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L^p Theory for Super-Parabolic Backward Stochastic Partial Differential Equations in the Whole Space

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Abstract This paper is concerned with semi-linear backward stochastic partial differential equations (BSPDEs for short) of super-parabolic type. An L^p-theory is given for the Cauchy problem of BSPDEs, separately for the case of $p \in (1, 2]$ and for the case of $p \in (2, \infty)$. A comparison theorem is also addressed.

Keywords Backward stochastic differential equation · Stochastic partial differential equation · Backward stochastic partial differential equation · Bessel potentials

1 Introduction

Since Bismut's pioneering work [2–4] and Pardoux and Peng's seminal work [26], the theory of backward stochastic differential equations (BSDEs) is rather complete now.

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See, among others, El Karoui et al. [17], and Delbaen and Tang [6] for a rather general L^p theory for BSDEs. As a natural generalization of BSDEs, backward stochastic partial differential equations (BSPDEs) arise in many applications of probability theory and stochastic processes, for instance in the optimal control of processes with incomplete information, as an adjoint equation of the Duncan-Mortensen-Zakai filtration equation (for instance, see [1, 14, 15, 32, 36, 37]), and naturally in the dynamic programming theory fully nonlinear BSPDEs as the so-called backward stochastic Hamilton-Jacobi-Bellman equations, are also introduced in the study of controlled non-Markovian processes (see Peng [27] and Englezos and Karatzas [13]).

In this paper, we consider the following semi-linear BSPDEs:

$$\begin{cases} -du(t,x) = [\mathcal{L}(t,x)u(t,x) + \mathcal{M}^r(t,x)v^r(t,x) + F(u,v,t,x)]\,dt \\ \qquad\qquad - v^r(t,x)\,dW_t^r, \quad (t,x) \in [0,T] \times \mathbb{R}^d; \\ u(T,x) = G(x), \quad x \in \mathbb{R}^d. \end{cases} \tag{1.1}$$

Here and throughout this paper, we denote

$$\mathcal{L}(t,x) := a^{ij}(t,x)\frac{\partial^2}{\partial x^i \partial x^j}, \qquad \mathcal{M}^r(t,x) := \sigma^{jr}(t,x)\frac{\partial}{\partial x^j}, \quad r = 1, 2, \ldots, m.$$

We use the Einstein summation convention and fix $T \in (0, \infty)$ as a finite deterministic time, which can be replaced by any bounded stopping time.

To the above BSPDEs, the method of stochastic flows was developed by Tang [33] which gives a probabilistic point of view and also gives classical solutions to BSPDEs (1.1). On the other hand, the L^2 theory for BSPDEs has been established in the framework of weak solutions (see [10, 14, 15, 36, 37], for example).

Still in the framework of weak solutions, we establish in this paper an L^p-theory for BSPDE (1.1) which seems to be the first study for the L^p-theory of BSPDEs. Motivated by Krylov's seminal work [20, 21] on forward stochastic partial differential equations, we establish an L^p-theory which includes as a particular case the L^p theory $(1 < p \le 2)$ for deterministic parabolic partial differential equations (PDEs for short).

This paper is organized as follows. In Sect. 2 we introduce the notions and define some spaces. We discuss a kind of Banach space-valued BSDEs in Sect. 3. In Sect. 4 we construct a stochastic Banach space $\mathcal{H}_p^n$ which plays the same role as spaces $W_p^{1,2}$ in the theory of second-order parabolic PDEs and we also give some basic properties of this space there. In Sect. 5 we present the L^p-theory of BSPDEs in the whole space for $p \in (1, 2]$. Specifically, we give the definition of the L^p solutions and list the assumptions. We first solve the BSPDEs with constant-field-valued leading coefficients and then solve the BSPDEs for the general case. In Sect. 6 we discus two related topics: a comparison theorem and an L^p-theory for $p > 2$. Finally, in Sect. 7 we give some comments.

2 Preliminaries

In most of this work, we shall denote by $|\cdot|$ (respectively, $\langle \cdot, \cdot \rangle$) the norm (respectively, scalar product) in finite-dimensional Hilbert space such as $\mathbb{R}$, $\mathbb{R}^k$, $\mathbb{R}^{k\times l}$ where

k, l are positive integers and

$$|x| := \left(\sum_{i=1}^{k} x_i^2\right)^{\frac{1}{2}} \quad \text{and} \quad |y| := \left(\sum_{i=1}^{k}\sum_{j=1}^{l} y_{ij}^2\right)^{\frac{1}{2}} \quad \text{for } (x, y) \in \mathbb{R}^k \times \mathbb{R}^{k\times l}.$$

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space on which is defined an m-dimensional standard Brownian motion $W = \{W_t : t \in [0, T]\}$ such that $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration generated by W and augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$. And we denote by $\mathscr{P}$ the σ-Algebra of the predictable sets on $\Omega \times [0, T]$ associated with $\{\mathcal{F}_t\}_{t\geq 0}$.

If $X = (X_t)_{t\in[0,T]}$ is an $\mathbb{R}^k$-valued, adapted and continuous processes, we denote $\sup_{t\in[0,T]} |X_t|$ by X_* or $\sup_t |X_t|$ simply. And for any $p \in (1, \infty)$, $\mathcal{S}^p(\mathbb{R}^k)$ denotes the set of all the $\mathbb{R}^k$-valued, adapted and continuous processes $(X_t)_{t\in[0,T]}$ such that

$$\|X\|_{\mathcal{S}^p} := \left\{E\left[\sup_t |X_t|^p\right]\right\}^{1/p} < \infty.$$

Then, $(\mathcal{S}^p(\mathbb{R}^k), \ \|\cdot\|_{\mathcal{S}^p})$ is a Banach space.

Define the set of multi-indices

$$\mathcal{A} := \{\alpha = (\alpha_1, \ldots, \alpha_d) : \alpha_1, \ldots, \alpha_d \text{ are nonnegative integers}\}.$$

For any $\alpha \in \mathcal{A}$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, denote

$$|\alpha| = \sum_{i=1}^{d} \alpha_i, \qquad x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}, \qquad D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_d^{\alpha_d}}.$$

For a positive integer number l, we denote by $C_c^\infty(\mathbb{R}^l)$ (respectively, $C_c^\infty(\mathcal{O})$ for each open set $\mathcal{O} \subset \mathbb{R}^l$) the set of all infinitely differentiable functions with compact supports on $\mathbb{R}^l$ ($\mathcal{O}$, respectively) and by $\mathcal{D}'$ the space of real-valued Schwartz distributions on $C_c^\infty(\mathbb{R}^d)$. For simplicity, we write C_c^∞ for the case $l = d$. On $\mathbb{R}^d$ we denote by $\mathscr{S}$ the set of all the Schwartz functions and by $\mathscr{S}'$ the set of all the tempered distributions. Note that C_c^∞ and $\mathscr{S}$ are endowed with matching topologies. For each positive integer N and multi-indices α, we define

$$\gamma_{\alpha,N}(f) := \sup_{|x|\leq N} \left|(\partial^\alpha f)(x)\right|, \quad f \in C_c^\infty$$

Then $C_c^\infty = \bigcup_{N=1}^\infty C_c^\infty(B(0, N))$ is the inductive limit of the complete metrizable spaces $(C_c^\infty(B(0, N)), \ \gamma_{\alpha,N})$, where $B(0, N)$ is the open ball of radius N centered at the origin. For each $\alpha, \beta \in \mathcal{A}$, we define

$$\rho_{\alpha,\beta}(f) := \sup_{x\in\mathbb{R}^d} \left|x^\alpha (\partial^\beta f)(x)\right|, \qquad d(f, g) := \sum_{j=1}^\infty 2^{-j} \frac{\rho_j(f - g)}{1 + \rho_j(f - g)}, \quad f, g \in \mathscr{S}$$

where ρ_j is an enumeration of all the seminorms $\rho_{\alpha,\beta}$ with $\alpha, \beta \in \mathcal{A}$. Then $\mathscr{S}$ is a Fréchet space equipped with the metric $d(\cdot, \cdot)$. It follows that $C_c^\infty \subset \mathscr{S}$ and $\mathscr{S}' \subset \mathcal{D}'$.

We shall denote by $(\cdot,\cdot)$ not only the duality between $\mathcal{D}'$ and C_c^∞ but also the duality between $\mathcal{S}$ and $\mathcal{S}'$. Then the Fourier transform $\mathcal{F}(f)$ of $f\in\mathcal{S}'$ is given by

$$\mathcal{F}(f)(\xi)=(2\pi)^{-d/2}\int_{\mathbb{R}^d}e^{-\sqrt{-1}\langle x,\xi\rangle}f(x)\,dx,\quad \xi\in\mathbb{R}^d,$$

and the inverse Fourier transform $\mathcal{F}^{-1}(f)$ is given by

$$\mathcal{F}^{-1}(f)(x)=(2\pi)^{-d/2}\int_{\mathbb{R}^d}e^{\sqrt{-1}\langle x,\xi\rangle}f(\xi)\,d\xi,\quad x\in\mathbb{R}^d.$$

It is well known that both $\mathcal{F}$ and $\mathcal{F}^{-1}$ map $\mathcal{S}'$ onto itself. As usual, for any $s\in\mathbb{R}$ and $f\in\mathcal{S}'$, we denote $I_s(f):=(1-\Delta)^{s/2}f=\mathcal{F}^{-1}((1+|\xi|^2)^{s/2}\mathcal{F}(f)(\xi))$.

For given $p\in(1,\infty)$ and $n\in(-\infty,\infty)$, we denote by H_p^n the space of Bessel potentials, that is

$$H_p^n:=\{\phi\in\mathcal{S}':(1-\Delta)^{\frac{n}{2}}\phi\in L^p(\mathbb{R}^d)\}$$

with the Sobolev norm

$$\|\phi\|_{n,p}:=\|(1-\Delta)^{\frac{n}{2}}\phi\|_p,\quad \phi\in H_p^n,$$

where $\|\cdot\|_p$ is the norm in $L^p(\mathbb{R}^d)$. It is well known that H_p^n is a Banach space with the norm $\|\cdot\|_{n,p}$ and the set C_c^∞ is dense in H_p^n. For any $p\in(1,\infty)$ and $n\in\mathbb{R}$, with a little notional abuse we still denote by $(\cdot,\ \cdot)$ the dual pairing between H_p^n and $H_{p'}^{-n}$ where $1/p'+1/p=1$, i.e., for any $(u,v)\in H_p^n\times H_{p'}^{-n}$

$$(u,v)=((1-\Delta)^{\frac{n}{2}}u,(1-\Delta)^{-\frac{n}{2}}v)=\int_{\mathbb{R}^d}(1-\Delta)^{\frac{n}{2}}u(x)(1-\Delta)^{-\frac{n}{2}}v(x)\,dx$$

where the last integral is a usual Lebesgue integral.

In contrast to H_p^n, we introduce the following so-called Besov space of functions (c.f. [34] or [35]).

Definition 2.1 Let $s>0,\ p\in(1,\infty)$, and $q\in[1,\infty)$. Define

$$B_{p,q}^s=\Bigg\{f\in L^p(\mathbb{R}^d):\|f\|_{B_{p,q}^s}=\|f\|_{H_p^{[s]^-}}+\sum_{|\alpha|=[s]^-}\bigg(\int_{\mathbb{R}^d}|h|^{-\{s\}^+q}\|D^\alpha f(\cdot+2h)-2D^\alpha f(\cdot+h)+D^\alpha f(\cdot)\|_p^q\frac{dh}{|h|^d}\bigg)^{1/q}<\infty\Bigg\}$$

where $s=[s]^-+\{s\}^+$, with $[s]^-$ being an integer and $\{s\}^+\in(0,1]$.

Let $\sigma>0,\ p\in(1,\infty),\ q\in[1,\infty)$, and $s\in\mathbb{R}$ such that $\sigma-s>0$. Then $I_s(B_{p,q}^\sigma)=B_{p,q}^{\sigma-s}$. In fact, we can introduce spaces $B_{p,q}^s$ with $s\le 0$ by defining $B_{p,q}^s=I_{-s+1}(B_{p,q}^1)$, although we prefer to define the Besov space through the Littlewood-Paley decomposition (for instance, see [35]). As to the specific structure and properties of Besov space, see [35] or [34]. In this paper, only the space $B_{p,p}^n$ is involved for

$p \in (1, \infty)$ and $n \in \mathbb{R}$ and for the reader's convenience, we define the norm which is equivalent to Definition 2.1 when $n > 0$:

$$\|f\|_{B^n_{p,p}} := \|I_{n-1} f\|_{B^1_{p,p}}, \quad f \in B^n_{p,p}.$$

The following lemma shows some embedding properties for the Besov spaces, whose proof is seen in [34].

Lemma 2.1 (i) *Let $n \in \mathbb{R}$ and $p \in (1, \infty)$, then*

$$B^n_{p,\min\{p,2\}} \hookrightarrow H^n_p \hookrightarrow B^n_{p,\max\{p,2\}},$$

where $\hookrightarrow$ stands for topological embedding throughout this work.

(ii) *Let $p, q, r \in (1, \infty)$, then*

$$B^s_{p,r} \hookrightarrow B^t_{p,q}, \quad -\infty < s < t < \infty.$$

Denote by $\mathfrak{S}$ the set of all $\mathcal{S}'$-valued functions defined on $\Omega \times [0, T]$ such that, for any $u \in \mathfrak{S}$ and $\phi \in \mathcal{S}$, the function (u, ϕ) is $\mathcal{P}$-measurable.

For $p \in (1, \infty)$, we define $\mathbb{H}^0_p := L^p(\Omega \times [0, T] \times \mathbb{R}^d, \mathcal{P} \times \mathcal{B}(\mathbb{R}^d), \mathbb{R})$. Denote by $\mathbb{H}^0_{p,2}$ the set of the functions which are defined on $\Omega \times [0, T] \times \mathbb{R}^d$ and $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable such that

$$E\left[\int_{\mathbb{R}^d} \left(\int_0^T |u(t, x)|^2 dt\right)^{\frac{p}{2}} dx\right] < \infty, \quad \forall u \in \mathbb{H}^0_{p,2}.$$

Observe that every element of $\mathbb{H}^0_p$ can be considered as an H^0_p-valued, $\mathcal{P}$-measurable process. For any $n \in \mathbb{R}$, we define

$$\mathbb{H}^n_p := \{f \in \mathfrak{S} : (1 - \Delta)^{\frac{n}{2}} f \in \mathbb{H}^0_p\},$$

equipped with the norm

$$\|f\|_{\mathbb{H}^n_p} := \left(E\left[\int_0^T \int_{\mathbb{R}^d} |(1 - \Delta)^{\frac{n}{2}} f(t, x)|^p \, dx dt\right]\right)^{1/p}.$$

Definition 2.2 Let $p \in (1, \infty)$ and $n \in \mathbb{R}$. Define

$$\mathbb{H}^n_{p,2} := \left\{u \in \mathfrak{S} : (1 - \Delta)^{\frac{n}{2}} u \in \mathbb{H}^0_{p,2}\right\}$$

equipped with the norm

$$\|u\|_{\mathbb{H}^n_{p,2}} := \left(E\left[\int_{\mathbb{R}^d} \left[\int_0^T |(1 - \Delta)^{\frac{n}{2}} u(t, x)|^2 dt\right]^{\frac{p}{2}} dx\right]\right)^{1/p}.$$

Definition 2.3 Let $p \in (1, \infty)$ and $n \in \mathbb{R}$. For a function $u \in \mathbb{H}^n_{p,2}$, we write $u \in \mathbb{H}^n_{p,\infty}$ if

(i) there exists $A(u) \in \mathcal{F}_T \times \mathcal{B}(\mathbb{R}^d)$, $\mathbb{P} \times \mathfrak{M}(A(u)) = 0$ where $\mathfrak{M}(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}^d$, such that for any $(\omega, x) \in \mathbb{R}^d \times \Omega \setminus A(u)$, $(1 - \Delta)^{n/2} u(\cdot, x)$ is continuous on $[0, T]$;
(ii) $\|u\|_{\mathbb{H}^n_{p,\infty}} := (E[\int_{\mathbb{R}^d} \sup_{t \in [0,T]} |(1-\Delta)^{\frac{n}{2}} u(t,x)|^p \, dx])^{1/p} < \infty$.

When we treat the general $\mathbb{R}^k$-valued function u for any integer $k > 1$, we still say $u \in \mathbb{H}^n_p$ if $u^l \in \mathbb{H}^n_p$ for $l = 1, 2, \ldots, k$. In this way, we generalize the real-valued function space $\mathbb{H}^n_p$ to $\mathbb{R}^k$-valued function space. And further, we define the norm

$$\|u\|_{\mathbb{H}^n_p} := \left(E\left[\int_0^T \int_{\mathbb{R}^d} |(1-\Delta)^{\frac{n}{2}} u(t,x)|^p \, dx dt \right] \right)^{1/p}.$$

By this means, not only can we generalize spaces H^n_p, $\mathbb{H}^n_p$, $\mathbb{H}^n_{p,2}$ and $\mathbb{H}^n_{p,\infty}$ from real-valued function spaces to any $\mathbb{R}^k$-valued ones, but also we can generalize these spaces from $\mathbb{R}^k$-valued to any Hilbert space-valued function spaces. And we do it when we need it.

Remark 2.1 One can check that the spaces $\mathbb{H}^n_p$, $\mathbb{H}^n_{p,2}$ and $\mathbb{H}^n_{p,\infty}$ are all Banach spaces under the norms $\|\cdot\|_{\mathbb{H}^n_p}$, $\|\cdot\|_{\mathbb{H}^n_{p,2}}$, and $\|\cdot\|_{\mathbb{H}^n_{p,\infty}}$, respectively. Moreover, for any $p \in (1, \infty)$ and $n \in \mathbb{R}$, $\mathbb{H}^n_p$ is a reflexive Banach space whose dual space is $\mathbb{H}^{-n}_{p/(p-1)}$, and it coincides with the space $\mathbb{H}^n_p(T)$ defined in [21] and [20]. On the other hand, for $s \in \mathbb{R}$, the operator $(1-\Delta)^{s/2}$ maps isometrically H^n_p to H^{n-s}_p and the same is true for spaces $\mathbb{H}^n_p$, $\mathbb{H}^n_{p,2}$, and $\mathbb{H}^n_{p,\infty}$.

In particular, as to the spaces $\mathbb{H}^n_p$ and $\mathbb{H}^n_{p,2}$, we have the following lemma whose proof is similar to that of [21, Theorem 3.10].

Lemma 2.2 *Let $p \in (1, \infty)$ and $n \in \mathbb{R}$. For $g \in \mathbb{H}^n_p$ ($\mathbb{H}^n_{p,2}$, respectively), there exists a sequence $\{g_j,\ j = 1, 2, \ldots\}$ in $\mathbb{H}^n_p$ ($\mathbb{H}^n_{p,2}$, respectively) such that $\|g - g_j\|_{\mathbb{H}^n_p} \to 0$ ($\|g - g_j\|_{\mathbb{H}^n_{p,2}} \to 0$, respectively) as $j \to \infty$ and*

$$g_j = \sum_{i=1}^{j} \mathbb{1}_{(\tau^j_{i-1}, \tau^j_i]}(t)\, g^i_j(x),$$

where $g^i_j \in C^\infty_c$ and τ^j_i are stopping times such that $\tau^j_{i-1} \le \tau^j_i \le T$.

For any $t \in [0, T)$, define

$$\|u\|_{\mathbb{H}^n_p(t)} := \|u \mathbb{1}_{[t,T]}\|_{\mathbb{H}^n_p} \quad \text{for } u \in \mathbb{H}^n_p.$$

In the same way, we define $\|\cdot\|_{\mathbb{H}^n_{p,2}(t)}$ in $\mathbb{H}^n_{p,2}$ and $\|\cdot\|_{\mathbb{H}^n_{p,\infty}(t)}$ in $\mathbb{H}^n_{p,\infty}$.

For an element u of spaces like $\mathbb{H}^n_p$, if it has a modification of higher regularity, then it is always considered to be this modification. However, elements of spaces like $\mathbb{H}^n_p$ are H^n_p-valued only for almost all (t,ω), not necessarily for all $(t,\omega)\in[0,T]\times\Omega$.

We end this section with introducing the Itô-Wentzell formula for distribution-valued processes established by Krylov [22].

