A Coupled System of Integrodifferential Equations Arising in Liquidity Risk Model

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Abstract We study the mathematical aspects of the portfolio/consumption choice problem in a market model with liquidity risk introduced in (Pham and Tankov, Math. Finance, 2006, to appear). In this model, the investor can trade and observe stock prices only at exogenous Poisson arrival times. He may also consume continuously from his cash holdings, and his goal is to maximize his expected utility from consumption. This is a mixed discrete/continuous time stochastic control problem, nonstandard in the literature. We show how the dynamic programming principle leads to a coupled system of Integro-Differential Equations (IDE), and we prove an analytic characterization of this control problem by adapting the concept of viscosity solutions. This coupled system of IDE may be numerically solved by a decoupling algorithm, and this is the topic of a companion paper (Pham and Tankov, Math. Finance, 2006, to appear).

1 Introduction

A fundamental assumption of the traditional portfolio/consumption choice paradigm of Merton [11] is that assets are liquid and readily continuously tradeable by eco-

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nomic agents. In reality, there are some restrictions on securities trade, and investors cannot buy and sell them immediately. We then usually speak about liquidity risk meaning that one may have to wait some time before being able to unwind a position in some financial assets.

There are various approaches to model liquidity risk since it is in fact related to many factors. A familiar approach in the academic literature is to measure illiquidity in terms of bid-ask spread and transaction costs, see e.g. [5] and many others. In this setting, potentially high cost is associated to frequent trading but the investors can trade whenever desired. On the other hand, there are some studies where illiquidity is represented by restrictions on trade times. For instance, Schwartz and Tebaldi [14] and Longstaff [9] assume in their model that illiquid assets can only be traded at the starting date and at a fixed terminal horizon. In a less extreme modelling, Rogers and Zane [13] and Matsumoto [10] consider random trade times by assuming that trade succeeds only at the jump times of a Poisson process, and study the impact on a portfolio choice problem. In these models, the price process is observed continuously, trading strategies are in continuous-time, and the corresponding portfolio/consumption problem leads to a standard jump-diffusion control problem, see also [15]. However, illiquidity is often viewed by practitioners as the situation where their ability to trade assets is limited or restricted to the times when a quote comes into the market.

In this paper, we consider a description of liquidity risk, which is consistent with the market-microstructure oriented modelling of high frequency financial data such as tick-by-tick stock prices. We assume that stock prices can be observed and traded only at random times of a Poisson process corresponding to quotes in the market. This setup is inspired by recent papers of Frey and Runggaldier [7] and Cvitanic, Liptser and Rozovskii [3], who assume in addition that there is an unobservable stochastic volatility, and are interested in the estimation of this volatility. In our liquidity risk context, we suppose that the investor is also allowed to consume continuously from the bank account, and we study the Merton's problem of maximizing the expected discounted utility of consumption.

From a mathematical viewpoint, the resulting optimization problem is a mixed discrete/continuous time stochastic control problem. The main feature is that one control component is decided only at random discrete times and based on discrete observation filtration, while the other control component is executed continuously in time. Moreover, we face some original state constraints required by the nonnegative budget condition at the observed random times. We first state a suitable version of the dynamic programming principle (DPP) for this control problem, and show how it leads, via the DPP, to a coupled system of nonlinear integro-partial differential equations (IPDE) for the corresponding value functions. Then, following the modern approach of stochastic control, and to overcome the possible lack of regularity of the value functions, we adapt the notion of viscosity solutions to our context, and prove a characterization (with a new uniqueness result) to this coupled system of IPDE. The numerical resolution of this IPDE is the purpose of a companion paper [12].

The plan of the paper is as follows. We formulate the liquidity risk model and the portfolio/consumption problem in Sect. 2. We show in Sect. 3 how it leads, via the dynamic programming principle, to a coupled system of IPDE. In Sect. 4, we

state some properties on the value function. We provide in Sect. 5 an analytic unique characterization of the value function by means of viscosity solutions.

2 Model and Problem Formulation

Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathbb{F}_t)_{t \geq 0}$ satisfying the usual conditions. All stochastic processes involved in this paper are defined on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

We consider a model of an illiquid market where the investor can observe the positive stock price process S and trade only at random times $\{\tau_k\}_{k \geq 0}$ with $\tau_0 = 0 < \tau_1 < \cdots < \tau_k < \cdots$. For simplicity, we assume that S_0 is known, and we denote

$$Z_k = \frac{S_{\tau_k} - S_{\tau_{k-1}}}{S_{\tau_{k-1}}}, \quad k \geq 1,$$

the observed return process valued in $(-1, \infty)$, where we set by convention Z_0 to some fixed constant.

The investor may also consume continuously from the bank account (interest rate is assumed w.l.o.g to be zero) between two trading dates. We introduce the continuous observation filtration $\mathbb{G}^c = (\mathcal{G}_t)_{t \geq 0}$ with

$$\mathcal{G}_t = \sigma\{(\tau_k, Z_k) : \tau_k \leq t\},$$

and the discrete observation filtration $\mathbb{G}^d = (\mathcal{G}_{\tau_k})_{k \geq 0}$. Notice that $\mathcal{G}_t$ is trivial for $t < \tau_1$.

A control policy is a mixed discrete-continuous process (α, c), where $\alpha = (\alpha_k)_{k \geq 1}$ is real-valued $\mathbb{G}^d$-predictable, i.e. α_k is $\mathcal{G}_{\tau_{k-1}}$-measurable, and $c = (c_t)_{t \geq 0}$ is a non-negative $\mathbb{G}^c$-predictable process: α_k represents the amount of stock invested for the period $(\tau_{k-1}, \tau_k]$ after observing the stock price at time τ_{k-1}, and c_t is the consumption rate at time t based on the available information. Starting from an initial capital $x \geq 0$, and given a control policy (α, c), we denote X_k^x the wealth of the investor at time τ_k defined by

$$X_k^x = x - \int_0^{\tau_k} c_t \, dt + \sum_{i=1}^k \alpha_i Z_i, \quad k \geq 1, \qquad X_0^x = x. \tag{2.1}$$

Given $x \geq 0$, we say that a control policy (α, c) is admissible, and we denote $(\alpha, c) \in \mathcal{A}(x)$ if

$$X_k^x \geq 0, \quad \text{a.s. } \forall k \geq 1. \tag{2.2}$$

We are interested in the optimal portfolio/consumption problem:

$$v(x) = \sup_{(\alpha, c) \in \mathcal{A}(x)} \mathbb{E}\left[\int_0^\infty e^{-\rho t} U(c_t) \, dt \right], \quad x \geq 0, \tag{2.3}$$

where $\rho > 0$ is a positive discount factor, and U is an utility function defined on $\mathbb{R}_+$, with l.o.g. $U(0) = 0$, nondecreasing, concave and C^1 on $(0, \infty)$ satisfying the Inada conditions $U'(0^+) = \infty$ and $U'(\infty) = 0$. We shall assume the following growth condition on U: there exists $\gamma \in (0, 1)$ s.t.

$$U(x) \leq K_1 x^\gamma, \quad x \geq 0, \tag{2.4}$$

for some positive constant K_1. We denote $\tilde{U}$ the convex conjugate of U i.e.:

$$\tilde{U}(y) = \sup_{x>0}[U(x) - xy], \quad y \geq 0. \tag{2.5}$$

Notice that $\tilde{U}$ is nonincreasing, $\tilde{U}(\infty) = U(0)$, and under (2.4) we have

$$\tilde{U}(y) \leq \tilde{K}_1 y^{-\tilde{\gamma}}, \quad y \geq 0, \text{ with } \tilde{\gamma} = \frac{\gamma}{1 - \gamma} > 0, \tag{2.6}$$

for some positive constant $\tilde{K}_1$ (actually $\tilde{K}_1 = \frac{(K_1 \gamma)^{\frac{1}{1-\gamma}}}{\tilde{\gamma}}$).

Remark 2.1 Denote by $\mu(dt, dz) = \sum_{k=1}^\infty \delta_{(\tau_k, Z_k)} dt\, dz$ the integer-valued random measure associated to the multivariate point process $(\tau_k, Z_k)_{k \geq 1}$. Let us then consider the piecewise deterministic controlled jump process:

$$\bar{X}_t^x = x - \int_0^t c_t dt + \int_0^t \int \bar{\alpha}_t z \mu(dt, dz), \tag{2.7}$$

where $\bar{\alpha} = (\bar{\alpha}_t)_{t \geq 0}$ is a $\mathbb{G}^c$-predictable control process, $c = (c_t)_{t \geq 0}$ is a nonnegative $\mathbb{G}^c$-predictable control process, and define the related standard continuous control problem:

$$\bar{v}(x) = \sup_{(\bar{\alpha}, c) \in \bar{\mathcal{A}}(x)} \mathbb{E}\left[\int_0^\infty e^{-\rho t} U(c_t) dt\right], \quad x \geq 0, \tag{2.8}$$

where $\bar{\mathcal{A}}(x)$ is the set of control processes $(\bar{\alpha}, c)$ s.t. $\bar{X}_t^x \geq 0$, for all $t \geq 0$. This control problem is interpreted as a consumption/investment problem where the investor may consume and trade continuously in a stock price whose return process $\bar{Z}$ is modelled as a pure jump process of dynamics $d\bar{Z}_t = \int z \mu(dt, dz)$. Problems of type (2.7), (2.8) belong to the class of piecewise deterministic jump Markov processes, see e.g. the book by Davis [4], and lead to integrodifferential equations for the corresponding value functions. The link with our original control problem (2.3) is the following. Given $(\alpha, c) \in \mathcal{A}(x)$, if we define the predictable process $\bar{\alpha}$ by $\bar{\alpha}_t = \sum_k \alpha_k 1_{(\tau_{k-1}, \tau_k]}(t)$, for $t \geq 0$, then it is easy to see that $(\bar{\alpha}, c) \in \bar{\mathcal{A}}(x)$, so that $v(x) \leq \bar{v}(x)$. Problem (2.3) is a mixed discrete/continuous-time stochastic control problem: this is a nonstandard control problem, and we cannot derive directly the Backward or Bellman equation associated to (2.3).

In the rest of the paper, the following conditions on (τ_k, Z_k) stand in force:

(H1) $\{\tau_k\}_{k\geq 1}$ is the sequence of jump times of a Poisson process with intensity λ.

(H2) (i) For all $k \geq 1$, conditionally on the interarrival time $\tau_k - \tau_{k-1} = t \in \mathbb{R}_+$, Z_k is independent from $\{\tau_i, Z_i\}_{i<k}$ and has a distribution denoted $p(t, dz)$.

    (ii) For all $t \geq 0$, the support of $p(t, dz)$ is:

- either an interval with interior equal to $(-\underline{z}, \bar{z})$, $\underline{z} \in (0, 1]$ and $\bar{z} \in (0, \infty]$,
- or is finite equal to $\{-\underline{z}, \ldots, \bar{z}\}$, $\underline{z} \in (0, 1]$ and $\bar{z} \in (0, \infty)$.

(H3) $\int z p(t, dz) \geq 0$, for all $t \geq 0$, and there exist some $\kappa \in \mathbb{R}_+$ and $b \in \mathbb{R}_+$ s.t.

$$\int (1 + z) p(t, dz) \leq \kappa e^{bt}, \quad \forall t \geq 0.$$

Remark 2.2 The assumption (H1) that random trading times occurs via a Poisson process is a simplified story for liquidity constraints, and could be extended by considering for instance Cox processes. Here, the modelling Poisson process simplifies the explicit derivation of the equations arising below from the dynamic programming principle, and in the limit as the intensity of the Poisson process increases to infinity, provides a valuable comparison with the original Merton problem. Assumption (H2)(i) means that the return process has stationary and independent increments, and is satisfied typically when it is extracted from a Lévy model for price process. The condition (H2)(ii) is not restrictive and is precised here simply for expliciting the a-posteriori bounds on the controls (see Remark 2.3). It is easy to see that if the support of Z_k is included in $(0, \infty)$, i.e. the sequence $(S_k)_k$ is increasing, or is included in $(-1, 0)$, i.e. $(S_k)_k$ is decreasing, then the value function v is infinite. Indeed, suppose that $\underline{z} > 0$. Then, one can consume as much as wanted, by buying enough stocks in order to satisfy the admissibility condition, so that v is infinite. A similar argument is valid (by selling actions) when $\bar{z} < 0$. The condition $\int z p(t, dz) \geq 0$ in (H3) is simply put for financial interpretation, but could be relaxed. The other condition in (H3) is a more crucial technical one.

