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# Semicontinuity and Supremal Representation in the Calculus of Variations

**Francesca Prinari**

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**Abstract** We study the weak* lower semicontinuity properties of functionals of the form

$$F(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(x, Du(x))$$

where $\Omega$ is a bounded open set of $\mathbf{R}^N$ and $u \in W^{1,\infty}(\Omega)$. Without a continuity assumption on $f(\cdot, \xi)$ we show that the *supremal* functional $F$ is weakly* lower semicontinuous if and only if it is a level convex functional (i.e. it has convex sub-levels). In particular if $F$ is weakly* lower semicontinuous, then it can be represented through a level convex function. Finally a counterexample shows that in general it is not possible to represent $F$ through the level convex envelope of $f$.

**Keywords** Supremal functionals · Calculus of variations in $L^\infty$ · Level convex function · Absolute minimizers

## 1 Introduction

In the last years a new class of functionals has been considered with growing interest in the mathematical literature: these functionals are represented in the so called *supremal form*

$$F(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(x, Du(x)) \tag{1.1}$$

where $\Omega$ is a bounded open set of $\mathbf{R}^N$ and $u \in W^{1,\infty}(\Omega)$. According to part of the already existing literature, we will refer to a functional of the type (1.1) as a *supremal*

F. Prinari (✉)
Dipartimento di Matematica "Ennio De Giorgi", Università del Salento,
Via Prov. le Lecce-Arnesano, 73100 Lecce, Italy
e-mail: francesca.prinari@unile.it

*functional* or $L^\infty$*-functional*, while we refer to the function $f$ which represents $F$ as an *admissible supremand*. The definition of this class is very important because in many situations one would like to minimize a quantity which cannot be expressed as an integral: for example, a quantity which does not express a mean property of a body or whose values can be relevant on sets of arbitrarily small measure. In these cases the problem could be formulated as the minimization of a supremal functional: see, for example, the classical problem of finding optimal Lipschitz extensions, first considered by McShane in [15] or the recent formulation of the first dielectric breakdown for composite conductors given in [13].

In order to apply the direct methods of Calculus of Variations to this class of functionals, the main issue to be solved is the identification of the qualitative conditions on the supremand $f$ which imply the lower semicontinuity of $F$ with respect to the weak* $W^{1,\infty}$ topology. In fact, under reasonable growth conditions for $f$, this is the right topology which gives the compactness of minimizing sequences. The characterization of lower semicontinuity of a functional expressed by a supremum requires a new notion of convexity: the level convexity. A function $f = f(\xi)$ is said to be level convex (or quasi-convex) if it has convex sub-levels. Namely $f$ is level convex if the set $\{\xi : f(\xi) \leq \lambda\}$ is convex for every $\lambda \in \mathbf{R}$; equivalently if

$$f(\theta\xi + (1-\theta)\eta) \leq f(\xi) \vee f(\eta)$$

for every $\xi, \eta \in \mathbf{R}^N$ and $\theta \in [0, 1]$. In [5] Barron, Jensen and Wang show the following sufficient condition:

**Theorem 1.1** (Sufficient condition, Theorem 3.4 in [5]) *Let $f : \Omega \times \mathbf{R}^N \to \mathbf{R}$ be a Borel function such that $f(x, \cdot)$ is lower semicontinuous and level convex. Then for any open subset $A \subset \Omega$ the functional $F(u, A) = \operatorname{ess\,sup}_{x\in A} f(x, Du(x))$ is sequentially weakly* lower semicontinuous on $W^{1,\infty}(\Omega)$.*

In the same paper, they show that this condition is also necessary.

**Theorem 1.2** (Necessary condition, Theorem 2.7 in [5]) *Let $f : \Omega \times \mathbf{R}^N \to \mathbf{R}$ be a Borel function such that there exists a function $w : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ which is continuous in its first variable with $w(0, s) = 0$ for every $s \in \mathbf{R}$ and non-decreasing in its second variable, such that*

$$|f(x_1, \xi) - f(x_2, \xi)| \leq w(|x_1 - x_2|, |\xi|)$$

*for any $x_1, x_2 \in \Omega$ and $\xi \in \mathbf{R}^N$. Let $F(u, A) = \operatorname{ess\,sup}_{x\in A} f(x, Du(x))$ for every open subset $A \subset \Omega$ and assume that $F(\cdot, A)$ is sequentially weakly* lower semicontinuous on $W^{1,\infty}(A)$. Then for every $x \in \Omega$ $f(x, \cdot)$ is a level convex function.*

