Appl Math Optim (2014) 70:539–563
DOI 10.1007/s00245-014-9249-1

Optimal Control and Controllability of a Phase Field System with One Control Force

F. D. Araruna · J. L. Boldrini · B. M. R. Calsavara

Published online: 22 April 2014

Abstract We investigate the relation between optimal control and controllability for a phase field system modeling the solidification process of pure materials in the case that only one control force is used. Such system is constituted of one energy balance equation, with a localized control associated to the density of heat sources and sinks to be determined, coupled with a phase field equation with the classical nonlinearity derived from the two-well potential. We prove that this system has a local controllability property and we establish that a sequence of solutions of certain optimal control problems converges to a solution of such controllability problem.

Keywords Phase field models · Solidification models · Optimal control · Controllability

Mathematics Subject Classification 82B26 · 49J20 · 93B05

F. D. Araruna
Departamento de Matemática, Universidade Federal da Paraíba,
58051-900 João Pessoa, PB, Brazil
e-mail: fagner@mat.ufpb.br

J. L. Boldrini
Instituto de Matemática, Estatística e Computação Científica, Universidade Estadual de Campinas,
13083-970 Campinas, SP, Brazil
e-mail: boldrini@ime.unicamp.br

B. M. R. Calsavara (✉)
Faculdade de Ciências Aplicadas, Universidade Estadual de Campinas,
R. Pedro Zaccaria, 1300, CEP 13484-350 Limeira, SP, Brazil
e-mail: biancamrc@yahoo.com

1 Introduction and Main Result

Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with a regular boundary Γ and $\mathcal{O} \subset \Omega$ be a (small) nonempty open subset. For $T > 0$, we consider the cylindrical domain $Q = \Omega \times (0, T)$ in $\mathbb{R}^4$ with lateral boundary $\Sigma = \Gamma \times (0, T)$; by $\nu = \nu(x)$ we denote the outward unit normal vector to Ω at a point $x \in \Gamma$.

Let us consider the following phase field system, which is usually used to model the solidification process of certain pure materials occurring in a region Ω:

$$\left| \begin{array}{ll} u_t - \Delta u + l\phi_t = v\mathbf{1}_{\mathcal{O}} & \text{in} \quad Q, \\ \phi_t - \Delta\phi - \left(a\phi + b\phi^2 - \phi^3\right) - u = 0 & \text{in} \quad Q, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial \phi}{\partial \nu} = 0 & \text{on} \quad \Sigma, \\ u(0) = u_0, \quad \phi(0) = \phi_0 & \text{in} \quad \Omega. \end{array} \right. \tag{1.1}$$

Here, the function $u = u(x, t)$ is related to the temperature of the material; $\phi = \phi(x, t)$ is the phase field functions used to identify the level of solid crystallization present at point $x \in \Omega$ and time $t \in [0, T]$; $\mathbf{1}_{\mathcal{O}}$ denotes the characteristic function of $\mathcal{O}$; v is a control function to be determined, which physically corresponds to the density of heat sources and sinks to be applied in $\mathcal{O}$ to control the solidification process; this is what we call control force. The given constants $l > 0$, $a > 0$ and b depend on the physical properties of the involved material; l in particular is related to the latent heat. The initial data u_0 and ϕ_0 are suitable given functions.

As for the boundary conditions in (1.1), it is important to remark that for the phase field ϕ the homogeneous Neumann boundary condition is the natural condition to be imposed. This is so because it corresponds to the physical requirement that there is no flux of phases at the boundary. Moreover, since the values of ϕ determine the material phase, the imposition of Dirichlet type boundary conditions for ϕ would mean that specific phases for material on the boundary should be maintained during the solidification process, which is not an usual situation in practice. As for the boundary conditions for the temperature, other possibilities could be considered, with similar analysis (see, for instance, Benicasa et al. [1–3] and Moroşanu [4,5]).

Our main goal is to establish the connection between a controllability property and optimal control problem for the system (1.1). More specifically, we want to show for this system that the control v of the local null controllability is actually the limit of a sequence of optimal controls. For this, we will also study a controllability problem for the mentioned system.

System (1.1) is said to be exactly controllable for trajectories at time $T > 0$ if, for any data $\{u_0, \phi_0\}$ and any globally defined trajectory $\{\widetilde{u}, \widetilde{\phi}\}$ (corresponding to the data $\{\widetilde{u}_0, \widetilde{\phi}_0\}$ and $\widetilde{v}$), there exists a control v such that the corresponding solution of (1.1) is also globally defined in $[0, T]$ and satisfies

$$u(\cdot, T) = \widetilde{u}(\cdot, T) \quad \text{and} \quad \phi(\cdot, T) = \widetilde{\phi}(\cdot, T) \quad \text{in} \quad \Omega. \tag{1.2}$$

When the trajectory is null, i.e., $\{\widetilde{u}, \widetilde{\phi}\} = \{0, 0\}$ (associated to $\widetilde{v} = 0$ and $\{\widetilde{u}_0, \widetilde{\phi}_0\} = \{0, 0\}$), the above definition is the formulation of the so-called null controllability for

(1.1). This is equivalent to say that, for each $\{u_0, \phi_0\}$, there exists v such that the corresponding solution $\{u, \phi\}$ satisfies

$$u(\cdot, T) = \phi(\cdot, T) = 0 \quad \text{in} \quad \Omega. \tag{1.3}$$

Before we describe in more detail our mathematical conclusions, let us comment on the importance of the phase field technique as a modeling strategy.

Phase field models are diffuse interface models for phase change, that is, those considering that the solid and liquid regions are separated by intermediate regions with positive width and their own physical structure. These intermediate regions are called *mushy zones* or *transitions layers* when the width may be small and are determined by the values of some specific variables called phase fields; this means in particular that the level sets of such fields separate the different phase regions. This kind of modeling is perhaps the most successful way to model phase change because there is incorporation of several important physical phenomena. A frequent realistic situation in which the separation among the phases involve complex geometries (dendrites, for instance) or low regularities.

Numerical simulations of such models, although difficult, are still possible. Some papers that dealt with several numerical aspects related to phase field models are for instance Benincasa and Moroşanu [1], Cheng and Warren [6], Hamide et al. [7], He and Kasagi [8], Moroşanu [4,9], Rosam et al. [10], Sun and Beckermann [11], Tan and Huang [12] and Zhao et al. [13].

Some papers showing the modeling flexibility of the phase field methodology and its mathematical richness are Ahmad et al. [14], Benincasa et al. [2,3], Boldrini and Vaz [15], Caginalp et al. [16–19], Cherfils et al. [20], Colli et al. [21], Gilardi et al. [22,23], Jiménez-Casas [24], Karma [25], Krejcí et al. [26,27], Laurençot et al. [28], McFadden et al. [29], Moroşanu [30], Nestler et al. [31], Penrose and Fife [32], Planas [33] and Stiner [34]. The asymptotic behavior in time of the solutions for some phase field models has been treated for instance in Aizicovici et al. [35,36], Bates and Zheng [37], Brochet et al. [38], Jiang [39], Kapustyan et al. [40], Röger and Tonegawa [41] and Sprekels and Zheng [42].

Next, we will briefly comment on some mathematically rigorous results that have some connections to ours.

The present model was studied by Hoffman and Jiang [43], where they were able to prove existence, uniqueness and regularity of solutions. Moreover, they also proved the existence a optimal control v minimizing the cost functional

$$\mathcal{F}_\epsilon(u, \phi; v) = \frac{1}{2}\int_\Omega \left(|u(x,T) - u_d|^2 + |\phi(x,T) - \phi_d|^2\right) dx + \frac{\epsilon}{2}\int_0^T \int_{\mathcal{O}} |v|^2 dx dt, \tag{1.4}$$

where $\epsilon > 0$ and the functions u_d, ϕ_d are given.

Related to this issue, and considering one control in each equation, we can mention the work [44] by Wang and Wang, which, by means of Carleman inequality, the authors have shown the existence of a pair of optimal time controls and the maximum principle for a phase field system.

Concerning controllability questions, to compare ours to other results, we must firstly observe that we can rewrite the equations of the system (1.1) as

$$\left| \begin{array}{ll} u_t - \Delta u + f(u, \phi) = -l\Delta\phi + v\mathbf{1}_{\mathcal{O}} & \text{in } Q, \\ \phi_t - \Delta\phi + g(\phi) = u & \text{in } Q, \end{array} \right. \tag{1.5}$$

where now

$$f(u, \phi) = lu + l\left(a\phi + b\phi^2 - \phi^3\right) \quad \text{and} \quad g(\phi) = -\left(a\phi + b\phi^2 - \phi^3\right). \tag{1.6}$$

In [45], Barbu proved local exact controllability of (1.5) to the stationary solutions by using two control forces. That is, he put an additional control term in the second equation in (1.5) (the phase field equation). However, the inclusion of a control term in the phase field equation can not be easily done in practice; in fact, in most usual realistic situations, only the temperature can be subjected to some control. Besides, Dirichlet boundary conditions were used in those papers.

In the more realistic setting of only one control force, but still with Dirichlet boundary conditions, Ammar-Khodja et al. [46] proved the exact controllability to the trajectories for a system similar to (1.5) in the case when $f \equiv 0$ and g is such that $g(0) = 0$ and

$$\lim_{|s|\to\infty} \frac{g(s)}{|s| \ln^{3/2}(1 + |s|)} = 0. \tag{1.7}$$

In [47], Gonzáles-Burgos and Pérez-García proved the null controllability, the exact controllability to the trajectories and the approximate controllability for (1.5) in the case when $f = f(u, \nabla u, \phi, \nabla\phi)$, under certain suitable conditions, and g also satisfying (1.7). So, they generalized the result in [46] and also improved the result in [45].

We point out that the function g given in (1.6) does not satisfy (1.7), and thus the results used in [47] does not seem directly applicable to our case. However, the authors in [47] commented that their controllability results, at least locally and with Dirichlet boundary conditions, may be adapted for system (1.5).

In this paper, before to establish a connection between a certain optimal control problem and the controllability one of the system (1.1), we state local controllability properties for (1.1) in the Theorems 4.1 and 4.2. Note that we are dealing with the more realistic Neumann boundary condition for the phase field and with a classical nonlinearity derived from the two-wells potential. This controllability result improves the one in [45], but we cannot say the same with respect to the results in [46] and [47], since theirs and ours nonlinearities and boundary conditions are different.

To prove Theorems 4.1 and 4.2, we will adapt the ideas from [47], whose strategy consists, firstly, to linearize the problem and then to introduce a fictitious control in the second equation (phase field equation) of this linearized one. So we will prove the null controllability of this linear system with two controls. To obtain this control property, we will get an observability estimate for the solution of the corresponding adjoint system as a consequence of suitable Carleman and energy inequalities. In a second step, we will eliminate the control in the phase field equation and will construct, by using the parabolic regularizing effect of the problem, a control v that gives a controllability of linearized problem. Finally, we will apply the Kakutani fixed point theorem and we will deduce the claimed controllability result for the nonlinear phase field system (1.1).

It is important to note that the control that leads to solution of a system to any desired target states may not be unique. In this way, to build the desired control v in Theorems 4.1 and 4.2 (see Sect. 3) we will make use of minimization of functionals J_ϵ (see 3.16).