Remark 2.3 Since $X_{k+1}^x = X_k^x - \int_{\tau_k}^{\tau_{k+1}} c_u du + \alpha_{k+1} Z_{k+1}$, and by the condition (H2) on the support of Z_{k+1}, we see that the admissibility condition 2.2) is written as:

$$X_k^x - \int_{\tau_k}^s c_u du + \alpha_{k+1} z \geq 0, \quad \forall k \geq 0, \ \forall s \geq \tau_k, \ \forall z \in \{-\underline{z}, \bar{z}\}$$

almost surely. This may be also formulated directly in terms of $(\alpha, c) \in \mathcal{A}(x)$ as:

$$-\frac{X_k^x}{\bar{z}} \leq \alpha_{k+1} \leq \frac{X_k^x}{\underline{z}}, \quad \forall k \geq 0, \tag{2.9}$$

$$\int_{\tau_k}^s c_u du \leq X_k^x - \ell(\alpha_{k+1}), \quad \forall k \geq 0, \ \forall s \geq \tau_k, \tag{2.10}$$

where we set for all $a \in \mathbb{R}$:

$$\ell(a) = \max(a\underline{z}, -a\bar{z}),$$

with the convention that $\max(a\underline{z}, -a\bar{z}) = a\underline{z}$ when $\bar{z} = \infty$. In particular, we see that for $x = 0$, $\mathcal{A}(0) = \{0, 0\}$ and so $v(0) = 0$. Notice that in the usual case of stock price with distribution support $(0, \infty)$, i.e. $\underline{z} = 1$ and $\bar{z} = \infty$, as in Example 2.1, we have $\ell(a) = a$, and the bounds in (2.9) is written as $\alpha_{k+1} \in [0, X_k^x]$.

The following simple but important examples illustrate these assumptions (H1)–(H3).

Example 2.1 S is extracted from a Black-Scholes model: $dS_t = bS_t dt + \sigma S_t dW_t$, with $b \geq 0$, $\sigma > 0$. Then $p(t, dz)$ is the distribution of

$$Z(t) = \exp\left[\left(b - \frac{\sigma^2}{2}\right)t + \sigma W_t\right] - 1,$$

with support $(-1, \infty)$, and (H3) is clearly satisfied, since in this case $\int(1 + z) \times p(t, dz) = \mathbb{E}[\exp((b - \sigma^2/2)t + \sigma W_t)] = e^{bt}$.

Example 2.2 Z_k is independent of the waiting times $\tau_k - \tau_{k-1}$, in which case its distribution $p(dz)$ does not depend on t. In particular, $p(dz)$ may be a discrete distribution with support $\{z_0, \ldots, z_d\}$ s.t. $\underline{z} = -z_0 \in (0, 1]$ and $z_d = \bar{z} \in (0, \infty)$.

3 A First-Order Coupled System of Nonlinear IPDE

In this section, we derive formally the coupled system of Integro Partial Differential Equation (IPDE) that will be satisfied by the value function of our control problem. The starting point is the following version of the dynamic programming principle (DPP), which takes this simple form in our context:

$$v(x) = \sup_{(\alpha,c) \in \mathcal{A}(x)} \mathbb{E}\left[\int_0^{\tau_1} e^{-\rho t} U(c_t)dt + e^{-\rho\tau_1} v(X_1^x)\right]. \tag{3.1}$$

This DPP is quite natural, but a precise reference is not easily found in the literature. For sake of completeness, we provide a rigorous proof in the Appendix. We shall then prove in Sect. 5 that the original value function is characterized as the unique (viscosity) solution to a coupled integrodifferential system arising from the DPP.

From the expression (2.1) of the wealth, and the measurability conditions on the control, the above dynamic programming relation is written as

$$v(x) = \sup_{(a,c) \in A_d(x)} \mathbb{E}\left[\int_0^{\tau_1} e^{-\rho t} U(c_t)dt + e^{-\rho\tau_1} v\left(x - \int_0^{\tau_1} c_t dt + aZ_1\right)\right], \tag{3.2}$$

where $A_d(x)$ is the set of pairs (a, c) with a deterministic constant, and c a deterministic nonnegative process s.t. (see Remark 2.3) $a \in [-x/\bar{z}, x/\underline{z}]$ and

$$\int_0^t c_u du \leq x - \ell(a) \quad \text{i.e.} \quad x - \int_0^t c_u du + az \geq 0, \quad \forall t \geq 0, \ \forall z \in (-\underline{z}, \bar{z}). \tag{3.3}$$

Given $a \in [-x/\bar{z}, x/\underline{z}]$, we denote by $\mathcal{C}_a(x)$ the set of deterministic nonnegative processes satisfying (3.3). Moreover, under conditions (H1) and H2), we may explicit (see also details in Lemma 4.1) the r.h.s. of (3.2) so that:

$$v(x) = \sup_{\substack{a \in [-\frac{x}{\bar{z}}, \frac{x}{\underline{z}}] \\ c \in \mathcal{C}_a(x)}} \int_0^\infty e^{-(\rho+\lambda)t} \left[U(c_t) + \lambda \int v(x - \int_0^t c_s ds + az) p(t, dz) \right] dt.$$

(3.4)

Let

$$\mathcal{D} = \mathbb{R}_+ \times \mathcal{X} \quad \text{with } \mathcal{X} = \left\{ (x, a) \in \mathbb{R}_+ \times \mathbb{R} : -\frac{x}{\bar{z}} \leq a \leq \frac{x}{\underline{z}} \right\}. \tag{3.5}$$

By setting $A = \mathbb{R}$ if $\bar{z} < \infty$, and $A = \mathbb{R}_+$ if $\bar{z} = \infty$, notice that $\mathcal{X}$ is written also as

$$\mathcal{X} = \{ (x, a) \in \mathbb{R}_+ \times A : x \geq \ell(a) \}.$$

Now, we introduce the dynamic auxiliary control problem: for $(t, x, a) \in \mathcal{D}$,

$$\hat{v}(t, x, a) = \sup_{c \in \mathcal{C}_a(t,x)} \int_t^\infty e^{-(\rho+\lambda)(s-t)} \left[U(c_s) + \lambda \int v(Y_s^{t,x} + az) p(s, dz) \right] ds,$$

(3.6)

where $\mathcal{C}_a(t, x)$ is the set of deterministic nonnegative processes $c = (c_s)_{s \geq t}$ s.t.

$$\int_t^s c_u du \leq x - \ell(a) \quad \text{i.e.} \quad Y_s^{t,x} + az \geq 0, \quad \forall s \geq t, \ \forall z \in (-\underline{z}, \bar{z}), \tag{3.7}$$

and $Y^{t,x}$ is the deterministic controlled process by $c \in \mathcal{C}_a(t, x)$:

$$Y_s^{t,x} = x - \int_t^s c_u du, \quad s \geq t. \tag{3.8}$$

We shall see later (see Proposition 4.2) that $\hat{v}$ lies in $C_+(\mathcal{D})$, the set of nonnegative continuous functions on $\mathcal{D}$. From (3.4)–(3.6), the original value function is then related to this auxiliary optimization problem by

$$v = \mathcal{H}\hat{v}, \tag{3.9}$$

where $\mathcal{H}$ is the operator mapping $C_+(\mathcal{D})$ into the set $\mathcal{B}_+(\mathbb{R}_+)$ of nonnegative measurable functions on $\mathbb{R}_+$ by

$$\mathcal{H}\hat{w}(x) = \sup_{a \in [-x/\bar{z}, x/\underline{z}]} \hat{w}(0, x, a). \tag{3.10}$$

Actually, we shall see in Proposition 4.2 that v is continuous on $\mathbb{R}_+$, and so lies in $C_+(\mathbb{R}_+)$ the set of nonnegative and continuous functions on $\mathbb{R}_+$.

Remark 3.1 For a given $a \in A$, $\hat{v}$ is the value function of an optimal consumption/problem over an infinite horizon in a certain environment:

$$\hat{v}(t, x, a) = \sup_{c \in \mathcal{C}_a(t,x)} \int_t^\infty e^{-(\rho+\lambda)(s-t)} V_a(s, Y_s^{t,x}, c_s) ds,$$

where V_a is a modified utility function depending not only on the current consumption rate c_s, but also on the cumulated consumption $\int c_s ds$.

At this stage, we may study the deterministic control problem (3.6) by standard dynamic programming methods: the associated Hamilton–Jacobi equation is

$$\sup_{c \geq 0} \left[-(\rho + \lambda)\hat{v} + \frac{\partial \hat{v}}{\partial t} - c\frac{\partial \hat{v}}{\partial x} + U(c) + \lambda \int v(x + az)p(t, dz) \right] = 0,$$

$$(t, x, a) \in \mathcal{D},$$

that may be rewritten as a first order Integro Partial Differential Equation (IPDE)

$$(\rho + \lambda)\hat{v} - \frac{\partial \hat{v}}{\partial t} - \tilde{U}\left(\frac{\partial \hat{v}}{\partial x}\right) - \lambda \int v(x + az)p(t, dz) = 0, \quad (t, x, a) \in \mathcal{D}. \quad (3.11)$$

Remark 3.2 In the particular case where the distribution $p(t, dz) = p(dz)$ does not depend on t, then the above IPDE reduces to the integro ordinary differential equation for $\hat{v}(x, a)$:

$$(\rho + \lambda)\hat{v} - \tilde{U}\left(\frac{\partial \hat{v}}{\partial x}\right) - \lambda \int v(x + az)p(dz) = 0, \quad (t, x, a) \in \mathcal{D},$$

with $v(x) = \sup_{a \in [-x/\bar{z}, x/\underline{z}]} \hat{v}(x, a)$.

We have then splitted our original stochastic optimization problem into two coupled tractable deterministic optimization problems: Problem (3.6) is a family over $a \in A$ of standard deterministic control problems on infinite horizon, which is stationary (i.e. $\hat{v}$ does not depend on t), whenever the distribution $p(t, dz)$ does not depend on t, and problem (3.9) is a classical one-dimensional extremum problem over a. Notice that these two optimization problems are coupled since the reward function appearing in the definition of problem (3.6) or in its IPDE (3.11) depends on the value function of problem (3.9) and vice-versa. However, this suggests a fixed point algorithm for numerically solving our original optimization problem, and this is the topic of the accompanying paper [12].

In this paper, we focus on the rigorous unique characterization of the value for the original control problem (2.3) by means of viscosity solutions to the coupled IPDE (3.11), (3.9).

4 Some Properties on the Value Functions

We state some preliminary properties on the value functions that will be used in the next section for the characterization by means of viscosity solutions. We start with the following two lemmas.