For $p=1,2$ we denote by $\mathfrak{S}^p$ the totality of $u\in\mathfrak{S}$ such that for any $R_1,R_2\in(0,\infty)$ and $\phi\in C_c^\infty$, we have

$$\int_0^{R_2}\sup_{|x|\le R_1}|(u(t,\cdot),\phi(\cdot-x))|^p\,dt<\infty \quad \text{a.s.}$$

In a similar way to spaces like H^n_p, we generalize $\mathfrak{S}$ to any Hilbert space-valued function space.

For $u,f,g\in\mathfrak{S}$, we say that the equality

$$du(t,x)=f(t,x)\,dt+g^k(t,x)\,dW^k_t,\qquad t\in[0,T], \tag{2.1}$$

holds in the sense of distribution if $f\mathbb{1}_{[0,T]}\in\mathfrak{S}^1$, $g\mathbb{1}_{[0,T]}\in\mathfrak{S}^2$ and for any $\phi\in C_c^\infty$ with probability one we have for all $t\in[0,T]$

$$(u(t,\cdot),\phi)=(u(0,\cdot),\phi)+\int_0^t(f(s,\cdot),\phi)\,ds+\int_0^t(g^k(s,\cdot),\phi)\,dW^k_s.$$

Let x_t be an $\mathbb{R}^d$-valued predictable process of the following form

$$x_t=\int_0^t b_s\,ds+\int_0^t\beta^k_s\,dW^k_s,$$

where b and β are predictable processes such that for all $\omega\in\Omega$ and $s\in[0,T]$, we have

$$\operatorname{tr}(\alpha_s)<\infty\quad\text{and}\quad\int_0^T[|b_t|+\operatorname{tr}(\alpha_t)]\,dt<\infty,$$

with $2\alpha^{ij}:=\beta^{ik}\beta^{jk}$.

Theorem 2.3 (Theorem 1 of [22]) *Assume that* (2.1) *holds in the sense of distributions and define*

$$v(t,x):=u(t,x+x_t).$$

Then we have

$$dv(t,x)=\bigg(f(t,x+x_t)+\alpha^{ij}\frac{\partial^2}{\partial x^i\partial x^j}v(t,x)+b^i(t,x)\frac{\partial}{\partial x^i}v(t,x)$$
$$+\frac{\partial}{\partial x^i}g^k(t,x+x_t)\beta^{ik}(t)\bigg)\,dt$$

$$+\left(g^k(t,x+x_t)+\frac{\partial}{\partial x^i}v(t,x)\beta^{ik}(t)\right)dW_t^k,\quad t\in[0,T]$$

holds in the sense of distribution.

Remark 2.2 In the Itô-Wentzell formula established by Krylov [22, Theorem 1], the Wiener process $(W_t)_{t\geq 0}$ can be any separable Hilbert space-valued process and the processes in $\mathfrak{S}$ therein were defined to be $\mathcal{D}'$-valued instead of being $\mathcal{S}'$-valued.

3 Banach Space-Valued BSDEs

This section is concerned with Banach space-valued BSDEs. Unless stated otherwise, we assume $p\in(1,\infty)$ and $n\in\mathbb{R}$ throughout this section. For $(F,G)\in\mathbb{H}_p^n\times L^p(\Omega,\mathcal{F}_T,H_p^n)$, consider the BSDE

$$\begin{cases}-du(t,x)=F(t,x)\,dt-v^k(t,x)\,dW_t^k, & (t,x)\in[0,T]\times\mathbb{R}^d,\\ u(T,x)=G(x), & x\in\mathbb{R}^d.\end{cases}\tag{3.1}$$

Or, equivalently

$$u(t,x)=G(x)+\int_t^T F(s,x)\,ds-\int_t^T v^k(s,x)\,dW_s^k,\quad (t,x)\in[0,T]\times\mathbb{R}^d.$$

Definition 3.1 Assume that $(F,G)\in\mathbb{H}_p^n\times L^p(\Omega,\mathcal{F}_T,H_p^n)$ with $p\in(1,\infty)$ and $n\in\mathbb{R}$. We say $(u,v)\in\mathbb{H}_p^n\times\mathbb{H}_{p,2}^n$ is a solution of (3.1) if for any $\phi\in C_c^\infty$ and $\tau\in[0,T]$, we have

$$(u(\tau,\cdot),\phi)=(G,\phi)+\int_\tau^T(F(s,\cdot),\phi)\,ds-\int_\tau^T(v^l(s,\cdot),\phi)\,dW_s^l,\quad\text{a.s.}\tag{3.2}$$

Remark 3.1 If $(u,v)\in\mathbb{H}_p^n\times\mathbb{H}_{p,2}^n$ is a solution to (3.1), then for any $\phi\in H_{p'}^{-n}$,

$$\begin{aligned}&E\left[\max_{t\in[0,T]}\left|\int_0^T(v^l,\phi)\,dW_s^l\right|\right]\\
&\leq CE\left[\left(\int_0^T|(v(s,\cdot),\phi)|^2\,ds\right)^{1/2}\right]\quad\text{(using the BDG inequality)}\\
&=CE\left[\left(\int_0^T\left|\int_{\mathbb{R}^d}(1-\Delta)^{-n/2}\phi(x)(1-\Delta)^{n/2}v(s,x)\,dx\right|^2ds\right)^{1/2}\right]\\
&\quad\text{(using Minkowski inequality)}\\
&\leq CE\left[\int_{\mathbb{R}^d}\left(\int_0^T|(1-\Delta)^{-n/2}\phi(x)(1-\Delta)^{n/2}v(s,x)|^2\,ds\right)^{1/2}dx\right]\end{aligned}$$

$$= CE\left[\int_{\mathbb{R}^d}\left(\int_0^T |(1-\Delta)^{n/2}v(s,x)|^2\,ds\right)^{1/2}\left|(1-\Delta)^{-n/2}\phi(x)\right|\,dx\right]$$
$$\le C\|v\|_{\mathbb{H}^n_{p,2}}\|\phi\|_{-n,p'}$$

where $1/p' + 1/p = 1$. So, the process

$$\int_0^t (v^l(s,\cdot),\phi)\,dW^l_s, \quad t\in[0,T]$$

is a continuous martingale. Note that, throughout the paper, unless stated otherwise, C is a positive constant and $C(\alpha,\beta,\dots,\gamma)$ is a constant only depending on $\alpha,\beta,\dots,$ and γ.

Lemma 3.1 *Assume that $(F,G)\in\mathbb{H}^n_p\times L^p(\Omega,\mathcal{F}_T,H^n_p)$ with $p\in(1,\infty)$ and $n\in\mathbb{R}$. We have*

(i) *Equation (3.1) has a unique solution $(u,v)\in(\mathbb{H}^n_p\cap\mathbb{H}^n_{p,\infty})\times\mathbb{H}^n_{p,2}$ which satisfies the following inequality*

$$\|u\|_{\mathbb{H}^n_{p,\infty}}+\|u\|_{\mathbb{H}^n_p}+\|v\|_{\mathbb{H}^n_{p,2}}\le c(p,T)[\|F\|_{\mathbb{H}^n_p}+\|G\|_{L^p(\Omega,\mathcal{F}_T,H^n_p)}]. \quad (3.3)$$

(ii) *For this solution, we have $u\in C([0,T],H^n_p)$ almost surely, and for any $\phi\in H^{-n}_{p/(p-1)}$ the following equality*

$$(u(\tau,\cdot),\phi)=(G,\phi)+\int_\tau^T(F(s,\cdot),\phi)\,ds-\int_\tau^T(v^l(s,\cdot),\phi)\,dW^l_s \quad (3.4)$$

holds for all $\tau\in[0,T]$ with probability 1.

Proof First, we prove the uniqueness of the solution. Suppose that (u_1,v_1) and (u_2,v_2) are two solutions of (3.1) in $\mathbb{H}^n_p\times\mathbb{H}^n_{p,2}$, and take $(u,v)=(u_1-u_2,v_1-v_2)$. Then, for any $\phi\in C_c^\infty$ and $t\in[0,T]$. We have

$$(u(t,\cdot),\phi)=\int_t^T(v(s,\cdot),\phi)\,dW_s, \quad \text{a.s.}$$

Then by the theory on BSDEs (c.f. [5, 17, 26]), we have

$$E\left[\int_{\tau_1}^{\tau_2}(u(t,\cdot),\phi)\,dt\right]=0 \quad\text{and}\quad E\left[\int_{\tau_1}^{\tau_2}(v(s,\cdot),\phi)\,ds\right]=0,$$

for any stopping times τ_1 *and* τ_2, $0\le\tau_1\le\tau_2\le T$. From Lemma 2.2, it follows that $(u,v)=0$ in $\mathbb{H}^n_p\times\mathbb{H}^n_{p,2}$. This verifies the uniqueness.

For the other assertions, it is sufficient to prove the lemma for $n=0$.

Indeed, assume that the lemma is true for $n=k$ with $k\in\mathbb{R}$. For $\forall\delta\in\mathbb{R}$, if $(F,G)\in\mathbb{H}^{k+\delta}_p\times L^p(\Omega,\mathcal{F}_T,H^{k+\delta}_p)$ then $(F',G')\in\mathbb{H}^k_p\times L^p(\Omega,\mathcal{F}_T,H^k_p)$ with

$$F':=(1-\Delta)^{\delta/2}F \quad\text{and}\quad G':=(1-\Delta)^{\delta/2}G.$$

From the induction assumption, there exists $(u', v') \in \mathbb{H}^k_{p,\infty} \times \mathbb{H}^k_{p,2}$, satisfying the following

$$\|u'\|_{\mathbb{H}^k_{p,\infty}} + \|u'\|_{\mathbb{H}^k_p} + \|v'\|_{\mathbb{H}^k_{p,2}} \le c(p,T)\left(\|F'\|_{\mathbb{H}^k_p} + \|G'\|_{L^p(\Omega,\mathcal{F}_T,H^k_p)}\right), \tag{3.5}$$

such that for any $\phi \in H^{-k}_{p/(p-1)}$ the equality

$$(u'(\tau,\cdot),\phi) = (G',\phi) + \int_\tau^T (F'(s,\cdot),\phi)\,ds - \int_\tau^T (v'^l(s,\cdot),\phi)\,dW^l_s,$$

holds for all $\tau \in [0,T]$ with probability 1. Take

$$u = (1-\Delta)^{-\delta/2}u' \quad \text{and} \quad v^l = (1-\Delta)^{-\delta/2}v'^l.$$

In view of Remark 2.1, $(u,v) \in \mathbb{H}^{k+\delta}_p \times L^p(\Omega,\mathcal{F}_T,H^{k+\delta}_p)$. Rewrite the last equality into the following

$$((1-\Delta)^{\delta/2}u(\tau,\cdot),\phi) = ((1-\Delta)^{\delta/2}G,\phi) + \int_\tau^T ((1-\Delta)^{\delta/2}F(s,\cdot),\phi)\,ds - \int_\tau^T ((1-\Delta)^{\delta/2}v^l(s,\cdot),\phi)\,dW^l_s, \quad \phi \in H^{-k}_{p/(p-1)}$$

which is equivalent to

$$(u(\tau,\cdot),(1-\Delta)^{\delta/2}\phi) = (G,(1-\Delta)^{\delta/2}\phi) + \int_\tau^T (F(s,\cdot),(1-\Delta)^{\delta/2}\phi)\,ds - \int_\tau^T (v^l(s,\cdot),(1-\Delta)^{\delta/2}\phi)\,dW^l_s, \quad \phi \in H^{-k}_{p/(p-1)}.$$

Hence, for any $\phi \in H^{-k-\delta}_{p/(p-1)}$ the equality

$$(u(\tau,\cdot),\phi) = (G,\phi) + \int_\tau^T (F(s,\cdot),\phi)\,ds - \int_\tau^T (v^l(s,\cdot),\phi)\,dW^l_s$$

holds for all $\tau \in [0,T]$ with probability 1. Then (u,v) solves BSDE (3.1) for $n = k+\delta$ in the sense of Definition 3.1, and satisfies the inequality (3.3) with $n := k+\delta$ which is exactly the inequality (3.5).

In what follows, we shall use the method of finite-dimensional approximation.

Since $\mathbb{H}^0_{p,\infty} \subset \mathbb{H}^0_p$ and $\|u\|_{\mathbb{H}^0_p} \le C(T,p)\|u\|_{\mathbb{H}^0_{p,\infty}}$ for $u \in \mathbb{H}^0_{p,\infty}$, it remains to prove the existence of the solution (u,v) in $\mathbb{H}^0_{p,\infty} \times \mathbb{H}^0_{p,2}$, the assertion (ii) and the following estimate

$$\|u\|_{\mathbb{H}^0_{p,\infty}} + \|v\|_{\mathbb{H}^0_{p,2}} \le c(p,T)\left(\|F\|_{\mathbb{H}^0_p} + \|G\|_{L^p(\Omega,\mathcal{F}_T,H^n_p)}\right). \tag{3.6}$$

It is known (see [16]) that the Banach space $L^p(\mathbb{R}^d)$ admits a Schauder basis for $p \in (1,\infty)$. Let $\{e_i : i = 1,2,3,\dots\}$ be a Schauder basis of $L^p(\mathbb{R}^d)$. Then there exists an $M \in (0, \infty)$ and a unique sequence bounded linear functional a_i such that for any $h \in L^p(\mathbb{R}^d)$, we have

$$\sup_{j\geq 1}\left\|\sum_{i=1}^{j} a_i(h)e_i\right\|_p \leq M\|h\|_p \quad \text{and} \quad \lim_{j\to\infty}\left\|h-\sum_{i=1}^{j} a_i(h)e_i\right\|_p = 0.$$

In particular, for convenient discussion, we consider $e_i(x)$ to be finite for every $x \in \mathbb{R}^d$ and $i = 1, 2, 3, \ldots$.

For each positive integer k, denote $\vec{G}_k := (G_{k1}, \ldots, G_{kk})^{\mathcal{T}}$ and $\vec{F}_k := (F_{k1}, \ldots, F_{kk})^{\mathcal{T}}$, with $G_{ki} = a_i(G)$ and $F_{ki} = a_i(F(t, \cdot))$, $i = 1, 2, \ldots, k$. By [5], there exist uniquely $U_k := (U_{k1}, \ldots, U_{kk})^{\mathcal{T}} \in \mathcal{S}^p(\mathbb{R}^k)$ and a $\mathcal{P}$-measurable process $V^l := (V^l_{ki}) \in L^p(\Omega, L^2([0, T], \mathbb{R}^{k\bigotimes m}))$ which solve the k-dimensional vector-valued BSDE:

$$U_k(t) = \vec{G}_k + \int_t^T \vec{F}_k(s)\,ds - \int_t^T V^l_k(s)\,dW^l_s, \quad t \in [0, T]. \tag{3.7}$$

Define

$$\begin{aligned} u_k &:= \sum_{i=1}^{k} U_{ki}e_i, \quad v^l_k := \sum_{i=1}^{k} V^l_{ki}e_i, \\ G_k &:= \sum_{i=1}^{k} G_{ki}e_i, \quad \text{and} \quad F_k := \sum_{i=1}^{k} F_{ki}e_i. \end{aligned} \tag{3.8}$$

We check that the pair (u_k, v_k) solves the Banach space-valued BSDE (3.1) with $(F, G) := (F_k, G_k)$ in the sense of Definition 3.1. Moreover, for any $x \in \mathbb{R}^d$, the pair $(u_k(\cdot, x), v(\cdot, x))$ solves the scalar valued BSDE

$$\begin{cases} -du_k(t, x) = F_k(t, x)\,dt - v^l_k(t, x)\,dW^l_t, & t \in [0, T], \\ u_k(T, x) = G_k(x), \end{cases} \tag{3.9}$$

and satisfies the following estimate

$$\begin{aligned} &E\Big[\sup_{t\leq T}|u_k(t, x)|^p\Big] + E\left[\int_0^T |v_k(t, x)|^2\,dt\right]^{\frac{p}{2}} \\ &\quad \leq C\left\{E\big[|G_k(x)|^p\big] + E\left[\int_0^T |F_k(t, x)|^p\,dt\right]\right\} \end{aligned} \tag{3.10}$$

where $C = C(T, p)$ does not depend on k since the constant in the BDG inequality is universal and does not depend on the dimension of the range space of the underlying local martingale. Integrating both sides of the last inequality on $\mathbb{R}^d$ and then applying the Fubini theorem, we get the pair $(u_k, v_k) \in \mathbb{H}^0_{p,\infty} \times \mathbb{H}^0_{p,2}$ satisfies the following inequality

$$\|u_k\|_{\mathbb{H}^0_{p,\infty}} + \|v_k\|_{\mathbb{H}^0_{p,2}} \leq c(p, T)\{\|F_k\|_{\mathbb{H}^0_p} + \|G_k\|_{L^p(\Omega,\mathcal{F}_T,H^0_p)}\}. \tag{3.11}$$

On the other hand, as $\|F_k(\omega,t)-F(\omega,t)\|_p \to 0$ and $\|F_k(\omega,t)-F(\omega,t)\| \le (M+1)\|F(\omega,t)\|_p$ for $(\omega,t)\in\Omega\times[0,T]$, a.e., by using the dominated convergence theorem we have $F_k \to F$ strongly in $\mathbb{H}^0_p$ as $k\to\infty$. Similarly, $G_k \to G$ strongly in $L^p(\Omega,\mathcal{F}_T,H^0_p)$ as $k\to\infty$. Hence, there exists $(u,v)\in\mathbb{H}^0_{p,\infty}\times\mathbb{H}^0_{p,2}$ such that it is the strong limit of the sequence $\{(u_k,v_k)\}$ in $\mathbb{H}^0_{p,\infty}\times\mathbb{H}^0_{p,2}$ as $k\to\infty$, and satisfies the estimate (3.6).

Furthermore, in view of (3.7) and (3.8), we conclude that, for any $\phi\in L^{p/(p-1)}(\mathbb{R}^d)$ the equality

$$(u_k(\tau,\cdot),\phi) = (G_k,\phi) + \int_\tau^T (F_k(s,\cdot),\phi)\,ds - \int_\tau^T (v_k^l(s,\cdot),\phi)\,dW_s^l \tag{3.12}$$

holds for all $\tau\in[0,T)$ with probability 1. Since

$$\begin{aligned}
&E\left[\int_0^T\left|\int_\tau^T (v_k^l - v^l(s,\cdot),\phi)\,dW_s^l\right| d\tau\right] \\
&\quad\le CE\left[\int_0^T\left(\int_\tau^T |(v_k - v(s,\cdot),\phi)|^2\,ds\right)^{1/2} d\tau\right] \\
&\quad\le C(T)E\left[\left(\int_0^T\left(\int_{\mathbb{R}^d}|(v_k-v)(s,x)\phi(x)|\,dx\right)^2 ds\right)^{1/2}\right] \\
&\qquad\text{(using Minkowski inequality)} \\
&\quad\le C(T)E\left[\int_{\mathbb{R}^d}\left(\int_0^T |(v_k-v)(s,x)\phi(x)|^2\,ds\right)^{1/2} dx\right] \\
&\quad\le C(T)\|v_k - v\|_{\mathbb{H}^0_{p,2}}\|\phi\|_{p/(p-1)} \to 0 \quad \text{as } k\to\infty, \\
&E\left[\int_0^T |(u_k(\tau,\cdot)-u(\tau,\cdot),\phi)|\,d\tau\right] \\
&\quad= E\left[\int_0^T\int_{\mathbb{R}^d}|(u_k-u)(s,x)\phi(x)|\,dx\,ds\right] \\
&\quad\le T\|u_k-u\|_{\mathbb{H}^0_{p,\infty}}\|\phi\|_{p/(p-1)} \to 0 \quad \text{as } k\to\infty,
\end{aligned}$$

and

$$E\left[\int_0^T |(G_k - G,\phi)|\,ds\right] + E\left[\int_0^T\left|\int_\tau^T ((F_k-F)(s,\cdot),\phi)\,ds\right| d\tau\right] \to 0 \quad \text{as } k\to\infty,$$

taking limits in $L^1(\Omega\times[0,T],\mathcal{F}_T\times\mathcal{B}(\mathbb{R}^d))$ on both sides of (3.12), we conclude that (3.4) holds almost everywhere in $[0,T]\times\Omega$.

Since, for any $\phi \in L^{p/(p-1)}(\mathbb{R}^d)$, (3.12) holds for all $\tau \le T$ with probability 1, the process $\{(u_k(t,\cdot),\phi), t\in[0,T]\}$ is continuous (a.s.). As

$$E \sup_{0\le t\le T} \|u-u_k\|^p_{L^p(\mathbb{R}^d)} \le \|u-u_k\|^p_{\mathbb{H}^0_{p,\infty}} \to 0,$$

the process $\{(u(t,\cdot),\phi), t\in[0,T]\}$ is continuous. This implies that, for any $\phi \in L^{p/(p-1)}(\mathbb{R}^d)$, equality (3.4) holds not only in $[0,T]\times\Omega$ almost everywhere but also for all $\tau\le T$ almost surely.