In this work, we will establish that the sequence of solutions of optimal control problem which minimize the functionals (1.4) converges to solution of the controllability problem obtained in the Theorems 4.1 and 4.2. This result is, in our opinion, interesting from a computational point of view, since in practice is easier to treat with the functionals $\mathcal{F}_\epsilon$ (see 1.4) than ones J_ϵ (see 3.16). In numerical approximations context and using the same type of ours cost functionals $\mathcal{F}_\epsilon$, we can cite the paper [48] by Cao, where this kind of limit behavior for an optimal control problem associated to linear parabolic equations, was analyzed and simulations were presented.

The rest of the present work is organized as follows. In Sect. 2, we fix the notation and recall certain known results to be used later on. Section 3 is dedicated to obtain null controllability for the linearized system. In the first part of Sect. 3 it is obtained null controllability for the linearized system by using two control functions. In the second part, we eliminate one of these controls to obtain null controllability for the linearized system by using only one control function. In Sect. 4 it is proved the local null controllability for the nonlinear system and then we get the local exact controllability to stationary trajectories. Finally, in Sect. 5 it is obtained a connection between the proposed optimal control problem and the controllability one.

2 Preliminaries

In this section some definitions, notations and technical results involving regularity of a linear parabolic system are presented. These results will be used latter.

Here we will use standard notations for Sobolev spaces, i.e., given $1 \leq p \leq +\infty$, $k \in \mathbb{N}$ and any open set $\mathcal{V} \in \mathbb{R}^n$, we denote the usual Sobolev space by

$$W_p^k(\mathcal{V}) = \{f \in L^p(\mathcal{V}) : D^\alpha f \in L^p(\mathcal{V}),\ |\alpha| \leq k\} \text{ and } H^k(\mathcal{V}) = W_2^k(\mathcal{V}).$$

Properties for such spaces can be found for instance in Adams [49].

To study the Eq. (1.1), we will need the following functional spaces: let any $1 \leq r \leq \infty$, $T \in (0, \infty)$, B a Banach space, $\delta \in [0, T)$ and any open set $\mathcal{V} \in \mathbb{R}^n$, we denote

$$\begin{aligned} L^r(0, T; B) &= \left\{f : (0, T) \to B : \|\, \|f(t)\|_B \,\|_{L^r((0,T))} < +\infty\right\}, \\ W_r^{2,1}(\delta, T; \mathcal{V}) &= \left\{f \in L^r(\delta, T; W_r^2(\mathcal{V})) :\ f_t \in L^r(\mathcal{V} \times (0, T))\right\}, \\ W_r^{2,1}(Q) &= W_r^{2,1}(0, T; \Omega). \end{aligned}$$

For results concerning the last two spaces, we refer for instance to Ladyzenskaja and Solonnikov [50] and Mikhaylov [51]. Here we recall a result that sometimes is called the Lions–Peetre embedding theorem (see [52, p. 24]); it is also consequence of Lemma 3.3, p. 80, in Ladyzhenskaya [50]:

Lemma 2.1 *Let $\mathcal{V} \subset \mathbb{R}^3$ an open and bounded domain satisfying the cone property and let $Q(\delta, T; \Omega) = \Omega \times (\delta, T)$, with $0 \leq \delta < T < \infty$. Then $W_r^{2,1}(Q(\delta, T; \Omega)) \subset L^p(Q(\delta, T; \Omega))$ with compact and continuous embedding for*

(i) $p < \left(\frac{1}{r} - \frac{2}{5}\right)^{-1}$ *if* $r < 5/2$,
(ii) $1 \leq p < \infty$ *if* $r = 5/2$,
(iii) $p = \infty$ *if* $r > 5/2$.

We will also need the following Hilbert spaces:

$$W(0, T) = \{f \in L^2(0, T; H^1(\Omega)) : f_t \in L^2(0, T; (H^1(\Omega))')\}$$

and

$$V = \left\{ f \in H^2(\Omega) : \frac{\partial f}{\partial \nu} = 0 \quad \text{on}\, \Gamma \right\}.$$

In the sequel, C denotes a generic positive constant; sometimes we will explicitly write its dependence on parameters; for instance, when we write $C = C(\Omega, T)$, we means that C only depends only on Ω and T.

Next, we present a result on existence, uniqueness and regularity of solutions of the following liner parabolic system:

$$\left| \begin{array}{ll} u_t - \Delta u + l u + \alpha \phi = -l \Delta \phi + f_1 & \text{in} \quad Q, \\ \phi_t - \Delta \phi + \beta \phi = u + f_2 & \text{in} \quad Q, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial \phi}{\partial \nu} = 0 & \text{on} \quad \Sigma, \\ u(0) = u_0, \quad \phi(0) = \phi_0 & \text{in} \quad \Omega. \end{array} \right. \tag{2.1}$$

Proposition 2.2 *Let $l > 0$ a constant and $\alpha, \beta \in L^\infty(Q)$ be given.*

(i) If $u_0, \phi_0 \in L^2(\Omega)$ and $f_1, f_2 \in L^2(0, T; (H^1(\Omega))')$, then there exists a unique weak solution $\{u, \phi\} \in [W(0, T)]^2$ of (2.1) in the following sense

$$\left| \begin{array}{ll} \displaystyle\int_0^T [\langle u_t, v \rangle + (\nabla u, \nabla v) + l(u, v) + (\alpha \phi, v)]\, dt & \\ \displaystyle\quad = \int_0^T [l(\nabla \phi, \nabla v) + \langle f_1, v \rangle]\, dt & \textit{in} \quad Q, \\ \displaystyle\int_0^T [\langle \phi_t, \varphi \rangle + (\nabla \phi, \nabla \varphi) + (\beta \phi, \varphi)]\, dt = \int_0^T [(u, \varphi) + \langle f_2, \varphi \rangle]\, dt & \textit{in} \quad Q, \\ u(0) = u_0, \quad \phi(0) = \phi_0 & \textit{in} \quad \Omega, \end{array} \right. \tag{2.2}$$

for all $\{v, \varphi\} \in [H^1(\Omega)]^2$*, where* $\langle \cdot, \cdot \rangle$ *denotes the pairing between* $(H^1(\Omega))'$ *and* $H^1(\Omega)$ *and* $(\cdot, \cdot)$ *denotes the inner product in* $L^2(\Omega)$.
Besides, this solution satisfies the estimate

$$\|\{u, \phi\}\|_{[W(0,T)]^2} \leq C\|\{u_0, \phi_0\}\|_{[L^2(\Omega)]^2} + \|\{f_1, f_2\}\|_{[L^2(0,T;(H^1(\Omega))')]^2},$$

for some constant $C = C\left(\Omega, T, l, \|\alpha\|_{L^\infty(Q)}, \|\beta\|_{L^\infty(Q)}\right) > 0$.

(ii) If $u_0, \phi_0 \in W_r^{2-2/r}(\Omega) \cap V$ *and* $f_1, f_2 \in L^r(Q)$ *for some* $r \in [2, \infty)$*, then the weak solution* $\{u, \phi\}$ *of (2.1) belongs to* $[W_r^{2,1}(Q)]^2$ *and satisfies the estimate*

$$\|\{u, \phi\}\|_{[W_r^{2,1}(Q)]^2} \leq C(\|\{u_0, \phi_0\}\|_{[W_r^{2-2/r}(\Omega)]^2} + \|\{f_1, f_2\}\|_{[L^r(Q)]^2}),$$

for some constant $C = C\left(\Omega, T, l, \|\alpha\|_{L^\infty(Q)}, \|\beta\|_{L^\infty(Q)}\right) > 0$.

The first part [item (i)] of the last proposition is proved by using Faedo-Galerkin method. The second part [item (ii)] is proved by using (i) and a 'bootstrap' argument.

By combining the local regularity of heat equation and a 'bootstrap' argument the following local regularity result is obtained for the linear parabolic system (2.1).

Proposition 2.3 *Let* $l > 0$ *a constant,* $\alpha \in L^\infty(Q)$*,* $\beta \in L^\infty(Q)$*,* $u_0, \phi_0 \in L^2(\Omega)$ *and* $f_1, f_2 \in L^2(0, T; (H^1(\Omega))')$ *be given and let* $\{u, \phi\} \in [W(0, T)]^2$ *be the correspondent unique solution of (2.1). Let* $\mathcal{V} \subset \Omega$ *and* $\mathcal{O} \subset\subset \Omega$ *be two open sets and let* $r \in [2, \infty)$ *given.*

(i) If $f_1, f_2 \in L^r(\delta, T; L^r(\Omega))$ *for some* $\delta \in (0, T)$*, then* $\{u, \phi\} \in [W_r^{2,1}(\delta', T; \Omega)]^2$ *for any* $\delta' \in (\delta, T)$ *and there exists constants* $C_1 > 0$*,* $C_2 > 0$ *depending on* Ω*,* l*,* $\|\alpha\|_{L^\infty(Q)}$*,* $\|\beta\|_{L^\infty(Q)}$*, independents of* T *such that*

$$\begin{aligned}&\|\{u, \phi\}\|_{[W_r^{2,1}(\delta', T; \Omega)]^2}\\ &\leq e^{C_1 T} C_2 \left(1 + \frac{1}{\delta' - \delta}\right)^{\mathcal{K}} \left(\|\{f_1, f_2\}\|_{[L^r(\delta, T; L^r(\Omega))]^2} + \|\{u, \phi\}\|_{[W(0,T)]^2}\right),\end{aligned}$$

where $\mathcal{K} > 0$ *is a constant.*

(ii) If $f_1, f_2 \in L^r(0, T; L^r(\mathcal{V}))$ *and* $u(x, 0) = \phi(x, 0) = 0$ *in* Ω*, then* $\{u, \phi\} \in [W_r^{2,1}(0, T; \mathcal{V}')]^2$ *for any* $\mathcal{V}' \subset\subset \mathcal{V} \subset \Omega$ *and there exists a constant* $C = C\left(\Omega, l, \|\alpha\|_{L^\infty(Q)}, \|\beta\|_{L^\infty(Q)}\right) > 0$ *independent of* T *such that*

$$\begin{aligned}&\|\{u, \phi\}\|_{[W_r^{2,1}(0,T;\mathcal{V}')]^2}\\ &\leq C(1 + T)\left(\|\{f_1, f_2\}\|_{[L^r(0,T;L^r(\mathcal{V}))]^2} + \|\{u, \phi\}\|_{[W(0,T)]^2}\right).\end{aligned}$$

(iii) If $f_1, f_2 \in L^r(0, T; L^r(\Omega\backslash\overline{\mathcal{O}'}))$ *and* $u(x, 0) = \phi(x, 0) = 0$ *in* Ω*, then* $\{u, \phi\} \in [W_r^{2,1}(0, T; \Omega\backslash\overline{\mathcal{O}})]^2$ *for any* $\mathcal{O} \subset\subset \mathcal{O}' \subset\subset \Omega$ *and there exists a constant* $C = C\left(\Omega, l, \|\alpha\|_{L^\infty(Q)}, \|\beta\|_{L^\infty(Q)}\right) > 0$ *independent of* T *such that*

$$\begin{aligned}&\|\{u, \phi\}\|_{[W_r^{2,1}(0,T;\Omega\backslash\overline{\mathcal{O}})]^2}\\ &\leq C(1 + T)\left(\|\{f_1, f_2\}\|_{[L^r(0,T;L^r(0,T;\Omega\backslash\overline{\mathcal{O}'}))]^2} + \|\{u, \phi\}\|_{[W(0,T)]^2}\right).\end{aligned}$$