Lemma 4.1 *Assume* (H1)–(H2) *hold. Let* $w \in \mathcal{B}_+(\mathbb{R}_+)$. *Then, for any* $x \geq 0$, $(\alpha, c) \in \mathcal{A}(x)$, $k \geq 0$, *we have*

$$\mathbb{E}\left[\int_{\tau_k}^{\tau_{k+1}} e^{-\rho(t-\tau_k)}U(c_t)dt + e^{-\rho(\tau_{k+1}-\tau_k)}w(X_{k+1}^x)\bigg|\mathcal{G}_{\tau_k}\right]$$

$$= \int_{\tau_k}^{\infty} e^{-(\rho+\lambda)(t-\tau_k)}\left[U(c_t) + \lambda\int w\left(X_k^x\right.\right.$$

$$\left.\left. - \int_{\tau_k}^{t} c_u du + \alpha_{k+1}z\right)p(t-\tau_k, dz)\right]dt.$$

Proof Since $X_{k+1}^x = X_k^x - \int_{\tau_k}^{\tau_{k+1}} c_u du + \alpha_{k+1}Z_{k+1}$, we have by the law of conditional toy expectations:

$$\mathbb{E}\left[\int_{\tau_k}^{\tau_{k+1}} e^{-\rho(t-\tau_k)}U(c_t)dt + e^{-\rho(\tau_{k+1}-\tau_k)}w(X_{k+1}^x)\bigg|\mathcal{G}_{\tau_k}\right]$$

$$= \mathbb{E}\left[\int_{\tau_k}^{\tau_{k+1}} e^{-\rho(t-\tau_k)}U(c_t)dt\right.$$

$$\left. + e^{-\rho(\tau_{k+1}-\tau_k)}\mathbb{E}\left[w\left(X_k^x - \int_{\tau_k}^{\tau_{k+1}} c_u du + \alpha_{k+1}Z_{k+1}\right)\bigg|\mathcal{G}_{\tau_k}, \tau_{k+1} - \tau_k\right]\bigg|\mathcal{G}_{\tau_k}\right]$$

$$= \mathbb{E}\left[\int_{\tau_k}^{\tau_{k+1}} e^{-\rho(t-\tau_k)}U(c_t)dt\right.$$

$$\left. + e^{-\rho(\tau_{k+1}-\tau_k)}\int w\left(X_k^x - \int_{\tau_k}^{\tau_{k+1}} c_u du + \alpha_{k+1}z\right)p(\tau_{k+1} - \tau_k, dz)\bigg|\mathcal{G}_{\tau_k}\right]$$

$$= \int_0^{\infty}\left[\int_{\tau_k}^{\tau_k+s} e^{-\rho(t-\tau_k)}U(c_t)dt\right.$$

$$\left. + e^{-\rho s}\int w\left(X_k^x - \int_{\tau_k}^{\tau_k+s} c_u du + \alpha_{k+1}z\right)p(s, dz)\right]\lambda e^{-\lambda s}ds,$$

where we used (H2) in the second equality and (H1) in the last one. We conclude with Fubini's theorem and the change of variable $s \to s + \tau_k$. □

Lemma 4.2 *Under* (H1)–(H3), *and* (2.4), *suppose that ρ satisfies*

$$\rho > b\gamma + \lambda\left(\frac{\kappa^\gamma}{\underline{z}^\gamma} - 1\right). \tag{4.1}$$

Then, for all $x \geq 0$, $(\alpha, c) \in \mathcal{A}(x)$, we have

$$\mathbb{E}\left[e^{-\rho\tau_n}(X_n^x)^\gamma\right] \leq x^\gamma \delta^n, \tag{4.2}$$

where

$$\delta = \frac{\lambda}{\rho - b\gamma + \lambda\frac{\kappa^\gamma}{\underline{z}^\gamma}} < 1. \tag{4.3}$$

In particular, $\mathbb{E}[e^{-\rho\tau_n}(X_n^x)^\gamma]$ converges to 0, as n goes to ∞.

Proof Observe from Jensen's inequality and conditions (H2), (H3) that for all $x \geq 0$, $(\alpha, c) \in \mathcal{A}(x)$, $n \geq 1$,

$$
\mathbb{E}\big[(X_{n-1}^x + \alpha_n Z_n)^\gamma \big| \mathcal{G}_{\tau_{n-1}}, \tau_n - \tau_{n-1}\big] \leq \left(X_{n-1}^x + \alpha_n \int z p(\tau_n - \tau_{n-1}, dz)\right)^\gamma
$$

$$
\leq \left(X_{n-1}^x + \frac{X_{n-1}^x}{\underline{z}}(\kappa e^{b(\tau_n - \tau_{n-1})} - 1)\right)^\gamma
$$

$$
\leq (X_{n-1}^x)^\gamma \frac{\kappa^\gamma}{\underline{z}^\gamma} e^{b\gamma(\tau_n - \tau_{n-1})}, \quad \text{a.s.} \quad (4.4)
$$

where we used also in the second inequality the bound (2.9) on α_n, and in the last one the fact that $\underline{z} \leq 1$. Thus, by writing that $X_n^x \leq X_{n-1}^x + \alpha_n Z_n$, and by the law of iterated conditional expectations, we get:

$$
\mathbb{E}\big[e^{-\rho \tau_k}(X_n^x)^\gamma\big] \leq \mathbb{E}\left[e^{-(\rho - b\gamma)(\tau_n - \tau_{n-1})} e^{-\rho \tau_{n-1}} \frac{(X_{n-1}^x)^\gamma}{\underline{z}^\gamma} \kappa^\gamma\right]
$$

$$
= \mathbb{E}\left[e^{-\rho \tau_{n-1}} \frac{(X_{n-1}^x)^\gamma}{\underline{z}^\gamma} \kappa^\gamma \int_0^\infty \lambda e^{-(\rho - b\gamma + \lambda)t} dt\right]
$$

$$
= \delta \mathbb{E}\big[e^{-\rho \tau_{n-1}}(X_{n-1}^x)^\gamma\big]
$$

where we used condition (H1) in the first equality. We obtain the required result by induction on n, and the convergence since $\delta < 1$ under (4.1). $\qquad\square$

Remark 4.1 In the case where $\int z p(t, dz) \leq 0$, and by assuming $-\int z p(t, dz) \leq \kappa e^{bt}$ for some $\kappa, b \in \mathbb{R}_+$, the inequality (4.4) should be replaced by:

$$
\mathbb{E}\big[(X_{n-1}^x + \alpha_n Z_n)^\gamma \big| \mathcal{G}_{\tau_{n-1}}, \tau_n - \tau_{n-1}\big] \leq (X_{n-1}^x)^\gamma \left(1 + \frac{\kappa^\gamma}{\overline{z}^\gamma} e^{b\gamma(\tau_n - \tau_{n-1})}\right), \quad \text{a.s.}
$$

Then, by same arguments as in the above lemma, we obtain $\mathbb{E}[e^{-\rho \tau_k}(X_n^x)^\gamma] \leq x^\gamma \delta^n$ with

$$
\delta = \frac{\lambda}{\rho + \lambda} + \frac{\lambda}{\rho - b\gamma + \lambda} \frac{\kappa^\gamma}{\overline{z}^\gamma}.
$$

Therefore, in this case, we get the convergence of $\mathbb{E}[e^{-\rho \tau_k}(X_n^x)^\gamma]$ to zero provided that

$$
\rho > b\gamma + \lambda \frac{\kappa^\gamma}{\overline{z}^\gamma}. \quad (4.5)
$$

The next result is a comparison principle for smooth solutions to the coupled IPDE (3.9)–(3.11).

Proposition 4.1 *Under* (H1)– (H3), (2.4) *and* (4.1), *suppose there exists* $\hat{w} \in C_+(\mathcal{D})$, C^1 *with respect to* (t, x), *and* $w \in C_+(\mathbb{R}_+)$ *satisfying:*

$$(\rho + \lambda)\hat{w} - \frac{\partial \hat{w}}{\partial t} - \tilde{U}\left(\frac{\partial \hat{w}}{\partial x}\right) - \lambda \int w(x + az) p(t, dz) \geq 0, \quad (t, x, a) \in \mathcal{D}, \quad (4.6)$$

$$w \geq \mathcal{H}\hat{w}, \quad (4.7)$$

together with the growth condition

$$w(x) \leq K(1 + x^\gamma), \quad \forall x \geq 0, \quad (4.8)$$

for some positive constant K. *Then*

$$\hat{v} \leq \hat{w} \quad and \quad v \leq w.$$

Proof 1. Given $x \in \mathbb{R}_+$, for all $(\alpha, c) \in \mathcal{A}(x)$, apply, for any $k \geq 0$, standard differential calculus to $e^{-(\rho + \lambda)(s - \tau_k)} \hat{w}(s - \tau_k, Y_s^{(k)}, \alpha_{k+1})$ between τ_k and T (to be sent to infinity) where $Y_s^{(k)} = X_k^x - \int_{\tau_k}^s c_u du$:

$$e^{-(\rho + \lambda)(T - \tau_k)} \hat{w}(T - \tau_k, Y_T^{(k)}, \alpha_{k+1})$$

$$= \hat{w}(0, X_k^x, \alpha_{k+1})$$

$$+ \int_{\tau_k}^T e^{-(\rho + \lambda)(s - \tau_k)} \left[-(\rho + \lambda)\hat{w} + \frac{\partial \hat{w}}{\partial t} - c_s \frac{\partial \hat{w}}{\partial x} \right](s - \tau_k, Y_s^{(k)}, \alpha_{k+1}) ds$$

$$\leq \hat{w}(0, X_k^x, \alpha_{k+1})$$

$$- \int_{\tau_k}^T e^{-(\rho + \lambda)(s - \tau_k)} \left[U(c_s) + \lambda \int w(Y_s^{(k)} + \alpha_{k+1}z) p(s - \tau_k, dz) \right] ds,$$

from (4.6). Now, since $\hat{w}$ is nonnegative, we get by sending T to infinity:

$$\int_{\tau_k}^\infty e^{-(\rho + \lambda)(s - \tau_k)} \left[U(c_s) + \lambda \int w(Y_s^{(k)} + \alpha_{k+1}z) p(s - \tau_k, dz) \right] ds$$

$$\leq \hat{w}(0, X_k^x, \alpha_{k+1}).$$

From Lemma 4.1, this is written as:

$$\mathbb{E}\left[\int_{\tau_k}^{\tau_{k+1}} e^{-\rho(s - \tau_k)} U(c_s) ds + e^{-\rho(\tau_{k+1} - \tau_k)} w(X_{k+1}^x) \,\bigg|\, \mathcal{G}_{\tau_k} \right] \leq \hat{w}(0, X_k^x, \alpha_{k+1})$$

$$\leq w(X_k^x),$$

where we used in the last inequality, (2.9) and the fact that $\mathcal{H}\hat{w} \leq w$. By induction on k and the law of iterated conditional expectations, we deduce

$$\mathbb{E}\left[\int_0^{\tau_n} e^{-\rho t} U(c_t) dt + e^{-\rho \tau_n} w(X_n^x) \right] \leq w(x),$$

for all n. Now, from the growth condition (4.8) and Lemma 4.2, we have

$$\mathbb{E}\left[e^{-\rho\tau_n}w(X_n^x)\right] \longrightarrow 0, \tag{4.9}$$

as n goes to infinity. Therefore, we obtain

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}U(c_t)dt\right] \leq w(x),$$

which proves from the arbitrariness of (α, c) that $w \geq v$.

2. Given $(t, x, a) \in \mathcal{D}$, apply standard differential calculus to $e^{-(\rho+\lambda)(s-t)}$ $\times \hat{w}(s, Y_s^{t,x}, a)$ between t and T (to be sent to infinity) where $Y_s^{t,x} = x - \int_t^s c_u du$, and c is arbitrary in $\mathcal{C}_a(t, x)$. Then, by similar arguments as in 1), we obtain

$$\hat{w}(t, x, a) \geq \int_t^\infty e^{-(\rho+\lambda)(s-t)}\left[U(c_s) + \lambda \int w(Y_s^{t,x} + az)p(s, dz)\right]ds$$

$$\geq \int_t^\infty e^{-(\rho+\lambda)(s-t)}\left[U(c_s) + \lambda \int v(Y_s^{t,x} + az)p(s, dz)\right]ds,$$

where we used in the second inequality the fact that $w \geq v$. From the arbitrariness of c, we conclude that $\hat{w} \geq \hat{v}$. $\qquad\square$

As a consequence of the above comparison principle, we state a growth condition on the value functions.