Besides, since $u_k \in C([0,T],L^p(\mathbb{R}^d))$(a.s.) and $E\sup_{0\le t\le T}\|u-u_k\|^p_{L^p(\mathbb{R}^d)} \to 0$ *as* $k\to\infty$, we have $u\in C([0,T],L^p(\mathbb{R}^d))$(a.s.). We complete the proof of the lemma. □

Remark 3.2 In view of Lemma 2.2, we can approximate in $\mathbb{H}^0_p \times L^p(\Omega,\mathcal{F}_T,H^0_p)$ for $p\in(1,2]$ during the proof (F,G) by a sequence (F_k,G_k) belonging to $\mathbb{H}^0_2\times L^2(\Omega,\mathcal{F}_T,H^0_2)$. Moreover, we can assume that $(F_k,G_k)(\omega,t)$ is uniformly compactly supported in $\mathbb{R}^d$ for $(\omega,t)\in\Omega\times[0,T]$ a.e. After finite-dimensional approximation of (F_k,G_k) in $\mathbb{H}^0_2\times L^2(\Omega,\mathcal{F}_T,H^0_2)$ where a Hilbert basis is a Schauder basis, the rest of our proof goes in a standard way (c.f. [5]) for $p\in(1,2]$, while not for $p\in(2,\infty)$.

Lemma 3.2 *Let* $(u,v)\in\mathbb{H}^n_{p,\infty}\times\mathbb{H}^n_{p,2}$ *be a solution of* (3.1) *for given* $F\in\mathbb{H}^n_p$ *and* $G=0$. *Then for any* $\varepsilon>0$, *there exists a positive constant* $c=c(p,T,\varepsilon)<\infty$ *such that*

$$\|v\|_{\mathbb{H}^n_{p,2}(t)} \le c(p,T,\varepsilon)\|u\|_{\mathbb{H}^n_p(t)} + \varepsilon\|F\|_{\mathbb{H}^n_p(t)}, \quad t\in[0,T]. \tag{3.13}$$

Proof First consider the case of $n=0$. Without loss of generality, we assume that the Brownian motion is one-dimensional.

Consider the approximation sequence $\{(u_k,v_k)\}$ defined in the proof of Lemma 3.1. For any fixed $x\in\mathbb{R}^d$ the pair $(u_k(\cdot,x),v_k(\cdot,x))$ solves the following scalar valued BSDE

$$\begin{cases} -du_k(t,x) = F_k(t,x)\,dt - v_k(t,x)\,dW_t, & t\in[0,T],\\ u_k(T,x)=0, \end{cases}$$

and satisfies the following inequality (see (3.9) and (3.10)).

$$E\Big[\sup_{t\le T}|u_k(t,x)|^p\Big] + E\left[\int_0^T |v_k(t,x)|^2\,dt\right]^{\frac{p}{2}} \le C(p,T)E\left[\int_0^T |F_k(t,x)|^p\,dt\right].$$

For each $\eta\in[0,T]$, we have by Itô's formula

$$\begin{aligned} &|u_k(\eta,x)|^2 + \int_\eta^T |v_k(r,x)|^2\,dr \\ &\quad = |u_k(T,x)|^2 + 2\int_\eta^T u_k(r,x)F_k(r,x)\,dr \\ &\qquad - 2\int_\eta^T u_k(r,x)v_k(r,x)\,dW_r, \quad \text{a.s.} \end{aligned}$$

Therefore,

$$\begin{aligned}&\left(\int_\eta^T |v_k(r,x)|^2\,dr\right)^{p/2}\\&\quad\le c(p)\left(\sup_{t\in[\eta,T]}|u_k(t,x)|^p+\left[\int_\eta^T |u_k(r,x)F_k(r,x)|\,dr\right]^{p/2}\right.\\&\qquad\left.+\left|\int_\eta^T u_k(r,x)v_k(r,x)\,dW_r\right|^{p/2}\right).\end{aligned}$$

Noting by the BDG inequality that

$$\begin{aligned}&E\left[\left|\int_\eta^T u_k(r,x)v_k(r,x)\,dW_r\right|^{p/2}\right]\\&\quad\le c_1(p)E\left[\left(\int_\eta^T |u_k(r,x)v_k(r,x)|^2\,dr\right)^{p/4}\right]\\&\quad\le c_1(p)E\left[\sup_{t\in[\eta,T]}|u_k(t,x)|^{p/2}\left(\int_\eta^T |v_k(r,x)|^2\,dr\right)^{p/4}\right],\end{aligned}$$

we have

$$\begin{aligned}&E\left[\left(\int_\eta^T |v_k(r,x)|^2\,dr\right)^{p/2}\right]\\&\quad\le c(p)E\left[\sup_{t\in[\eta,T]}|u_k(t,x)|^p+\left(\int_\eta^T|u_k(r,x)F_k(r,x)|\,dr\right)^{p/2}\right.\\&\qquad\left.+\left|\int_\eta^T u_k(r,x)v_k(r,x)\,dW_r\right|^{p/2}\right]\\&\quad\le c(p)E\left[\sup_{t\in[\eta,T]}|u_k(t,x)|^p+\left(\int_\eta^T |u_k(r,x)F_k(r,x)|\,dr\right)^{p/2}\right]\\&\qquad+c_1(p)E\left[\sup_{t\in[\eta,T]}|u_k(t,x)|^{p/2}\left(\int_\eta^T |v_k(r,x)|^2\,dr\right)^{p/4}\right]\\&\quad\le c(p)E\left[\sup_{t\in[\eta,T]}|u_k(t,x)|^p+\left(\int_\eta^T |u_k(r,x)F_k(r,x)|\,dr\right)^{p/2}\right]\\&\qquad+\frac{1}{2}E\left[\left(\int_\eta^T |v_k(r,x)|^2\,dr\right)^{p/2}\right],\end{aligned}$$

and, for any $\varepsilon_1 > 0$, there is $c = c(p, \varepsilon_1, T) > 0$ such that

$$
\begin{aligned}
&E\left[\left(\int_\eta^T |v_k(r,x)|^2\,dr\right)^{p/2}\right] \\
&\quad \le c(p)E\left[\sup_{t\in[\eta,T]}|u_k(t,x)|^p + \left(\int_\eta^T |u_k(r,x)F_k(r,x)|\,dr\right)^{p/2}\right] \\
&\quad \le c(p)E\left[\sup_{t\in[\eta,T]}|u_k(t,x)|^p + \sup_{t\in[\eta,T]}|u_k(t,x)|^{p/2}\left(\int_\eta^T |F_k(r,x)|\,dr\right)^{p/2}\right] \\
&\quad \le c(p,\varepsilon_1,T)E\Big[\sup_{t\in[\eta,T]}|u_k(t,x)|^p\Big] + \varepsilon_1 E\left[\int_\eta^T |F_k(r,x)|^p\,dr\right]. \qquad (3.14)
\end{aligned}
$$

On the other hand, using Corollary 2.3 of Briand et al. [5], we have almost surely

$$
\begin{aligned}
&|u_k(t,x)|^p + c_0(p)\int_t^T |u_k(s,x)|^{p-2}\mathbb{1}_{\{u_k(s,x)\neq 0\}}|v_k(s,x)|^2\,ds \\
&\quad \le p\int_t^T |u_k(s,x)|^{p-1}|F_k(s,x)|\,ds \\
&\qquad - p\int_t^T |u_k(s,x)|^{p-2}u_k(s,x)v_k(s,x)\,dW_s, \quad t\in[0,T] \qquad (3.15)
\end{aligned}
$$

where $c_0(p) = p[(p-1)\wedge 1]/2$.

As $(u_k, v_k) \in \mathbb{H}^0_{p,\infty} \times \mathbb{H}^0_{p,2}$, from the preceding inequality, we have almost surely

$$
\int_t^T |u_k(s,x)|^{p-2}\mathbb{1}_{\{u_k(s,x)\neq 0\}}|v_k(s,x)|^2\,ds < \infty, \quad t\in[0,T],
$$

and further,

$$
\begin{aligned}
&c_0(p)E\left[\int_t^T |u_k(s,x)|^{p-2}\mathbb{1}_{\{u_k(s,x)\neq 0\}}|v_k(s,x)|^2\,ds\right] \\
&\quad \le pE\left[\int_t^T |u_k(s,x)|^{p-1}|F_k(s,x)|\,ds\right], \quad t\in[0,T]. \qquad (3.16)
\end{aligned}
$$

From (3.15) and (3.16), using the BDG inequality we have

$$
\begin{aligned}
&E\Big[\sup_{s\in[t,T]}|u_k(s,x)|^p\Big] \\
&\quad \le pE\left[\int_t^T |u_k(s,x)|^{p-1}|F_k(s,x)|\,ds\right] \\
&\qquad + pE\left[\left|\int_t^T |u_k(s,x)|^{p-2}u_k(s,x)v_k(s,x)\,dW_s\right|\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ pE\left[\sup_{r\in[t,T]}\left|\int_t^r |u_k(s,x)|^{p-2}u_k(s,x)v_k(s,x)\,dW_s\right|\right]\\
&\le pE\left[\int_t^T |u_k(s,x)|^{p-1}|F_k(s,x)|\,ds\right]\\
&\quad + c(p)E\left[\left(\int_t^T (|u_k(s,x)|^{p-1}|v_k(s,x)|)^2\,ds\right)^{1/2}\right]\\
&\le pE\left[\int_t^T |u_k(s,x)|^{p-1}|F_k(s,x)|\,ds\right]\\
&\quad + c(p)E\left[\sup_{s\in[t,T]}|u_k(s,x)|^{p/2}\right.\\
&\quad \times \left.\left(\int_t^T |u_k(s,x)|^{p-2}\mathbb{1}_{\{u_k(s,x)\neq 0\}}|v_k(s,x)|^2\,ds\right)^{1/2}\right]\\
&\le pE\left[\int_t^T |u_k(s,x)|^{p-1}|F_k(s,x)|\,ds\right] + \frac{1}{2}E\Big[\sup_{s\in[t,T]}|u_k(s,x)|^p\Big]\\
&\quad + \frac{c(p)^2}{2}E\left[\int_t^T |u_k(s,x)|^{p-2}\mathbb{1}_{\{u_k(s,x)\neq 0\}}|v_k(s,x)|^2\,ds\right]\\
&\le c'(p)E\left[\int_t^T |u_k(s,x)|^{p-1}|F_k(s,x)|\,ds\right] + \frac{1}{2}E\Big[\sup_{s\in[t,T]}|u_k(s,x)|^p\Big].
\end{aligned}
$$

Thus, for any $\varepsilon_2 > 0$, we have

$$
\begin{aligned}
&E\Big[\sup_{s\in[t,T]}|u_k(s,x)|^p\Big]\\
&\quad\le 2c'(p)E\left[\int_t^T |u_k(s,x)|^{p-1}|F_k(s,x)|\,ds\right]\\
&\quad\le c(p,\varepsilon_2)E\left[\int_t^T |u_k(s,x)|^p\,ds\right] + \varepsilon_2 E\left[\int_t^T |F_k(s,x)|^p\,ds\right]. \qquad (3.17)
\end{aligned}
$$

Combining the last inequality with (3.14), and letting ε_1 and ε_2 be small enough such that $\varepsilon_2 c(p,\varepsilon_1,T)+\varepsilon_1 < \varepsilon$, we get

$$
\begin{aligned}
&E\left[\left(\int_t^T |v_k(r,x)|^2\,dr\right)^{p/2}\right]\\
&\qquad\le CE\left[\int_t^T |u_k(s,x)|^p\,ds\right] + \varepsilon E\left[\int_t^T |F_k(s,x)|^p\,ds\right]. \qquad (3.18)
\end{aligned}
$$

Here the constant $C = C(p,T,\varepsilon)$ is independent of k.

Now, integrating on $\mathbb{R}^d$ both sides of the preceding inequality and letting $k \to \infty$, we get (3.13) for $n = 0$. The general case can be proved by induction. The proof is complete. □

Remark 3.3 The arguments in the proof of Lemma 3.2 are more or less standard (see pp. 115–118 of [5]).

4 A Stochastic Banach Space

In this section we shall define a stochastic Banach space which will play a crucial role in L^p theory of BSPDEs.

Definition 4.1 For $n \in \mathbb{R}$, $p \in (1, \infty)$ and a $\mathcal{D}'$-valued function $u \in \mathbb{H}_p^n$, we say $u \in \mathcal{H}_p^n$ if $u_{xx} \in \mathbb{H}_p^{n-2}$, $u(T, \cdot) \in L^p(\Omega, \mathcal{F}_T, B_{p,p}^{n-2/p})$, and there exists $(F, v) \in \mathbb{H}_p^{n-2} \times \mathbb{H}_p^{n-1}$ such that, $\forall \phi \in C_c^\infty$, the following equality

$$(u(t,\cdot), \phi) = (u(T,\cdot), \phi) + \int_t^T (F(s,\cdot), \phi)\, ds - \sum_{r=1}^m \int_t^T (v^k(s,\cdot), \phi)\, dW_s^k, \quad (4.1)$$

holds for all $t \le T$ with probability 1.

Define $\mathcal{H}_{p,0}^n := \mathcal{H}_p^n \cap \{u : u(T,\cdot) = 0\}$, and for $u \in \mathcal{H}_p^n$

$$\|u\|_{\mathcal{H}_p^n} := \|u_{xx}\|_{\mathbb{H}_p^{n-2}} + \|F\|_{\mathbb{H}_p^{n-2}} + \|v_x\|_{\mathbb{H}_p^{n-2}} + \left(E\|u(T,\cdot)\|_{B_{p,p}^{n-2/p}}^p \right)^{\frac{1}{p}}. \quad (4.2)$$

Remark 4.1 By Lemma 2.1, there holds $B_{p,p}^{n-2/p} \hookrightarrow B_{p,\min\{2,p\}}^{n-2} \hookrightarrow H_p^{n-2}$ for $p \in (1, \infty)$. Thus, $L^p(\Omega, \mathcal{F}_T, B_{p,p}^{n-2/p})$ is continuously embedded into $L^p(\Omega, \mathcal{F}_T, H_p^{n-2})$. If $u \in \mathcal{H}_p^n$, it follows from Lemma 3.1 that $u \in \mathbb{H}_{p,\infty}^{n-2}$, $v \in \mathbb{H}_{p,2}^{n-2}$ and

$$\begin{aligned}
&\left(E \sup_{t \le T} \|u(t,\cdot)\|_{H_p^{n-2}}^p \right)^{1/p} + \|v\|_{\mathbb{H}_{p,2}^{n-2}} \\
&\quad \le \|u\|_{\mathbb{H}_{p,\infty}^{n-2}} + \|v\|_{\mathbb{H}_{p,2}^{n-2}} \\
&\quad \le c(p,T) \left(\|F\|_{\mathbb{H}_p^{n-2}} + \|u(T,\cdot)\|_{L^p(\Omega, \mathcal{F}_T, H_p^{n-2})} \right) \\
&\quad \le c(p,T) \left(\|F\|_{\mathbb{H}_p^{n-2}} + \|u(T,\cdot)\|_{L^p(\Omega, \mathcal{F}_T, B_{p,p}^{n-2/p})} \right).
\end{aligned}$$

Remark 4.2 From Remarks 4.1 and 3.1, the fact that $u \in \mathcal{H}_p^n$ implies, in some sense $\{u(t,x)\}_{0 \le t \le T}$ is a semi-martingale of drift $F(t,x)_{0 \le t \le T}$ and diffusion $v(t,x)_{0 \le t \le T}$. Further, by Lemma 2.2 and the estimates in Remark 4.1, Doob-Meyer decomposition theorem implies the uniqueness of (F, v). Therefore, the norm (4.2) is well defined.

Without confusions, we shall always say that F and v are the drift term and diffusion term of u respectively. In the following, we denote the diffusion term v of u by $\mathbb{D}u$.

On the other hand, it is worth noting that the elements of $\mathcal{H}_p^n$ are assumed to be defined for all (ω, t) and take values in $\mathcal{D}'$, and that $\mathcal{H}_p^n$ is a normed linear space in which we identify two elements u_1 and u_2 if $\|u_1 - u_2\|_{\mathcal{H}_p^n} = 0$. In view of Definition 4.1, for any $p, q \in (1, \infty)$ and $n, r \in \mathbb{R}$, if $u \in \mathcal{H}_p^n$ and $\|u\|_{\mathcal{H}_q^r} < \infty$, one can check that $u \in \mathcal{H}_q^r$ and that, in particular, $\|u\|_{\mathcal{H}_p^n} = 0$ implies $\|u\|_{\mathcal{H}_q^r} = 0$.

Theorem 4.1 *The spaces $\mathcal{H}_p^n$ and $\mathcal{H}_{p,0}^n$ equipped with norm (4.2) are Banach spaces. Moreover, we have*

$$\|u\|_{\mathbb{H}_p^n} \le C(p,T)\|u\|_{\mathcal{H}_p^n}, \qquad E\Big[\sup_{t\le T}\|u(t,\cdot)\|_{H_p^{n-2}}^p\Big] \le C(p,T)\|u\|_{\mathcal{H}_p^n}^p. \tag{4.3}$$

Proof The second inequality of (4.3) is given in Remark 4.1. Since

$$\begin{aligned}\|u\|_{\mathbb{H}_p^n} &= \|(1-\Delta)u\|_{\mathbb{H}_p^{n-2}} \\ &\le \|u\|_{\mathbb{H}_p^{n-2}} + \|u\|_{\mathcal{H}_p^n} \\ &\le T^{1/p}\Big(E\Big[\sup_{t\le T}\|u(t,\cdot)\|_{H_p^{n-2}}^p\Big]\Big)^{1/p} + \|u\|_{\mathcal{H}_p^n},\end{aligned}$$

we have the first inequality of (4.3).

It remains for us to show the completeness of $\mathcal{H}_p^n$. Let $\{u_j\}$ be a Cauchy sequence in $\mathcal{H}_p^n$. Then it is also a Cauchy sequence in $\mathbb{H}_p^n$, and there exists $u \in \mathbb{H}_p^n$ such that

$$\lim_{j\to\infty}\|u - u_j\|_{\mathbb{H}_p^n} = 0.$$

Furthermore, $\{u_{jxx}\}$ is a Cauchy sequence in $\mathbb{H}_p^{n-2}$ and

$$\lim_{j\to\infty}\|u_{jxx} - u_{xx}\|_{\mathbb{H}_p^{n-2}} = 0.$$

For $u_j(T)$, F_j, and the corresponding u_j, there exist $u(T) \in L^p(\Omega, \mathcal{F}_T, B_{p,p}^{n-2/p}) \subset L^p(\Omega, \mathcal{F}_T, H_p^{n-2})$ and $F \in \mathbb{H}_p^{n-2}$ such that

$$\begin{aligned}&\lim_{j\to\infty}\|u(T) - u_j(T)\|_{L^p(\Omega,\mathcal{F}_T,B_{p,p}^{n-2/p})} = 0, \\ &\lim_{j\to\infty}\|u(T) - u_j(T)\|_{L^p(\Omega,\mathcal{F}_T,H_p^{n-2})} = 0,\end{aligned}$$

and

$$\lim_{j\to\infty}\|F - F_j\|_{\mathbb{H}_p^{n-2}} = 0.$$

Let v_j be the diffusion term of u_j. Using the argument from Remark 4.1, we conclude that there is $v \in \mathbb{H}_p^{n-1} \cap \mathbb{H}_{p,2}^{n-2}$ such that

$$\lim_{j\to\infty}\|v_x - (v_j)_x\|_{\mathbb{H}_p^{n-2}} = 0 \quad \text{and} \quad \lim_{j\to\infty}\|v - v_j\|_{\mathbb{H}_{p,2}^{n-2}} = 0.$$

Since for any $\phi \in C_c^\infty$ the equality

$$(u_j(t,\cdot),\phi) = (u_j(T,\cdot),\phi) + \int_t^T (F_j(s,\cdot),\phi)\,ds - \sum_{k=1}^m \int_t^T (v_j^k(s,\cdot),\phi)\,dW_s^k \quad (4.4)$$

holds for all $t \le T$ with probability 1, by taking on both sides limits in $L^1([0,T] \times \Omega, \mathcal{F}_T \times \mathcal{B}(\mathbb{R}^d))$, we show that for any $\phi \in C_c^\infty$ equality (4.1) holds in $[0,T] \times \Omega$ almost everywhere.