(iv) Assume that hypothesis in part (ii) are satisfied. Suppose also that $f_2,\ u \in L^r(0,T;W^{1,r}(\mathcal{V}))$, $\nabla\beta \in L^\gamma(Q)^3$, *with* γ *given by*

$$\gamma = \begin{cases} \max\{r, 5/2\} & \text{if } r \neq 5/2, \\ \epsilon + 5/2 \ (\text{for any } \epsilon > 0) & \text{if } r = 5/2. \end{cases}$$

Then $\phi \in L^r(0,T;W_r^3(\mathcal{V}'))$ *and* $\phi_t \in L^r(0,T;W_r^1(\mathcal{V}'))$ *for any* $\mathcal{V}' \subset\subset \mathcal{V} \subset \Omega$ *and there exists a constant* $C = C\left(\Omega, l, \|\alpha\|_{L^\infty(Q)}, \|\beta\|_{L^\infty(Q)}\right) > 0$ *independent of* T *such that*

$$\begin{aligned} &\|\phi\|_{L^r(0,T;W_r^3(\mathcal{V}'))} + \|\phi_t\|_{L^r(0,T;W_r^1(\mathcal{V}'))} \leq C(1+T)^2(1+\|\nabla\beta\|_{L^\gamma(Q)}) \\ &\times\left(\|f_1\|_{L^r(0,T;L^r(\mathcal{V})} + \|\{f_2,u\}\|_{L^r(0,T;W_r^{1,r}(\mathcal{V})} + \|\phi\|_{W(0,T)}\right). \end{aligned}$$

3 Null Controllability for the Linear Phase Field System

In this section we will adapt the ideas from [47] to prove the null controllability for a linear phase field system. Here, the strategy consists in to introduce a fictitious control in the second equation (phase field equation) of the linearized system and prove the null controllability of this linear system with two controls. To obtain this control property, we will get an observability estimate for the solution of the corresponding adjoint system as a consequence of suitable Carleman and energy inequalities. In a second step, we will eliminate the control in the phase field equation and will construct, by using the parabolic regularizing effect of the problem, a control v that gives a controllability of linearized problem.

The aim of this section is to prove the null controllability property for the following linear phase field system:

$$\left| \begin{array}{ll} u_t - \Delta u + lu + \alpha\phi = -l\Delta\phi + v\mathbf{1}_{\mathcal{O}} & \text{in } Q, \\ \phi_t - \Delta\phi + \beta\phi = u & \text{in } Q, \\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial\phi}{\partial\nu} = 0 & \text{on } \Sigma, \\ u(0) = u_0, \quad \phi(0) = \phi_0 & \text{in } \Omega, \end{array} \right. \tag{3.1}$$

where $\alpha, \beta \in L^\infty(Q)$. For this, we will study, initially, the null controllability for the auxiliary linear system with two control forces:

$$\left| \begin{array}{ll} u_t - \Delta u + lu + \alpha\phi = -l\Delta\phi + v_1\mathbf{1}_{\mathcal{O}} & \text{in } Q, \\ \phi_t - \Delta\phi + \beta\phi = u + v_2\mathbf{1}_{\mathcal{O}} & \text{in } Q, \\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial\phi}{\partial\nu} = 0 & \text{on } \Sigma, \\ u(0) = u_0, \quad \phi(0) = \phi_0 & \text{in } \Omega. \end{array} \right. \tag{3.2}$$

After that we will eliminate the control in the phase field equation $(3.2)_2$ and will construct a control v that gives a null controllability of (3.1). We divide it into two subsection.

3.1 Null Controllability for Linear Phase Field System with Two Controls

For the linear phase field system (3.2) we have the following null controllability result:

Theorem 3.1 *Let $T > 0$ and initial data $u_0, \phi_0 \in L^2(\Omega)$ be given, then there exist two control functions $v_1, v_2 \in L^2(Q)$ such that the solution $\{u, \phi\}$ of (3.2) satisfies (1.3). Furthermore, there exists a constant $C = C(T, \|\alpha\|_{L^\infty(Q)}, \|\beta\|_{L^\infty(Q)}) > 0$ such that*

$$\|\{v_1, v_2\}\|_{[L^2(\mathcal{O}\times(0,T))]^2} \leq C \|\{u_0, \phi_0\}\|_{[L^2(\Omega)]^2}. \tag{3.3}$$

The proof of this theorem is a consequence of the suitable observability property for the following adjoint system:

$$\left|\begin{array}{ll} -\varphi_t - \Delta\varphi + \beta\varphi + \alpha\psi = -l\Delta\psi & \text{in } \; Q, \\ -\psi_t - \Delta\psi + l\psi = \varphi & \text{in } \; Q, \\ \dfrac{\partial\varphi}{\partial\nu} = \dfrac{\partial\psi}{\partial\nu} = 0 & \text{on } \; \Sigma, \\ \varphi(T) = \varphi_0, \quad \psi(T) = \psi_0 & \text{in } \; \Omega. \end{array}\right. \tag{3.4}$$

Precisely, we have to prove the observability result:

Theorem 3.2 *Let $T > 0$ and initial data $\varphi_0, \psi_0 \in L^2(\Omega)$ be given, then there exists a positive constant C such that*

$$\|\{\varphi(0), \psi(0)\}\|^2_{[L^2(\Omega)]^2} \leq C \int\limits_{\mathcal{O}\times(0,T)} \left(|\varphi|^2 + |\psi|^2\right) dxdt. \tag{3.5}$$

Proof The proof of this observability result combines a suitable Carleman and energy estimates for the solution of the adjoint system. We set

$$\begin{array}{l} \xi(x,t) = \dfrac{e^{\lambda\eta(x)}}{t(T-t)}, \quad \rho(x,t) = \dfrac{e^{\lambda\eta(x)} - e^{2\lambda\|\eta\|_{C(\overline{\Omega})}}}{t(T-t)}, \\ \widetilde{\rho}(x,t) = \dfrac{e^{-\lambda\eta(x)} - e^{2\lambda\|\eta\|_{C(\overline{\Omega})}}}{t(T-t)}, \end{array} \tag{3.6}$$

where $\lambda > 0$ and the function $\eta \in C^2(\overline{\Omega})$ is such that

$$\eta > 0 \text{ in } \Omega, \quad \eta = 0 \text{ on } \Gamma \quad \text{and} \quad \nabla\eta \neq 0 \text{ in } \Omega\backslash\mathcal{O}', \tag{3.7}$$

with $\mathcal{O}' \subset\subset \mathcal{O}$ being an arbitrary fixed subdomain of Ω. We refer to [53] for the proof of the existence of a function satisfying (3.7).

The following Carleman estimate for the solutions of (3.4) holds:

Lemma 3.3 *Let functions ξ, ρ and $\widetilde{\rho}$ be defined as in (3.6). Then there exists a number $\widehat{\lambda} > 0$ such that for an arbitrary $\lambda > \widehat{\lambda}$, there exists a positive function $s_0 = s_0(\lambda)$ such that for every $s \geq s_0$ the solutions of problem (3.4) satisfy the following inequality*

$$\int_Q \left[\left(s^3\xi - C_1 - C_2T^2 - C_1C_2T^2l^2 \left\| e^{\lambda\eta} \right\|_\infty s \right) |\varphi|^2 + \frac{s}{\xi} |\nabla\varphi|^2 \right] \left(e^{2s\rho} + e^{2s\widetilde{\rho}} \right) dxdt$$
$$+ \int_Q \left[\left(s^3\xi - C_2T^4 \|\alpha\|_\infty \right) |\psi|^2 + s\xi |\nabla\psi|^2 \right] \left(e^{2s\rho} + e^{2s\widetilde{\rho}} \right) dxdt$$
$$\leq C_2T^4 \int_{\mathcal{O}\times[0,T]} s^3\xi^3 |\varphi|^2 \left(e^{2s\rho} + e^{2s\widetilde{\rho}} \right) dxdt + C_1 \int_{\mathcal{O}\times[0,T]} s^3\xi^3 |\psi|^2 \left(e^{2s\rho} + e^{2s\widetilde{\rho}} \right) dxdt$$
$$+ C_1C_2T^2l^2 \left\| e^{\lambda\eta} \right\|_\infty^2 \int_{\mathcal{O}\times[0,T]} s^4\xi^3 |\psi|^2 \left(e^{2s\rho} + e^{2s\widetilde{\rho}} \right) dxdt, \tag{3.8}$$

where the constants $C_1 > 0$ depends continuously on λ and the constant $C_2 > 0$ depends continuously on λ and $\|\beta\|_\infty$.

Proof of the Lemma 3.3 By applying the Carleman inequality (see [53]) in $(3.4)_2$ we can guarantee the existence of a number $\widehat{\lambda}_1 > 0$ and of a positive function $s_1 = s_1(\lambda)$ such that

$$\int_Q \left[\frac{1}{s\xi} \left(|\psi_t|^2 + |\Delta\psi|^2 \right) + s\xi |\nabla\psi|^2 + s^3\xi^3 |\psi|^2 \right] \left(e^{2s\rho} + e^{2s\widetilde{\rho}} \right) dxdt$$
$$\leq C_1 \left(\int_Q |\varphi|^2 \left(e^{2s\rho} + e^{2s\widetilde{\rho}} \right) dxdt + \int_{\mathcal{O}\times[0,T]} s^3\xi^3 |\psi|^2 \left(e^{2s\rho} + e^{2s\widetilde{\rho}} \right) dxdt \right), \tag{3.9}$$

for $\lambda > \widehat{\lambda}_1$ and $s \geq s_1$, where the positive constant C_1 depends continuously on λ.

On the other hand, we see that the first equation of (3.4) is equivalent to

$$y_t - \Delta y + \beta y = -\left[t(T-t) \right]_t \varphi - \alpha t(T-t)\psi - lt(T-t)\Delta\psi, \tag{3.10}$$

where $y = t(T-t)\varphi$.

Thus, by applying once again the Carleman inequality to Eq. (3.10), we obtain the existence of a number $\widehat{\lambda}_2 > 0$ and of a positive function $s_2 = s_2(\lambda)$ such that

$$\int_Q \left[\frac{1}{s\xi} \left(|y_t|^2 + |\Delta y|^2 \right) + s\xi |\nabla y|^2 + s^3\xi^3 |y|^2 \right] \left(e^{2s\rho} + e^{2s\widetilde{\rho}} \right) dxdt$$
$$\leq C_2T^2 \int_Q |\varphi|^2 \left(e^{2s\rho} + e^{2s\widetilde{\rho}} \right) dxdt + C_2T^4 \|\alpha\|_\infty \int_Q |\psi|^2 \left(e^{2s\rho} + e^{2s\widetilde{\rho}} \right) dxdt$$

$$+ C_2 T^2 l^2 \int_Q t\,(T-t)\, |\Delta\psi|^2 \left(e^{2s\rho} + e^{2s\widetilde{\rho}}\right) dxdt$$
$$+ C_2 T^4 \int_{\mathcal{O}\times[0,T]} s^3 \xi^3 |\varphi|^2 \left(e^{2s\rho} + e^{2s\widetilde{\rho}}\right) dxdt, \tag{3.11}$$

for $\lambda > \widehat{\lambda}_2$ and $s \geq s_2$, where the positive constant C_2 depends continuously on λ and $\|\beta\|_\infty$.