Corollary 4.1 *Under* (H1)–(H3), (2.4), *and* (4.1), *there exists some positive constant K s.t.*

$$\hat{v}(t, x, a) \leq K(e^{bt}x)^\gamma, \quad \forall (t, x, a) \in \mathcal{D}, \tag{4.10}$$

$$v(x) \leq Kx^\gamma, \qquad \forall x \geq 0. \tag{4.11}$$

Proof For ρ large enough, actually satisfying (4.1), we claim that one may find some constants $K \geq 0$ and β s.t.

$$\hat{w}(t, x, a) = Ke^{\beta t}x^\gamma, \quad (t, x, a) \in \mathcal{D}, \tag{4.12}$$

$$w = \mathcal{H}\hat{w}, \tag{4.13}$$

satisfies (4.6), (4.7). Indeed, similarly as in (4.4), we notice from Jensen's inequality and conditions (H2), (H3) that for all $(t, x, a) \in \mathcal{D}$,

$$\int (x + az)^\gamma p(t, dz) \leq \left(x + a\int zp(t, dz)\right)^\gamma \leq x^\gamma \frac{\kappa^\gamma}{\underline{z}^\gamma}e^{b\gamma t}. \tag{4.14}$$

Then, with this choice of $\hat{w}$, noting that $w(x) = Kx^\gamma$, and recalling (2.6), we have for all $(t, x, a) \in \mathcal{D}$,

$$(\rho + \lambda)\hat{w} - \frac{\partial\hat{w}}{\partial t} - \tilde{U}\left(\frac{\partial\hat{w}}{\partial x}\right) - \lambda\int w(x + az)p(t, dz)$$

$$\geq K e^{\beta t} x^\gamma (\rho + \lambda - \beta) - \tilde{K}_1 (K \gamma e^{\beta t} x^{\gamma - 1})^{-\tilde{\gamma}} - \lambda K \int (x + az)^\gamma p(t, dz)$$

$$\geq x^\gamma \left[K(\rho + \lambda - \beta) e^{\beta t} - K^{-\tilde{\gamma}} \tilde{K}_1 \gamma^{-\tilde{\gamma}} e^{-\beta \tilde{\gamma} t} - \lambda K \frac{\kappa^\gamma}{\underline{z}^\gamma} e^{b \gamma t} \right]. \tag{4.15}$$

By choosing $\beta = b\gamma$, we then get

$$(\rho + \lambda) \hat{w} - \frac{\partial \hat{w}}{\partial t} - \tilde{U} \left(\frac{\partial \hat{w}}{\partial x} \right) - \lambda \int w(x + az) p(t, dz)$$

$$\geq (K e^{\beta t})^{-\tilde{\gamma}} x^\gamma \left[K^{\frac{1}{1-\gamma}} \left(\rho - b\gamma + \lambda - \frac{\lambda \kappa^\gamma}{\underline{z}^\gamma} \right) e^{\frac{b\gamma}{1-\gamma} t} - \tilde{K}_1 \gamma^{-\tilde{\gamma}} \right].$$

Therefore, under (4.1), and by taking K positive s.t.

$$K^{\frac{1}{1-\gamma}} \left(\rho - b\gamma + \lambda - \frac{\lambda \kappa^\gamma}{\underline{z}^\gamma} \right) \geq \tilde{K}_1 \gamma^{-\tilde{\gamma}}, \tag{4.16}$$

the pair of functions $(\hat{w}, w)$ defined in (4.12), (4.13) is a supersolution to (4.6), (4.7), satisfying the growth condition (4.8). We conclude with Proposition 4.1. $\qquad \square$

Remark 4.2

1. In the case of Example 2.1, we have $\underline{z} = 1$ and $\kappa = 1$. Hence, from (4.1) and (4.16), we may take ρ and K large enough but independently of λ so that $v(x) \leq K x^\gamma$ for all $x \geq 0$. We then have a bound on v uniformly with respect to the intensity λ of the Poisson process. This is important once we want to study the asymptotic analysis of v when λ goes to infinity.
2. Similarly as in Remark 4.1, in the case where $\int z p(t, dz) \leq 0$, and by assuming $-\int z p(t, dz) \leq \kappa e^{bt}$ for some $\kappa, b \in \mathbb{R}_+$, the inequality (4.14) should be replaced by

$$\int (x + az)^\gamma p(t, dz) \leq \left(x + a \int z p(t, dz) \right)^\gamma \leq x^\gamma \left(1 + \frac{\kappa^\gamma}{\bar{z}^\gamma} e^{b \gamma t} \right).$$

Hence, by same arguments as above, we obtain the growth condition (4.10), (4.11) provided that ρ satisfies (4.5) and with K s.t.

$$K^{\frac{1}{1-\gamma}} \left(\rho - b\gamma - \frac{\lambda \kappa^\gamma}{\bar{z}^\gamma} \right) \geq \tilde{K}_1 \gamma^{-\tilde{\gamma}},$$

The point is that in this case ρ and K have to be chosen large enough, depending on λ.

We next prove the continuity of v and $\hat{v}$, and in particular the boundary condition imposed by the state constraint (3.7).

Proposition 4.2 *Assume that* (H1)–(H3), (2.4), *and* (4.1) *hold.*

(1) *The value function v is nondecreasing, concave and continuous on $\mathbb{R}_+$, with $v(0) = 0$.*

(2) *The value function $\hat{v}$ defined in* (3.6) *is continuous on $\mathcal{D}$, and*

$$\hat{v}(t, x, a) = \lambda \int_t^\infty e^{-(\rho+\lambda)(s-t)} \int v(x + az) p(s, dz) ds,$$

$$\forall t \geq 0, \ \forall (x, a) \in \partial \mathcal{X}. \tag{4.17}$$

Proof (1) Notice that for any $0 \leq x \leq x'$, and any given mixed control (α, c), we have $X_k^x \leq X_k^{x'}, k \geq 1$. This implies $\mathcal{A}(x) \subset \mathcal{A}(x')$ and so the nondecreasing property of v. The concavity property of v also follows by standard arguments using the linearity of X_k^x on x, (α, c), and the concavity of U.

Moreover, since v is finite on $\mathbb{R}_+$, it is continuous on $(0, \infty)$. Observe also from the growth condition (4.11) on the nonnegative value function v, that $v(0^+) = 0 = v(0)$. This shows that v is also continuous on $x = 0$.

(2i) We first prove the concavity of $\hat{v}(t, ., .)$ in $(x, a) \in \mathcal{X}$ for any $t \in \mathbb{R}_+$. Indeed, this follows from the linearity of the dynamics $Y^{t,x}$ in (3.8) in x, the linearity in (x, a) of the admissibility condition (3.7), and the concavity of the reward functions U and v appearing in the definition (3.6) of $\hat{v}$. Since we have also showed in (4.10) that $\hat{v}$ is finite on $\mathcal{D}$, this implies the continuity of v on the interior $\mathrm{int}(\mathcal{X})$ of $\mathcal{X}$.

(ii) We now show the continuity of $\hat{v}$ on $\partial \mathcal{X}$. Fix some $t \in \mathbb{R}_+$, and take some $(x_0, a_0) \in \partial \mathcal{X}$, i.e. $a_0 \in A$ and $x_0 = \ell(a_0)$. Since $\mathcal{C}_{a_0}(t, x_0) = \{0\}$ by (3.7), we have

$$\hat{v}(t, x_0, a_0) = \lambda \int_t^\infty e^{-(\rho+\lambda)(s-t)} \int v(x_0 + a_0 z) p(s, dz) ds. \tag{4.18}$$

Fix now some arbitrary $\varepsilon > 0$. By continuity of the function $a \in A \mapsto \ell(a)$, one can find some $\delta > 0$ s.t. for all $(x, a) \in \mathcal{X}_\delta = \{(x, a) \in \mathcal{X} : |x - x_0| + |a - a_0| < \delta\}$, we have $x - \ell(a) < \varepsilon^{1+1/\tilde{\gamma}}$, and so by (3.7)

$$\int_t^\infty c_s ds < \varepsilon^{1+\frac{1}{\tilde{\gamma}}}, \quad \forall c \in \mathcal{C}_a(t, x), \tag{4.19}$$

where $\tilde{\gamma}$ was defined in (2.6). Now, by choosing y s.t. $\tilde{K}_1 y^{-\tilde{\gamma}} = \varepsilon$ in (2.6), we have for any $c \geq 0$, $U(c) \leq \tilde{U}(y) + cy \leq \varepsilon + c(\tilde{K}_1/\varepsilon)^{\frac{1}{\tilde{\gamma}}}$. Hence, for all $(x, a) \in \mathcal{X}_\delta$, we have

$$\int_t^\infty e^{-(\rho+\lambda)(s-t)} U(c_s) ds \leq \int_t^\infty e^{-(\rho+\lambda)(s-t)} \left(\varepsilon + c_s (\tilde{K}_1/\varepsilon)^{\frac{1}{\tilde{\gamma}}} \right) ds$$

$$\leq \varepsilon \left(\frac{1}{\rho+\lambda} + \tilde{K}_1^{\frac{1}{\tilde{\gamma}}} \right), \quad \forall c \in \mathcal{C}_a(t, x),$$

by (4.19). We deduce for all $(x, a) \in \mathcal{X}_\delta$:

$$|\hat{v}(t, x, a) - \hat{v}(t, x_0, a_0)|$$

$$\leq \sup_{c \in \mathcal{C}_a(t, x)} \left[\int_t^\infty e^{-(\rho+\lambda)(s-t)} U(c_s) ds \right.$$

$$+ \lambda \int_t^\infty e^{-(\rho+\lambda)(s-t)} \int |v(Y_s^{t,x} + az) - v(x_0 + a_0 z)| p(s, dz) ds \Bigg]$$

$$\leq \varepsilon \left(\frac{1}{\rho + \lambda} + \tilde{K}_1^{\frac{1}{\gamma}} \right)$$

$$+ \sup_{c \in \mathcal{C}_a(t,x)} \lambda \int_t^\infty e^{-(\rho+\lambda)(s-t)} \int |v(Y_s^{t,x} + az) - v(x_0 + a_0 z)| p(s, dz) ds. \tag{4.20}$$

Consider first the case where $\bar{z} < \infty$. By noting that for all $c \in \mathcal{C}_a(t, x)$, $s \geq t$, $|Y_s^{t,x} - x_0| \leq |x - x_0| + |x - \ell(a)|$, and by continuity of the function v, one may still choose $\delta > 0$ small enough so that for all $(x, a) \in \mathcal{X}_\delta$:

$$\sup_{z \in [-\underline{z}, \bar{z}]} \left| v(Y_s^{t,x} + az) - v(x_0 + a_0 z) \right| \leq \varepsilon, \quad \forall s \geq t, \ \forall c \in \mathcal{C}_a(t, x). \tag{4.21}$$

By plugging into (4.20), we obtain for all $(x, a) \in \mathcal{X}_\delta$:

$$|\hat{v}(t, x, a) - \hat{v}(t, x_0, a_0)| \leq \varepsilon \left(\frac{1 + \lambda}{\rho + \lambda} + \tilde{K}_1^{\frac{1}{\gamma}} \right), \tag{4.22}$$

which proves the continuity of $\hat{v}$ on (t, x_0, a_0).

Consider now the case where $\bar{z} = \infty$. Then $A = \mathbb{R}_+$ and $x_0 = a_0 \underline{z}$. Suppose that $a_0 = 0$, and so $\hat{v}(t, x_0, a_0) = 0$ by (4.18). Recalling that v is nondecreasing, and from the growth condition (4.11) on v together with (4.14), we have for all $(x, a) \in \mathcal{X}_\delta$:

$$\int_{-\underline{z}}^\infty v(Y_s^{t,x} + az) p(s, dz) \leq \int_{-\underline{z}}^\infty v(x + az) p(s, dz) \leq K x^\gamma \frac{\kappa^\gamma}{\underline{z}^\gamma} e^{b\gamma s}$$

$$\leq K \varepsilon e^{b\gamma s}, \quad \forall s \geq t, \ \forall c \in \mathcal{C}_a(t, x),$$

by choosing δ s.t. $(\delta \kappa / \underline{z})^\gamma < \varepsilon$. By plugging into (4.20), we obtain for all $(x, a) \in \mathcal{X}_\delta$:

$$|\hat{v}(t, x, a) - \hat{v}(t, 0, 0)| \leq \varepsilon \left(\frac{1}{\rho + \lambda} + \tilde{K}_1^{\frac{1}{\gamma}} + \frac{\lambda K e^{b\gamma t}}{\rho + \lambda - b\gamma} \right), \tag{4.23}$$

which proves the continuity of $\hat{v}$ on $(t, 0, 0)$. Suppose $a_0 > 0$, i.e. $x_0 > 0$, so that w.l.o.g. we may assume that $\delta + \varepsilon^{1+1/\tilde{\gamma}} < x_0 / 2$. Hence, for all $(x, a) \in \mathcal{X}_\delta$, $c \in \mathcal{C}_a(t, x)$, we have by (4.19), $Y_s^{t,x} + az \geq x_0 / 2$, for any $t \leq s$, $z \geq 0$. Moreover, since the function v is concave and finite on $\mathbb{R}_+$, it is Lipschitz on $[x_0 / 2, \infty)$. Thus, for all $(x, a) \in \mathcal{X}_\delta$, $c \in \mathcal{C}_a(t, x)$, there exists some positive constant C_0 s.t.

$$\left| v(Y_s^{t,x} + az) - v(x_0 + a_0 z) \right| \leq C_0 \big(|Y_s^{t,x} - x_0| + |a - a_0| z \big)$$

$$\leq C_0 \big(\delta + \varepsilon^{1 + \frac{1}{\gamma}} + \delta z \big)$$

$$\leq C_0 \varepsilon (2 + z), \quad t \leq s, z \geq 0, \tag{4.24}$$

for $0 < \delta < \varepsilon < 1$. On the other hand, similarly as in (4.21), we have

$$\sup_{z \in [-\underline{z}, 0]} \left| v(Y_s^{t,x} + az) - v(x_0 + a_0 z) \right| \leq \varepsilon, \quad \forall s \geq t, \ \forall c \in \mathcal{C}_a(t, x). \tag{4.25}$$

By plugging (4.24), (4.25) into (4.20), we obtain for all $(x, a) \in \mathcal{X}_\delta$:

$$|\hat{v}(t, x, a) - \hat{v}(t, x_0, a_0)| \leq \varepsilon \left(\frac{1 + \lambda + \lambda C_0}{\rho + \lambda} + \tilde{K}_1^{\frac{1}{\gamma}} + \frac{\lambda C_0 \kappa e^{bt}}{\rho + \lambda - b} \right), \quad (4.26)$$

which proves the continuity of $\hat{v}$ on (t, x_0, a_0).