Furthermore, (4.3) implies that for u (at least for a modification of u), we have

$$\lim_{j\to\infty} E\Big[\sup_{t\le T} \|u(t,\cdot) - u_j(t,\cdot)\|^p_{H_p^{n-2}}\Big] = 0.$$

Since the processes $\{(u_j(t,\cdot),\phi), t \in [0,T]\}$, $j = 1,2,\dots$ are all continuous, it follows that $\{(u(t,\cdot),\phi), t \in [0,T]\}$ is continuous. Therefore, for any $\phi \in C_c^\infty$, equality (4.3) not only holds in $[0,T] \times \Omega$ almost everywhere but also for all $\tau \le T$ almost surely. Hence, $u \in \mathcal{H}_p^n$ and u_j converges to u in $\mathcal{H}_p^n$. So, $\mathcal{H}_p^n$ is a Banach space.

In a similar way, we can check the completeness of $\mathcal{H}_{p,0}^n$. The proof is complete. □

Remark 4.3 The estimate (4.3) can be verified for $u\mathbb{1}_{(t,T]}$, $t \in [0,T)$. Especially, we have

$$E \sup_{s\in(t,T]} \|u(s,\cdot)\|^p_{H_p^{n-2}} \le C(p,T)\|u\|^p_{\mathcal{H}_p^n(t)}$$

with $\|u\|_{\mathcal{H}_p^n(t)} := \|u\mathbb{1}_{(t,T]}\|_{\mathcal{H}_p^n}$.

Now, we show an embedding result about the stochastic Banach space $\mathcal{H}_p^n$.

Proposition 4.2 *For $u \in \mathcal{H}_p^n$ and $v = \mathbb{D}u$, the following assertions hold*:

(i) *If $\beta := n - d/p > 0$, then $u \in L^p((0,T], \mathcal{P}, \mathcal{C}^\beta(\mathbb{R}^d))$ satisfying*

$$E\left[\int_0^T \|u(t,\cdot)\|^p_{\mathcal{C}^\beta(\mathbb{R}^d)}\,dt\right] \le C(n,d,p)\|u\|^p_{\mathbb{H}_p^n} \le C(T,n,d,p)\|u\|^p_{\mathcal{H}_p^n},$$

where $\mathcal{C}^\beta(\mathbb{R}^d)$ is the Zygmund space which is different from the ordinary Hölder spaces $C^\beta(\mathbb{R}^d)$ only if β is an integer. In particular, if $p \in (1,2]$ and $\beta > 1$, we also have

$$E\left[\int_0^T \|v(t,\cdot)\|^p_{\mathcal{C}^{\beta-1}(\mathbb{R}^d)}\,dt\right] \le C(T,n,d,p)\|u\|^p_{\mathcal{H}_p^n}.$$

(ii) *If $n > l$ and $n - d/p = l - d/q$, then*

$$E\left[\int_0^T \|u(t,\cdot)\|^p_{l,q}\,dt\right] \le C(l,n,d,p)\|u\|^p_{\mathbb{H}_p^n} \le C(T,l,n,d,p)\|u\|^p_{\mathcal{H}_p^n}.$$

In particular, if $p \in (1,2]$, *we also have*

$$E\left[\int_0^T \|v(t,\cdot)\|_{l-1,q}^p\,dt\right] \le C(T,l,n,d,p)\|u\|_{\mathcal{H}_p^n}^p.$$

(iii) *If* $q \ge p$ *and* $\theta \in (0,1)$, *then for*

$$n \ge l - \frac{d}{q} + \frac{d}{p} + 2(1-\theta),$$

we have $u \in L^{p/\theta}((0,T], H_q^l)$ *(a.s.) and*

$$E\left[\left(\int_0^T \|u(t,\cdot)\|_{l,q}^{p/\theta}\,dt\right)^\theta\right] \le C(T,n,l,q,d,p,\theta)\|u\|_{\mathcal{H}_p^n}^p.$$

In particular, if

$$q > p \quad \text{and} \quad n \ge l + \frac{d}{p} + \frac{2q-2p-d}{q},$$

by taking $\theta = pq^{-1}$, *we have*

$$E\left[\left(\int_0^T \|u(t,\cdot)\|_{l,q}^q\,dt\right)^{p/q}\right] \le C(T,n,l,q,d,p)\|u\|_{\mathcal{H}_p^n}^p.$$

Proof By Lemma 3.1 and Theorem 4.1, the assertions (i) and (ii) are straightforward in view of the classical Sobolev embedding theorems, which say that under conditions in (i) and (ii), we have $H_p^n \subset \mathcal{C}^\beta(\mathbb{R}^d)$ and $H_p^n \subset H_q^l$, respectively. On the other hand, from the Sobolev embedding theorems, we get

$$\|f\|_{l,q} \le C(l,d,q,p)\|f\|_{l+d/p-d/q,p} \le C(l,d,q,p,\theta)\|f\|_{n'-2,p}^{1-\theta}\|f\|_{n',p}^\theta,$$

where $n' := l + d/p - d/q + 2(1-\theta) \le n$. Hence,

$$\begin{aligned}
E\left[\left(\int_0^T \|u(t,\cdot)\|_{l,q}^{p/\theta}\,dt\right)^\theta\right] &\le CE\left[\left(\int_0^T \|u(t,\cdot)\|_{n'-2,p}^{(1-\theta)p/\theta}\|u(t,\cdot)\|_{n',p}^p\,dt\right)^\theta\right]\\
&\le CE\left[\left(\int_0^T \|u(t,\cdot)\|_{n-2,p}^{(1-\theta)p/\theta}\|u(t,\cdot)\|_{n,p}^p\,dt\right)^\theta\right]\\
&\le CE\left[\sup_{t\le T}\|u\|_{n-2,p}^{(1-\theta)p}\left(\int_0^T \|u(t,\cdot)\|_{n,p}^p\,dt\right)^\theta\right]\\
&\le C\left(E\left[\sup_{t\le T}\|u\|_{n-2,p}^p\right] + \|u\|_{\mathbb{H}_p^n}^p\right)\\
&\le C\|u\|_{\mathcal{H}_p^n}^p.
\end{aligned}$$

The last inequality is derived from Theorem 4.1, and $C = C(T,n,l,q,d,p,\theta)$. The proof is complete. □

5 L^p Solution of BSPDEs

5.1 Assumptions and the Notion of the Solution to BSPDEs

Let $B(\mathbb{R}^d)$ be the Banach spaces of bounded and continuous functions on $\mathbb{R}^d$, $C^{|n|-1,1}(\mathbb{R}^d)$ the Banach space of all $(|n|-1)$ times continuously differentiable functions with all the (up to the $(|n|-1)$th order) partial derivatives being bounded and all the $(|n|-1)$th order partial derivatives being Lipschitz continuous on $\mathbb{R}^d$, and $C^{|n|+\gamma}(\mathbb{R}^d)$ the usual Hölder space. The space $B^{|n|+\gamma}$ of Krylov [21] is defined as follows.

$$B^{|n|+\gamma} = \begin{cases} B(\mathbb{R}^d) & \text{if } n = 0, \\ C^{|n|-1,1}(\mathbb{R}^d) & \text{if } n = \pm 1, \pm 2, \dots, \\ C^{|n|+\gamma}(\mathbb{R}^d) & \text{otherwise.} \end{cases}$$

Here, $n \in (-\infty, \infty)$, and $\gamma \in [0, 1)$ is fixed such that $\gamma = 0$ if n is an integer; $\gamma > 0$ otherwise is so small that there is no integer in $[|n|, |n| + \gamma]$.

Consider the following semi-linear BSPDE:

$$\begin{cases} -du(t,x) = [a^{ij}(t,x)u_{x^ix^j}(t,x) + \sigma^{ik}(t,x)v^k_{x^i}(t,x) + F(u,v,t,x)]\,dt \\ \qquad\qquad - v^l(t,x)dW^l_t, \quad (t,x) \in [0,T] \times \mathbb{R}^d; \\ u(T,x) = G(x), \quad x \in \mathbb{R}^d. \end{cases} \tag{5.1}$$

Here and in the following, denote

$$u_{x^ix^j} := \frac{\partial^2}{\partial x^i \partial x^j} u, \qquad u_{x^i} := \frac{\partial}{\partial x^i} u, \qquad v^k_{x^i} := \frac{\partial}{\partial x^i} v^k,$$
$$u_x := \nabla u = (u_{x^1}, \dots, u_{x^d}), \qquad u_{xx} := (u_{x^ix^j})_{1 \le i,j \le d},$$

and

$$\alpha^{ij} := \frac{1}{2} \sum_{k=1}^m \sigma^{ik} \sigma^{jk}.$$

Assumption 5.1 (Super-parabolicity) There exists a positive constant λ such that

$$[a^{ij}(t,x) - \alpha^{ij}(t,x)]\xi^i\xi^j \ge \lambda|\xi|^2 \tag{5.2}$$

holds almost surely for all $x, \xi \in \mathbb{R}^d$ and $t \in [0,T]$.

Assumption 5.2 There exists an increasing function $\kappa : [0,\infty) \to [0,\infty)$ such that $k(s) \downarrow 0$ as $s \downarrow 0$ and

$$\sum_{i,j=1}^d |a^{ij}(t,x) - a^{ij}(t,y)| + \sum_{i=1}^d \sum_{k=1}^m |\sigma^{ik}(t,x) - \sigma^{ik}(t,y)| \le \kappa(|x-y|) \tag{5.3}$$

holds almost surely for all $(t,x,y) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d$.

Assumption 5.3 The functions $a^{ij}(t,x)$ and $\sigma^{ik}(t,x)$ are real-valued $\mathscr{P}\times\mathcal{B}(\mathbb{R}^d)$-measurable, such that

$$a^{ij}(t,\cdot),\sigma^{ik}(t,\cdot)\in B^{|n|+\gamma},\quad \text{and}\quad \|a^{ij}(t,\cdot)\|_{B^{|n|+\gamma}}+\|\sigma^{ik}(t,\cdot)\|_{B^{|n|+\gamma}}\le\Lambda, \tag{5.4}$$

almost surely for $i,j=1,\dots,d$, $k=1,\dots,m$, and $t\in[0,T]$.

Assumption 5.4 $F(0,0,\cdot,\cdot)\in\mathbb{H}^n_p$. For $(u,v)\in H^{n+2}_p\times H^{n+1}_p$, $F(u,v,t,\cdot)$ is an H^n_p-valued $\mathscr{P}$-measurable process such that there is a continuous and decreasing function $\varrho:(0,\infty)\to[0,\infty)$ such that for any $\varepsilon>0$, we have

$$\begin{aligned}&\|F(u_1,v_1,t,\cdot)-F(u_2,v_2,t,\cdot)\|_{n,p}\\&\quad\le\varepsilon(\|u_1-u_2\|_{n+2,p}+\|v_1-v_2\|_{n+1,p})+\varrho(\varepsilon)(\|u_1-u_2\|_{n,p}+\|v_1-v_2\|_{n,p}),\\&\qquad u_1,u_2\in H^{n+2}_p \text{ and } v_1,v_2\in H^{n+1}_p,\end{aligned}\tag{5.5}$$

holds for any $(t,\omega)\in[0,T]\times\Omega$.

Remark 5.1 Assumption 5.4 holds for the following (u,v)-linear functional:

$$F(u,v,t,x):=b^i(t,x)u_{x^i}(t,x)+c(t,x)u(t,x)+\varsigma^k(t,x)v^k(t,x)$$

for $(u,v,t,x)\in H^{n+2}_p\times H^{n+1}_p\times[0,T]\times\mathbb{R}^d$, if all the coefficients b^i,c and ς^k satisfy Assumption 5.3 like the components of a and σ. For any $f\in C^\infty_c(\mathbb{R}^{1+d+2m})$, the following nonlinear functional:

$$F(u,v,t,x):=I_{-n}[f(I_nu,\ I_n\nabla u,\ I_nv,\ I_{n+1/2}\,v)](t,x)$$

for $(u,v,t,x)\in H^{n+2}_p\times H^{n+1}_p\times[0,T]\times\mathbb{R}^d$, also satisfies Assumption 5.4. In fact, we recall that $I_s=(1-\Delta)^{s/2}$ for $s\in\mathbb{R}$ and note that, for any $(\phi,\varphi)\in H^{n+2}_p\times H^{n+1}_p$

$$\begin{aligned}\|I_{n+1/2}\varphi\|_p&\le\|\varphi\|^{1/2}_{n,p}\|\varphi\|^{1/2}_{n+1,p}\le\epsilon\|\varphi\|_{n+2,p}+\epsilon^{-1}\|\varphi\|_{n,p},\\ \|I_n\nabla\phi\|_p&\le C(n,d)\|\phi\|_{n+1,p}\le C(n,d)\|\phi\|^{1/2}_{n,p}\|\phi\|^{1/2}_{n+2,p}\\&\le\epsilon\|\phi\|_{n+2,p}+\epsilon^{-1}C(n,p)\|\phi\|_{n,p},\quad\forall\varepsilon>0.\end{aligned}$$

Therefore, for any $u_1,u_2\in H^{n+2}_p$ and $v_1,v_2\in H^{n+1}_p$, we have

$$\begin{aligned}&\|I_{-n}[f(I_nu_1,\ I_n\nabla u_1,\ I_nv_1,\ I_{n+1/2}\,v_1)](t,\cdot)\\&\quad-I_{-n}[f(I_nu_2,\ I_n\nabla u_2,\ I_nv_2,\ I_{n+1/2}\,v_2)](t,\cdot)\|_{n,p}\\&=\|[f(I_nu_1,\ I_n\nabla u_1,\ I_nv_1,\ I_{n+1/2}\,v_1)](t,\cdot)\\&\quad-f(I_nu_2,\ I_n\nabla u_2,\ I_nv_2,\ I_{n+1/2}\,v_2)](t,\cdot)\|_p\\&\le L\left(\|I_n(u_1-u_2)\|_p+\|I_n\nabla(u_1-u_2)\|_p\right.\\&\quad\left.+\|I_n(v_1-v_2)\|_p+\|I_{n+1/2}(v_1-v_2)\|_p\right)\end{aligned}$$

$$\leq L\big\{\big[1+L(1+C(n,p))\varepsilon^{-1}\big](\|u_1-u_2\|_{n,p}+\|v_1-v_2\|_{n,p})$$
$$+L^{-1}\varepsilon\left(\|u_1-u_2\|_{n+2,p}+\|v_1-v_2\|_{n+1,p}\right)\big\}$$
$$\leq \varepsilon\left(\|u_1-u_2\|_{n+2,p}+\|v_1-v_2\|_{n+1,p}\right)$$
$$+L\big[1+L(1+C(n,p))\varepsilon^{-1}\big]\left(\|u_1-u_2\|_{n,p}+\|v_1-v_2\|_{n,p}\right),$$

with L being a positive constant depending only on f, and $\varepsilon > 0$ being arbitrary. We expect that the L^p-theory to be established in this work includes some fully nonlinear BSPDEs. In the proof of Theorem 5.5, the well-posedness of our BSPDE (5.1) only requires (5.5) to hold for some small positive number ε. Assumption 5.4 is therefore formulated to expose this fact, though it seems quite technical.

Remark 5.2 Assumption 5.4 implies that $F(u, v, t, x)$ is Lipchitz continuous with respect to $(u, v) \in H_p^{n+2} \times H_p^{n+1}$ for any $(t, \omega) \in (0, T] \times \Omega$, that is there is $C > 0$ such that

$$\|F(u_1,v_1,t,\cdot)-F(u_2,v_2,t,\cdot)\|_{n,p}$$
$$\leq C(\|u_1-u_2\|_{n+2,p}+\|v_1-v_2\|_{n+1,p}), \quad u_1,u_2\in H_p^{n+2} \text{ and } v_1,v_2\in H_p^{n+1}.$$

Further, with $\varrho \equiv 0$, it implies that F does not depend on the pair (u, v).

Definition 5.1 We call $u \in \mathcal{H}_p^{n+2}$ a solution of BSPDE (5.1) if for any $\phi \in C_c^\infty$, the equality

$$(u(\tau,\cdot),\phi)=(G,\phi)+\int_\tau^T (a^{ij}(t,\cdot)u_{x^ix^j}(t,\cdot)+\sigma^{ik}(t,\cdot)(\mathbb{D}u)^k_{x^i}(t,\cdot)$$
$$+F(u,\mathbb{D}u,t,\cdot),\phi)\,dt-\int_\tau^T ((\mathbb{D}u)^l(t,\cdot),\phi)\,dW_t^l, \tag{5.6}$$

holds for all $\tau \in [0, T]$ with probability 1. As usual, we also call $(u, \mathbb{D}u)$ a solution pair of BSPDE (5.1).

Remark 5.3 Assume that (u, v) belongs to $\mathbb{H}_p^{n+2} \times \mathbb{H}_p^{n+1}$ with $u(T,\cdot) \in L^p(\Omega, \mathcal{F}_T, B_{p,p}^{n+2-2/p})$, and further that the equality

$$(u(\tau,\cdot),\phi)=(G,\phi)+\int_\tau^T (a^{ij}(t,\cdot)u_{x^ix^j}(t,\cdot)+\sigma^{ik}(t,\cdot)v^k_{x^i}(t,\cdot)+F(u,v,t,\cdot),\phi)\,dt$$
$$-\int_\tau^T (v^l(t,\cdot),\phi)\,dW_t^l, \quad \forall (t,\phi)\in[0,T)\times C_c^\infty \tag{5.7}$$

holds with probability 1. Then by Lemma 3.1, u has a modification, still denoted by itself, such that the pair $(u, v) \in \mathbb{H}_{p,\infty}^n \times \mathbb{H}_{p,2}^n$ solves the Banach space-valued BSDE (3.1) with $F(t,\cdot) := a^{ij}(t,\cdot)u_{x^ix^j}(t,\cdot) + \sigma^{ik}(t,\cdot)v^k_{x^i}(t,\cdot) + F(u(t,\cdot), v(t,\cdot), t,\cdot), t \in [0, T]$, belonging to $\mathbb{H}_p^{n+2}$. Hence, by Lemma 3.1 for any $\phi \in C_c^\infty$, (5.7) holds for all $\tau \in [0, T]$ with probability 1. Hence $u \in \mathcal{H}_p^n$.

Note that Definition 5.1 includes as a particular case the notion of weak solution to deterministic parabolic PDEs. For example, consider the particular case:

$$\begin{cases} -\frac{\partial}{\partial t}u = \Delta u + f, \\ u(T) = u_T. \end{cases} \tag{5.8}$$

By reversing the time, we have the following proposition (see [24]).

Proposition 5.1 *For any $f \in L^p([0,T] \times \mathbb{R}^d)$, and $u_T \in B^{2-2/p}_{p,p}$ with $p \in (1,\infty)$, there exists a unique weak solution $u \in W^{1,2}_p(T)$ to (5.8) with terminal data $u(T) = u_T$. In addition,*

$$\|u\|_{W^{1,2}_p} \le C(d,p,T)(\|f\|_{L^p((0,T)\times\mathbb{R}^d)} + \|u_T\|_{B^{2-2/p}_{p,p}}), \tag{5.9}$$

where

$$\|u\|_{W^{1,2}_p} := \|u_{xx}\|_{L^p((0,T)\times\mathbb{R}^d)} + \|u_x\|_{L^p((0,T)\times\mathbb{R}^d)} + \|u\|_{L^p((0,T)\times\mathbb{R}^d)} + \|u_t\|_{L^p((0,T)\times\mathbb{R}^d)}.$$

In Proposition 5.1, the space $W^{1,2}_p$ can be replaced with $\mathcal{H}^2_p$ in an equivalent way. This fact also explains why the Besov space $B^n_{p,p}$ is used for the terminal value in Definition 4.1.

5.2 The Case of Space-Invariant Leading Coefficients

Consider the following BSPDE

$$\begin{cases} -du(t,x) = [a^{ij}(t)u_{x^ix^j}(t,x) + \sigma^{ik}(t)v^k_{x^i}(t,x) + F(t,x)]dt \\ \qquad\qquad - v^l(t,x)\,dW^l_t, \quad (t,x) \in [0,T] \times \mathbb{R}^d; \\ u(T,x) = G(x), \quad x \in \mathbb{R}^d \end{cases} \tag{5.10}$$

where $(F,G) \in \mathbb{H}^n_p \times L^p(\Omega, \mathcal{F}_T, H^{n+1}_p)$, with $p \in (1,2]$ and $n \in \mathbb{R}$.