Since $t\,(T-t) = e^{\lambda\eta}/\xi$ with $\lambda > 0$ and $\eta \geq 0$ in $\overline{\Omega}$, we have by (3.9) that

$$\int_Q t\,(T-t)\, |\Delta\psi|^2 \left(e^{2s\rho} + e^{2s\widetilde{\rho}}\right) dxdt \leq \left\|e^{\lambda\eta}\right\|_\infty s \int_Q \frac{1}{s\xi} |\Delta\psi|^2 \left(e^{2s\rho} + e^{2s\widetilde{\rho}}\right) dxdt$$
$$\leq C_1 \left\|e^{\lambda\eta}\right\|_\infty \left(\int_Q s\, |\varphi|^2 \left(e^{2s\rho} + e^{2s\widetilde{\rho}}\right) dxdt + \int_{\mathcal{O}\times[0,T]} s^4 \xi^3 |\psi|^2 \left(e^{2s\rho} + e^{2s\widetilde{\rho}}\right) dxdt \right), \tag{3.12}$$

for $\lambda > \widehat{\lambda}_1$ and $s \geq s_1$.

Taking $\widehat{\lambda} = \max\left\{\widehat{\lambda}_1, \widehat{\lambda}_2\right\}$ and $s_0 = \max\{s_1, s_2\}$, for $\lambda > \widehat{\lambda}$ and $s \geq s_0$ we can replace (3.12) in (3.11) to obtain

$$\int_Q \left(\frac{s}{\xi} |\nabla\varphi|^2 + s^3 \xi\, |\varphi|^2\right) \left(e^{2s\rho} + e^{2s\widetilde{\rho}}\right) dxdt$$
$$\leq C_2 T^2 \int_Q |\varphi|^2 \left(e^{2s\rho} + e^{2s\widetilde{\rho}}\right) dxdt + C_2 T^4 \|\alpha\|_\infty \int_Q |\psi|^2 \left(e^{2s\rho} + e^{2s\widetilde{\rho}}\right) dxdt$$
$$+ C_1 C_2 T^2 l^2 \left\|e^{\lambda\eta}\right\|_\infty \left(\int_Q s\, |\varphi|^2 \left(e^{2s\rho} + e^{2s\widetilde{\rho}}\right) dxdt \right.$$
$$\left. + \int_{\mathcal{O}\times[0,T]} s^4 \xi^3 |\psi|^2 \left(e^{2s\rho} + e^{2s\widetilde{\rho}}\right) dxdt \right)$$
$$+ C_2 T^4 \int_{\mathcal{O}\times[0,T]} s^3 \xi^3 |\varphi|^2 \left(e^{2s\rho} + e^{2s\widetilde{\rho}}\right) dxdt. \tag{3.13}$$

By combining (3.9) and (3.13), we get the inequality (3.8). □

In order to complete to proof of Theorem 3.2, it is sufficient to show the existence of a positive constant C such that

$$\|\{\varphi(0), \psi(0)\}\|^2_{[L^2(\Omega)]^2} \leq C \int_Q \left(|\varphi|^2 + |\psi|^2\right) dxdt. \tag{3.14}$$

This can be easily done by means of classical energy estimates. In fact, by multiplying the Eq. $(3.4)_1$ by $\epsilon\varphi$ ($\epsilon > 0$ to be chosen), $(3.4)_2$ by ψ, integrating in Ω and adding the obtained equations we can deduce that

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega \left(\epsilon|\varphi|^2+|\psi|^2\right)dx+\int_\Omega\left[\frac{\epsilon}{2}|\nabla\varphi|^2+\left(1-\frac{\epsilon l^2}{2}\right)|\nabla\psi|^2\right]dx+l\int_\Omega|\psi|^2\,dxdt$$
$$\le\left(\frac{1+\epsilon\|\alpha\|_\infty}{2}+\epsilon\|\beta\|_\infty\right)\int_\Omega|\varphi|^2\,dxdt+\frac{1+\epsilon\|\alpha\|_\infty}{2}\int_\Omega|\psi|^2\,dx. \tag{3.15}$$

Taking $\epsilon < 2/l^2$ and integrating (3.15) in time from 0 to t, it follows that

$$\int_\Omega\left(|\varphi(0)|^2+|\psi(0)|^2\right)dx\le C\int_\Omega\left(|\varphi(t)|^2+|\psi(t)|^2\right)dx$$
$$+C\int_0^t\int_\Omega\left(|\varphi(x,\tau)|^2+|\psi(x,\tau)|^2\right)dxd\tau.$$

By majorizing the last integral by taking $t = T$ in it, and then integrating in time from 0 to T, we obtain (3.14) and this concludes the proof of Theorem 3.2.

Now, we return to the proof of Theorem 3.1.

Proof of the Theorem 3.1 In view of the observability inequality (3.5), the null controllability result of the linearized system can be proved as the limit of an approximate controllability property. Let us first discuss the approximate controllability property.

Given $u_0, \phi_0 \in L^2(\Omega)$ and $\epsilon > 0$, we introduce the following functional:

$$J_\epsilon : \left[L^2(\Omega)\right]^2 \to \mathbb{R}$$

$$J_\epsilon\{\varphi_0,\psi_0\}=\frac{1}{2}\int_0^T\int_{\mathcal{O}}\left(\varphi^2+\psi^2\right)dxdt+\epsilon\|\{\varphi_0,\psi_0\}\|_{[L^2(\Omega)]^2}$$
$$+\int_\Omega\left[u_0\varphi(0)+\phi_0\psi(0)\right]dx, \tag{3.16}$$

where $\{\varphi,\psi\}$ is the solution of the adjoint system (3.4) with data $\{\varphi_0,\psi_0\}$.

The functional J_ϵ is continuous and strictly convex in $L^2(\Omega)$, and it is also coercive. More precisely, arguing as in [54], it can be seen that

$$\liminf_{\|\{\varphi_0,\psi_0\}\|_{[L^2(\Omega)]^2}\to\infty}\frac{J_\epsilon\{\varphi_0,\psi_0\}}{\|\{\varphi_0,\psi_0\}\|_{[L^2(\Omega)]^2}}\ge\epsilon. \tag{3.17}$$

In this way, J_ϵ has a unique minimizer in $[L^2(\Omega)]^2$. Let us denote it by $\{\widehat{\varphi}_{0,\epsilon},\widehat{\psi}_{0,\epsilon}\}$. Proceeding as in [54], it can be checked that the pair of controls $\left\{v_{1,\epsilon}, v_{2,\epsilon}\right\} =$

$\{\widehat{\varphi}_\epsilon, \widehat{\psi}_\epsilon\}$, where $\{\widehat{\varphi}_\epsilon, \widehat{\psi}_\epsilon\}$ is the solution of (3.4) associated to minimizer $\{\widehat{\varphi}_{0,\epsilon}, \widehat{\psi}_{0,\epsilon}\}$ is such that the solution $\{u, \phi\}$ of (3.2) satisfies

$$\|\{u_\epsilon (T), \phi_\epsilon (T)\}\|_{[L^2(\Omega)]^2} \leq \epsilon. \tag{3.18}$$

The null-controllability property can be obtained as the limit when ϵ tends to zero of the approximate controllability property above. However, to pass to the limit, we need a uniform bound on the control. To get this bound, we observe that, by (3.5),

$$\begin{aligned} J_\epsilon \{\varphi_0, \psi_0\} = & \frac{1}{2} \int_0^T \int_{\mathcal{O}} \left(\varphi^2 + \psi^2\right) dxdt \\ & - \sqrt{C} \left(\int_0^T \int_{\mathcal{O}} \left(\varphi^2 + \psi^2\right) dxdt \right)^{1/2} \|\{u_0, \phi_0\}\|_{[L^2(\Omega)]^2}. \end{aligned} \tag{3.19}$$

On the other hand,

$$J_\epsilon\{\widehat{\varphi}_{0,\epsilon}, \widehat{\psi}_{0,\epsilon}\} \leq J_\epsilon\{0, 0\} = 0. \tag{3.20}$$

By writing (3.19) for $\{\widehat{\varphi}_{0,\epsilon}, \widehat{\psi}_{0,\epsilon}\}$, the minimizer of J_ϵ in $L^2(\Omega)$, and combining with (3.20), we deduce that

$$\left\| \left\{ v_{1,\epsilon}, v_{2,\epsilon} \right\} \right\|_{[L^2(\mathcal{O}\times(0,T))]^2} \leq 2\sqrt{C} \, \|\{u_0, \phi_0\}\|_{[L^2(\Omega)]^2}, \quad \forall \epsilon > 0.$$

In other words, $\left\{v_{1,\epsilon}, v_{2,\epsilon}\right\}$ remains bounded in $[L^2 (\mathcal{O} \times (0, T))]^2$ as $\epsilon \to 0$. In this way, by extracting subsequences, we deduce that, as $\epsilon \to 0$,

$$\left\{v_{1,\epsilon}, v_{2,\epsilon}\right\} \to \{v_1, v_2\} \quad \text{weakly in} \quad [L^2 (\mathcal{O} \times (0, T))]^2, \tag{3.21}$$

for some pair $\{v_1, v_2\} \in [L^2 (\mathcal{O} \times (0, T))]^2$.