(iii) We next prove the continuity[1] of $\hat{v}(t, x, a)$ (and in fact equivalently of $e^{-(\rho+\lambda)t}\hat{v}(t, x, a)$) in $t \in \mathbb{R}_+$ for fixed $(x, a) \in \mathcal{X}$. From the dynamic programming principle applied to $\hat{v}(t, x, a)$, we have for $s > t$:

$$e^{-(\rho+\lambda)t}\hat{v}(t, x, a) = \sup_{c \in \mathcal{C}_a(t,x)} \left\{ \int_t^s e^{-(\rho+\lambda)u} \left[U(c_u) + \lambda \int v(Y_u^{t,x} + az)p(u, dz) \right] du \right.$$
$$\left. + e^{-(\rho+\lambda)s}\hat{v}(s, Y_s^{t,x}, a) \right\}.$$

By choosing in particular $c = 0$, and recalling that U, v are nonnegative, this shows that $e^{-(\rho+\lambda)t}\hat{v}(t, x, a)$ is nonincreasing in t. Moreover, since $Y^{t,x} \leq x$ and v (hence $\hat{v}$) is nondecreasing, this yields for all $0 \leq t < s$:

$$0 \leq e^{-(\rho+\lambda)t}\hat{v}(t, x, a) - e^{-(\rho+\lambda)s}\hat{v}(s, x, a)$$
$$\leq \sup_{c \in \mathcal{C}_a(t,x)} \left\{ \int_t^s e^{-(\rho+\lambda)u} \left[U(c_u) + \lambda \int v(x + az)p(u, dz) \right] du \right.$$
$$\left. + e^{-(\rho+\lambda)s}\hat{v}(s, Y_s^{t,x}, a) - e^{-(\rho+\lambda)s}\hat{v}(s, x, a) \right\}$$
$$\leq \sup_{c \in \mathcal{C}_a(t,x)} \left\{ \int_t^s e^{-(\rho+\lambda)u} \left[U(c_u) + \lambda \int v(x + az)p(u, dz) \right] du \right\}. \quad (4.27)$$

Now by Jensen's inequality and concavity of U, we get

$$\int_t^s e^{-(\rho+\lambda)u} U(c_u) du \leq (s - t) \frac{1}{s - t} \int_t^s U(c_u) du \leq (s - t) U \left(\frac{1}{s - t} \int_t^s c_u du \right).$$

Since for each control $c \in \mathcal{C}_a(t, x)$, $\int_t^s c_u du \leq x - \ell(a)$, then we get from (4.27) and recalling that U is nondecreasing:

$$0 \leq e^{-(\rho+\lambda)t}\hat{v}(t, x, a) - e^{-(\rho+\lambda)s}\hat{v}(s, x, a)$$
$$\leq (s - t) U \left(\frac{x - \ell(a)}{s - t} \right) + \lambda \int_t^s e^{-(\rho+\lambda)u} \int v(x + az)p(u, dz) du$$
$$\leq (s - t) \left(U \left(\frac{x - \ell(a)}{s - t} \right) + \lambda K x^\gamma \frac{\kappa^\gamma}{\underline{z}^\gamma} \right) := \omega(s - t), \quad (4.28)$$

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[1] We thank F. Gozzi and A. Cretarola for pointing out an error in a previous version in the proof for the continuity in time of $\hat{v}$, and for showing us the correct arguments.

where we used in the last inequality the growth condition (4.11) on v and (4.14). By using the growth condition (2.4) on U, we see that $\omega(s-t) \leq K_0(s-t)^{1-\gamma}$, for some $K_0 > 0$. This proves the continuity in t of $e^{-(\rho+\lambda)t}\hat{v}(t,x,a)$, and so of $\hat{v}(t,x,a)$.

(iv) Finally, by combining inequalities (4.22), (4.23), (4.26), and (4.28), we have the continuity of $\hat{v}$ in $(t,x,a) \in \mathcal{D}$. □

Remark 4.3 The arguments in the above proposition for proving the continuity of $\hat{v}$ on the boundary $\partial\mathcal{X}$ show that this boundary is absorbing: indeed, when $(x,a) \in \partial\mathcal{X}$, i.e. $x = \ell(a)$, the only admissible control for $c \in \mathcal{C}_a(t,x)$ is $c = 0$, so that the state process $Y^{t,x}$ remains at $\ell(a)$ once it reaches this threshold.

Remark 4.4 Notice from (4.17) that $\hat{v}$ is differentiable in t for $(x,a) \in \partial\mathcal{X}$, and so this boundary condition may be also formulated as:

$$\lim_{t\to\infty} e^{-(\rho+\lambda)t}\hat{v}(t,x,a) = 0, \quad \forall(x,a) \in \partial\mathcal{X},$$

$$(\rho+\lambda)\hat{v}(t,x,a) - \frac{\partial\hat{v}}{\partial t}(t,x) - \lambda\int v(x+az)p(t,dz) = 0, \qquad (4.29)$$

$$\forall t \geq 0, \ \forall(x,a) \in \partial\mathcal{X}.$$

5 Viscosity Characterization

We adapt now the notion of viscosity solutions to our context, i.e. for the coupled IPDE:

$$(\rho+\lambda)\hat{w} - \frac{\partial\hat{w}}{\partial t} - \tilde{U}\left(\frac{\partial\hat{w}}{\partial x}\right) - \lambda\int w(x+az)p(t,dz) = 0, \quad (t,x,a) \in \mathcal{D}, \ (5.1)$$

$$w = \mathcal{H}\hat{w}. \qquad (5.2)$$

Definition 5.1 A pair of functions $(w,\hat{w}) \in C_+(\mathbb{R}_+) \times C_+(\mathcal{D})$ is a viscosity solution to (5.1), (5.2) if:

(i) *viscosity supersolution property*: $w \geq \mathcal{H}\hat{w}$, and for all $a \in A$,

$$(\rho+\lambda)\hat{w}(\bar{t},\bar{x},a) - \frac{\partial\varphi}{\partial t}(\bar{t},\bar{x}) - \tilde{U}\left(\frac{\partial\varphi}{\partial x}(\bar{t},\bar{x})\right) - \lambda\int w(\bar{x}+az)p(\bar{t},dz) \geq 0,$$

for any test function $\varphi \in C^1(\mathbb{R}_+ \times (\ell(a),\infty))$, and $(\bar{t},\bar{x}) \in \mathbb{R}_+ \times (\ell(a),\infty)$, which is a local minimum of $(\hat{w}(.,.,a) - \varphi)$.

(ii) *viscosity subsolution property*: $w \leq \mathcal{H}\hat{w}$, and for all $a \in A$,

$$(\rho+\lambda)\hat{w}(\bar{t},\bar{x},a) - \frac{\partial\varphi}{\partial t}(\bar{t},\bar{x}) - \tilde{U}\left(\frac{\partial\varphi}{\partial x}(\bar{t},\bar{x})\right) - \lambda\int w(\bar{x}+az)p(\bar{t},dz) \leq 0,$$

for any test function $\varphi \in C^1(\mathbb{R}_+ \times (\ell(a),\infty))$, and $(\bar{t},\bar{x}) \in \mathbb{R}_+ \times (\ell(a),\infty)$, which is a local maximum of $(\hat{w}(.,.,a) - \varphi)$.

Our main result is a viscosity characterization of the value functions to our original control problem by means of viscosity solution to the coupled IPDE. This is achieved in two steps. We first prove, as usual, the viscosity property as a consequence of the dynamic programming principle. We then prove a new comparison principle for the coupled IPDE (5.1), (5.2). We make an additional continuity assumption on the measure $p(t, dz)$:

$$\lim_{t \to t_0} \int w(z) p(t, dz) = \int w(z) p(t_0, dz), \quad \forall t_0 \geq 0, \tag{5.3}$$

for all measurable functions w on $(-\underline{z}, \bar{z})$ with linear growth condition.

Theorem 5.1 *Assume that* (H1)–(H3), (2.4), (4.1) *and* (5.3) *hold. The pair of value functions* $(v, \hat{v})$ *defined in* (2.3), (3.6) *is the unique viscosity solution to* (5.1), (5.2), *satisfying the growth condition* (4.10), (4.11), *and the boundary condition* (4.17).

Proof 1. From the dynamic programming principle (3.1) proved in the Appendix, and following the arguments in (3.2)–(3.9), we prove that $v = \mathcal{H}\hat{v}$. Moreover, for each $a \in A$, $\hat{v}(., ., a)$ is the value function of a deterministic time-dependent control problem with state Y. Hence, standard dynamic programming principle in this context, see e.g. [6], yields the viscosity property of $\hat{v}(., ., a)$ to (5.1), and so the viscosity property of $(v, \hat{v})$ to (5.1), (5.2). The growth condition (4.10), (4.11), and the boundary condition (4.17) are proved in Corollary 4.1 and Proposition 4.2.

2. The main task is to prove the following comparison principle: if $(w_1, \hat{w}_1)$ (resp. $(w_2, \hat{w}_2)) \in C_+(\mathbb{R}_+) \times C_+(\mathcal{D})$ is a viscosity subsolution (resp. supersolution) to (5.1), (5.2), satisfying the growth condition (4.10), (4.11), and

$$\hat{w}_1(t, x, a) = \lambda \int_t^\infty e^{-(\rho+\lambda)(s-t)} \int w_1(x + az) p(s, dz) ds, \quad \forall t \geq 0, \ \forall (x, a) \in \partial \mathcal{X},$$

then $w_1 \leq w_2$, and $\hat{w}_1 \leq \hat{w}_2$. Uniqueness result is then a direct corollary.