Theorem 5.2 *Assume that the coefficients a^{ij} and σ^{il} $i,j = 1,\dots,d, l = 1,\dots,m$, are $\mathcal{P}$-measurable real-valued functions which are defined on $\Omega \times [0,T]$ and bounded by a positive constant Λ, and also that they satisfy the super-parabolicity condition 5.1. Take $(F,G) \in \mathbb{H}^n_p \times L^p(\Omega, \mathcal{F}_T, H^{n+1}_p)$, $p \in (1,2], n \in \mathbb{R}$. Then, we have*

(i) *BSPDE (5.10) has a unique solution $u \in \mathcal{H}^{n+2}_p$ and for this solution, we have*

$$\|u\|_{\mathcal{H}^{n+2}_p} \le C(T,n,d,p,\lambda,\Lambda)(\|G\|_{L^p(\Omega,\mathcal{F}_T,H^{n+1}_p)} + \|F\|_{\mathbb{H}^n_p});$$

(ii) *we have $u \in C([0,T], H^n_p)$ almost surely and*

$$\|u\|_{\mathbb{H}^n_{p,\infty}} + \|\mathbb{D}u\|_{\mathbb{H}^n_{p,2}} \le C(T,n,d,p,\lambda,\Lambda)(\|F\|_{\mathbb{H}^n_p} + \|G\|_{L^p(\Omega,\mathcal{F}_T,H^{n+1}_p)});$$

(iii) *in particular, for the case* $G \equiv 0$, *there is a constant* $C(d, p, \lambda, \Lambda)$ *which does not depend on* T, *such that*

$$\|u_{xx}\|_{\mathbb{H}_p^n} + \|(\mathbb{D}u)_x\|_{\mathbb{H}_p^n} \le C(d, p, \lambda, \Lambda)\ \|F\|_{\mathbb{H}_p^n},$$
$$\|u\|_{\mathcal{H}_p^{n+2}} \le C(d, p, \lambda, \Lambda)\ \|F\|_{\mathbb{H}_p^n}.$$

In view of Lemma 3.1 and Remark 5.3, the assertions for $p = 2$ can be deduced from [10, 14, 36], while Theorem 5.2 for $p \in (1, 2)$ seems to be new. The proof of Theorem 5.2 will appeal to a harmonic analysis result which is due to Krylov [19, Theorem 2.1].

Lemma 5.3 *Let* H *be a Hilbert space,* $p \in [2, \infty)$, $-\infty \le a < b \le \infty$, $g \in L^p((a, b) \times \mathbb{R}^d, H)$. *Then*

$$\int_{\mathbb{R}^d} \int_a^b \left[\int_a^t |\nabla T_{t-s} g(s, \cdot)(x)|_H^2 ds \right]^{p/2} dt dx$$
$$\le C(d, p) \int_{\mathbb{R}^d} \int_a^b |g(t, x)|_H^p dt dx \tag{5.11}$$

where $T_t := e^{\Delta t}$, $t \ge 0$, *is the semigroup corresponding to the heat equation* $\frac{\partial u}{\partial t} = \Delta u$ *in* $\mathbb{R}^d$.

Remark 5.4 The assertion of Lemma 5.3 is not true for $p < 2$.

We have the following more general version.

Proposition 5.4 *Let* $a^{ij}(t)$ *satisfy the strong ellipticity condition, i.e. there exist two positive constants* λ_1 *and* Λ_1 *such that*

$$\Lambda_1 |\xi|^2 \ge a^{ij}(t)\xi^i \xi^j \ge \lambda_1 |\xi|^2 \tag{5.12}$$

holds for all $\xi \in \mathbb{R}^d, t \ge 0$ *with probability* 1. *Assume that* $g \in \mathbb{H}_p^n$ *with* $p \in [2, \infty)$ *and* $n \in \mathbb{R}$. *Then, the SPDE*

$$\begin{cases} d\eta(t, x) = a^{ij}(t)\eta_{x^i x^j}(t, x)dt + g^l(t, x)dW_t^l, & (t, x) \in [0, T] \times \mathbb{R}^d, \\ \eta(0, x) = 0, & x \in \mathbb{R}^d, \end{cases} \tag{5.13}$$

has a unique solution $\eta \in \mathbb{H}_p^{n+1}$ *such that for any* $\phi \in C_c^\infty$, *the equality*

$$(\eta(\tau, \cdot), \phi) = \int_0^\tau (a^{ij}(t)\eta_{x^i x^j}(t, \cdot), \phi)dt + \int_0^\tau (g^l(t, \cdot), \phi)dW_t^l, \tag{5.14}$$

holds for all $\tau \in (0, T]$ *with probability* 1, *and there holds the following estimate*

$$\|\eta_x\|_{\mathbb{H}_p^n} \le C(d, p, \lambda_1, \Lambda_1)\|g\|_{\mathbb{H}_p^n}. \tag{5.15}$$

Proof In view of [21, Theorem 4.10], SPDE (5.13) has a unique solution. It remains to prove the estimate (5.15). It is sufficient to prove the estimate for $n = 0$, and other cases can be proved by induction.

We follow a standard procedure which is due to Krylov (for instance, see [21, Theorem 4.10, pp. 205–206]).

First, for the model case $a := (a_{ij})_{1\le i,j\le d} = I$, it can be checked that

$$\eta(t,x) = \int_0^t T_{t-r} g^l(r,x)dW_r^l \quad \text{a.s.},$$

and thus,

$$\eta_x(t,x) = \int_0^t \nabla T_{t-r} g^l(r,x)dW_r^l \quad \text{a.s.},$$

where $T_t := e^{\Delta t}$, $t \ge 0$, is the semigroup corresponding to the heat equation $\frac{\partial u}{\partial t} = \Delta u$ in $\mathbb{R}^d$. From Lemma 5.3, we get

$$\begin{aligned}\|\eta_x\|_{\mathbb{H}_p^0} &= E\int_{\mathbb{R}^d}\int_0^T \left|\int_0^t \nabla T_{t-r} g^l(s,x)dW_s^l\right|^p dtdx \\ &\le C(p)E\int_{\mathbb{R}^d}\int_0^T \left[\int_0^t |\nabla T_{t-s} g(s,\cdot)(x)|^2 ds\right]^{p/2} dtdx \\ &\le C(d,p)\|g\|_{\mathbb{H}_p^0}.\end{aligned}$$

For the general case, we can take $a \ge I$, otherwise we take a nonrandom time change. Take $\sigma(t) = \sigma^*(t) \ge 0$ as a solution of the matrix equation $\sigma^2(t) + 2I = 2a$. Furthermore, we also assume that there is a d-dimensional Wiener process $(B_t)_{t\ge 0}$ independent of $(\mathcal{F}_t)_{0\le T}$.

Then, like the model case, the equation

$$d\zeta(t,x) = \Delta\zeta(t,x)dt + g^l\left(t, x - \int_0^t \sigma(s)\,dB_s\right)dW_t^l,$$

with the zero initial condition has a unique solution $\zeta \in \mathbb{H}_p^0$ satisfying (5.14) and (5.15). Note that the predictable σ-algebra $\mathcal{P}$ is replaced by σ-algebra generated by $\mathcal{F}_t \vee \sigma(B_s; s \le t)$ here. In particular, as our norms are all translation invariant with respect to the space variable, we have

$$\|\zeta_x\|_{\mathbb{H}_p^0} \le C(d,p)\|g\|_{\mathbb{H}_p^0}.$$

From Theorem 2.3 it follows that the field $Y(t,x) := \zeta(t, x + \int_0^t \sigma(s)dB_s)$, $(t,x) \in [0,T] \times \mathbb{R}^d$ solves the SPDE

$$dY(t,x) = a^{ij}(t)Y_{x^i x^j}(t,x)\,dt + g^l(t,x)\,dW_t^l + Y_{x^i}(t,x)\sigma^{ij}(t)\,dB_t^j, \quad Y(0,x) = 0.$$

For any $\phi \in C_c^\infty$ and $t \ge 0$,

$$(\eta(t,\cdot),\phi) = E[(Y(t,\cdot),\phi)|\mathcal{F}_t] = E\left[\left(\zeta\left(t, \cdot + \int_0^t \sigma(s)dB_s\right), \phi\right)\Big|\mathcal{F}_t\right] \quad \text{a.s.}$$

Therefore,

$$(\eta_x(t,\cdot),\phi)=E\bigg[\bigg(\zeta_x\bigg(t,\cdot+\int_0^t\sigma(s)dB_s\bigg),\phi\bigg)\bigg|\mathcal{F}_t\bigg]\quad\text{a.s.}$$

As C_c^∞ is separable and dense in $H^0_{p/(p-1)}$, it follows that

$$\|\eta_x(t,\cdot)\|^p_{H^0_p}\le E\big[\|\zeta_x(t,\cdot)\|^p_{H^0_p}|\mathcal{F}_t\big]\quad\text{a.s.}$$

Hence,

$$\|\eta_x\|_{\mathbb{H}^0_p}\le\|\zeta_x\|_{\mathbb{H}^0_p}\le C(d,p)\|g\|_{\mathbb{H}^0_p}.$$

By considering the possible nonrandom time change, we get (5.15) for $n=0$. □

Proof of Theorem 5.2 Without loss of generality, assume that $m=1$.

Step 1. We use the duality method to prove assertion (i).

Consider the following SPDE:

$$\begin{cases}d\eta(t,x)=[a^{ij}(t)\eta_{x^ix^j}(t,x)+f(t,x)]dt\\ \qquad\qquad +[-\sigma^i(t)\eta_{x^i}(t,x)+g(t,x)]dW_t,\quad (t,x)\in[0,T]\times\mathbb{R}^d,\\ \eta(0,x)=0,\quad x\in\mathbb{R}^d,\end{cases}\tag{5.16}$$

where $(f,g)\in(\mathbb{H}_2^{-n-2}\cap\mathbb{H}_{p'}^{-n-2})\times(\mathbb{H}_2^{-n-1}\cap\mathbb{H}_{p'}^{-n-1})$, and $1/p+1/p'=1$. In view of [21, Theorem 4.10], SPDE (5.16) has a unique solution $u\in\mathbb{H}_q^{-n}$ such that

$$\begin{aligned}\|\eta\|_{\mathbb{H}_q^{-n}}&\le C(T,d,q,\lambda,\Lambda)(\|f\|_{\mathbb{H}_q^{-n-2}}+\|g\|_{\mathbb{H}_q^{-n-1}}),\\ \|\eta_{xx}\|_{\mathbb{H}_q^{-n-2}}&\le C(d,q,\lambda,\Lambda)(\|f\|_{\mathbb{H}_q^{-n-2}}+\|g\|_{\mathbb{H}_q^{-n-1}}),\\ E\sup_{t\in[0,T]}\|\eta(t,\cdot)\|_{H_q^{-n-1}}&\le C(T,d,q,\lambda,\Lambda)(\|f\|_{\mathbb{H}_q^{-n-2}}+\|g\|_{\mathbb{H}_q^{-n-1}})\end{aligned}\tag{5.17}$$

for $q=2,p'$. For the moment, assume that

$$(F,G)\in(\mathbb{H}_p^n\cap\mathbb{H}_2^n)\times(L^p(\Omega,\mathcal{F}_T,H_p^{n+1})\cap L^2(\Omega,\mathcal{F}_T,H_2^{n+1})).$$

For $p=2$, BSPDE (5.10) has a unique pair $(u,v)\in\mathbb{H}_2^{n+2}\times\mathbb{H}_2^{n+1}$ such that (see [36])

$$\|u\|_{\mathbb{H}_2^{n+2}}+\|v\|_{\mathbb{H}_2^{n+1}}\le C(T,d,\lambda,\Lambda)[\|F\|_{\mathbb{H}_2^n}+\|G\|_{L^2(\Omega,\mathcal{F}_T,H_2^{n+1})}],$$

and for any $\phi\in C_c^\infty$ and $\tau\in[0,T]$

$$\begin{aligned}(u(\tau,\cdot),\phi)=(G,\phi)+\int_\tau^T(a^{ij}(t)u_{x^ix^j}(t,\cdot)+\sigma^i(t)v_{x^i}(t,\cdot)+F(t,\cdot),\phi)dt\\ \times\int_\tau^T(v(t,\cdot),\phi)dW_t,\quad\text{a.s.}\end{aligned}\tag{5.18}$$

Furthermore, keeping in mind the existence of $(u, v) \in \mathbb{H}_2^{n+2} \times \mathbb{H}_2^{n+1}$, we conclude that (at least for a modification of u) for any $\phi \in C_c^\infty$, the equality (5.18) holds for all $\tau \in [0, T]$ with probability 1. From Remark 5.3, we have $u \in \mathcal{H}_2^{n+2}$.

The parallelogram rule yields the following

$$\begin{aligned}
&\int_{\mathbb{R}^d} (1-\Delta)^{\frac{n+1}{2}} u(t,x)(1-\Delta)^{-\frac{n+1}{2}} \eta(t,x)dx \\
&\quad = \frac{1}{4}\{\|(1-\Delta)^{\frac{n+1}{2}} u(t,\cdot) + (1-\Delta)^{-\frac{n+1}{2}} \eta(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} \\
&\qquad \times \|(1-\Delta)^{\frac{n+1}{2}} u(t,\cdot) - (1-\Delta)^{-\frac{n+1}{2}} \eta(t,\cdot)\|^2_{L^2(\mathbb{R}^d)}\}.
\end{aligned}$$

Applying Itô's formula to compute the square of the norm (see [23, Theorem 3.1]), we get

$$\begin{aligned}
&E\int_0^T (u(t,\cdot), f(t,\cdot)) + (v(t,\cdot), g(t,\cdot))dt \\
&\quad = (G, \eta(T,\cdot)) + E\int_0^T (F(t,\cdot), \eta(t,\cdot))dt \\
&\quad \le \|G\|_{L^p(\Omega,\mathcal{F}_T,H_p^{n+1})} \|\eta(T)\|_{L^{p'}(\Omega,\mathcal{F}_T,H_{p'}^{-n-1})} + \|F\|_{\mathbb{H}_p^n} \|\eta\|_{\mathbb{H}_{p'}^{-n}} \\
&\quad \le (\|G\|_{L^p(\Omega,\mathcal{F}_T,H_p^{n+1})} + \|F\|_{\mathbb{H}_p^n})(\|\eta(T)\|_{L^{p'}(\Omega,\mathcal{F}_T,H_{p'}^{-n-1})} + \|\eta\|_{\mathbb{H}_{p'}^{-n}}) \\
&\quad \le C(T,\lambda,\Lambda,d,p)(\|G\|_{L^p(\Omega,\mathcal{F}_T,H_p^{n+1})} + \|F\|_{\mathbb{H}_p^n})(\|f\|_{\mathbb{H}_{p'}^{-n-2}} + \|g\|_{\mathbb{H}_{p'}^{-n-1}}).
\end{aligned}$$

Note that $(F, G) \in (\mathbb{H}_p^n \cap \mathbb{H}_2^n) \times (L^p(\Omega, \mathcal{F}_T, H_p^{n+1}) \cap L^2(\Omega, \mathcal{F}_T, H_2^{n+1}))$.

For $(F, G) \in \mathbb{H}_p^n \times L^p(\Omega, \mathcal{F}_T, H_p^{n+1})$, we choose a sequence $(F^k, G^k) \in (\mathbb{H}_p^n \cap \mathbb{H}_2^n) \times (L^p(\Omega, \mathcal{F}_T, H_p^{n+1}) \cap L^2(\Omega, \mathcal{F}_T, H_2^{n+1}))$, $k = 1, 2, \ldots$, such that

$$\|F^k - F\|_{\mathbb{H}_p^n} + \|G^k - G\|_{L^p(\Omega,\mathcal{F}_T,H_p^{n+1})} \to 0 \quad \text{as } k \to \infty. \tag{5.19}$$

Denote by (u_k, v_k) the unique solution pair to BSPDE (5.10) for $(F, G) := (F^k, G^k)$. Thus,

$$\begin{aligned}
&E\int_0^T (u^k(t,\cdot), f(t,\cdot)) + (v^k(t,\cdot), g(t,\cdot))dt \\
&\quad = (G^k, \eta(T,\cdot)) + E\int_0^T (F^k(t,\cdot), \eta(t,\cdot))dt \\
&\quad \le C(T,d,p,\lambda,\Lambda)(\|G^k\|_{L^p(\Omega,\mathcal{F}_T^k,H_p^{n+1})} + \|F^k\|_{\mathbb{H}_p^n})(\|f\|_{\mathbb{H}_{p'}^{-n-2}} + \|g\|_{\mathbb{H}_{p'}^{-n-1}}),
\end{aligned}$$

where $C(T,d,p,\lambda,\Lambda)$ is independent of k. Noting that $\mathbb{H}_2^{-n-2} \cap \mathbb{H}_{p'}^{-n-2}$ and $\mathbb{H}_2^{-n-1} \cap \mathbb{H}_{p'}^{-n-1}$ are dense in $\mathbb{H}_{p'}^{-n-2}$ and $\mathbb{H}_{p'}^{-n-1}$ respectively, and that $(f, g) \in$

$(\mathbb{H}_2^{-n-2} \cap \mathbb{H}_{p'}^{-n-2}) \times (\mathbb{H}_2^{-n-1} \cap \mathbb{H}_{p'}^{-n-1})$ is arbitrary, from the last inequality, we have

$$\|u^k\|_{\mathbb{H}_p^{n+2}} + \|v^k\|_{\mathbb{H}_p^{n+1}} \le C(T, d, p, \lambda, \Lambda)[\|F^k\|_{\mathbb{H}_p^n} + \|G^k\|_{L^p(\Omega, \mathscr{F}_T^k, H_p^{n+1})}]. \quad (5.20)$$

Moreover,

$$\begin{aligned}
\|u^k\|_{\mathcal{H}_p^{n+2}} &= \|u_{xx}^k\|_{\mathbb{H}_p^n} + \|a^{ij} u_{x^i x^j}^k + \sigma^i v_{x^i}^k + F^k\|_{\mathbb{H}_p^n} \\
&\quad + \|v_x^k\|_{\mathbb{H}_p^n} + \|G^k\|_{L^p(\Omega, \mathscr{F}_T, B_{p,p}^{n+2-2/p})} \\
&\le C(n, d, p, \lambda, \Lambda)[\|u_{xx}^k\|_{\mathbb{H}_p^n} + \|v^k\|_{\mathbb{H}_p^{n+1}} + \|F^k\|_{\mathbb{H}_p^n} \\
&\quad + \|G^k\|_{L^p(\Omega, \mathscr{F}_T, H_p^{n+1})}] \\
&\le C(n, d, p, \lambda, \Lambda)[\|u^k\|_{\mathbb{H}_p^{n+2}} + \|v^k\|_{\mathbb{H}_p^{n+1}} \\
&\quad + \|F^k\|_{\mathbb{H}_p^n} + \|G^k\|_{L^p(\Omega, \mathscr{F}_T, H_p^{n+1})}] \\
&\le C(T, n, d, p, \lambda, \Lambda)[\|F^k\|_{\mathbb{H}_p^n} + \|G^k\|_{L^p(\Omega, \mathscr{F}_T, H_p^{n+1})}]
\end{aligned}$$

and this combined with $u^k \in \mathcal{H}_2^{n+2}$, implies $u^k \in \mathcal{H}_p^{n+2}$ for $k = 1, 2, 3, \ldots$.

From (5.19), (5.20) and the last inequality, it follows that u^k is a Cauchy sequence in $\mathcal{H}_p^{n+2}$. By Theorem 4.1, there exists $u \in \mathcal{H}_p^{n+2}$ such that $\|u^k - u\|_{\mathcal{H}_p^{n+2}} \to 0$, *as* $k \to \infty$, and there holds the following estimate

$$\|u\|_{\mathcal{H}_p^{n+2}} \le C(T, n, p, d, \lambda, \Lambda)[\|G\|_{L^p(\Omega, \mathscr{F}_T, H_p^{n+1})} + \|F\|_{\mathbb{H}_p^n}].$$

Denote $v := \mathbb{D}u$. It is obvious that $\|v^k - v\|_{\mathbb{H}_p^{n+1}} \to 0$, *as* $k \to \infty$. In view of Remark 4.1, one can check that $v \in \mathbb{H}_p^{n+1} \cap \mathbb{H}_{p,2}^n$. By taking limits one can check that $u \in \mathcal{H}_p^{n+2}$ is a solution of BSPDE (5.10).