It is easy to see that the limit $\{v_1, v_2\}$ is such that the solution $\{u, \phi\}$ of (3.2) satisfies (1.3). Moreover, by the lower semicontinuity of the norm with respect to the weak topology and in view of (3.21), we get

$$\begin{aligned} \|\{v_1, v_2\}\|_{[L^2(\mathcal{O}\times(0,T))]^2} & \leq \liminf_{\epsilon\to 0} \left\| \left\{ v_{1,\epsilon}, v_{2,\epsilon} \right\} \right\|_{[L^2(\mathcal{O}\times(0,T))]^2} \\ & \leq 2\sqrt{C} \, \|\{u_0, \phi_0\}\|_{[L^2(\Omega)]^2}, \end{aligned}$$

this completes the proof of Theorem 3.1. □

3.2 Null Controllability for the Linear Phase Field System with One Control

For the linear phase field system (3.1), the following null controllability result holds:

Theorem 3.4 *let $T > 0$, $r \geq 2$ and data $u_0, \phi_0 \in W_r^{2-2/r}(\Omega) \cap V$ be given, then there exists a control function $v \in L^r(Q)$ such that the solution $\{u, \phi\}$ of (3.1) belongs to $[W_r^{2,1}(Q)]^2$ and satisfies (1.3). Moreover, there exists a constant $C = C(T, \|\alpha\|_{L^\infty(Q)}, \|\beta\|_{L^\infty(Q)}) > 0$ such that*

$$\|v\|_{L^r(\mathcal{O}\times(0,T))} \leq C \|\{u_0, \phi_0\}\|_{[L^2(\Omega)]^2} \tag{3.22}$$

and

$$\|\{u, \phi\}\|_{[W_r^{2,1}(Q)]^2} \leq C \|\{u_0, \phi_0\}\|_{[W_r^{2-2/r}(\Omega)]^2}. \tag{3.23}$$

Proof Let with $\mathcal{O}_1$ be a regular nonempty open subset of Ω such that $\mathcal{O}_1 \subset\subset \mathcal{O} \subset \Omega$. Let $\widehat{v}_1, \widehat{v}_2 \in L^2(Q)$ be two controls provided by Theorem 3.1 associated to $\mathcal{O}_1$ and the solution $\{\widehat{u}, \widehat{\phi}\}$ of (3.2). We consider a function $\zeta \in C^\infty([0, T])$ such that $\zeta \equiv 1$ in $[0, T/3]$, $\zeta \equiv 0$ in $[2T/3, T]$, $0 \leq \zeta \leq 1$ and $\left|\zeta'(t)\right| \leq C/T$ in $[0, T]$, and we introduce the change of variables

$$u = U + \zeta\overline{u} \quad \text{and} \quad \phi = \Phi + \zeta\overline{\phi},$$

where $\left\{\overline{u}, \overline{\phi}\right\}$ solves

$$\left|\begin{array}{ll} \overline{u}_t - \Delta\overline{u} + l\overline{u} + \alpha\overline{\phi} = -l\Delta\overline{\phi} & \text{in} \quad Q, \\ \overline{\phi}_t - \Delta\overline{\phi} + \beta\overline{\phi} = \overline{u} & \text{in} \quad Q, \\ \dfrac{\partial\overline{u}}{\partial\nu} = \dfrac{\partial\overline{\phi}}{\partial\nu} = 0 & \text{on} \quad \Sigma, \\ \overline{u}(0) = u_0, \quad \overline{\phi}(0) = \phi_0 & \text{in} \quad \Omega. \end{array}\right.$$

Notice that, the proof of the null controllability of (3.1) is reduced to find a control v that solves the null controllability for the following system:

$$\left|\begin{array}{ll} U_t - \Delta U + lU + \alpha\Phi = -l\Delta\Phi - \zeta'\overline{u} + v\mathbf{1}_{\mathcal{O}} & \text{in} \quad Q, \\ \Phi_t - \Delta\Phi + \beta\Phi = U - \zeta'\overline{\phi} & \text{in} \quad Q, \\ \dfrac{\partial U}{\partial\nu} = \dfrac{\partial\Phi}{\partial\nu} = 0 & \text{on} \quad \Sigma, \\ U(0) = \Phi(0) = U(T) = \Phi(T) = 0 & \text{in} \quad \Omega. \end{array}\right. \tag{3.24}$$

Let us consider $\mathcal{O}_2$, $\mathcal{O}_3$ and $\mathcal{O}_4$ three regular open subsets of Ω such that $\mathcal{O}_1 \subset\subset \mathcal{O}_2 \subset\subset \mathcal{O}_3 \subset\subset \mathcal{O}_4 \subset\subset \mathcal{O}$ and a function $\theta \in \mathcal{D}(\mathcal{O}_4)$ satisfying $\theta \equiv 1$ in $\mathcal{O}_2$. We Take

$$\Phi = (1-\theta)\widehat{\Phi}, \qquad U = (1-\theta)\widehat{U} + \theta\zeta'\overline{\phi} + 2\nabla\theta\cdot\nabla\widehat{\Phi} + \widehat{\Phi}\Delta\theta \tag{3.25}$$

and

$$\begin{aligned} v = {} & \theta\zeta'\overline{u} - 2l\nabla\theta\cdot\nabla\widehat{\Phi} - l\widehat{\Phi}\Delta\theta + 2\nabla\theta\cdot\nabla\widehat{U} + \widehat{U}\Delta\theta \\ & + \left(\frac{d}{dt} - \Delta + l\right)\left(\theta\zeta'\overline{\phi} + 2\nabla\theta\cdot\nabla\widehat{\Phi} + \widehat{\Phi}\Delta\theta\right), \end{aligned} \tag{3.26}$$

where $\{\widehat{U}, \widehat{\Phi}\}$ defined by

$$\widehat{u} = \widehat{U} + \zeta\overline{u} \quad \text{and} \quad \widehat{\phi} = \widehat{\Phi} + \zeta\overline{\phi}$$

solves the problem

$$\left|\begin{array}{ll} \widehat{U}_t - \Delta\widehat{U} + l\widehat{U} + \alpha\widehat{\Phi} = -l\Delta\widehat{\Phi} - \zeta'\overline{u} + v_1\mathbf{1}_{\mathcal{O}_1} & \text{in} \quad Q, \\ \widehat{\Phi}_t - \Delta\widehat{\Phi} + \beta\widehat{\Phi} = \widehat{U} - \zeta'\overline{\phi} + v_2\mathbf{1}_{\mathcal{O}_1} & \text{in} \quad Q, \\ \dfrac{\partial\widehat{U}}{\partial\nu} = \dfrac{\partial\widehat{\Phi}}{\partial\nu} = 0 & \text{on} \quad \Sigma, \\ \widehat{U}(0) = \widehat{\Phi}(0) = \widehat{U}(T) = \widehat{\Phi}(T) = 0 & \text{in} \quad \Omega, \end{array}\right. \tag{3.27}$$

and $\{\widehat{u}, \widehat{\phi}\}$ being the solution of (3.2) (with null initial data and $\mathcal{O}_1$ instead $\mathcal{O}$) satisfying (1.3). Notice that the solution $\{\widehat{u}, \widehat{\phi}\}$ can be determined by Theorem 3.1.

It is easy to see that the control function v given by (3.26) (together with $\{U, \Phi\}$) solves (3.24). Consequently, the control v (together with $u = U + \zeta\overline{u}$ and $\phi = \Phi + \zeta\overline{\phi}$) gives the null controllability of system (3.1).

Proceeding as in [47], we can obtain the estimates (3.22) and (3.23) as a consequence of the parabolic results in Sect. 2. □

4 Controllability for the Nonlinear Phase Field System

The main aim of this section is to obtain controllability properties for the nonlinear phase field system (1.1).

In order to use the results of the previous section, we begin by considering the following local null controllability result:

Theorem 4.1 *Let $T > 0$ be given, then there exists $r_0 > 0$ such that for any data $\{u_0, \phi_0\} \in [W_s^{2-2/s}(\Omega) \cap V]^2$, with $s > 5/2$, satisfying*

$$\|\{u_0, \phi_0\}\|_{[W_s^{2-2/s}(\Omega)]^2} < r_0, \tag{4.1}$$

there exists a control function $v \in L^2(\mathcal{O} \times (0, T))$ such that the solution $\{u, \phi\}$ of (1.1) satisfies (1.3).

Before starting the proof of the theorem, let us observe that system (1.1) can be rewritten as

$$\left|\begin{array}{ll} u_t - \Delta u + lu + h(\phi)\phi = -l\Delta\phi + v\mathbf{1}_{\mathcal{O}} & \text{in} \quad Q, \\ \phi_t - \Delta\phi + \widetilde{g}(\phi)\phi = u & \text{in} \quad Q, \\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial\phi}{\partial\nu} = 0 & \text{on} \quad \Sigma, \\ u(0) = u_0, \quad \phi(0) = \phi_0 & \text{in} \quad \Omega, \end{array}\right. \tag{4.2}$$

with

$$h(\sigma) = l\left(a + b\sigma - \sigma^2\right) \quad \text{and} \quad \widetilde{g}(\sigma) = -\left(a + b\sigma - \sigma^2\right).$$

Next, we consider

$$X := L^2(0, T; H^1(\Omega)) \times L^{5/2}(0, T; W^1_{5/2}(\Omega))$$

and

$$\Pi_R(\sigma) = \begin{cases} \sigma & \text{if} \quad |\sigma| \leq R, \\ R\, sgn(\sigma) & \text{if} \quad |\sigma| > R, \end{cases} \tag{4.3}$$

where $R > 0$ is an arbitrary constant.

For each $\{z, \xi\} \in X$, let us consider the linear system

$$\left| \begin{array}{ll} u_t - \Delta u + lu + \alpha_\xi \phi = -l\Delta\phi + v\mathbf{1}_{\mathcal{O}} & \text{in} \quad Q, \\ \phi_t - \Delta\phi + \beta_\xi \phi = u & \text{in} \quad Q, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial \phi}{\partial \nu} = 0 & \text{on} \quad \Sigma, \\ u\,(0) = u_0, \quad \phi\,(0) = \phi_0 & \text{in} \quad \Omega, \end{array} \right. \tag{4.4}$$

where

$$\alpha_\xi = h(\Pi_R(\xi)) \quad \text{and} \quad \beta_\xi = \widetilde{g}(\Pi_R(\xi)).$$

We note that

$$\alpha_\xi \in L^\infty(Q) \quad \text{and} \quad \beta_\xi \in L^\infty(Q) \cap L^{5/2}(0, T; W^1_{5/2}(\Omega)),$$

with

$$\begin{array}{l} \|\alpha_\xi\|_{L^\infty(Q)} = \max_{|\sigma|\leq R}\{|h(\sigma)|\} =: \alpha_R, \\ \|\beta_\xi\|_{L^\infty(Q)} = \max_{|\sigma|\leq R}\{|\widetilde{g}(\sigma)|\} =: \beta_R. \end{array}$$

By using the previous notations and remarks, we return to Theorem 4.1 to prove it.

Proof of Theorem 4.1 First of all, for each pair $\{z, \xi\} \in X$ let us apply Theorem 3.4 to find a control v such that the solution $(u_{z,\xi}, \phi_{z,\xi})$ of system (4.4) with control v and potentials α_ξ and β_ξ is such that $u_{z,\xi}(x, T) = \phi_{z,\xi}(x, T) = 0$ in Ω.

Since $u_0, \phi_0 \in W_s^{2-2/s}(\Omega) \cap V$ $(s > 5/2)$ and, for each $\{z, \xi\} \in X$, we have $\alpha_\xi \in L^\infty(Q)$ and $\beta_\xi \in L^\infty(Q) \cap L^{5/2}(0, T; W^1_{5/2}(\Omega))$, we can apply Theorem 3.4, with $r = 2$ (this implies that $\gamma = 5/2$), to guarantee the existence of control $v_{z,\xi} \in L^2(Q)$ such that the corresponding solution $\{u_{z,\xi}, \phi_{z,\xi}\}$ of system (4.4) satisfies

$$u_{z,\xi} \in L^s(0, T; W^1_s(\Omega)) \cap C^0(\overline{Q}), \quad \phi_{z,\xi} \in W^{2,1}_s(Q),$$

with

$$\|u_{z,\xi}\|_{L^s(0,T;W^1_s(\Omega))\cap C^0(\overline{Q})} + \|\phi_{z,\xi}\|_{W^{2,1}_s(Q)} \leq C_1 \|\,\{u_0, \phi_0\}\,\|_{[W_s^{2-2/s}(\Omega)]^2}, \tag{4.5}$$

where $C_1 = C_1(\Omega, \mathcal{O}, T, \|\alpha_\xi\|_{L^\infty(Q)}, \|\beta_\xi\|_{L^\infty(Q)}) > 0$ and

$$u_{z,\xi}(T) = \phi_{z,\xi}(T) = 0 \quad \text{in } \Omega.$$

Besides, the control function v satisfies the estimate

$$\|v_{z,\xi}\|_{L^s(0,T;W^1_s(\mathcal{O}))\cap C^0(\overline{\mathcal{O}}\times[0,T])} \le C_2\| \{u_0, \phi_0\} \|_{[L^2(Q)]^2}, \tag{4.6}$$

where $C_2 = C_2(\Omega, \mathcal{O}, T, \|\alpha_\xi\|_{L^\infty(Q)}, \|\beta_\xi\|_{L^\infty(Q)}, \|\beta_\xi\|_{L^{5/2}(0,T;W^1_{5/2}(\Omega))}) > 0$.