Note that the last theorem requires that the localized functional $F(\cdot, A)$ is weakly* lower semicontinuous for every open subset $A \subset \Omega$. Moreover, in the case in which only $F(\cdot, \Omega)$ is weakly* lower semicontinuous, we cannot apply the previous result in order to deduce some convexity property for $f$. The proof of Theorem 1.2 heavily relies on the continuity assumption on $f(\cdot, \xi)$. If one drops this assumption, then the statement of Theorem 1.2 can be false as it is shown in Remark 3.1 of [14]. This counterexample is based on the fact that, in general, a supremal functional does not admit a unique representation. More precisely, the Authors construct a dense open set $A \subset \Omega$ with $|A| > 0$, $|\Omega \setminus A| > 0$, two admissible level convex supremands $\varphi_1, \varphi_2$ with $\varphi_1 \leq \varphi_2$ on $\Omega$ and $\varphi_1 < \varphi_2$ on $(\Omega \setminus A) \times \mathbf{R}^N$ such that $\operatorname{ess\,sup}_\Omega \varphi_1(x, Du(x)) = \operatorname{ess\,sup}_\Omega \varphi_2(x, Du(x)) := F(u)$ for every $u \in W^{1,\infty}(\Omega)$. Moreover $F$ turns out to be weakly* lower semicontinuous since it admits a level convex supremand, but it can be also represented by any function $f$, possibly non level convex, such that $\varphi_1 \leq f \leq \varphi_2$. This means that, without having a continuity property on $f(\cdot, \xi)$, one cannot expect that any admissible supremand for a weakly* l.s.c. $L^\infty$ functional is a level convex function.

Despite to these facts, in most of the results concerning the class of supremal functionals it is assumed that a priori that the weakly* lower semicontinuous functional $F$ is represented by a level convex function. For example, the existence of absolute minimizers (the so called AML) shown in [6], the $\Gamma$-convergence result given in [9], the homogenization theorem in [7], the principles of comparison with distance functions for AML stated in [10], all assume such a representation. Thus the question whether a weakly* lower semicontinuous functional always admits a level convex supremand turns out to be interesting and useful for applications.

The first positive answer to this problem is given in the 1-dimensional case in [16] where in Corollary 3.1 it is shown that if $F(u) = \operatorname{ess\,sup}_\Omega f(x, u'(x))$ is weakly* lower semicontinuous on $W^{1,\infty}(\Omega)$ then there exists a level convex supremand $\tilde{f}$ which represents $F$.

The main contribution of this paper is the extension of this result to the $N$-dimensional case (see Sect. 2, Theorems 2.5–2.7). With a completely different technique, under mild assumptions on $f(x, \cdot)$ and without requiring a continuity property on $f(\cdot, \xi)$, we prove that all weakly* lower semicontinuous supremal functionals:

- $F : W^{1,\infty}(\Omega) \to \mathbf{R}$ of the form (1.1)
- $F : W^{1,\infty}(\Omega) \times \mathcal{A} \to \mathbf{R}$ of the form

$$F(u, A) = \operatorname*{ess\,sup}_A f(x, Du(x)) \tag{1.2}$$

(where $\mathcal{A}$ is the class of the open subsets of $\Omega$) can be represented by a level convex supremand. The proofs of these results are given in Sect. 6 and are achieved in two steps. First we show that if a supremal functional of the form (1.1) is weakly* lower semicontinuous on $W^{1,\infty}(\Omega)$ then $F$ is *a level convex functional* on $W^{1,\infty}(\Omega)$, i.e. the sub-level sets

$$E_\lambda := \{u \in W^{1,\infty}(\Omega) : \ F(u) \leq \lambda\}$$

are convex. The strategy used to prove this property follows the metric approach used in [14] where, among other results, it is shown that a 1-homogeneous supremal functional can be written in terms of intrinsic distances associated with the functionals (see Sect. 3). The second step concerns the representation in terms of a level convex supremand. Since we know that the representation is not unique, the main issue is to identify a good candidate. As shows the example constructed in Sect. 8, given a weakly* lower semicontinuous $F$ of the form (1.1), in general it is not possible to choose as an admissible level convex supremand of $F$ the level convex envelope $f^{lc}$ of $f$ given by

$$f^{lc}(x,\cdot) = \sup\{h : \mathbf{R}^N \to \overline{\mathbf{R}} : \ h \text{ lower semicontinuous and level convex}, \ h(\cdot) \le f(x,\cdot)\}.$$

