Indeed, when $\gamma < 0$, for $x \in \{x; \underline{z}(x) \leq 0\}$ $\tau\underline{z}(x) \geq \underline{z}(x) > -C$ for each $\tau \in (0, 1]$ since $\underline{z}(x)$ is bounded in $\overline{B}_R$. $\tau\underline{z}(x) > -C$ holds as well in $x \in \{x; \underline{z}(x) \leq 0\}^c$. Moreover, $-C$ becomes a sub-solution of (6.3) by taking C to be sufficiently large. Therefore we see (6.10) owing to Lemma 3.6 in [7]. Similarly, (6.10) holds also in the case of $0 < \gamma < 1$.

Further, owing to Theorem 3.9 in Hata and Sheu [7], we have

$$-\rho E[\sigma_R] \leq z_0^{(\tau)} \leq \log(1 + E[\sigma_R]), \tag{6.11}$$

where $\sigma_R = \inf\{t; |X_t| = R\}$, and X_t is the solution to the stochastic differential equation:

$$dX_t = \tau\beta(X_t)dt + \lambda(X_t)dW_t, \quad X_0 = x.$$

Therefore, from Theorem 4.1 and 6.1, Chapter 4 in [13], we obtain the estimates

$$\sup_{B_R} |z^{(\tau)}| \leq M, \quad \sup_{B_R} |z_0^{(\tau)}| \leq M$$

for a positive constant M independent of τ, $z^{(\tau)}$ and $z_0^{(\tau)}$. Then, from Theorem 4.1 and 6.1, Chapter 4 in [13], we obtain

$$\sup_{B_R} |\nabla z^{(\tau)}| \leq M_1, \ \sup_{B_R} |\nabla z_0^{(\tau)}| \leq M_1,$$
$$\sum_{i=1}^n \| D_i z^{(\tau)} \|_{C^{1,\mu'}(\overline{B}_R)} \leq M_2, \ \sum_{i=1}^n \| D_i z_0^{(\tau)} \|_{C^{1,\mu'}(\overline{B}_R)} \leq M_2,$$

where the constants M_1, M_2 and μ' are determined by n, M, c_1, c_2, and c_3. Thus, we can consider the operator $T(w; \tau)$ only on the space

$$Z := \{z \in C^{1,\mu}(\overline{B}_R); \sup_{B_R} |z| \leq M + \epsilon, \ \sup_{B_R} |\nabla z| \leq M_1 + \epsilon, \ \sum_{i=1}^n \| D_i z \|_{C^{1,\mu'}(\overline{B}_R)} \leq M_2 + \epsilon\}$$

for $\epsilon > 0$. Then, we see that there exists $\tilde{M}$ such that

$$\| z^{(\tau)} \|_{C^{1,\mu'}(\overline{B}_R)}, \ \| z_0^{(\tau)} \|_{C^{1,\mu'}(\overline{B}_R)} < \tilde{M}$$

for any fixed points $z^{(\tau)}$ and $z_0^{(\tau)}$. Hence, Theorem 6.1 applies and we see that $z \in C^{2,\mu\mu'}(\overline{B}_R)$ such that $z = T(z, 1)$.

Step 2. Proof of the existence of the solution to (6.1).

Let us take a sequence $\{R_n\}$ such that $R_n \to \infty$, as $n \to \infty$, and a sequence of solutions z_{R_n} to (6.2). Since $\underline{z}$ is a subsolution to (6.1), we can see that z_{R_n} is nondecreasing because of the maximum principle. Further, we can see that z_{R_n} is dominated by the super-solution $\overline{z}(x)$ again by the maximum principle. Therefore, there exists $z(x)$ to which z_{R_n} converges as $n \to \infty$. Note that there exists a constant M independent of n such that

$$\sup_{x \in B_r} |\nabla z_{R_n}| \leq M(1 + r)$$

for $r < R_n$, which can be seen in a similar manner to the proof of Proposition 3.2 in [17] (cf. Appendix 2 and also (3.8)). Therefore, we can see that $z_{R_n} \to z$, $W^{1,p}_{loc}$ weakly $\forall p > 1$ by taking a subsequence if necessary. The convergence can be strengthen as ∇z_{R_n} converges in L^2_{loc} strongly to ∇z. As a result we can see that $z \in W^{1,p}_{loc}$ is a weak solution to (6.1). Then, from the regularity theorem we see that $z \in C^{2,\mu''}$ and that it is a classical solution to (6.1). □

7 Appendix 2

Let us give the gradient estimates for the solution to H–J–B equation (2.11).