In the sequel, for each fixed pair $\{z, \xi\} \in X$, we will denote by $\{u_v, \phi_v\}$ the solution of (4.4) associated to control v and potentials α_ξ and β_ξ. By simplicity, the dependence of $\{z, \xi\}$ will be omitted here.

Now, for each pair $\{z, \xi\} \in X$, we define the family of control functions

$$\mathcal{A}_R\{z, \xi\} = \{v \in L^2(Q) : \{u_v, \phi_v\} \in \left(L^s(0, T; W^1_s(\Omega)) \cap C^0(\overline{Q})\right) \times W^{2,1}_s(Q) u_v(x, T) = \phi_v(x, T) = 0 \text{ in } \Omega \text{ and } v \text{ satisfies } (4.6)\}.$$

Then, we can consider the multi-valued mapping

$$\Lambda_R : X \to X$$

where $\Lambda_R\{z, \xi\}$ is the family of pairs $\{u_v, \phi_v\} \in \left(L^s(0, T; W^1_s(\Omega)) \cap C^0(\overline{Q})\right) \times W^{2,1}_s(Q)$ such that $v \in \mathcal{A}_R\{z, \xi\}$ and $\{u_v, \phi_v\}$ satisfies (4.5).

Let us apply Kakutani's fixed point theorem to Λ_R. For this, we observe that:

(i) $\Lambda_R\{z, \xi\} \neq \emptyset$ for all $\{z, \xi\} \in X$.
Let be given $\{z, \xi\} \in X$; by our previous discussion on the linear system (4.4) and the fact that $\left[\left(L^s(0, T; W^1_s(\Omega)) \cap C^0(\overline{Q})\right) \times W^{2,1}_s(Q)\right] \subset X$, we can conclude that there exists $\{u_v, \phi_v\} \in \Lambda_R(z, \xi)$.

(ii) $\Lambda_R\{z, \xi\}$ is a convex set for all $\{z, \xi\} \in X$.
This follows directly from linearity of system (4.4).

(iii) $\Lambda_R\{z, \xi\}$ is closed for all $\{z, \xi\} \in X$.
For this, let us fix an arbitrary pair $\{z, \xi\} \in X$. Let $\{u_n, \phi_n\} \in \Lambda_R\{z, \xi\}$, $\forall n \in \mathbb{N}$, such that $\{u_n, \phi_n\} \to \{u, \phi\}$ in X. Let us denote by v_n the control function associated to $\{u_n, \phi_n\}$ for each $n \in \mathbb{N}$. We have that $\{u_n, \phi_n\}$ satisfies (4.5) and consequently there exists a subsequence (that we will denote again by $(\{u_n, \phi_n\})$ by simplicity) such that

$$\{u_n, \phi_n\} \to \{u, \phi\} \quad \text{weakly in} \quad \left(L^s(0, T; W^1_s(\Omega)) \cap C^0(\overline{Q})\right)) \times W^{2,1}_s(Q).$$

Since v_n satisfies (4.6), for all $n \in \mathbb{N}$, we have the existence of a subsequence (that we will denote again by (v_n) by simplicity) such that

$$v_n \to v \quad \text{weakly in} \quad L^2(\mathcal{O} \times (0, T)).$$

By using the previous convergences to pass to the limit $n \to \infty$ (in a subsequence if necessary) in (4.4) we obtain that $\{u, \phi\}$ is the solution of (4.4) with control v and potentials α_ξ and β_ξ. Furthermore, we get $u(T) = \phi(T) = 0$ in Ω and $\{u, \phi\} \in \Lambda_R\{z, \xi\}$. So we conclude that $\Lambda_R\{z, \xi\}$ is closed.

(iv) $\Lambda_R\{z, \xi\}$ is uniformly bounded for all $\{z, \xi\} \in X$.
This follows from estimate (4.5).

(v) $\Lambda_R : X \to X$ is a compact mapping.
Let $\mathcal{B} \subset X$ a bounded set and $\mathcal{A}_R(\mathcal{B}) = \cup\{\mathcal{A}_R\{z, \xi\} : (z, \xi) \in \mathcal{B}\}$.
Let us consider an arbitrary $\{u_v, \phi_v\} \in \Lambda_R(\mathcal{B})$. Since $\mathcal{B}$ is bounded, it follows, by estimate (4.6), that $\mathcal{A}_R(\mathcal{B})$ is uniformly bounded in $L^2(Q)$. By applying Proposition 2.2 (with $r = 2$, $f_1 = v\mathbf{1}_{\mathcal{O}}$, $f_2 \equiv 0$) and using estimate (4.6), we obtain

$$\| \{u_v, \phi_v\} \|_{[W_2^{2,1}(Q)]^2} \le C \| \{u_0, \phi_0\} \|_{[H^1(\Omega)]^2},$$

with $C = C(\Omega, \mathcal{O}, T, \alpha_R, \beta_R)$. Besides, by (4.5),

$$\|\phi_v\|_{[W_s^{2,1}(Q)]^2} \le C \| \{u_0, \phi_0\} \|_{[W_s^{2-2/s}(\Omega)]^2},$$

with $C = C(\Omega, \mathcal{O}, T, \alpha_R, \beta_R)$.
Since $\{u_v, \phi_v\} \in \Lambda_R(\mathcal{B})$ is arbitrary, the previous inequalities imply that $\Lambda_R(\mathcal{B})$ is bounded in $[W_2^{2,1}(Q)]^2$. Moreover, $W_2^{2,1}(Q) \subset L^2(0, T; H^1(\Omega))$ and $W_s^{2,1}(Q) \subset L^{5/2}(0, T; W^1_{5/2}(\Omega))$ with compact imbeddings, because $s > 5/2$. Consequently, $\Lambda_R(\mathcal{B})$ is relatively compact in X. Thus, $\Lambda_R(\mathcal{B}) : X \to X$ is compact.

(vi) $\Lambda_R : X \to X$ is upper hemicontinuous.

Here we need to show that for each bounded linear real-valued function μ on X, i.e., $\mu \in X'$, the real-valued function

$$\begin{aligned} X &\to \mathbb{R} \\ \{z, \xi\} &\mapsto \sup_{\{u,\phi\} \in \Lambda_R\{z,\xi\}} \langle \mu, \{u, \phi\} \rangle \end{aligned}$$

is upper semicontinuous. In other words, we need to show that the set

$$\mathcal{B}_{k,\mu} = \left\{ \{z, \xi\} \in X : \sup_{\{u,\phi\} \in \Lambda_R\{z,\xi\}} \langle \mu, \{u, \phi\} \rangle \ge k \right\}$$

is closed in X, for all $\mu \in X'$ and all $k \in \mathbb{R}$ (see Doubova et al. [55]).

For this, let us fix arbitraries $\mu \in X'$ and $k \in \mathbb{R}$. Let be $(\{z_n, \xi_n\})$ a sequence in $\mathcal{B}_{k,\mu}$ such that $\{z_n, \xi_n\} \to \{z, \xi\}$ in X. In this way, we have

$$\begin{aligned} &\xi_n \to \xi \quad \text{strongly in} \quad L^{5/2}(Q), \\ &\alpha_{\xi_n} = l\left(a + b\Pi_R(\xi_n) - \Pi_R(\xi_n)^2\right) \to \alpha_\xi \quad \text{strongly in} \quad L^{5/4}(Q), \\ &\beta_{\xi_n} = -\left(a + b\Pi_R(\xi_n) - \Pi_R(\xi_n)^2\right) \to \beta_\xi \quad \text{strongly in} \quad L^{5/4}(Q). \end{aligned}$$

Observe that $\alpha_\xi, \beta_\xi \in L^\infty(Q)$ by definition of α, β.

Now, for each $n \in \mathbb{N}$, $\Lambda_R\{z_n, \xi_n\}$ is closed and relatively compact in X, then it is a compact set. Thus, there exists $\{u_n, \phi_n\} \in \Lambda_R(z_n, \xi_n)$ such that

$$k \leq \sup_{(u,\phi)\in\Lambda_R\{z_n,\xi_n\}} \langle \mu, \{u, \phi\}\rangle = \langle \mu, \{u_n, \phi_n\}\rangle \tag{4.7}$$

By definitions of $\Lambda_R\{z_n, \xi_n\}$ and $\mathcal{A}_R\{z_n, \xi_n\}$, there exists a control function $v_n \in L^2(Q)$ such that $\{u_n, \phi_n\} \in \left(L^s(0, T; W_s^1(\Omega)) \cap C^0(\overline{Q})\right) \times W_s^{2,1}(Q)$ is solution of (4.4) with control v_n and potentials $\alpha_{\xi_n} = h(\Pi_R(\xi_n))$ and $\beta_{\xi_n} = \widetilde{g}(\Pi_R(\xi_n))$. Furthermore $u_n(T) = \phi_n(T) = 0$ in Ω and v_n, $\{u_n, \phi_n\}$ satisfy (4.6) and (4.5), respectively.

Since (v_n) and $(\{u_n, \phi_n\})$ are uniformly bounded sequences in $L^2(Q)$ and $\left(L^s(0, T; W_s^1(\Omega)) \cap C^0(\overline{Q})\right) \times W_s^{2,1}(Q)$, respectively, then, considering a subsequence if it is necessary, we have that

$$\begin{aligned}
&\{u_n, \phi_n\} \to \{\widehat{u}, \widehat{\phi}\} \quad \text{weakly in} \quad \left(L^s(0, T; W_s^1(\Omega)) \cap C^0(\overline{Q})\right) \times W_s^{2,1}(Q),\\
&\quad v_n \to \widehat{v} \quad \text{weakly in} \quad L^2(\mathcal{O} \times (0, T)).
\end{aligned}$$

By using the previous convergences to pass to the limit, as $n \to \infty$, in problem (4.4), we obtain that $(\widehat{u}, \widehat{\phi})$ satisfies this problem with control $\widehat{v}$ and potentials $\alpha_\xi = h(\Pi_R(\xi))$ and $\beta_\xi = \widetilde{g}(\Pi_R(\xi))$. Besides, $\widehat{u}(T) = \widehat{\phi}(T) = 0$ in Ω and $\widehat{v}$, $\{\widehat{u}, \widehat{\phi}\}$ satisfy (4.6) and (4.5), respectively. Thus,

$$\widehat{v} \in \mathcal{A}_R\{z, \xi\} \quad \text{and} \quad \{\widehat{u}, \widehat{\phi}\} \in \Lambda_R\{z, \xi\}.$$

Next, by passing to the limit in (4.7), since $\{\widehat{u}, \widehat{\phi}\} \in \Lambda_R\{z, \xi\}$, we obtain

$$k \leq \langle \mu, \{\widehat{u}, \widehat{\phi}\}\rangle \leq \sup_{\{u,\phi\}\in\Lambda_R(z,\xi)} \langle \mu, \{u, \phi\}\rangle.$$

Thus $\{z, \xi\} \in \mathcal{B}_{k,\mu}$, which implies that $\mathcal{B}_{k,\mu}$ is a closed set and then Λ_R is upper hemicontinuous.