Now we prove the uniqueness of the solution. Suppose that $F = 0$, $G = 0$ and $u \in \mathcal{H}_p^{n+2}$ solving BSPDE (5.10). It is sufficient to show $u = 0$, which is immediate from the last estimate with $F = 0$ and $G = 0$.

Step 2. We prove assertion (ii).

Note that $L^p(\Omega, \mathscr{F}_T, H_p^{n+1})$ is continuously embedded into $L^p(\Omega, \mathscr{F}_T, H_p^n)$. From Lemma 3.1, it follows that $u \in \mathbb{H}_{p,\infty}^n$, $v \in \mathbb{H}_{p,2}^n$, and $u \in C([0, T], H_p^n)$ almost surely. In fact, in view of Lemma 3.1 and Theorem 4.1, we have

$$\begin{aligned}
&\|u\|_{\mathbb{H}_{p,\infty}^n} + \|v\|_{\mathbb{H}_{p,2}^n} \\
&\quad \le C(T, p)(\|a^{ij} u_{x^i x^j} + \sigma^{ik} v_{x^i}^k + F\|_{\mathbb{H}_p^n} + \|G\|_{L^p(\Omega, \mathscr{F}_T, H_p^n)}) \\
&\quad \le C(T, p, \lambda, \Lambda)(\|u_{xx}\|_{\mathbb{H}_p^n} + \|v_x\|_{\mathbb{H}_p^n} + \|F\|_{\mathbb{H}_p^n} + \|G\|_{L^p(\Omega, \mathscr{F}_T, H_p^n)}) \\
&\quad \le C(T, p, \lambda, \Lambda)(\|u\|_{\mathcal{H}_p^{n+2}} + \|F\|_{\mathbb{H}_p^n} + \|G\|_{L^p(\Omega, \mathscr{F}_T, H_p^n)}) \\
&\quad \le C(T, n, d, p, \lambda, \Lambda)(\|F\|_{\mathbb{H}_p^n} + \|G\|_{L^p(\Omega, \mathscr{F}_T, H_p^{n+1})}).
\end{aligned}$$

Step 3. We prove assertion (iii) using the duality method.

Consider $G=0$. For $(f,g)\in(\mathbb{H}^{-n}_{p'}\cap\mathbb{H}^{-n}_2)\times(\mathbb{H}^{-n+1}_{p'}\cap\mathbb{H}^{-n+1}_2)$, the Hessian η_{xx} of the corresponding solution solves SPDE (5.16) with (f,g) being replaced with (f_{xx},g_{xx}). For the SPDE with (f,g), we have the following analogue to (5.17):

$$\|\eta_{xx}\|_{\mathbb{H}^{-n}_{p'}}\le C(d,p,\lambda,\Lambda)(\|f\|_{\mathbb{H}^{-n}_{p'}}+\|g\|_{\mathbb{H}^{-n+1}_{p'}}).$$

Furthermore, proceeding identically as in the proof of assertion (i), we have

$$\begin{aligned}&E\int_0^T(u_{x^ix^j}(t,\cdot),f(t,\cdot))+(v_{x^ix^j}(t,\cdot),g(t,\cdot))dt\\&\quad=E\int_0^T(u(t,\cdot),f_{x^ix^j}(t,\cdot))+(v(t,\cdot),g_{x^ix^j}(t,\cdot))dt\\&\quad=E\int_0^T(F(t,\cdot),\eta_{x^ix^j}(t,\cdot))dt\\&\quad\le\|F\|_{\mathbb{H}^n_p}\|\eta_{xx}\|_{\mathbb{H}^{-n}_{p'}}\\&\quad\le C(\lambda,\Lambda,d,p)\|F\|_{\mathbb{H}^n_p}(\|f\|_{\mathbb{H}^{-n}_{p'}}+\|g\|_{\mathbb{H}^{-n+1}_{p'}}),\quad\text{for } i,j=1,\ldots,d.\end{aligned}$$

Hence, by the arbitrariness of (f,g) and the denseness of $(\mathbb{H}^{-n}_{p'}\cap\mathbb{H}^{-n}_2)\times(\mathbb{H}^{-n+1}_{p'}\cap\mathbb{H}^{-n+1}_2)$ in $\mathbb{H}^{-n}_{p'}\times\mathbb{H}^{-n+1}_{p'}$ it follows that

$$\|u_{xx}\|_{\mathbb{H}^n_p}+\|v_{xx}\|_{\mathbb{H}^{n-1}_p}\le C(d,p,\lambda,\Lambda)\,\|F\|_{\mathbb{H}^n_p}.\tag{5.21}$$

On the other hand, let $\zeta(t,x):=u(t,x+\int_0^t\sigma(s)dW_s)$. By Theorem 2.3, we have

$$\begin{cases}-d\zeta(t,x)=[(a^{ij}(t)-\alpha^{ij}(t))\zeta(t,x)_{x^ix^j}+F(t,x+\int_0^t\sigma(s)dW_s)]dt\\\qquad\qquad-[\sigma^i\zeta_{x^i}(t,x)+v(t,x+\int_0^t\sigma(s)dW_s)]dW_t,\quad(t,x)\in[0,T]\times\mathbb{R}^d;\\\zeta(T,x)=0,\quad x\in\mathbb{R}^d.\end{cases}\tag{5.22}$$

We consider the dual SPDE

$$\begin{cases}d\psi(t,x)=(a^{ij}(t)-\alpha^{ij}(t))\psi_{x^ix^j}(t,x)\,dt\\\qquad\qquad+h(t,x)\,dW_t,\quad(t,x)\in[0,T]\times\mathbb{R}^d;\\\psi(0,x)=0,\quad x\in\mathbb{R}^d\end{cases}\tag{5.23}$$

where $h\in\mathbb{H}^{-n}_{p'}\cap\mathbb{H}^{-n}_2$, $1/p'+1/p=1$. In view of Proposition 5.4, we conclude that SPDE (5.23) has a unique solution $\psi\in\mathbb{H}^{-n+1}_{p'}$ satisfying

$$\|\psi_x\|_{\mathbb{H}^{-n}_{p'}}\le C(d,p,\lambda,\Lambda)\|h\|_{\mathbb{H}^{-n}_{p'}}.$$

Moreover, we have

$$E\int_0^T\left(\sigma^i\zeta_{x^ix^j}(t,\cdot)+v_{x^j}\left(t,\cdot+\int_0^t\sigma(s)\,dW_s\right),h(t,\cdot)\right)dt$$

$$= E\int_0^T \left(\sigma^i \zeta_{x^i}(t,\cdot) + v\left(t,\cdot+\int_0^t \sigma(s)\,dW_s\right), h_{x^j}(t,\cdot)\right) dt$$

$$= E\int_0^T \left(\psi_{x^j}(t,\cdot), F\left(t,\cdot+\int_0^t \sigma(s)\,dW_s\right)\right) dt$$

$$\le \|\psi_x\|_{\mathbb{H}_{p'}^{-n}} \left\| F\left(\cdot,\cdot+\int_0^t \sigma(s)\,dW_s(\cdot)\right)\right\|_{\mathbb{H}_p^n}$$

$$\le C(d,p,\lambda,\Lambda)\|h\|_{\mathbb{H}_{p'}^{-n}} \|F\|_{\mathbb{H}_p^n} \quad \text{for } j=1,\dots,d. \tag{5.24}$$

Since h is arbitrary and $\mathbb{H}_{p'}^{-n} \cap \mathbb{H}_2^{-n}$ is dense in $\mathbb{H}_{p'}^{-n}$, we have

$$\left\|\sigma^i \zeta_{x^i x}(\cdot,\cdot) + v_x\left(\cdot,\cdot+\int_0^t \sigma(s)\,dW_s\right)\right\|_{\mathbb{H}_p^n} \le C(d,p,\lambda,\Lambda)\|F\|_{\mathbb{H}_p^n},$$

which yields

$$\|\sigma^i u_{x^i x} + v_x\|_{\mathbb{H}_p^n} \le C(d,p,\lambda,\Lambda)\|F\|_{\mathbb{H}_p^n}.$$

Therefore,

$$\|v_x\|_{\mathbb{H}_p^n} \le \|\sigma^i u_{x^i x}\|_{\mathbb{H}_p^n} + \|\sigma^i u_{x^i x} + v_x\|_{\mathbb{H}_p^n} \le C(d,p,\lambda,\Lambda)\|F\|_{\mathbb{H}_p^n}, \tag{5.25}$$

which, combined with (5.21), implies the assertion (iii).

The proof is complete. □

Remark 5.5 Our assumptions listed in Theorem 5.2 are required by [21, Theorem 4.10] for the dual SPDE (5.16) to have a unique solution. In the whole proof, Proposition 5.4 is used only in *Step 3* to prove assertion (iii) of Theorem 5.2. In particular, using the sharp harmonic analysis result (Lemma 5.3), Krylov established the L^q-theory ([21, Theorem 4.10]) for $q \ge 2$, which implies via duality the assertions of Theorem 5.2.

Remark 5.6 If the assumptions of Theorem 5.2 are satisfied for both q_1 and q_2 instead of p, where $q_1, q_2 \in (1,2]$, then the solutions in $\mathcal{H}_{q_1}^{n+2}$ and $\mathcal{H}_{q_2}^{n+2}$ coincide. Indeed, we need only to take $(F^k, G^k) \in (\mathbb{H}_{q_1}^n \cap \mathbb{H}_2^n \cap \mathbb{H}_{q_2}^n) \times (L^{q_1}(\Omega, \mathscr{F}_T, H_{q_1}^{n+1}) \cap L^2(\Omega, \mathscr{F}_T, H_2^{n+1}) \cap L^{q_2}(\Omega, \mathscr{F}_T, H_{q_2}^{n+1}))$ during the proof of Theorem 5.2. Then the approximating solutions in $\mathcal{H}_{q_1}^{n+2}$ and $\mathcal{H}_{q_2}^{n+2}$ coincide in $\mathcal{H}_2^{n+2}$. This implies the solutions to (5.10) in $\mathcal{H}_{q_1}^{n+2}$ and $\mathcal{H}_{q_2}^{n+2}$ coincide.

Remark 5.7 For the case $p \in (2,\infty)$, consider the following BSPDE

$$\begin{cases} -du(t,x) = [a^{ij}(t)u_{x^i x^j}(t,x) + \sigma^{ik}(t)v_{x^i}(t,x) + F(t,x)]dt \\ \qquad\qquad - v^k(t,x)dW_t^k, \qquad (t,x) \in [0,T]\times\mathbb{R}^d, \\ u(T,x) = G(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{5.26}$$

and SPDE:

$$\begin{cases} d\eta(t,x) = [a^{ij}(t)\eta_{x^ix^j}(t,x) + f(t,x)]dt \\ \qquad\qquad - \sigma^{ik}(t)\eta_{x^i}(t,x)dW_t^k, \quad (t,x)\in[0,T]\times\mathbb{R}^d, \\ \eta(0,x)=0, \quad x\in\mathbb{R}^d, \end{cases} \tag{5.27}$$

where $f \in \mathbb{H}_{p'}^{-n-2}$, $(F,G)\in \mathbb{H}_p^n \times L^p(\Omega,\mathcal{F}_T,H_p^{n+1})$, $1/p+1/p'=1$, $n\in\mathbb{R}$ and $(a^{ij})_{1\le i,j\le d}$ and $(\sigma^{ik})_{1\le i\le d,1\le k\le m}$ are the same as Theorem 5.2. If $a=I$ and $\sigma=0$, one can check that $\eta(t,x)=\int_0^t e^{\Delta(t-s)}f(s,x)ds\in\mathbb{H}_{p'}^{-n}$ is the unique solution of (5.27) in the sense of [20, 21]. For the general a and σ, by applying the Itô-Wentzell formula and the technical method used in Proposition 5.4, we can conclude that (5.27) has a unique solution $\eta\in\mathbb{H}_{p'}^{-n}$. It is crucial that σ is invariant in the space variable.

Then through a procedure similar to the proof of Theorem 5.2, we can conclude that BSPDE (5.26) has a unique solution pair (u,v) such that $u\in\mathbb{H}_p^{n+2}\cap\mathbb{H}_{p,\infty}^n$, $v(\cdot,\cdot+\int_0^\cdot\sigma^k(s)dW_s^k)\in\mathbb{H}_{p,2}^n$ and for any $\phi\in C_c^\infty$, the equality

$$(u(\tau,\cdot),\phi) = (G,\phi) + \int_\tau^T (a^{ij}(t)u_{x^ix^j}(t,\cdot)+\sigma^{ik}(t)v_{x^i}^k(t,\cdot)+F(t,\cdot),\phi)dt \\ - \int_\tau^T (v^k(t,\cdot),\phi)dW_t^k,$$

holds for all $\tau\in[0,T]$ with probability 1. For this solution pair, we have $u\in C([0,T],H_p^n)$ almost surely and

$$\|u\|_{\mathbb{H}_p^{n+2}} + \|u\|_{\mathbb{H}_{p,\infty}^n} + \|v'\|_{\mathbb{H}_{p,2}^n} \le C(T,n,d,p,\lambda,\Lambda)\left(\|G\|_{L^p(\Omega,\mathcal{F}_T,H_p^{n+1})} + \|F\|_{\mathbb{H}_p^n}\right)$$

where $v' = v(\cdot,\cdot+\int_0^\cdot\sigma^k(s)dW_s^k)$. In particular, when $G=0$, we have $\|u\|_{\mathbb{H}_p^{n+2}} \le C(d,p,\lambda,\Lambda)\,\|F\|_{\mathbb{H}_p^n}$.

5.3 The Case of General Variable Leading Coefficients

Now we deal with the general case.

Theorem 5.5 *Suppose that the Assumptions 5.1–5.4 are all satisfied. Consider $G\in L^p(\Omega,\mathcal{F}_T,H_p^{n+1})$ with $p\in(1,2]$ and $n\in\mathbb{R}$. Then BSPDE (5.1) has a unique solution $u\in\mathcal{H}_p^{n+2}$, satisfying the following inequality*

$$\|u\|_{\mathcal{H}_p^{n+2}} \le C(T,n,\kappa,\varrho,d,p,\lambda,\Lambda)\left(\|F(0,0,\cdot,\cdot)\|_{\mathbb{H}_p^n} + \|G\|_{L^p(\Omega,\mathcal{F}_T,H_p^{n+1})}\right). \tag{5.28}$$

The following lemma can be found in [21, Lemma 5.2].

Lemma 5.6 *Let $\zeta\in C_c^\infty(\mathbb{R}^d)$ be a nonnegative function such that $\int\zeta(x)dx=1$ and define $\zeta_k(x)=k^d\zeta(kx)$, $k=1,2,3,\ldots$. Then for any $u\in H_p^n$, $p\in(1,\infty)$, and any $n\in\mathbb{R}$, we have*

(i) $\|au\|_{n,p} \le C\|a\|_{B^{|n|+\gamma}}\|u\|_{n,p}$ *where* $C = C(d, p, n, \gamma)$;
(ii) $\|u * \zeta_k\|_{n,p} \le \|u\|_{n,p}$, $\|u - u * \zeta_k\|_{n,p} \to 0$ *as* $k \to \infty$.

Applying Lemma 3.2, we get a priori result about the solution of BSPDE (5.1), which is given in the following lemma. It will play a key role in the proof of Theorem 5.5 and distinguish our proof of BSPDEs from that of SPDEs in Krylov [20, 21].

Lemma 5.7 *Let* $u \in \mathcal{H}^{n+2}_{p,0}$ *be a solution to BSPDE* (5.1). *Let the Assumptions* 5.1–5.4 *be satisfied. Then for any* $\varepsilon > 0$, *there exists a constant* $C = C(T, p, \varepsilon)$ *such that*

$$\begin{aligned}\|\mathbb{D}u\|_{\mathbb{H}^n_{p,2}(t)} &\le \varepsilon[\|u_{xx}\|_{\mathbb{H}^n_p(t)} + \|(\mathbb{D}u)_x\|_{\mathbb{H}^n_p(t)} + \|F(0,0,\cdot,\cdot)\|_{\mathbb{H}^n_p(t)}] \\ &\quad + C(T, p, \varepsilon, \varrho, \Lambda)\|u\|_{\mathbb{H}^n_p(t)}, \quad t \in [0, T).\end{aligned}$$

Proof Denote $v := \mathbb{D}u$. By Lemma 3.2, for any $\bar\varepsilon > 0$, there exists a constant $C = C(T, p, \bar\varepsilon)$ such that

$$\begin{aligned}\|v\|_{\mathbb{H}^n_{p,2}} &\le \bar\varepsilon\|\mathcal{L}u + \mathcal{M}^k v^k + F(u, v, \cdot,\cdot)\|_{\mathbb{H}^n_p} + C(T, p, \bar\varepsilon)\|u\|_{\mathbb{H}^n_p} \\ &\le \bar\varepsilon C(\Lambda)(\|u_{xx}\|_{\mathbb{H}^n_p} + \|v_x\|_{\mathbb{H}^n_p} + \|F(0,0,\cdot,\cdot)\|_{\mathbb{H}^n_p}) \\ &\quad + \bar\varepsilon(\varrho(1)+1)(\|u\|_{\mathbb{H}^n_p} + \|v\|_{\mathbb{H}^n_p}) + C(T, p, \bar\varepsilon)\|u\|_{\mathbb{H}^n_p}.\end{aligned}$$

Since

$$\|v\|_{\mathbb{H}^n_p} \le T^{(2-p)/2}\|v\|_{\mathbb{H}^n_{p,2}},$$

we choose $\bar\varepsilon$ sufficiently small so that $1 - \bar\varepsilon(\varrho(1)+1)T^{(2-p)/2} > 1/2$. Therefore,

$$\begin{aligned}\|v\|_{\mathbb{H}^n_{p,2}} &\le 2\bar\varepsilon C(\Lambda)[\|u_{xx}\|_{\mathbb{H}^n_p} + \|v_x\|_{\mathbb{H}^n_p} + \|F(0,0,\cdot,\cdot)\|_{\mathbb{H}^n_p}] \\ &\quad + 2\bar\varepsilon(\varrho(1)+1)\|u\|_{\mathbb{H}^n_p} + 2C(T, p, \bar\varepsilon)\|u\|_{\mathbb{H}^n_p} \\ &\le 2\bar\varepsilon C(\Lambda)[\|u_{xx}\|_{\mathbb{H}^n_p} + \|v_x\|_{\mathbb{H}^n_p} + \|F(0,0,\cdot,\cdot)\|_{\mathbb{H}^n_p}] + C(T, p, \bar\varepsilon, \varrho(1))\|u\|_{\mathbb{H}^n_p}.\end{aligned}$$

This shows that the lemma is true for $t = 0$. Replacing $\mathbb{H}^n_p$ with $\mathbb{H}^n_p(t)$, we can prove the lemma for any $t \in [0, T)$ similarly. □

We have the following result about the perturbed leading coefficients.

Theorem 5.8 *Let Assumptions* 5.1–5.4 *be satisfied. Then there exists a constant* $\varepsilon \in (0, 1)$ *depending only on* d, p, λ *and* Λ *such that if the inequality*

$$\begin{aligned}&\|(a(t,\cdot) - \bar a(t))^{ij}(u_1)_{x^i x^j}\|_{n,p} + \|(\sigma(t,\cdot) - \bar\sigma(t,\cdot))^{ik}(v_1)^k_{x^i}\|_{n,p} \\ &\quad \le \varepsilon(\|(u_1)_{xx}\|_{n,p} + \|(v_1)_x\|_{n,p}) + K_0(\|u_1\|_{n,p} + \|v_1\|_{n,p}), \\ &\quad \forall (u_1, v_1) \in H^{n+2}_p \times H^{n+1}_p, t \ge 0, \end{aligned} \tag{5.29}$$

holds for some constant K_0 *and some pair* $(\bar a, \bar\sigma)$ *which satisfies the assumptions in Theorem* 5.2, *there exists a unique solution* $u \in \mathcal{H}^{n+2}_{p,0}$ *to* (5.1) *with* $G = 0$. *Moreover,* *we have*

$$\|u\|_{\mathcal{H}_p^{n+2}} \le C(T, K_0, \varrho, d, p, \lambda, \Lambda)\|F(0,0,\cdot,\cdot)\|_{\mathbb{H}_p^n}. \tag{5.30}$$

In particular, C is independent of T if $K_0 = 0$ and $\varrho \equiv 0$.