Lemma 7.1 *Under assumptions*(2.13)–(2.15), *the solution* z *to* (2.11) *satisfies the following estimate.*

$$|\nabla z| \leq C(1 + |x|) \tag{7.1}$$

for some positive constant $C > 0$.

The proof of this estimate is almost the same as the one of Proposition 3.2 in [17]. Here we only give some remarks that the proof could proceed in almost parallel to it. One could see [17] to be more precise.

Proof Set $Q^{ij} := (\lambda N_\gamma^{-1} \lambda^*)^{ij}$ and differentiate (2.11). Then, we have

$$\begin{aligned} 0 = &\frac{1}{2}(\lambda\lambda^*)^{ij} D_{ijk}z + \beta^i_\gamma D_{ik}z + (D_i z)Q^{ij} D_{jk}z + D_k U_\gamma - D_k z e^{-\frac{z}{1-\gamma}} \\ &+ \frac{1}{2} D_k(\lambda\lambda^*)^{ij} D_{ij}z + (D_k \beta^i_\gamma) D_i z + \frac{1}{2}(D_i z)(D_k Q^{ij}) D_j z \end{aligned} \tag{7.2}$$

Set $F = |\nabla z|^2 = \sum_{k=1}^{n} |D_k z|^2$ and

$$\Gamma(F) := \frac{1}{2}(\lambda\lambda^*)^{ij} D_{ij} F + \beta_\gamma^i D_i F + Q^{ij} D_i z D_j F.$$

Then, we have

$$\begin{aligned}
\Gamma(F) &= (\lambda\lambda^*)^{ij} D_{ik} z D_{jk} z + D_k z\{(\lambda\lambda^*)^{ij} D_{ijk} z + 2\beta_\gamma^i D_{ik} z + 2Q^{ij} D_i z D_{jk} z\} \\
&= (\lambda\lambda^*)^{ij} D_{ik} z D_{jk} z + D_k z\{-2D_k U_\gamma + 2D_k z e^{-\frac{z}{1-\gamma}} - D_k(\lambda\lambda^*)^{ij} D_{ij} z \\
&\quad -2(D_k \beta_\gamma^i) D_i z - (D_i z)(D_k Q^{ij}) D_j z\} \\
&\geq \frac{1}{2nc_2}\{(\lambda\lambda^*)^{ij} D_{ij} z\}^2 + \frac{1}{2}(\lambda\lambda^*)^{ij} D_{ik} z D_{jk} z - \frac{|\nabla z|^2}{2\delta} - \frac{\delta C}{2}|D^2 z|^2 \\
&\quad -2|\nabla z||\nabla U_\gamma| + 2|\nabla z|^2 e^{-\frac{z}{1-\gamma}} - 2|\nabla z|^2|\nabla \beta_\gamma| - |\nabla z|^3|\nabla Q| \\
&\geq \frac{1}{2nc_2}\{(\lambda\lambda^*)^{ij} D_{ij} z\}^2 - \frac{|\nabla z|^2}{2\delta} - 2|\nabla z||\nabla U_\gamma| + 2|\nabla z|^2 e^{-\frac{z}{1-\gamma}} \\
&\quad -2|\nabla z|^2|\nabla \beta_\gamma| - |\nabla z|^3|\nabla Q|.
\end{aligned}$$

Here we have used (7.2) and the matrix inequality $(\mathrm{tr}[AB])^2 \leq nC\mathrm{tr}[AB^2]$, for symmetric matrix B and nonnegative definite symmetric matrix A having the maximum eigenvalue C. Thus, in a similar manner to the proof of Proposition 3.2 in [17] (cf. also [9,10,15]) we can obtain estimate (7.1) by using H–J–B equation (2.11) again in the last line in the above. Although we have the term $(1-\gamma)e^{-\frac{z}{1-\gamma}}$ in the equation it does not affect the proof since it is nonnegative. □

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