Since hypotheses of Kakutani's fixed point theorem are satisfied by $\Lambda_R : X \to X$, we conclude that there exists at least a fixed point $\{u, \phi\} \in X$ of Λ_R, that is, there exists a control $v \in L^2(\mathcal{O} \times (0, T))$ such that $\{u, \phi\} \in \left(L^s(0, T; W_s^1(\Omega)) \cap C^0(\overline{Q})\right) \times W_s^{2,1}(Q)$ is the corresponding solution of problem (4.4) with potentials $c_\phi = h(\Pi_R(\phi))$ and $e_\phi = \widetilde{g}(\Pi_R(\phi))$. Moreover, v and $\{u, \phi\}$ satisfy (4.6) and (4.5), resp., and $u(x, T) = \phi(x, T) = 0$ in Ω.

To complete the proof, we just need to show that $\Pi_R(u) = u$ and $\Pi_R(\phi) = \phi$. By using (4.5) we have

$$\begin{aligned}
\|u\|_{L^\infty(Q)} + \|\phi\|_{L^\infty(Q)} &\leq C(\|u\|_{C^0(\overline{Q})} + \|\phi\|_{W_s^{2,1}(Q)})\\
&\leq C_1(\Omega, \mathcal{O}, T, R)\|\{u_0, \phi_0\}\|_{[W_s^{2-2/s}(\Omega)]^2}.
\end{aligned}$$

By taking $\delta = R/C_1(\Omega, \mathcal{O}, T, R) > 0$, we obtain from last inequality that if $\|\{u_0, \phi_0\}\|_{[W_s^{2-2/s}(\Omega)]^2} \leq \delta$, then $\Pi_R(u) = u$ and $\Pi_R(\phi) = \phi$. This proves the theorem. □

Notice that the null controllability can be read as the exact controllability to the trajectory $\{\widetilde{u}, \widetilde{\phi}\} = \{0, 0\}$ (associated to $\widetilde{v} = 0$ and $\{\widetilde{u}_0, \widetilde{\phi}_0\} = \{0, 0\}$). In particular for the null stationary trajectory $\{\widetilde{u}, \widetilde{\phi}\} = \{0, 0\}$ (associated to $\widetilde{v} = 0$).

Considering $\{u_d, \phi_d\}$ a steady-state (equilibrium) solution to system (1.1), i.e.,

$$\left|\begin{array}{ll} -\Delta u_d = 0 & \text{in } \Omega, \\ -\Delta \phi_d - \left(a\phi_d + b\phi_d^2 - \phi_d^3\right) - u_d = 0 & \text{in } \Omega, \\ \dfrac{\partial u_d}{\partial \nu} = \dfrac{\partial \phi_d}{\partial \nu} = 0 & \text{on } \Gamma, \end{array}\right. \tag{4.8}$$

we can proceed as in the proof of Theorem 4.1 to obtain the following result:

Theorem 4.2 *Let be $T > 0$ given. If $\{u_d, \phi_d\}$ is a solution of (4.8), then there exists $r_0 > 0$ such that for any data $\{u_0, \phi_0\} \in [W_s^{2-2/s}(\Omega) \cap V]^2$, with $s > 5/2$, satisfying*

$$\|\{u_0 - u_d, \phi_0 - \phi_d\}\|_{[W_s^{2-2/s}(\Omega)]^2} < r_0, \tag{4.9}$$

there exists a control function $v \in L^2(\mathcal{O} \times (0, T))$ satisfying

$$\|v\|_{L^2(\mathcal{O}\times(0,T))} \leq C\, \|\{u_0 - u_d, \phi_0 - \phi_d\}\|_{[L^2(\Omega)]^2} \tag{4.10}$$

such that the associated solution $\{u, \phi\}$ of (1.1) satisfies

$$u(\cdot, T) = u_d \quad \textit{and} \quad \phi(\cdot, T) = \phi_d \quad \textit{in} \quad \Omega. \tag{4.11}$$

5 Optimal Control and Controllability

Our goal in this section is to establish a connection between optimal control problem obtained in [43] and the exact controllability for trajectories achieved in the Theorem 4.2.

According to Remark 5.1 in [43], we have the following result:

Theorem 5.1 *Let $T > 0$ and $\{u_0, \phi_0\} \in [W_s^{2-2/s}(\Omega) \cap V]^2$ be given. If $\{u_d, \phi_d\} \in [L^2(\Omega)]^2$, then for each fixed $\epsilon > 0$ there exists a optimal control $v_\epsilon \in L^2(\mathcal{O}\times(0, T))$, which minimizes the cost functional $\mathcal{F}_\epsilon$ given in (1.4). That is, $v_\varepsilon \in L^2(\mathcal{O} \times (0, T))$ is such that the corresponding solution $\{u_\epsilon, \phi_\epsilon\}$ of (1.1) satisfies*

$$\mathcal{F}_\epsilon(u_\epsilon, \phi_\epsilon; v_\epsilon) = \inf_{v \in L^2(\mathcal{O})} \mathcal{F}_\epsilon(u, \phi; v), \tag{5.1}$$

where $\{u, \phi\}$ is the corresponding solution of (1.1) with v.

The next result gives us a relation between optimal control problem and exact controllability one. More precisely, it establishes that a sequence of solutions of optimal control problems (5.1) converges to solution of the exact controllability obtained in Theorem 4.2.

Theorem 5.2 *Let $T > 0$ and $\{u_0, \phi_0, u_d, \phi_d\}$ satisfying the conditions of Theorem 4.2. For each $\epsilon > 0$ consider $\{u_\epsilon, \phi_\epsilon, v_\epsilon\}$ the solution of optimal control problem (5.1) associated to $\{u_0, \phi_0, u_d, \phi_d\}$. Then, as $\epsilon \to 0$, the following convergences hold:*

$$\{u_\epsilon, \phi_\epsilon, v_\epsilon\} \to \left\{u^*, \phi^*, v^*\right\} \quad \text{weakly in} \quad [W_2^{2,1}(Q)]^2 \times L^2(\mathcal{O} \times (0, T)) \tag{5.2}$$

and

$$\{u_\epsilon, \phi_\epsilon\} \to \left\{u^*, \phi^*\right\} \quad \text{strongly in} \quad [L^9(Q)]^2, \tag{5.3}$$

where $\{u^*, \phi^*, v^*\}$ is the solution of the exact controllability problem obtained in Theorem 4.2.

Proof We fix an arbitrary (small) $\epsilon > 0$ and let us take $\{u_d, \phi_d\}$ and $\{u_0, \phi_0\}$ satisfying the hypotheses of Theorem 4.2. In this way, we have a control function $v \in L^2(\mathcal{O} \times (0, T))$ such that the associated solution $\{u, \phi\}$ of (1.1) satisfies (4.11). By other hand, Theorem 5.1 assures us the existence of a optimal control $v_\epsilon \in L^2(\mathcal{O} \times (0, T))$. Let us denote by $\{u_\epsilon, \phi_\epsilon\}$ the corresponding solution of (1.1). Then, by (5.1), we have

$$\mathcal{F}_\epsilon(u_\epsilon, \phi_\epsilon; v_\epsilon) \le \mathcal{F}_\epsilon(u, \phi; v).$$

By the last inequality and (4.11)

$$\begin{aligned} &\|\{u_\epsilon, \phi_\epsilon\}(T) - \{u_d, \phi_d\}\|^2_{[L^2(\Omega)]^2} + \epsilon\|v_\epsilon\|^2_{L^2(\mathcal{O}\times(0,T))} \\ &\quad \le \|\{u, \phi\}(T) - \{u_d, \phi_d\}\|^2_{[L^2(\Omega)]^2} + \epsilon\|v\|^2_{L^2(\mathcal{O}\times(0,T))} \\ &\quad = \epsilon\|v\|^2_{L^2(\mathcal{O}\times(0,T))}. \end{aligned}$$

The last inequality and (4.10) imply that

$$\begin{aligned} u_\epsilon(T) &\to u_d \quad \text{strongly in} \quad L^2(\Omega), \\ \phi_\epsilon(T) &\to \phi_d \quad \text{strongly in} \quad L^2(\Omega) \end{aligned}$$

and that $\|v_\epsilon\|_{L^2(\mathcal{O}\times(0,T))}$ is bounded. Then, there exists $v^* \in L^2(\mathcal{O} \times (0, T))$ such that

$$v_\epsilon \to v^* \quad \text{weakly in} \quad L^2(\mathcal{O} \times (0, T))$$

Next, by Theorem 3.1 in [43], we have that $\|\{u_\epsilon, \phi_\epsilon\}\|_{[W_2^{2,1}(Q)]^2}$ is bounded. Then, there exists $\{u^*, \phi^*\} \in [W_2^{2,1}(Q)]^2$

$$
\begin{aligned}
u_\epsilon &\to u^* \quad \text{weakly in} \quad W_2^{2,1}(Q),\\
\phi_\epsilon &\to \phi^* \quad \text{weakly in} \quad W_2^{2,1}(Q),
\end{aligned}
$$

which implies

$$
\begin{aligned}
u_\epsilon &\to u^* \quad \text{strongly in} \quad L^9(Q),\\
\phi_\epsilon &\to \phi^* \quad \text{strongly in} \quad L^9(Q).
\end{aligned}
$$

To conclude the proof of the theorem, it remains to show that $u^*(T) = u_d$ and $\phi^*(T) = \phi_d$. In fact, let us consider any function $w \in C^\infty\left(\overline{Q}\right)$ such that supp $w \subset\subset \overline{\Omega} \times (0, T]$.

By multiplying first equation of the system satisfied by $\{u_\epsilon, \phi_\epsilon\}$, i.e (1.1), by w and integrating by parts in Q, we have

$$
\int_\Omega u_\epsilon(T)w(T)dx - \int_0^T\int_\Omega u_\epsilon w_t dxdt + \int_0^T\int_\Omega \nabla u_\epsilon \cdot \nabla w dxdt + l\int_0^T\int_\Omega (\phi_\epsilon)_t w dxdt
$$
$$
= \int_0^T\int_{\mathcal{O}} v_\epsilon w dxdt.
$$

By using the previous weak convergences, we obtain

$$
\int_\Omega u_d w(T)dx - \int_0^T\int_\Omega u^* w_t dxdt + \int_0^T\int_\Omega \nabla u^* \cdot \nabla w dxdt + l\int_0^T\int_\Omega \phi_t^* w dxdt
$$
$$
= \int_0^T\int_{\mathcal{O}} v^* w dxdt.
$$

Since $\{u^*, \phi^*\}$ satisfies (1.1), by multiplying it by w and integrating by parts in Q, we have

$$
\int_\Omega u^*(T)w(T)dx - \int_0^T\int_\Omega u^* w_t dxdt + \int_0^T\int_\Omega \nabla u^* \cdot \nabla w dxdt + l\int_0^T\int_\Omega \phi_t^* w dxdt
$$
$$
= \int_0^T\int_{\mathcal{O}} v^* w dxdt.
$$

Being w arbitrary, we conclude that $u^*(T) = u_d$. By similar procedure we obtain $\phi^*(T) = \phi_d$. This concludes the proof. □

Acknowledgments Part of this research was done while the first author was visiting Universidade Estadual de Campinas, and he expresses his thanks for their kind hospitality. The first author's study was partially supported by INCTMat, CAPES, CNPq (Brazil). The second author's study was partially supported by FAPESP and CNPq (Brazil). The third author's study was partially supported by FAPESP (Brazil).