Step 1. In a first step, we deal with the noncompactness of the domain (regarding the growth condition of $\hat{w}_1$, $\hat{w}_2$ in (t, x)) by constructing a suitable perturbation of the viscosity supersolution $(w_2, \hat{w}_2)$. Under (4.1), we can choose $\gamma' \in (\gamma, 1)$, and $\rho' > 0$ s.t.

$$b\gamma' < \rho' \leq \rho - \lambda \left(\frac{\kappa^{\gamma'}}{\underline{z}^{\gamma'}} - 1 \right). \tag{5.4}$$

Now, let for all $n \geq 1$, $\hat{w}_{2,n} = \hat{w}_2 + \frac{1}{n}\hat{\psi}$, $w_{2,n} = w_2 + \frac{1}{n}\psi$, , with $\hat{\psi}(t, x) = e^{\rho't}x^{\gamma'}$ and $\psi(x) = \mathcal{H}\hat{\psi}(x) = x^{\gamma'}$. From condition (H3), and by similar calculations as in (4.15), we see that for all $(t, x, a) \in \mathcal{D}$,

$$(\rho + \lambda)\hat{\psi} - \frac{\partial \hat{\psi}}{\partial t} - \lambda \int \psi(x + az) p(t, dz) \geq x^{\gamma'} \left[(\rho + \lambda - \rho')e^{\rho't} - \lambda \frac{\kappa^{\gamma'} e^{b\gamma't}}{\underline{z}^{\gamma'}} \right]$$

$$\geq x^{\gamma'} e^{\rho't} \left[\rho - \rho' + \lambda - \lambda \frac{\kappa^{\gamma'}}{\underline{z}^{\gamma'}} \right] \geq 0,$$

by (5.4). By noting also that $\tilde{U}$ is nonincreasing, we then deduce that $(w_{2,n}, \hat{w}_{2,n})$ is a viscosity supersolution to (5.1), (5.2). Moreover, from the growth condition (4.10) on $\hat{w}_1$, and $\hat{w}_2$, and since $\gamma' > \gamma$, $\rho' > b\gamma'$, we have for all $n \geq 1$:

$$\lim_{|(t,x)| \to \infty} \sup_{a \in A} (\hat{w}_1 - \hat{w}_{2,n})(t, x, a) = -\infty. \tag{5.5}$$

Step 2. We show that for all $n \geq 1$, $\hat{w}_1 \leq \hat{w}_{2,n}$ on $\mathcal{D}$. We argue by contradiction, and assume on the contrary that there exists some $n \geq 1$ s.t.

$$M := \sup_{(t,x,a) \in \mathcal{D}} (\hat{w}_1 - \hat{w}_{2,n})(t, x, a) > 0.$$

In this case, from (5.5) and by continuity of $\hat{w}_1$ and $\hat{w}_2$, there exists some compact subset $\mathcal{D}_0$ of $\mathcal{D}$, which may be chosen in the form $\mathcal{D}_0 = [0, T_0] \times \mathcal{X}_0$ with

$$\mathcal{X}_0 = \{(x, a) \in \mathcal{X} : x \leq x_0\} = \left\{ (x, a) \in \mathbb{R}_+ \times \left[-\frac{x_0}{\bar{z}}, \frac{x_0}{\underline{z}} \right] : x \in [\ell(a), x_0] \right\}$$

for some finite positive $T_0 > 0$ and $x_0 > 0$ (depending on n), and $(\bar{t}, \bar{x}, \bar{a}) \in \mathcal{D}_0$ with $\bar{t} < T_0$, $\bar{x} < x_0$ s.t.

$$M = \max_{(t,x,a) \in \mathcal{D}_0} (\hat{w}_1 - \hat{w}_{2,n})(t, x, a) = (\hat{w}_1 - \hat{w}_{2,n})(\bar{t}, \bar{x}, \bar{a}).$$

We distinguish the two cases depending on $(\bar{x}, \bar{a}) \in \partial \mathcal{X}$, i.e. $\bar{x} = \ell(\bar{a})$, or $(\bar{x}, \bar{a}) \notin \partial \mathcal{X}$, i.e. $\bar{x} > \ell(\bar{a})$.

Case 1: $\bar{x} > \ell(\bar{a})$. Following the general technique for comparison principle, we then consider, for any $\varepsilon > 0$, the function defined by

$$\Phi_\varepsilon(t, s, x, y) = \hat{w}_1(t, x, \bar{a}) - \hat{w}_{2,n}(s, y, \bar{a}) - \phi_\varepsilon(t, s, x, y),$$

$$\phi_\varepsilon(t, s, x, y) = \frac{|t - \bar{t}|^2}{2} + \frac{|x - \bar{x}|^3}{3} + \frac{|t - s|^2}{2\varepsilon} + \frac{|x - y|^2}{2\varepsilon}. \tag{5.6}$$

Since Φ_ε is continuous on the compact set $[0, T_0]^2 \times [\ell(\bar{a}), x_0]^2$, there exists $(t_\varepsilon, s_\varepsilon, x_\varepsilon, y_\varepsilon) \in [0, T_0]^2 \times [\ell(\bar{a}), x_0]^2$ s.t.

$$M_\varepsilon := \sup_{[0,T_0]^2 \times [\ell(\bar{a}), x_0]^2} \Phi_\varepsilon(t, s, x, y) = \Phi_\varepsilon(t_\varepsilon, s_\varepsilon, x_\varepsilon, y_\varepsilon),$$

and a subsequence, still denoted $(t_\varepsilon, s_\varepsilon, x_\varepsilon, y_\varepsilon)_{\varepsilon > 0}$, converging to some $(\bar{t}', \bar{s}', \bar{x}', \bar{y}')$ when ε goes to zero. Actually, by standard arguments in viscosity solutions theory, we have

$$(\bar{t}', \bar{s}', \bar{x}', \bar{y}') = (\bar{t}, \bar{t}, \bar{x}, \bar{x}). \tag{5.7}$$

For sake of completeness, we recall these arguments: by writing that $\Phi_\varepsilon(\bar{t}, \bar{t}, \bar{x}, \bar{x}) \leq M_\varepsilon = \Phi_\varepsilon(t_\varepsilon, s_\varepsilon, x_\varepsilon, y_\varepsilon)$, we have

$$M = \hat{w}_1(\bar{t}, \bar{x}, \bar{a}) - \hat{w}_{2,n}(\bar{t}, \bar{x}, \bar{a})$$

$$\leq M_\varepsilon = \hat{w}_1(t_\varepsilon, x_\varepsilon, \bar{a}) - \hat{w}_{2,n}(s_\varepsilon, y_\varepsilon, \bar{a}) - \frac{|t_\varepsilon - \bar{t}|^2}{2} - \frac{|x_\varepsilon - \bar{x}|^3}{3} - R_\varepsilon \quad (5.8)$$

$$\leq \hat{w}_1(t_\varepsilon, x_\varepsilon, \bar{a}) - \hat{w}_{2,n}(s_\varepsilon, y_\varepsilon, \bar{a}) - \frac{|t_\varepsilon - \bar{t}|^2}{2} - \frac{|x_\varepsilon - \bar{x}|^3}{3}, \quad (5.9)$$

where we set $R_\varepsilon = \frac{|t_\varepsilon - s_\varepsilon|^2}{2\varepsilon} + \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon}$. From the boundedness of $\hat{w}_1(.,.,\bar{a})$, $\hat{w}_{2,n}(.,.,\bar{a})$ on $[0, T_0] \times [\ell(\bar{a}), x_0]$, we deduce by inequality (5.8) the boundedness of the sequence $(R_\varepsilon)_\varepsilon$, which implies $\bar{t}' = \bar{s}'$, and $\bar{x}' = \bar{y}'$. Then, by sending ε to zero into (5.9), we obtain $M \leq \hat{w}_1(\bar{t}', \bar{x}', \bar{a}) - \hat{w}_{2,n}(\bar{t}', \bar{x}', \bar{a}) - \frac{|\bar{t}' - \bar{t}|^2}{2} - \frac{|\bar{x}' - \bar{x}|^3}{3} \leq M - \frac{|\bar{t}' - \bar{t}|^2}{2} - \frac{|\bar{x}' - \bar{x}|^3}{3}$ by definition of M. This shows $\bar{t}' = \bar{t}$, $\bar{x}' = \bar{x}$, and so (5.7). In particular, for ε small enough, we have $(t_\varepsilon, s_\varepsilon) \in [0, T_0)^2$ and $(x_\varepsilon, y_\varepsilon) \in (\ell(\bar{a}), x_0)^2$. Hence, Φ_ε admits a local maximum at $(t_\varepsilon, s_\varepsilon, x_\varepsilon, y_\varepsilon)$. This implies that the function $(t, x) \to \hat{w}_1(t, x, \bar{a}) - \varphi_1(t, x)$, with $\varphi_1(t, x) = \frac{|t - \bar{t}|^2}{2} + \frac{|x - \bar{x}|^3}{3} + \frac{|t - s_\varepsilon|^2}{2\varepsilon} + \frac{|x - y_\varepsilon|^2}{2\varepsilon}$, admits a local maximum at $(t_\varepsilon, x_\varepsilon)$. By writing the viscosity subsolution property of $(w_1, \hat{w}_1)$ to (5.1), (5.2) at $(t_\varepsilon, x_\varepsilon, \bar{a})$ with this test function φ_1, we have

$$(\rho + \lambda)\hat{w}_1(t_\varepsilon, x_\varepsilon, \bar{a}) - (t_\varepsilon - \bar{t}) - \frac{(t_\varepsilon - s_\varepsilon)}{\varepsilon} - \tilde{U}\left(|x_\varepsilon - \bar{x}|^2 + \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\right)$$

$$- \lambda \int w_1(x_\varepsilon + \bar{a}z)p(t_\varepsilon, dz) \leq 0. \quad (5.10)$$

Likewise, the function $(s, y) \to \hat{w}_{2,n}(s, y, \bar{a}) - \varphi_2(s, y)$, with $\varphi_2(s, y) = -\frac{|t_\varepsilon - s|^2}{2\varepsilon} - \frac{|x_\varepsilon - y|^2}{2\varepsilon}$, admits a local minimum at $(s_\varepsilon, y_\varepsilon)$. By writing the viscosity supersolution property of $(w_{2,n}, \hat{w}_{2,n})$ to (5.1), (5.2) at $(s_\varepsilon, y_\varepsilon, \bar{a})$ with this test function φ_2, we have

$$(\rho + \lambda)\hat{w}_{2,n}(s_\varepsilon, y_\varepsilon, \bar{a}) - \frac{(t_\varepsilon - s_\varepsilon)}{\varepsilon} - \tilde{U}\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\right)$$

$$- \lambda \int w_{2,n}(y_\varepsilon + \bar{a}z)p(s_\varepsilon, dz) \geq 0. \quad (5.11)$$

By subtracting (5.10) and (5.11), and since $\tilde{U}$ is nonincreasing, we obtain

$$(\rho + \lambda)\left(\hat{w}_1(t_\varepsilon, x_\varepsilon, \bar{a}) - \hat{w}_{2,n}(s_\varepsilon, y_\varepsilon, \bar{a})\right)$$

$$\leq (t_\varepsilon - \bar{t}) + \lambda\left[\int w_1(x_\varepsilon + \bar{a}z)p(t_\varepsilon, dz) - \int w_{2,n}(y_\varepsilon + \bar{a}z)p(s_\varepsilon, dz)\right].$$

By sending ε to zero, and from (5.3), (5.7), we get:

$$(\rho + \lambda)M = (\rho + \lambda)(\hat{w}_1 - \hat{w}_{2,n})(\bar{t}, \bar{x}, \bar{a}) \leq \lambda \int (w_1 - w_{2,n})(\bar{x} + \bar{a}z)p(\bar{t}, dz)$$

$$\leq \lambda \int (\mathcal{H}\hat{w}_1 - \mathcal{H}\hat{w}_{2,n})(\bar{x} + \bar{a}z)p(\bar{t}, dz),$$

since $w_1 \leq \mathcal{H}\hat{w}_1$ and $w_{2,n} \geq \mathcal{H}\hat{w}_{2,n}$. Finally, by noting from the definition of $\mathcal{H}$ and M that $\mathcal{H}\hat{w}_1 - \mathcal{H}\hat{w}_{2,n} \leq M$, we get the required contradiction $(\rho + \lambda)M \leq \lambda M$.