Proof Step 1. We first prove that there is a generic constant $\varepsilon \in (0,1)$ such that the inequality (5.29) yields the estimate (5.30) for any solution $u \in \mathcal{H}_{p,0}^{n+2}$ to BSPDE (5.1). Denote $v := \mathbb{D}u$ and rewrite BSPDE (5.1) into the following form:

$$\begin{cases} -du(t,x) = [\bar{\mathcal{L}}u(t,x) + \bar{\mathcal{M}}^k v^k(t,x) + (\mathcal{L}-\bar{\mathcal{L}})u(t,x) + (\mathcal{M}-\bar{\mathcal{M}})^k v^k(t,x) \\ \qquad\qquad + F(u,v,t,x)]dt - v^k(t,x)dW_t^k, \quad (t,x) \in [0,T]\times\mathbb{R}^d; \\ u(T,x) = 0, \quad x \in \mathbb{R}^d \end{cases} \tag{5.31}$$

where

$$\bar{\mathcal{L}} = \bar{a}^{ij}\frac{\partial^2}{\partial x^i \partial x^j}, \qquad \bar{\mathcal{M}}^k = \bar{\sigma}^{ik}\frac{\partial}{\partial x^i}, \quad k = 1,\dots,m.$$

In view of Theorem 5.2, we have

$$\begin{aligned} &\|u\|_{\mathcal{H}_p^{n+2}} \\ &\quad \le C(d,p,\lambda,\Lambda)\|(\mathcal{L}-\bar{\mathcal{L}})u + (\mathcal{M}-\bar{\mathcal{M}})^k v^k + F(u,v,\cdot,\cdot)\|_{\mathbb{H}_p^n} \\ &\quad \le C(d,p,\lambda,\Lambda)[\varepsilon(\|u_{xx}\|_{\mathbb{H}_p^n} + \|v_x\|_{\mathbb{H}_p^n}) + K_0(\|u\|_{\mathbb{H}_p^n} + \|v\|_{\mathbb{H}_p^n}) \\ &\qquad + \|F(0,0,\cdot,\cdot)\|_{\mathbb{H}_p^n} + \varepsilon_1(\|u_{xx}\|_{\mathbb{H}_p^n} + \|v_x\|_{\mathbb{H}_p^n}) + (\varrho(\varepsilon_1)+\varepsilon_1)(\|u\|_{\mathbb{H}_p^n} + \|v\|_{\mathbb{H}_p^n})] \\ &\quad \le C(d,p,\lambda,\Lambda)[(\varepsilon+\varepsilon_1)(\|u_{xx}\|_{\mathbb{H}_p^n} + \|v_x\|_{\mathbb{H}_p^n}) \\ &\qquad + (K_0 + \varrho(\varepsilon_1)+\varepsilon_1)(\|u\|_{\mathbb{H}_p^n} + \|v\|_{\mathbb{H}_p^n}) + \|F(0,0,\cdot,\cdot)\|_{\mathbb{H}_p^n}]. \end{aligned}$$

Note that the above still holds if $\mathbb{H}_p^n$ is replaced by $\mathbb{H}_p^n(t)$ for $t \in [0,T)$. Furthermore, if $K_0 = 0$ and $\varrho \equiv 0$, the map F does not depend on (u,v) and we get instead that

$$\|u\|_{\mathcal{H}_p^{n+2}} \le C(d,p,\lambda,\Lambda)[\varepsilon(\|u_{xx}\|_{\mathbb{H}_p^n} + \|v_x\|_{\mathbb{H}_p^n}) + \|F\|_{\mathbb{H}_p^n}],$$

which implies the last assertion of Theorem 5.8 by taking ε small enough such that $C(d,p,\lambda,\Lambda)\varepsilon < 1/2$.

Now, fix $t \in [0,T)$. Then, noting that $\|v\|_{\mathbb{H}_p^n(t)} \le T^{(2-p)/2}\|v\|_{\mathbb{H}_{p,2}^n}(t)$, from Lemma 5.7, we conclude that for any $\varepsilon_2 > 0$, there exists a constant $C = C(T,p,\varepsilon_2,\varrho)$ such that

$$\begin{aligned} &\|v\|_{\mathbb{H}_p^n(t)} \\ &\quad \le \varepsilon_2(\|u_{xx}\|_{\mathbb{H}_p^n(t)} + \|v_x\|_{\mathbb{H}_p^n(t)} + \|F(0,0,\cdot,\cdot)\|_{\mathbb{H}_p^n(t)}) + C_2(T,p,\varepsilon_2,\varrho,\Lambda)\|u\|_{\mathbb{H}_p^n(t)}. \end{aligned}$$

Thus, it follows that

$$\begin{aligned}
&\|u\|_{\mathcal{H}_p^{n+2}(t)} \\
&\quad \le C_1(d,p,\lambda,\Lambda)\{[\varepsilon+\varepsilon_1+(K_0+\varrho(\varepsilon_1)+\varepsilon_1)\varepsilon_2](\|u_{xx}\|_{\mathbb{H}_p^n(t)}+\|v_x\|_{\mathbb{H}_p^n(t)}) \\
&\qquad +[(K_0+\varrho(\varepsilon_1)+\varepsilon_1)\varepsilon_2+1]\|F(0,0,\cdot,\cdot)\|_{\mathbb{H}_p^n(t)} \\
&\qquad +(K_0+\varrho(\varepsilon_1)+\varepsilon_1)(1+C_2(T,p,\varepsilon_2,\varrho))\|u\|_{\mathbb{H}_p^n(t)}\}.
\end{aligned}$$

Taking $\varepsilon_1=\varepsilon$, $\varepsilon_2=\varepsilon/(K_0+\varrho(\varepsilon)+\varepsilon+1)$ and $\varepsilon=1/(4C_1+1)$, we get

$$\|u\|_{\mathcal{H}_p^{n+2}(t)} \le 5\|F(0,0,\cdot,\cdot)\|_{\mathbb{H}_p^n(t)}+C(T,p,d,\lambda,\Lambda,K_0,\varrho(\varepsilon))\|u\|_{\mathbb{H}_p^n(t)},$$

which immediately implies the following inequality

$$\|u\|^p_{\mathcal{H}_p^{n+2}(t)} \le C(T,p,d,\lambda,\Lambda,K_0,\varrho(\varepsilon))(\|F(0,0,\cdot,\cdot)\|^p_{\mathbb{H}_p^n(t)}+\|u\|^p_{\mathbb{H}_p^n(t)}).$$

Since (see Remark 4.3)

$$E\sup_{s\in[t,T]}\|u(s,\cdot)\|^p_{H_p^n}\le C(p,T)\|u\|^p_{\mathcal{H}_p^{n+2}(t)},$$

we have

$$\begin{aligned}
&\|u\|^p_{\mathcal{H}_p^{n+2}(t)} \\
&\quad \le C(T,p,d,\lambda,K_0,\Lambda,\varrho(\varepsilon))\left(\|F(0,0,\cdot,\cdot)\|^p_{\mathbb{H}_p^n(t)}+E\int_t^T\|u(s,\cdot)\|^p_{H_p^n}ds\right) \\
&\quad \le C(T,p,d,\lambda,K_0,\Lambda,\varrho(\varepsilon))\left(\|F(0,0,\cdot,\cdot)\|^p_{\mathbb{H}_p^n(t)}+\int_t^T\|u\|^p_{\mathcal{H}_p^{n+2}(s)}ds\right).
\end{aligned}$$

Using Gronwall inequality, we get the desired estimation (5.30).

Step 2. We use the standard method of continuity to prove the existence of the solution $u\in\mathcal{H}_p^{n+2}$. For $\theta\in[0,1]$, we consider the BSPDE

$$\begin{cases}-du=(\mathcal{L}_\theta u+\mathcal{M}_\theta^k v^k+(1-\theta)F(u,v,t,x))dt-v^k dW_t^k\\ u(T,\cdot)=0\end{cases} \tag{5.32}$$

where

$$\mathcal{L}_\theta:=\theta\bar{\mathcal{L}}+(1-\theta)\mathcal{L}\quad\text{and}\quad\mathcal{M}_\theta^k=\theta\bar{\mathcal{M}}^k+(1-\theta)\mathcal{M}^k.$$

Note that the priori estimate (5.30) holds with the constant C being independent of θ. Assume that BSPDE (5.32) has a unique solution $u\in\mathcal{H}_{p,0}^{n+2}$ for $\theta=\theta_0$. Theorem 5.2 shows that this assumption is true for $\theta_0=1$. For any $u_1\in\mathcal{H}_{p,0}^{n+2}$, the following BSPDE

$$\begin{cases}-du=\{\mathcal{L}_{\theta_0}u+\mathcal{M}_{\theta_0}^k v^k+(1-\theta_0)F(u,v,t,x)+(\theta-\theta_0)[(\bar{\mathcal{L}}-\mathcal{L})u_1\\ \qquad +(\bar{\mathcal{M}}^k-\mathcal{M}^k)(\mathbb{D}u_1)^k+F(u_1,\mathbb{D}u_1,t,x)]\}dt-v^k dW_t^k,\\ u(T,\cdot)=0,\end{cases} \tag{5.33}$$

has a unique solution u in $\mathcal{H}^{n+2}_{p,0}$, and we can define the solution map as follows

$$\mathfrak{R}_{\theta_0}:\ \mathcal{H}^{n+2}_{p,0}\to\mathcal{H}^{n+2}_{p,0},\quad u_1\mapsto u.$$

Then for any $u_i\in\mathcal{H}^{n+2}_{p,0}$, $i=1,2$, we have

$$\begin{aligned}
&\|\mathfrak{R}_{\theta_0}u_2-\mathfrak{R}_{\theta_0}u_1\|_{\mathcal{H}^{n+2}_p}\\
&\quad\le C|\theta-\theta_0|\|(\bar{\mathcal{L}}-\mathcal{L})(u_2-u_1)+(\bar{\mathcal{M}}^k-\mathcal{M}^k)(\mathbb{D}u_2-\mathbb{D}u_1)^k\\
&\qquad+F(u_2,\mathbb{D}u_2,t,x)-F(u_1,\mathbb{D}u_1,t,x)\|_{\mathbb{H}^n_p}\\
&\quad\le\bar{C}|\theta-\theta_0|\|u_1-u_2\|_{\mathcal{H}^{n+2}_p},
\end{aligned}$$

where $\bar{C}$ does not depend on θ and θ_0. If $\bar{C}|\theta-\theta_0|<1/2$, $\mathfrak{R}_{\theta_0}$ is a contraction mapping and it has a unique fixed point $u\in\mathcal{H}^{n+2}_{p,0}$ which solves BSPDE (5.32). In this way if (5.32) is solvable for θ_0, then it is solvable for θ satisfying $\bar{C}|\theta-\theta_0|<1/2$. In finite number of steps starting from $\theta=1$, we get to $\theta=0$. This completes the proof. □

Lemma 5.9 *Under the Assumptions 5.1–5.4, there exists an $\varepsilon=\varepsilon(n,\gamma,d,p,\lambda,\Lambda)>0$ such that if $\kappa(\infty-)<\varepsilon$, then the condition of Theorem 5.8 is satisfied . In this case, by Theorem 5.8 we conclude that there exists a unique solution $u\in\mathcal{H}^{n+2}_{p,0}$ to BSPDE (5.1) with the zero terminal condition satisfying the following inequality*

$$\|u\|_{\mathcal{H}^{n+2}_p}\le C(T,\varrho,d,p,\lambda,\Lambda)\|F(0,0,\cdot,\cdot)\|_{\mathbb{H}^n_p}.$$

Proof Define $\bar{a}(t)=a(t,0)$ and $\bar{\sigma}(t)=\sigma(t,0)$. It follows from Lemma 5.6 that, for any $(u_1,v_1)\in H^{n+2}_p\times H^{n+1}_p$, we have

$$\begin{aligned}
&\|(a-\bar{a})^{ij}(t,\cdot)(u_1)_{ij}\|_{n,p}+\|(\sigma-\bar{\sigma})^{ik}(t,\cdot)(v_1)^k_i\|_{n,p}\\
&\quad\le C(n,d,p,\gamma)\left(\|(a-\bar{a})(t,\cdot)\|_{B^{|n|+\gamma}}\|(u_1)_{xx}\|_{n,p}\right.\\
&\qquad\left.+\|(\sigma-\bar{\sigma})(t,\cdot)\|_{B^{|n|+\gamma}}\|(v_1)_x\|_{n,p}\right).
\end{aligned}\tag{5.34}$$

In view of (5.29), there exists a constant $\varepsilon_1=\varepsilon_1(n,\gamma,d,p,\lambda,\Lambda)$ such that if

$$\begin{aligned}
&\|a(t,\cdot)-\bar{a}(t)\|_{B^{|n|+\gamma}}\|(u_1)_{xx}\|_{n,p}+\|\sigma(t,\cdot)-\bar{\sigma}(t)\|_{B^{|n|+\gamma}}\|(v_1)_x\|_{n,p}\le\varepsilon_1,\\
&\forall t\in[0,T],
\end{aligned}\tag{5.35}$$

the condition (5.29) in Theorem 5.8 is satisfied. With a standard method (c.f. [21, Lemma 6.6, pp. 215–216]), we can check that if ε in our lemma is sufficiently small, (5.35) holds true. This complete the proof. □

To prove Theorem 5.5, we need a generalization of the Littlewood-Paley inequality, which is due to Krylov [19].

Lemma 5.10 *Let* $p \in (1,\infty)$, $n \in (-\infty,+\infty)$, $\delta > 0$, *and* $\zeta_k \in C^\infty$, $k = 1,2,3,\ldots$. *Assume that for any multi-index* α *and* $x \in \mathbb{R}^d$

$$\sup_{x\in\mathbb{R}^d}\sum_k |D^\alpha \zeta_k(x)| \le M(\alpha),$$

where $M(\alpha)$ *is constant. Then there exists a constant* $C = C(d,n,M)$ *such that, for any* $f \in H_p^n$,

$$\sum_k \|\zeta_k f\|_{n,p}^p \le C\|f\|_{n,p}^p.$$

If in addition

$$\sum_k |\zeta_k(x)|^p \ge \delta,$$

then for any $f \in H_p^n$,

$$\|f\|_{n,p}^p \le C(d,n,M,\delta)\sum_k \|\zeta_k f\|_{n,p}^p.$$

Proof of Theorem 5.5 *Step 1.* Without loss of generality, assume that $G = 0$.

In fact, by Theorem 5.2, there exists a unique solution $\bar{u} \in \mathcal{H}_p^{n+2}$ for the equation

$$\begin{cases} -du = \Delta u dt - v^k dW_t^k, & t \in [0,T]; \\ u(T,x) = G(x) \end{cases} \tag{5.36}$$

satisfying the estimate

$$\|\bar{u}\|_{\mathcal{H}_p^{n+2}} \le C(T,p,d,\lambda,\Lambda)\|G\|_{L^p(\Omega,\mathcal{F}_T,H_p^{n+1})}.$$

Without loss of generality, we consider $(\bar{u}(t,\cdot),\mathbb{D}\bar{u}(t,\cdot)) \in H_p^{n+2} \times H_p^{n+1}$ for any $(t,\omega) \in [0,T]\times\Omega$. Setting $(u,v) := (\tilde{u}+\bar{u},\tilde{v}+\bar{v})$, we need only to consider the BSPDE

$$\begin{aligned} -d\tilde{u}(t,x) = {} & [a^{ij}(t,x)\tilde{u}_{x^ix^j}(t,x) + \sigma^{ik}(t,x)\tilde{v}^k_{x^i}(t,x) + \bar{F}(\tilde{u},\tilde{v},t,x)]dt \\ & - \tilde{v}^k(t,x)dW_t^k \end{aligned}$$

where

$$\begin{aligned} \bar{F}(\tilde{u},\tilde{v},t,x) = {} & F(\tilde{u}+\bar{u},\tilde{v}+\bar{v},t,x) + a^{ij}(t,x)\bar{u}_{x^ix^j}(t,x) \\ & + \sigma^{ik}(t,x)\bar{v}^k_{x^i}(t,x) - \Delta\bar{u}(t,x). \end{aligned}$$

It can be checked that $\bar{F}$ satisfies the same condition as F.

Step 2. We give a priori estimate for the solution $u \in \mathcal{H}_{p,0}^{n+2}$ to BSPDE (5.1).

For $\varepsilon > 0$ in Lemma 5.9, by Assumption 5.2, there exists $\varepsilon_0 > 0$ such that $\kappa(s) < \varepsilon$ for any $s \in [0,\varepsilon_0]$. Let $\{\zeta_l : l = 1,2,3,\ldots\}$ be a standard partition of unity in $\mathbb{R}^d$ such that, for any l, the support of ζ_l lies in the ball $B(x_l,\varepsilon_0/4)$. For any l, take a function $\eta_l \in C_c^\infty$ valued in $[0,1]$ such that the support of η_l lies in $B_l(x_l,\varepsilon_0/2)$ and $\eta_l = 1$ on B_l. Denote $v := \mathbb{D}u$. Then we get

$$\begin{cases} -d(u\zeta_l)(t,x) = [\tilde{\mathcal{L}}_l(t,x)(\zeta_l u)(t,x) + \tilde{\mathcal{M}}_l^k(t,x)(\zeta_l v^k)(t,x) + \tilde{F}(t,x)]dt \\ \qquad\qquad - \zeta_l(x) v^k(t,x)dW_t^k \\ (u\zeta_l)(T,x) = 0 \end{cases} \tag{5.37}$$

where

$$\begin{aligned} \tilde{\mathcal{L}}_l(t,x) &:= \eta_l(x)\mathcal{L}(t,x) + (1-\eta_l(x))\mathcal{L}(t,x_l), \\ \tilde{\mathcal{M}}_l^k(t,x) &:= \eta_l(x)\mathcal{M}^k(t,x) + (1-\eta_l(x))\mathcal{M}^k(t,x_l), \\ \tilde{F}(t,x) &:= -2(\zeta_l)_{x^i} a^{ij} u_{x^j}(t,x) - (\zeta_l)_{x^i x^j} a^{ij} u(t,x) \\ &\quad - (\zeta_l)_{x^i}\sigma^{ik} v^k(t,x) + \zeta_l F(u,v,t,x). \end{aligned}$$

From Theorem 4.1 and Lemma 5.9, we get

$$\|u\zeta_l\|_{\mathbb{H}_p^{n+2}} + \|v\zeta_l\|_{\mathbb{H}_p^{n+1}} \le C(T,\varrho,\lambda,\Lambda,d,p)\|\tilde{F}\|_{\mathbb{H}_p^n}.$$

Applying Lemma 5.10 and 5.6, we can get such conclusions as

$$\sum_l \|\zeta_l F(\omega,t)\|_{n,p}^p \le C\|F(\omega,t)\|_{n,p}^p,$$

$$\sum_l \|(\zeta_l)_{x^i x^j} a^{ij} u(\omega,t)\|_{n,p}^p \le C\|a^{ij} u(\omega,t)\|_{n,p}^p \le C\|u\|_{n,p}^p,$$

$$\|u(\omega,t)\|_{n,p} \le C\sum_l \|\zeta_l u(\omega,t)\|_{n,p} \le C\|u(\omega,t)\|_{n,p}, \quad (\omega,t)\in\Omega\times[0,T] \text{ a.e.}$$

Integrating each term on $\Omega\times[0,T]$, we have

$$\begin{aligned} &\|u\|_{\mathcal{H}_p^{n+2}} \\ &\quad \le C(T,n,\kappa,d,p,\lambda,\Lambda)(\|F(u,v,\cdot,\cdot)\|_{\mathbb{H}_p^n} + \|u\|_{\mathbb{H}_p^{n+1}} + \|v\|_{\mathbb{H}_p^n}) \\ &\quad \le C_1(T,\kappa,n,d,p,\lambda,\Lambda)(\varepsilon_1\|u\|_{\mathcal{H}_p^{n+2}} + \|F(0,0,\cdot,\cdot)\|_{\mathbb{H}_p^n} \\ &\qquad + (1+\varrho(\varepsilon_1))\|u\|_{\mathbb{H}_p^n} + \|v\|_{\mathbb{H}_p^n}) \end{aligned}$$

where $\varepsilon_1 > 0$ is arbitrary. Then, noting that $\|v\|_{\mathbb{H}_p^n} \le T^{(2-p)/2}\|v\|_{\mathbb{H}_{p,2}^n}$, from Lemma 5.7, we conclude that for any $\varepsilon_2 > 0$, there exists a constant $C = C(T,p,\varepsilon_2,\varrho)$ such that

$$\|v\|_{\mathbb{H}_p^n} \le \varepsilon_2(\|u\|_{\mathcal{H}_p^{n+2}} + \|F(0,0,\cdot,\cdot)\|_{\mathbb{H}_p^n}) + C_2(T,p,\varepsilon_2,\varrho,\Lambda)\|u\|_{\mathbb{H}_p^n}.$$

By choosing $\varepsilon_1+\varepsilon_2$ small enough such that $C_1(T,\kappa,n,d,p,\lambda,\Lambda)(\varepsilon_1+\varepsilon_2) < 1/2$, we get

$$\|u\|_{\mathcal{H}_p^{n+2}} \le C(T,\kappa,\varrho,n,d,p,\lambda,\Lambda)(\|F(0,0,\cdot,\cdot)\|_{\mathbb{H}_p^n} + \|u\|_{\mathbb{H}_p^n}). \tag{5.38}$$

In view of Theorem 4.1 and Remark 4.3, we can show in a similar way the following inequality

$$E\|u(t,\cdot)\|_{n,p}^p \le C\|F(0,0,\cdot,\cdot)\|_{\mathbb{H}_p^n}^p + C\int_t^T E\|u(s,\cdot)\|_{n,p}^p ds$$

for all $t \in [0,T]$. Using Gronwall's inequality, we have

$$\|u\|_{\mathbb{H}_p^n}^p \le C\|F(0,0,\cdot,\cdot)\|_{\mathbb{H}_p^n}^p,$$

which along with (5.38) implies the following estimate

$$\|u\|_{\mathcal{H}_p^{n+2}} \le C(T,\kappa,\varrho,n,d,p,\lambda,\Lambda)\|F(0,0,\cdot,\cdot)\|_{\mathbb{H}_p^n}. \tag{5.39}$$

Step 3. In the end, proceeding identically as in *Step 2* in the proof of Theorem 5.8, we can prove the existence and uniqueness of the solution. The proof is complete. □

Corollary 5.11 *Let the assumptions of Theorem 5.5 be satisfied. We assume that the assumptions are not only satisfied for p but also for* $q \in (1,2]$. *Then the solution u in Theorem 5.5 belongs to* $\mathcal{H}_q^{n+2}$.