References

1. Benincasa, T., Moroşanu, C.: Fractional steps scheme to approximate the phase-field transition system with non-homogeneous Cauchy–Neumann boundary conditions. Numer. Funct. Anal. Optim. **30**(3–4), 199–213 (2009)
2. Benincasa, T., Favini, A., Moroşanu, C.: A product formula approach to a nonhomogeneous boundary optimal control problem governed by nonlinear phase-field transition system. Part I: a phase-field model. J. Optim. Theory Appl. **148**(1), 14–30 (2011)
3. Benincasa, T., Favini, A., Moroşanu, C.: A product formula approach to a nonhomogeneous boundary optimal control problem governed by nonlinear phase-field transition system. Part II: Lie–Trotter product formula. J. Optim. Theory Appl. **148**(1), 14–30 (2011)
4. Moroşanu, C.: Boundary optimal control problem for the phase-field transition system using fractional steps method. Control Cybern. **32**(1), 5–32 (2003)
5. Moroşanu, C.: The phase-field transition system with non-homogeneous Cauchy–Stefan–Boltzmann and homogeneous Neumann boundary conditions and non-constant thermal conductivity. Nonlinear Anal. **87**, 22–32 (2013)
6. Cheng, M., Warren, J.A.: An efficient algorithm for solving the phase field crystal model. J. Comput. Phys. **227**(12), 6241–6248 (2008)
7. Hamide, M., Massoni, E., Bellet, M.: Adaptive mesh technique for thermal–metallurgical numerical simulation of arc welding processes. Int. J. Numer. Methods Eng. **73**(5), 624–641 (2008)
8. He, Q., Kasagi, N.: Phase-field simulation of small capillary-number two-phase flow in a microtube. Fluid Dyn. Res. **40**(7–8), 497–509 (2008)
9. Moroşanu, C., Wang, G.: State-constrained optimal control for the phase-field transition system. Numer. Funct. Anal. Optim. **28**(3–4), 379–403 (2007)
10. Rosam, J., Jimack, P.K., Mullis, A.A.: Fully implicit, fully adaptive time and space discretization method for phase-field simulation of binary alloy solidification. J. Comput. Phys. **225**(2), 1271–1287 (2007)
11. Sun, Y., Beckermann, C.: Phase-Field Simulation of Two-Phase Micro-flows in a Hele-Shaw Cell, Computational Methods in Multiphase Flow III, WIT Trans. Eng. Sci., vol. 50. WIT Press, Southampton (2005)
12. Tan, Z., Huang, Y.: An alternating Crank–Nicolson method for the numerical solution of the phase-field equations using adaptive moving meshes. Int. J. Numer. Methods Fluids **56**(9), 1673–1693 (2008)
13. Zhao, P., Heinrich, J.C., Poirier, D.R.: Dendritic solidification of binary alloys with free and forced convection. Int. J. Numer. Methods Fluids **49**(3), 233–266 (2005)
14. Ahmad, N.A., Wheeler, A.A., Boettinger, W.J., Mcfadden, G.B.: Solute trapping and solute drag in a phase-field model of rapid solidification. Phys. Rev. E Stat. Phys. Plasmas Fluids **58**(3B), 3436–3450 (1998)
15. Boldrini, J.L., Vaz, C.L.D.: Existence and regularity of solutions of a phase field model for solidification with convection of pure materials in two dimensions. Electron. J. Differ. Equ. **109**, 1–25 (2003)
16. Caginalp, G.: An analysis of phase field model of a free boundary. Arch. Ration. Mech. Anal. **92**, 205–245 (1986)
17. Caginalp, G.: Stefan and Hele-Shaw type models as assymptotic limits of the phase-field equations. Phys. Rev. A **39**(11), 5887–5896 (1989)
18. Caginalp, G.: Phase field computations of single-needle crystals, crystal growth and motion by mean curvature. SIAM J. Sci. Comput. **15**(1), 106–126 (1994)
19. Caginalp, G., Jones, J.: A derivation and analysis of phase field models of thermal alloys. Annal. Phys. **237**, 66–107 (1995)
20. Cherfils, L., Gatti, S., Miranville, A.: Existence of global solutions to the Caginalp phase-field system with dynamic boundary conditions and singular potentials. J. Math. Anal. Appl. **343**(1), 557–566 (2008)

21. Colli, P., Grasselli, M., Ito, A.: On a parabolic–hyperbolic Penrose–Fife phase-field system. Electron. J. Differ. Equ. **100**, 1–30 (2002)
22. Gilardi, G., Marson, A.: On a Penrose–Fife type system with Dirichlet boundary conditions for temperature. Math. Methods Appl. Sci. **26**(15), 1303–1325 (2003)
23. Gilardi, G., Rocca, E.: Convergence of phase field to phase relaxation models governed by an entropy equation with memory. Math. Methods Appl. Sci. **29**(18), 2149–2179 (2006)
24. Jiménez-Casas, A.: Invariant regions and global existence for a phase field model. Discret. Contin. Dyn. Syst. Ser. S **1**(2), 273–281 (2008)
25. Karma, A.: Phase-field models of microstructural pattern formation. Thermodyn. Microstruct. Plast. NATO Sci. Ser. II Math. Phys. Chem. **108**, 65–89 (2003)
26. Krejcí, P., Rocca, E., Sprekels, J.: Non-local temperature-dependent phase-field models for non-isothermal phase transitions. J. Lond. Math. Soc. **76**(1), 197–210 (2007)
27. Krejcí, P., Sprekels, J., Stefanelli, U.: One-dimensional thermo-visco-plastic processes with hysteresis and phase transitions. Adv. Math. Sci. Appl. **13**(2), 695–712 (2003)
28. Laurençot, P., Schimperna, G., Stefanelli, U.: Global existence of a strong solution to the one-dimensional full model for irreversible phase transitions. J. Math. Anal. Appl. **271**(2), 426–442 (2002)
29. McFadden, G.B., Wheeler, A.A., Anderson, D.M.: Thin interface asymptotics for an energy/entropy approach to phase-field models with unequal conductivities. Phys. D **144**(1–2), 154–168 (2000)
30. Moroşanu, C.: Analysis and optimal control of phase-field transition system. Nonlinear Funct. Anal. Appl. **8**(3), 433–460 (2003)
31. Nestler, B., Garcke, H., Stinner, B.: Multicomponent alloy solidification: phase-field modeling and simulations. Phys. Rev. E **71**(4), 1–6 (2005)
32. Penrose, O., Fife, P.C.: Thermodynamically consistent models of phase field type for the kinetics of phase transitions. Phys. D **43**, 44–62 (1990)
33. Planas, G.: Existence of solutions to a phase-field model with phase-dependent heat absorption. Electron. J. Differ. Equ. **28**, 1–12 (2007)
34. Stinner, B.: Weak solutions to a multi-phase field system of parabolic equations related to alloy solidification. Adv. Math. Sci. Appl. **17**(2), 589–638 (2007)
35. Aizicovici, S., Feireisl, E.: Long-time stabilization of solutions to a phase-field model with memory. J. Evol. Equ. **1**, 69–84 (2001)
36. Aizicovici, S., Feireisl, E., Issard-Roch, F.: Long time convergence of solutions to a phase-field system. Math. Methods Appl. Sci. **24**, 277–287 (2001)
37. Bates, P.W., Zheng, S.: Inertial manifolds and inertial sets for phase-field equations. J. Dyn. Differ. Equ. **4**, 375–397 (1992)
38. Brochet, D., Chen, X., Hilhorst, D.: Finite dimensional exponential attractor for the phase-field model. Appl. Anal. **49**, 197–212 (1993)
39. Jiang, J.: Convergence to equilibrium for a parabolic–hyperbolic phase field model with Cattaneo heat flux law. J. Math. Anal. Appl. **341**(1), 149–169 (2008)
40. Kapustyan, A.V., Melnik, V.S., Valero, J.: Attractors of multivalued dynamical processes generated by phase-field equations. Int. J. Bifurc. Chaos Appl. Sci. Eng. **13**(7), 1969–1983 (2003)
41. Röger, M., Tonegawa, Y.: Convergence of phase-field approximations to the Gibbs–Thomson law. Calc. Var. Partial Differ. Equ. **32**(1), 111–136 (2008)
42. Sprekels, J., Zheng, S.: Global existence and asymptotic behaviour for a nonlocal phase-field model for non-isothermal phase transitions. J. Math. Anal. Appl. **279**(1), 97–110 (2003)
43. Hoffman, K., Jiang, L.: Optimal control of a phase field model for solidification. Numer. Funct. Anal. Optim. **13**, 11–27 (1992)
44. Wang, L., Wang, G.: The optimal time control of a phase-field system. SIAM J. Control Optim. **42**(4), 1483–1508 (2003)
45. Barbu, V.: Local controllability of the phase field system. Nonlinear Anal. **50**(3), 363–372 (2002)
46. Ammar-Khodja, F., Benabdallah, A., Dupaix, C., Kostin, I.: Controllability to the trajectories of phase-field models by one control force. SIAM J. Control Optim. **42**(5), 1661–1680 (2003)
47. González-Burgos, M., Pérez-García, R.: Controllability results for some nonlinear coupled parabolic systems by one control force. Asymptot. Anal. **46**(2), 123–162 (2006)
48. Cao, Y.: Numerical approximations of exact controllability problems by optimal control problems for parabolic differential equations. Appl. Math. Comput. **119**(2–3), 127–145 (2001)
49. Adams, R.: Sobolev Spaces. Academic Press, New York (1975)

50. Ladyzenskaja, O.A., Solonnikov, V.A., Ural'ceva, N.N.: Linear and Quasilinear Equations of Parabolic Type, Translations of Mathematical Monographs, vol. 23. American Mathematical Society, Providence (1968)
51. Mikhaylov, V.P.: Partial Differential Equations. Mir Publishers, Moscow (1978)
52. Lions, J.L.: Control of Distributed Singular Systems. Gauthier-Villars, Paris (1985)
53. Chae, D., Imanuvilov, O.Y., Kim, S.M.: Exact controllability for semilinear parabolic equations with Neumann boundary conditions. J. Dyn. Control Syst. **2**(4), 449–483 (1996)
54. Fabre, C., Puel, J.-P., Zuazua, E.: Approximate controllability of the semilinear heat equation. Proc. R. Soc. Edinb. **125**(1), 31–61 (1995)
55. Doubova, A., Fernández-Cara, E., González-Burgos, M., Zuazua, E.: On the controllability of parabolic systems with a nonlinear term involving the state and the gradient. SIAM J. Control Optim. **41**(3), 798–819 (2002)