Case 2: $\bar{x} = \ell(\bar{a})$. Notice that (4.29) implies that the viscosity subsolution property for $(v, \hat{v})$ holds also at any $(t, x, a) \in \mathbb{R}_+ \times \partial \mathcal{X}$. However, this is not true for the viscosity supersolution property, and we have to modify the test function Φ_ε in (5.6) in order to ensure that for the local minimum point $(s_\varepsilon, y_\varepsilon)$, $(y_\varepsilon, \bar{a}) \notin \partial \mathcal{X}$, i.e. $y_\varepsilon > \ell(\bar{a})$. We follow arguments in Barles [1] (see its appendix paragraph 7.1.2). By continuity of $\hat{w}_{2,n}$ on $\mathcal{D}$, there exists a sequence $(\bar{t}_\varepsilon, \bar{x}_\varepsilon)_{\varepsilon > 0}$, with $\bar{x}_\varepsilon > \bar{x} = \ell(\bar{a})$, $\bar{t}_\varepsilon \neq \bar{t}$, converging to $(\bar{t}, \bar{x})$ s.t. $\hat{w}_{2,n}(\bar{t}_\varepsilon, \bar{x}_\varepsilon, \bar{a})$ tends to $\hat{w}_{2,n}(\bar{t}, \bar{x}, \bar{a})$ as ε goes to zero. We then consider the function

$$\Psi_\varepsilon(t, s, x, y) = \hat{w}_1(t, x, \bar{a}) - \hat{w}_{2,n}(s, y, \bar{a}) - \psi_\varepsilon(t, s, x, y)$$

$$\psi_\varepsilon(t, s, x, y) = \frac{|t - \bar{t}|^2}{2} + \frac{|x - \bar{x}|^3}{3} + \frac{|t - s|^2}{2|\bar{t}_\varepsilon - \bar{t}|} + \frac{|x - y|^2}{2|\bar{x}_\varepsilon - \bar{x}|} + \frac{1}{3}\left|\frac{y - \ell(\bar{a})}{\bar{x}_\varepsilon - \ell(\bar{a})} - 1\right|^3.$$

Since Ψ_ε is continuous on the compact set $[0, T_0]^2 \times [\ell(\bar{a}), x_0]^2$, there exists $(t_\varepsilon, s_\varepsilon, x_\varepsilon, y_\varepsilon) \in [0, T_0]^2 \times [\ell(\bar{a}), x_0]^2$ s.t.

$$N_\varepsilon := \sup_{[0, T_0]^2 \times [\ell(\bar{a}), x_0]^2} \Psi_\varepsilon(t, s, x, y) = \Psi_\varepsilon(t_\varepsilon, s_\varepsilon, x_\varepsilon, y_\varepsilon),$$

and a subsequence, still denoted $(t_\varepsilon, s_\varepsilon, x_\varepsilon, y_\varepsilon)_{\varepsilon > 0}$, converging to some $(\bar{t}', \bar{s}', \bar{x}', \bar{y}')$ when ε goes to zero. We claim that

$$(\bar{t}', \bar{s}', \bar{x}', \bar{y}') = (\bar{t}, \bar{t}, \bar{x}, \bar{x}), \tag{5.12}$$

$$N_\varepsilon \longrightarrow M, \tag{5.13}$$

$$\frac{|t_\varepsilon - s_\varepsilon|^2}{2|\bar{t}_\varepsilon - \bar{t}|} + \frac{|x_\varepsilon - y_\varepsilon|^2}{2|\bar{x}_\varepsilon - \bar{x}|} + \frac{1}{3}\left|\frac{y_\varepsilon - \ell(\bar{a})}{\bar{x}_\varepsilon - \ell(\bar{a})} - 1\right|^3 \longrightarrow 0. \tag{5.14}$$

Indeed, by writing that $\Psi_\varepsilon(\bar{t}, \bar{t}_\varepsilon, \bar{x}, \bar{x}_\varepsilon) \leq N_\varepsilon = \Psi_\varepsilon(t_\varepsilon, s_\varepsilon, x_\varepsilon, y_\varepsilon)$, we have

$$\hat{w}_1(\bar{t}, \bar{x}, \bar{a}) - \hat{w}_{2,n}(\bar{t}_\varepsilon, \bar{x}_\varepsilon, \bar{a}) - \frac{1}{2}(|\bar{t}_\varepsilon - \bar{t}| + |\bar{x}_\varepsilon - \bar{x}|) \tag{5.15}$$

$$\leq N_\varepsilon = \hat{w}_1(t_\varepsilon, x_\varepsilon, \bar{a}) - \hat{w}_{2,n}(s_\varepsilon, y_\varepsilon, \bar{a}) - \frac{|t_\varepsilon - \bar{t}|^2}{2} - \frac{|x_\varepsilon - \bar{x}|^3}{3} - R_\varepsilon \tag{5.16}$$

$$\leq \hat{w}_1(t_\varepsilon, x_\varepsilon, \bar{a}) - \hat{w}_{2,n}(s_\varepsilon, y_\varepsilon, \bar{a}) - \frac{|t_\varepsilon - \bar{t}|^2}{2} - \frac{|x_\varepsilon - \bar{x}|^3}{3}, \tag{5.17}$$

where we set

$$R_\varepsilon = \frac{|t_\varepsilon - s_\varepsilon|^2}{2|\bar{t}_\varepsilon - \bar{t}|} + \frac{|x_\varepsilon - y_\varepsilon|^2}{2|\bar{x}_\varepsilon - \bar{x}|} + \frac{1}{3}\left|\frac{y_\varepsilon - \ell(\bar{a})}{\bar{x}_\varepsilon - \ell(\bar{a})} - 1\right|^3.$$

From the boundedness of $\hat{w}_1(., ., \bar{a})$, $\hat{w}_{2,n}(., ., \bar{a})$ on $[0, T_0] \times [\ell(\bar{a}), x_0]$, we deduce by inequality (5.16) the boundedness of the sequence $(B_\varepsilon)_\varepsilon$, which implies $\bar{t}' = \bar{s}'$,

and $\bar{x}' = \bar{y}'$. Then, by sending ε to zero into (5.15) and (5.17), we obtain

$$
\begin{aligned}
M &= \hat{w}_1(\bar{t}, \bar{x}, \bar{a}) - \hat{w}_{2,n}(\bar{t}, \bar{x}, \bar{a}) \\
&\leq \hat{w}_1(\bar{t}', \bar{x}', \bar{a}) - \hat{w}_{2,n}(\bar{t}', \bar{x}', \bar{a}) - \frac{|\bar{t}' - \bar{t}|^2}{2} - \frac{|\bar{x}' - \bar{x}|^3}{3} \\
&\leq M - \frac{|\bar{t}' - \bar{t}|^2}{2} - \frac{|\bar{x}' - \bar{x}|^3}{3}
\end{aligned}
$$

by definition of M. This shows (5.12). Moreover, by sending again ε to zero into (5.15), (5.16) and (5.17), we get $M \leq \lim_\varepsilon N_\varepsilon = M - \lim_\varepsilon R_\varepsilon \leq M$, which implies (5.13) and (5.14). In particular, for ε small enough, we have $(t_\varepsilon, s_\varepsilon) \in [0, T_0)^2$ and $|y_\varepsilon - \ell(\bar{a})| > |\bar{x}_\varepsilon - \ell(\bar{a})|/2 > 0$ and so $y_\varepsilon > \ell(\bar{a})$. We can then write the viscosity subsolution property of $(w_1, \hat{w}_1)$ to (5.1), (5.2) at $(t_\varepsilon, x_\varepsilon, \bar{a})$ with the test function

$$
(t, x) \mapsto \frac{|t - \bar{t}|^2}{2} + \frac{|x - \bar{x}|^3}{3} + \frac{|t - s_\varepsilon|^2}{2|\bar{t}_\varepsilon - \bar{t}|} + \frac{|x - y_\varepsilon|^2}{2|\bar{x}_\varepsilon - \bar{x}|},
$$

and the viscosity subsolution property of $(w_{2,n}, \hat{w}_{2,n})$ to (5.1), (5.2) at $(s_\varepsilon, y_\varepsilon, \bar{a})$ with the test function

$$
(s, y) \mapsto -\frac{|t_\varepsilon - s|^2}{2|\bar{t}_\varepsilon - \bar{t}|} - \frac{|x_\varepsilon - y|^2}{2|\bar{x}_\varepsilon - \bar{x}|} - \frac{1}{3} \left| \frac{y - \ell(\bar{a})}{\bar{x}_\varepsilon - \ell(\bar{a})} - 1 \right|^3.
$$

This means:

$$
\begin{aligned}
(\rho + \lambda) \hat{w}_1(t_\varepsilon, x_\varepsilon, \bar{a}) - (t_\varepsilon - \bar{t}) - \frac{(t_\varepsilon - s_\varepsilon)}{\varepsilon} - \tilde{U} \left(|x_\varepsilon - \bar{x}|^2 + \frac{x_\varepsilon - y_\varepsilon}{|\bar{x}_\varepsilon - \bar{x}|} \right) \\
- \lambda \int w_1(x_\varepsilon + \bar{a}z) p(t_\varepsilon, dz) \leq 0,
\end{aligned} \tag{5.18}
$$

and

$$
\begin{aligned}
(\rho + \lambda) \hat{w}_{2,n}(s_\varepsilon, y_\varepsilon, \bar{a}) - \frac{(t_\varepsilon - s_\varepsilon)}{\varepsilon} - \tilde{U} \left(\frac{x_\varepsilon - y_\varepsilon}{|\bar{x}_\varepsilon - \bar{x}|} - \frac{1}{\bar{x}_\varepsilon - \ell(\bar{a})} \left| \frac{y - \ell(\bar{a})}{\bar{x}_\varepsilon - \ell(\bar{a})} - 1 \right|^2 \right) \\
- \lambda \int w_{2,n}(y_\varepsilon + \bar{a}z) p(s_\varepsilon, dz) \geq 0.
\end{aligned} \tag{5.19}
$$

Again by subtracting these two inequalities and since $\tilde{U}$ is nonincreasing, we get

$$
\begin{aligned}
(\rho + \lambda) & \left(\hat{w}_1(t_\varepsilon, x_\varepsilon, \bar{a}) - \hat{w}_{2,n}(s_\varepsilon, y_\varepsilon, \bar{a}) \right) \\
& \leq (t_\varepsilon - \bar{t}) + \lambda \left[\int w_1(x_\varepsilon + \bar{a}z) p(t_\varepsilon, dz) - \int w_{2,n}(y_\varepsilon + \bar{a}z) p(s_\varepsilon, dz) \right].
\end{aligned}
$$

We then get the required contradiction similarly as in case 1.

Step 3. Now, since $\hat{w}_1 \leq \hat{w}_{2,n}$ for all n, we obtain by sending n to infinity: $\hat{w}_1 \leq \hat{w}_2$. Therefore, we finally get: $w_1 \leq \mathcal{H}\hat{w}_1 \leq \mathcal{H}\hat{w}_2 \leq w_2$. This ends the proof. $\square$

Remark 5.1 Once we have characterized the value function through its dynamic programming equation by means of viscosity solutions, another question is to characterize the optimal control as in classical verification theorem when the value function was supposed to be smooth. This can be done with smooth solutions of the dynamic programming equation replaced by viscosity solutions, and derivatives involved replaced by super and subdifferentials, as described in Theorem 3.9 in [16], see also [8].

Appendix: Dynamic Programming Principle

In the Appendix, we derive the dynamic programming principle for the *weak* formulation of the stochastic control problem (2.3) where one varies the probability spaces as well as controls.

Definition A.1 The space $\mathcal{U}^w$ of controls is the set of all 7-uples

$$(\Omega, \mathcal{F}, \mathbb{P}, (\tau_k)_{k \geq 1}, (Z_k)_{k \geq 1}, (\alpha_k)_{k \geq 1}, (c_t)_{t \geq 0})$$

satisfying the following:

(i) $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space.
(ii) $(\tau_k)_{k \geq 1}$ and $(Z_k)_{k \geq 1}$ satisfy the hypotheses (H1) and (H2) under $\mathbb{P}$. Let $\mathcal{G}_t$ denote the filtration of the marked point process $(\tau_k, Z_k)_{k \geq 1}$. This means in particular that $\mathcal{G}_{\tau_n} = \sigma\{(\tau_k, Z_k) : k \leq n\}$, for all $n \geq 1$ (cf. Theorem T30 in Appendix A2 in [2]). By convention, $\tau_0 = 0$.
(iii) For each k, (α_k) is $\mathcal{G}_{\tau_{k-1}}$-measurable.
(iv) $(c_t)_{t \geq 0}$ is a nonnegative $\mathcal{G}_t$-predictable process.

The admissible consumption processes are characterized by the following result from [2, Theorem T34 in Appendix A2].

Lemma A.1 *A process $(c_t)_{t \geq 0}$ is $\mathcal{G}_t$-predictable if and only if it admits the representation*

$$c_t = \sum_{n \geq 0} C_n(t, \omega) 1_{\tau_n < t \leq \tau_{n+1}},$$

where, for every $n \geq 0$, the mapping $(t, \omega) \to C_n(t, \omega)$ is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{G}_{\tau_n}$-measurable.

Let x be a.s. deterministic under $\mathbb{P}$. We denote by $\mathcal{A}^w(x)$ the set of all x-admissible controls: the subset of $\mathcal{U}^w$ containing all controls for which $\mathbb{P}[X_k^x \geq 0, \forall k \geq 1] = 1$, where X_k^x is defined by (2.1). $\mathcal{A}^w(x)$ is clearly non-empty for all $x \geq 0$.