Proof We can prove our corollary by completing the *Step 3* of the proof of Theorem 5.5. The difference from *Step 2* in the proof of Theorem 5.8 lies that we use the Picard iteration this time instead of the contraction mapping principle. Assume that for some $\theta = \theta_0$, equation (5.32) with zero terminal condition admits a unique solution $u \in \mathcal{H}_{p,0}^{n+2}(T) \cap \mathcal{H}_{q,0}^{n+2}$. By the way, this assumption is satisfied for $\theta_0 = 1$ by Theorem 5.2 and Remark 5.6. Set $u_0 = 0$ and take iterations $u_l = \mathfrak{R}_{\theta_0} u_{l-1}$, $l = 1,2,3,\ldots$. Then there exists a constant $\delta > 0$ independent of θ_0 such that if $\theta \in [\theta_0 - \delta, \theta_0 + \delta] \cap [0,1]$, u_l is a cauchy sequence both in $\mathcal{H}_{p,0}^{n+2}$ and $\mathcal{H}_{p,0}^{n+2}$ and for these θs the solutions in $\mathcal{H}_{p,0}^{n+2}$ and $\mathcal{H}_{p,0}^{n+2}$ coincide. In finite steps from $\theta = 1$, we get to $\theta = 0$. This completes the proof. □

6 Two Related Topics

The proofs of the following results are similar to that of the SPDE in [21], and will be sketched only.

6.1 Comparison Theorem

The following theorem shows that the solution to BSPDE (5.1) is continuous w.r.t. the leading coefficients a^{ij} and σ^{ik}, the non-homogeneous drift term F, and the terminal value G.

Theorem 6.1 *Assume that for $l=1,2,3,\dots$, we are given a_l^{ij}, σ_l^{ik}, F_l, and G_l verifying the same assumptions as a^{ij}, σ^{ik}, F and G in Theorem 5.5 with the same constants λ, Λ and the same functions κ, ϱ. Let $\zeta(x)\in C_c^\infty$ be a real function taking values in $[0,1]$ such that $\zeta(x)=1$ if $|x|\le 1$ and $\zeta(x)=0$ if $|x|\ge 2$. Define $\zeta_r(x)=\zeta(x/r)$ for $r=1,2,3,\dots$. And we also assume that, for $r=1,2,3,\dots,i,j=1,\dots,d,k=1,\dots,m,t\in[0,T]$, and $\omega\in\Omega$,*

$$\|\zeta_r\{a^{ij}(t,\cdot)-a_l^{ij}(t,\cdot)\}\|_{n,p}+\|\zeta_r\{\sigma^{ik}(t,\cdot)-\sigma_l^{ik}(t,\cdot)\}\|_{n,p}\to 0 \tag{6.1}$$

as $l\to\infty$. Furthermore, assume $E\|G_l-G\|_{n+1,p}^p\to 0$ and

$$\|F(u,v,\cdot,\cdot)-F_l(u,v,\cdot,\cdot)\|_{\mathbb{H}_p^n}\to 0, \tag{6.2}$$

whenever $u\in\mathcal{H}_p^{n+2}$ and $v:=\mathbb{D}u$. If we take the function u from Theorem 5.5 and for any l define $u_l\in\mathcal{H}_p^{n+2}$ as the unique solution of the following BSPDE

$$\begin{cases}-du_l(t,x)=[a_l^{ij}(t,x)u_{lx^ix^j}(t,x)+\sigma_l^{ik}(t,x)v_{lx^i}^k(t,x)+F_l(u_l,v_l,t,x)]dt\\ \qquad\qquad -v_l^k(t,x)dW_t^k,\quad (t,x)\in[0,T]\times\mathbb{R}^d,\\ u_l(T,x)=G_l(x),\quad x\in\mathbb{R}^d,\end{cases} \tag{6.3}$$

where $v_l:=\mathbb{D}u_l$, then we have $\|u-u_l\|_{\mathcal{H}_p^{n+2}}\to 0$ as $l\to\infty$.

Proof Let $\bar{u}_l=u-u_l$ and $\bar{v}_l=v-v_l$. Then we have

$$\begin{cases}-d\bar{u}_l(t,x)=[a_l^{ij}(t,x)\bar{u}_{lx^ix^j}(t,x)+\sigma_l^{ik}(t,x)\bar{v}_{lx^i}^k(t,x)+f_l(\bar{u}_l,\bar{v}_l)]dt\\ \qquad\qquad -\bar{v}_l^k(t,x)dW_t^k,\quad (t,x)\in[0,T]\times\mathbb{R}^d,\\ \bar{u}_l(T,x)=\bar{G}_l(x),\quad x\in\mathbb{R}^d,\end{cases} \tag{6.4}$$

where

$$f_l(\bar{u}_l,\bar{v}_l)=(a^{ij}-a_l^{ij})u_{x^ix^j}+(\sigma^{ik}-\sigma_l^{ik})v_{x^i}+F(u,v)-F_l(u-\bar{u},v-\bar{v}).$$

Then by Theorem 5.5, we obtain

$$\|u-u_l\|_{\mathcal{H}_p^{n+2}}\le CJ_l,$$

where C is independent of l and

$$\begin{aligned}J_l&=\|(a^{ij}-a_l^{ij})u_{x^ix^j}\|_{\mathbb{H}_p^n}+\|(\sigma^{ik}-\sigma_l^{ik})v_{x^i}\|_{\mathbb{H}_p^n}\\&\quad+\|F(u,v)-F_l(u,v)\|_{\mathbb{H}_p^n}+(E\|G_l-G\|_{n+1,p}^p)^{1/p}.\end{aligned} \tag{6.5}$$

By our assumptions, we have

$$\limsup_{l\to\infty}J_l\le\limsup_{l\to\infty}\{\|(a^{ij}-a_l^{ij})u_{x^ix^j}\|_{\mathbb{H}_p^n}+\|(\sigma^{ik}-\sigma_l^{ik})v_{x^i}\|_{\mathbb{H}_p^n}\}. \tag{6.6}$$

Then the following is standard (for reference, see the proof of Theorem 5.7 of [21] pp. 209–210).

For any $\phi \in C_c^\infty$, let r be so large that $\phi\zeta_r = \phi$. Then, by Lemma 5.6, we get

$$\|(a^{ij} - a_l^{ij})u_{x^ix^j}\|_{n,p} \le C\|(u-\phi)_{x^ix^j}\|_{n,p} + \|(a^{ij} - a_l^{ij})\phi_{x^ix^j}\|_{n,p},$$

$$\|(a^{ij} - a_l^{ij})\phi_{x^ix^j}\|_{n,p} = \|(a^{ij} - a_l^{ij})\zeta_r\phi_{x^ix^j}\|_{n,p} \le C\|(a^{ij} - a_l^{ij})\zeta_r\|_{n,p}\|\phi\|_{B^{|n|+2+\gamma}}, \tag{6.7}$$

where the constants C's are independent of r and l. Thus,

$$\limsup_{l\to\infty} \|(a^{ij} - a_l^{ij})u_{x^ix^j}\|_{n,p} \le C\|(u-\phi)_{x^ix^j}\|_{n,p} \quad \text{for } (t,\omega) \in [0.T] \times \Omega, \text{a.e.},$$

and the arbitrariness of ϕ implies the left-hand side above is zero. Then by Lemma 5.6 and the dominated convergence theorem, we conclude that

$$\lim_{l\to\infty} \|(a^{ij} - a_l^{ij})u_{x^ix^j}\|_{\mathbb{H}_p^n} = 0.$$

Similarly, we can get $\lim_{l\to\infty} \|(\sigma^{ik} - \sigma_l^{ik})v_{x^i}\|_{\mathbb{H}_p^n} = 0$. □

Remark 6.1 From Lemma 2.2, it follows that the condition (6.2) holds for any $u \in \mathcal{H}_p^{n+2}$ if and only if it is satisfied for $u(t,x) \equiv \phi$, $v^k(t,x) \equiv \phi^k$ with any $\phi, \phi^k \in C_c^\infty$, $k = 1, \dots, m$.

Corollary 6.2 *Take ζ_l from Lemma 5.6. Under the assumptions of Theorem 5.5, for $l = 1, 2, 3, \dots$, we define*

$$(a_l, \sigma_l) = (a, \sigma)(t, \cdot) * \zeta_l(x), \qquad G_l = G * \zeta_l(x),$$

and also

$$F_l(u, v, t, x) = F(u, v, t, \cdot) * \zeta_l(x) = \int_{\mathbb{R}^d} F(u(x), v(x), t, x-y)\zeta_l(y)dy.$$

Then the assumptions of Theorem 6.1 are satisfied, and if we take $u_l \in \mathcal{H}_p^{n+2}$ as the unique solution of BSPDE (6.4), we have $\|u - u_l\|_{\mathcal{H}_p^{n+2}} \to 0$ as $l \to \infty$.

As the proof of the corollary is just a verification, which is very similar to [21, Corollary 5.10], it is omitted here.

Theorem 6.3 *Under the assumptions of Theorem 5.5, let u be the solution of BSPDE (5.1) for $n = 0$. And further, assume that*

$$F(u, v, t, x) = b^i(t,x)u_{x^i} + c_0(t,x)u(t,x) + c_k(t,x)v^k(t,x) + f(t,x),$$

where $b^i(t,x), c_0(t,x), c_k(t,x), k = 1, \dots, m$ are bounded $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable functions on $[0,T] \times \Omega \times \mathbb{R}^d$ and $f(t,x) \ge 0$. Also assume that $G \ge 0$ almost surely. Then $u(t,\cdot) \ge 0$ for all $t \in [0,T]$ almost surely.

Proof First, we take two nonnegative sequences $(f^l)_{l\geq 1}$ in $L^\infty(\Omega\times[0,T],\mathscr{P},H_2^0)\cap\mathbb{H}_p^0$ and $(G^l)_{l\geq 1}$ in $L^2(\Omega,\mathscr{F}_T,H_2^1)\cap L^p(\Omega,\mathscr{F}_T,H_p^1)$ such that $\|f^l-f\|_{\mathbb{H}_p^0}\to 0$ and $\|G^l-G\|_{L^p(\Omega,\mathscr{F}_T,H_p^1)}\to 0$ as $l\to\infty$. Next, Corollary 6.2 allows us to assume that G^l, f^l and all the other coefficients are infinitely differentiable in x.

After those above, by Theorem 6.1 we get an approximating solutions u^l of u. In this case the assumptions of Theorem 5.5 are satisfied for $p=2$, and any $n\geq 0$. Then, Corollary 5.11, yields $u^l\in\mathscr{H}_2^r$ for any $r\geq 0$. Furthermore, in this case the assumptions of [10, Theorem 5.1], [15, Theorem 6.1] and [33, Theorem 6.1] are all satisfied, and the comparison theorems there all imply $u^l\geq 0$ (a.e. (t,x,ω)). By taking limits, we get $u\geq 0$ (a.e. (t,x,ω)). On the other hand, in light of Lemma 3.1, it follows that $u\in C([0,T],H_p^0)$ a.s., which implies $u\geq 0$ (at least for a modification of u) for all $t\in[0,T]$ almost surely. □

6.2 L^p Theory for $p>2$

When $p<2$, the assertion of Lemma 5.3 is not true in general. This fact makes the L^p-theory we have established in Sect. 5 require the assumption $p\in(1,2]$ and Krylov's seminal work [20, 21] require $p\in[2,\infty)$. However, if we consider SPDEs (5.27) where the diffusion is homogeneous in the unknown variable, the harmonic result (Lemma 5.3) can be avoided, which could allow us to get further results.

Consider the following BSPDE

$$\begin{cases}-du(t,x)=[a^{ij}(t,x)u_{x^ix^j}(t,x)+\sigma^{ik}(t)v^k_{x^i}(t,x)+F(u,\sigma^iu_{x^i}+v,t,x)]dt\\ \qquad\qquad -v^k(t,x)dW_t^k,\quad (t,x)\in[0,T]\times\mathbb{R}^d;\\ u(T,x)=G(x),\quad x\in\mathbb{R}^d.\end{cases}\tag{6.8}$$

Definition 6.1 We call (u,v) a solution pair of BSPDE (6.8) in $\mathbb{H}_p^{n+2}\times\mathbb{H}_{p,2}^n$ if $u\in\mathbb{H}_p^n$, $v(\cdot,\cdot+\int_0^\cdot\sigma^k(s)dW_s^k)\in\mathbb{H}_{p,2}^n$ and for any $\phi\in C_c^\infty$, the equality

$$\begin{aligned}(u(\tau,\cdot),\phi)=(G,\phi)+\int_\tau^T(a^{ij}(t,\cdot)u_{x^ix^j}(t,\cdot)+\sigma^{ik}(t)v^k_{x^i}(t,\cdot)\\ +F(u,\sigma^iu_{x^i}+v,t,\cdot),\phi)dt-\int_\tau^T(v^k(t,\cdot),\phi)dW_t^k,\\ \forall(t,\phi)\in[0,T)\times C_c^\infty\end{aligned}\tag{6.9}$$

holds for all $\tau\in[0,T]$ with probability 1.

For the case $p>2$, we have presented some results in Remark 5.7 on BSPDEs with constant-field-valued coefficients. Through a procedure similar to the case $p\in(1,2]$ we get the following result.

Proposition 6.4 *For $p>2$ and $n\in\mathbb{R}$, suppose that a and σ satisfy Assumptions 5.1–5.3 with σ being invariant in the space variable. Let $F(0,0,\cdot,\cdot)\in\mathbb{H}_p^n$. For any*

$(h,g)\in\mathbb{H}_p^{n+2}\times\mathbb{H}_{p,2}^n$, $F(h,g,t,\cdot)$ is an H_p^n-valued $\mathcal{P}$-measurable process such that there is a continuous and decreasing function $\varrho:(0,\infty)\to[0,\infty)$ and a constant $L>0$ such that for any $\varepsilon>0$, we have

$$\begin{aligned}&\|F(\bar h,\bar g,\cdot,\cdot)-F(h',g',\cdot,\cdot)\|_{\mathbb{H}_p^n(t)}\\&\quad\le\varepsilon\|\bar h-h'\|_{\mathbb{H}_p^{n+2}(t)}+\varrho(\varepsilon)\|\bar h-h'\|_{\mathbb{H}_p^n(t)}+L\|\bar g-g'\|_{\mathbb{H}_{p,2}^n(t)},\\&\qquad\bar h,h'\in\mathbb{H}_p^{n+2}\text{ and }\bar g,g'\in\mathbb{H}_{p,2}^n,\end{aligned}\tag{6.10}$$

holds for any $t\in[0,T)$. Consider $G\in L^p(\Omega,\mathcal{F}_T,H_p^{n+1})$. Then BSPDE (6.8) has a unique solution pair (u,v) in $\mathbb{H}_p^{n+2}\times\mathbb{H}_{p,2}^n$. For this solution pair, we have

$$\begin{aligned}&\|u\|_{\mathbb{H}_p^{n+2}}+\|v'\|_{\mathbb{H}_{p,2}^n}\\&\quad\le C(T,n,\kappa,\varrho,d,p,\lambda,\Lambda)(\|F(0,0,\cdot,\cdot)\|_{\mathbb{H}_p^n}+\|G\|_{L^p(\Omega,\mathcal{F}_T,H_p^{n+1})})\end{aligned}$$

where $v'(t,x)=v(t,x+\int_0^t\sigma^k(s)dW_s^k)$, $(t,x)\in[0,T]\times\mathbb{R}^d$.

Here, we only give a sketch of the proof. First, take $\zeta(t,x)=u(t,x+\int_0^t\sigma^k(s)dW_s^k)$. By Theorem 2.3, we can rewrite the BSPDE (6.8)

$$\begin{cases}-d\zeta(t,x)=[\bar a^{ij}(t,x)\zeta_{x^ix^j}(t,x)+F(\zeta,\sigma^i\zeta_{x^i}+v',t,x+\int_0^t\sigma^k(s)dW_s^k)]dt\\\qquad\qquad-(\sigma^{ki}\zeta_{x^i}(t,x)+v'^k(t,x))dW_t^k,\quad(t,x)\in[0,T]\times\mathbb{R}^d;\\\zeta(T,x)=\bar G(x),\quad x\in\mathbb{R}^d,\end{cases}\tag{6.11}$$

where $\bar a(t,x):=a(t,x+\int_0^t\sigma^k(s)dW_s^k)-\frac{1}{2}\sigma\sigma^{\mathcal{T}}$ and $\bar G=G(x+\int_0^T\sigma^k(s)dW_s^k)$. Actually the estimate about v are deduced from Lemma 3.1. The proof of the other assertions are very similar to those seen in Sect. 5.3.

7 Comments

In the framework of weak solutions, we establish in this paper an L^p-theory for BSPDE (1.1) which seems to be the first study for the L^p-theory of BSPDEs. Motivated by Krylov's seminal work [20, 21] on forward stochastic partial differential equations, we establish an L^p-theory which includes as a particular case the L^p-theory ($1<p\le2$) for deterministic parabolic PDEs. As the L^p-theory for SPDEs essentially relies on the sharp harmonic result Lemma 5.3 for the case $p\in[2,\infty)$, the L^p-theory for the Cauchy problem of BSDPEs obtained via duality herein requires $p\in(1,2]$. For $p>2$, we give some separate results.

Finally, we would like to make the following remarks:

(1) In this work, we establish the L^p-theory for BSPDEs (1.1) with the unknown random field u being scalar valued. In a similar way, we can also present the corresponding L^p-theory for systems of backward stochastic partial differential equations with the unknown random field u taking values in a Euclidean space,

on the basis of the L^p-theory for systems of SPDEs obtained by Mikulevicius and Rozovskii [25], which is left as an exercise to the interested reader. As a particular nonlinear system of BSPDEs, the two-dimensional backward stochastic Navier-Stokes equation is studied by Qiu, Tang and You [31].

(2) The super-parabolicity and bounded assumptions on the coefficients are required in this work. For the degenerate, unbounded and irregular cases, refer to [8, 12, 15, 28–30, 33].

(3) In this work, our BSPDE (1.1) is addressed in the whole space. For BSPDEs in domains, refer to [7, 11, 28]. In particular, strong solution to the Cauchy-Dirichlet problem of BSPDEs is considered in [11]. However, to the best of our knowledge, when $p \neq 2$ the corresponding L^p-theory for BSPDEs in domains is still blank.

(4) For the critical case $p = 1$, L^p-theory for BSDEs is given in [5]. In [29], we discuss the L^1-theory for BSPDEs and obtain some similar properties to those for BSDEs.

(5) In unbounded control with stochastic coefficients, quadratic BSPDE arises. Borrowing the techniques of Kobylanski's seminal work [18], Du and Chen [9] gives a fairly complete theory for quadratic BSPDEs.

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