The value function of the stochastic control problem (2.3) is now defined by

$$v(x) = \sup_{\mathcal{A}^w(x)} \mathbb{E}\left[\int_0^\infty e^{-\rho t} U(c_t) dt\right] \tag{A.1}$$

Theorem A.1 (Dynamic programming principle) *The value function defined in* (A.1) *satisfies*

$$v(x) \le \sup_{\mathcal{A}^w(x)} \mathbb{E}\left[\int_0^{\tau_1} e^{-\rho t} U(c_t)dt + e^{-\rho \tau_1} v(X_1^x)\right], \quad x \ge 0. \tag{A.2}$$

If, in addition, the hypotheses (H3), (2.4) *and* (4.1) *are satisfied then*

$$v(x) = \sup_{\mathcal{A}^w(x)} \mathbb{E}\left[\int_0^{\tau_1} e^{-\rho t} U(c_t)dt + e^{-\rho \tau_1} v(X_1^x)\right], \quad x \ge 0. \tag{A.3}$$

Proof 1. *First part.* Since $\mathbb{P}[Z_1 < 0] > 0$, the only admissible policy for $x = 0$ is $c_t \equiv 0$ and $\alpha_k \equiv 0$. Therefore, $v(0) = 0$ and (A.2) is trivially satisfied for $x = 0$. On the other hand, since by Proposition 4.2, v is nondecreasing and concave on $\mathbb{R}_+$, either $v(x) < \infty$ for all $x > 0$ or $v(x) = \infty$ for all $x > 0$, and in the latter case (A.2) is once again trivially satisfied (take the control $c_t \equiv 0$ and $\alpha_k \equiv 0$). Therefore, in this proof we suppose w.l.o.g. that $v(x) < \infty$, all $x \ge 0$.

Denote the right-hand side of (A.2) by $V(x)$. In this part we want to show that $v(x) \le V(x)$, all $x \ge 0$.

Let $\varepsilon > 0$, $x \ge 0$. There is an element

$$u := \big(\Omega, \mathcal{F}, \mathbb{P}, (\tau_k)_{k\ge 1}, (Z_k)_{k\ge 1}, (\alpha_k)_{k\ge 1}, (c_t)_{t\ge 0}\big) \in \mathcal{A}^w(x),$$

such that

$$v(x) - \varepsilon \le \mathbb{E}\left[\int_0^\infty e^{-\rho t} U(c_t)dt\right]$$
$$= \mathbb{E}\left[\int_0^{\tau_1} e^{-\rho t} U(c_t)dt\right] + \mathbb{E}\left\{e^{-\rho \tau_1} \mathbb{E}\left[\int_0^\infty e^{-\rho t} U(c_{\tau_1+t})dt \Big| \mathcal{G}_{\tau_1}\right]\right\}.$$

Let $\tilde{\tau}_k = \tau_{k+1} - \tau_1$, $\tilde{Z}_k = Z_{k+1}$, $\tilde{\alpha}_k = \alpha_{k+1}$ and $\tilde{c}_t = c_{\tau_1+t}$. If we are able to show that $X_1^x = x - \int_0^{\tau_1} c_t dt + \alpha_1 Z_1$ is a.s. deterministic under $\mathbb{P}(\cdot|\mathcal{G}_{\tau_1})$ and that

$$\tilde{u} := \big(\Omega, \mathcal{F}, \mathbb{P}(\cdot|\mathcal{G}_{\tau_1}), (\tilde{\tau}_k)_{k\ge 1}, (\tilde{Z}_k)_{k\ge 1}, (\tilde{\alpha}_k)_{k\ge 1}, (\tilde{c}_t)_{t\ge 0}\big) \in \mathcal{A}^w(X_1^x),$$

it will follow that

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} U(c_{\tau_1+t})dt \Big| \mathcal{G}_{\tau_1}\right] \le v(X_1^x), \quad \mathbb{P}(\cdot|\mathcal{G}_{\tau_1})\text{-a.s.},$$

and therefore $v(x) \le V(x)$.

By Lemma A.1,

$$X_1^x = x - \int_0^{\tau_1} C_0(t)dt + \alpha_1 Z_1$$

for some measurable deterministic function C_0. Therefore, X_1^x is a.s. deterministic under $\mathbb{P}(\cdot|\mathcal{G}_{\tau_1})$. Conditions (i) and (ii) of Definition A.1 are clearly satisfied. Since Z_1 and τ_1 are almost surely deterministic under $\mathbb{P}(\cdot|\mathcal{G}_{\tau_1})$, $\tilde{\alpha}_n$ is measurable with respect to $\sigma\{(\tau_k, Z_k) : 2 \le k \le n+1\}$, and so with respect to $\mathcal{G}_{\tau_{n+1}}$, which proves condition (iii). To prove condition (iv), fix some $n \ge 0$. By Lemma A.1,

$$\tilde{c}_t 1_{\tilde{\tau}_n < t \le \tilde{\tau}_{n+1}} = c_{t+\tau_1} 1_{\tau_{n+1} < t+\tau_1 \le \tau_{n+2}} = C^{n+1}(t + \tau_1, \omega) 1_{\tau_{n+1} < t+\tau_1 \le \tilde{\tau}_{n+2}},$$

where C^{n+1} is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{G}_{\tau_{n+1}}$-measurable. Therefore (cf. Theorem 1.7 in [16]), there exists a measurable mapping $f^{n+1} : \mathbb{R}^{2n+3} \to \mathbb{R}_+$ such that

$$\tilde{c}_t 1_{\tilde{\tau}_n < t \leq \tilde{\tau}_{n+1}} = f^{n+1}(t + \tau_1, \tau_1, Z_1, \ldots, \tau_{n+1}, Z_{n+1}) 1_{\tau_{n+1} < t + \tau_1 \leq \tilde{\tau}_{n+2}}.$$

Since Z_1 and τ_1 are $\mathbb{P}(\cdot | \mathcal{G}_{\tau_1})$-a.s. deterministic, $f^{n+1}(t + \tau_1, \tau_1, Z_1, \ldots, \tau_{n+1}, Z_{n+1})$ is $\mathcal{B}(\mathbb{R}_+) \otimes \tilde{\mathcal{G}}_{\tilde{\tau}_n}$-measurable and we conclude, once again by Lemma A.1, that condition (iv) of Definition A.1 is satisfied and $\tilde{u} \in \mathcal{U}^w$. Finally, from the admissibility of $u \in \mathcal{A}^w(x)$, it is straightforward to check that $\tilde{u} \in \mathcal{A}(X_1^x)$.

2. *Second part.* Let us now prove that $v(x) \geq V(x)$, all $x \geq 0$ under (H3), (2.4) and (4.1). First, we notice under these conditions, and by the arguments of Corollary 4.1, that $V(x) \leq K x^\gamma$, for all $x \geq 0$, and in particular is finite. Hence, for all $x \geq 0$, $\varepsilon > 0$, one may find

$$u = \left(\Omega, \mathcal{F}, \mathbb{P}, (\tau_k)_{k \geq 1}, (Z_k)_{k \geq 1}, (\alpha_k)_{k \geq 1}, (c_t)_{t \geq 0} \right) \in \mathcal{A}^w(x)$$

such that

$$V(x) \leq \frac{\varepsilon}{3} + \mathbb{E}\left[\int_0^{\tau_1} e^{-\rho t} U(c_t) + e^{-\rho \tau_1} v(X_1^x) \right]$$

with $X_1^x = x + \alpha_1 Z_1 - \int_0^{\tau_1} c_t dt$.

Since the value function is nondecreasing and continuous on $[0, \infty)$, one can choose a sequence of measurable sets $\{B_j\}_{j \geq 1}$ such that $\bigcup_{j \geq 1} B_j = \mathbb{R}_+$, $B_i \cap B_j = \emptyset$ for $i \neq j$ and whenever $x, y \in B_j$, $|v(x) - v(y)| \leq \frac{\varepsilon}{3}$. For every j, put $x_j = \inf B_j$ and a choose a control

$$u_j = \left(\Omega_j, \mathcal{F}_j, \mathbb{P}_j, (\tau_k^j)_{k \geq 1}, (Z_k^j)_{k \geq 1}, (\alpha_k^j)_{k \geq 1}, (c_t^j)_{t \geq 0} \right) \in \mathcal{A}^w(x_j)$$

such that

$$v(x_j) \leq \frac{\varepsilon}{3} + \mathbb{E}\left[\int_0^\infty e^{-\rho t} U(c_t^j) dt \right].$$

Note that $u_j \in \mathcal{A}^w(x')$ for every $x' \in B_j$.

By the same argument as in the proof of part 1, for every j, one can find a sequence $(f_n^j)_{n \geq 0}$, $f_n^j : \mathbb{R}^{2n+1} \to \mathbb{R}_+$ measurable such that

$$c_t^j = \sum_{n \geq 0} f_n^j(t, \tau_1^j, Z_1^j, \ldots, \tau_n^j, Z_n^j) 1_{\tau_n^j < t \leq \tau_{n+1}^j}$$

and a sequence $(g_n^j)_{n \geq 1}$, $g_n^j : \mathbb{R}^{2n-1} \to \mathbb{R}$ measurable such that

$$\alpha_n^j = g_n^j(\tau_1^j, Z_1^j, \ldots, \tau_{n-1}^j, Z_{n-1}^j).$$

Now define the new control $\tilde{u}$ via

$$\tilde{u} = \left(\Omega, \mathcal{F}, \mathbb{P}, (\tau_k)_{k \geq 1}, (Z_k)_{k \geq 1}, (\tilde{\alpha}_k)_{k \geq 1}, (\tilde{c}_t)_{t \geq 0} \right),$$

where

$$\tilde{\alpha}_1 = \alpha_1,$$

$$\tilde{\alpha}_n = \sum_j 1_{X_1^x \in B_j} g_{n-1}^j(\tau_2, Z_2, \dots, \tau_{n-1}, Z_{n-1}), \quad n \geq 2,$$

$$\tilde{c}_t = c_t 1_{t \leq \tau_1} + \sum_j 1_{X_1^x \in B_j} \sum_{n \geq 0} f_n^j(t - \tau_1, \tau_2 - \tau_1, Z_2, \dots, \tau_{n+1} - \tau_1, Z_{n+1})$$

$$\times 1_{\tau_{n+1} < t \leq \tau_{n+2}}.$$

By construction, $\tilde{u} \in \mathcal{A}^w(x)$. Finally,

$$v(x) \geq \mathbb{E}\left[\int_0^\infty e^{-\rho t} U(\tilde{c}_t) dt\right]$$

$$= \mathbb{E}\left[\int_0^{\tau_1} e^{-\rho t} U(\tilde{c}_t) dt + e^{-\rho \tau_1} \mathbb{E}\left\{\int_0^\infty e^{-\rho t} U(\tilde{c}_{\tau_1 + t}) dt \Big| \mathcal{G}_{\tau_1}\right\}\right]$$

$$\geq \mathbb{E}\left[\int_0^{\tau_1} e^{-\rho t} U(\tilde{c}_t) dt + e^{-\rho \tau_1} \sum_j v(x_j) 1_{X_1^x \in B_j}\right] - \frac{\varepsilon}{3}$$

$$\geq \mathbb{E}\left[\int_0^{\tau_1} e^{-\rho t} U(\tilde{c}_t) dt + e^{-\rho \tau_1} v(X_1^x)\right] - \frac{2\varepsilon}{3}$$

$$\geq V(x) - \varepsilon.$$

Since the choice of $\varepsilon > 0$ was arbitrary, the proof is complete. □

Remark A.1 A straightforward modification of the above proof allows to establish the following modified version of the dynamic programming principle: for every $n \geq 1$,

$$v(x) = \sup_{\mathcal{A}^w(x)} \mathbb{E}\left[\int_0^{\tau_n} e^{-\rho t} U(c_t) dt + e^{-\rho \tau_n} v(X_n^x)\right], \quad x \geq 0. \tag{A.4}$$

Remark A.2 Finally, we note that the dynamic programming principles (A.3) can be formulated on a single probability space. Indeed, from lemma A.1 and the measurability condition on (α_k), for every admissible control

$$u := (\Omega, \mathcal{F}, \mathbb{P}, (\tau_k)_{k \geq 1}, (Z_k)_{k \geq 1}, (\alpha_k)_{k \geq 1}, (c_t)_{t \geq 0}),$$

α_1 is a deterministic constant and $c_t = \tilde{c}(t) 1_{t < \tau_1}$ for some deterministic function $\tilde{c}(t)$. Therefore, we can fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the conditions (i) and (ii) of Definition A.1 and Eq. (A.3) will take the form

$$v(x) = \sup_{A_d(x)} \mathbb{E}\left[\int_0^{\tau_1} e^{-\rho t} U(c_t) dt + e^{-\rho \tau_1} v(X_1^x)\right], \quad x \geq 0, \tag{A.5}$$

where $A_d(x)$ is the set of deterministic controls defined in (3.2).

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