In the choice of a suitable supremand we have been inspired by the one constructed in [8] (see Theorem 2.2 and Lemma 3.4 therein). In this paper, given an abstract functional $F : W^{1,\infty}(\Omega) \times \mathcal{A} \to \mathbf{R}$ such that $F(\cdot, A)$ is weakly* lower semicontinuous for every $A \in \mathcal{A}$, the Authors construct a function $\tilde{f}$ in the following way

$$\tilde{f}(x,\xi) := \inf\{F(u, B_r(x)) \mid r > 0, \ u \in W^{1,\infty}(\Omega) \text{ s.t. } x \in \widehat{u}, \text{ with } Du(x) = \xi\} \tag{1.3}$$

where

$$\widehat{u} := \{x \in \Omega : x \text{ is a differentiability point of } u \text{ and a Lebesgue point of } Du\}$$

and under some suitable assumptions on $F$, they represent the functional in the supremal form

$$F(u, A) = \operatorname*{ess\,sup}_A \tilde{f}(x, Du(x)), \tag{1.4}$$

but they cannot deduce that $\tilde{f}$ is a level convex function. Inspired by the above result, we devote Sect. 5 to show that if $F$ is a coercive supremal functional of the form (1.2) (possibly non weak* lower semicontinuous), then the function $\tilde{f}$ defined by (1.3) is an admissible supremand of $F$ (see Theorem 5.4). In the case in which $F$ is weakly* l.s.c. on $W^{1,\infty}(\Omega)$, we show that $\tilde{f}$ is a level convex supremand of $F$. As an easy consequence we obtain also that the function

$$\varphi(x,\xi) := \inf\{F(u) \mid u \in W^{1,\infty}(\Omega) \text{ s.t. } x \in \widehat{u}, \text{ with } Du(x) = \xi\} \tag{1.5}$$

is an admissible level convex supremand of a weakly* l.s.c functional $F$ of the form (1.1). Finally, as a special case we deal with the class of the 1-homogeneous supremal functionals already considered by Garroni, Ponsiglione and Prinari in [14] (see Theorem 2.6).

As a consequence of these results, in Sect. 7 we show the existence of absolute minimizers for a weakly* l.s.c. supremal functional. An absolute minimizer (or AML) of the functional (1.2) is a function $u \in W^{1,\infty}(\Omega)$ such that for all subdomain $V \subset \Omega$ one has

$$F(u, V) \le F(v, V)$$

for all $v$ in $W^{1,\infty}(V)$ such that $v = u$ on $\partial V$. In [6, 9] it is shown that if the functional (1.2) is coercive and represented by a level convex function $f$ then there exists at least one absolute minimizer of $F$. Thanks to Theorem 2.7 we can give a result of existence of AML under the natural assumptions that $F$ is weakly* l.s.c. and coercive (see Theorem 7.2). Moreover, we discuss the problem of characterizing the AMLs by extending the principle of comparison with cones introduced by Crandall, Evans and Gariepy in [11] for the minimizing Lipschitz Extension Problem. In [10]

Champion and De Pascale give a comparison principle with distances but they confine themselves to the case where $f$ is globally l.s.c. Now if $f$ is not globally l.s.c. the possibility of giving an analogous principle is an open problem. We obtain a partial result by showing that if $u$ satisfies a comparison principle with the distance functions introduced in Sect. 7 and associated with the supremal functional $F$ then $u$ is an AML of $F$.

Finally, a paper in preparation (see [17]) is devoted to extending the previous results to the class of the supremal functionals of the form

$$F(u) = \operatorname*{ess\,sup}_{\Omega} f(x, u(x), Du(x))$$

under a continuity assumption on $f(x, \cdot, \xi)$. Moreover we study the weak* l.s.c. envelope of a supremal functional in order to show that the lower semicontinuous envelope of a supremal functional is a level convex functional.

Let us fix some notations useful in the sequel.

**Notations**

- We denote by $\Omega$ an open bounded domain of $\mathbf{R}^N$ and by $\mathcal{A}$ the family of all open subsets of $\Omega$.
- For every $x \in \mathbf{R}^N$ and $r > 0$ we denote by $B_r(x)$ the open ball $\{y \in \mathbf{R}^N : |x - y| < r\}$ where $|\cdot|$ is the euclidean norm on $\mathbf{R}^N$.
- For any set $B \subset \mathbf{R}^N$ we denote by $\mathcal{H}^1(B)$ its one dimensional Hausdorff measure. Moreover if $B \subset \mathbf{R}^N$ is a measurable set then $|B|$ denotes its Lebesgue measure.
- A modulus of continuity is any continuous function $w : [0, +\infty) \to [0, +\infty)$ such that $w(0) = 0$.
  For every $u \in W^{1,\infty}(\Omega)$ we denote by $\widehat{u}$ the set

$$\widehat{u} := \{x \in \Omega : x \text{ is a differentiability point of } u \text{ and a Lebesgue point of } Du\}.$$

## 2 Necessary and Sufficient Conditions for the w* Lower Semicontinuity

Before stating the main results of this paper, we introduce the following definitions.

**Definition 2.1** A function $f : \Omega \times \mathbf{R}^N \to \overline{\mathbf{R}}$ is said to be

(a) a *normal supremand* if:
  (i) $f$ is a Borel function;
  (ii) for a. a. $x \in \Omega$ the function $\xi \mapsto f(x, \xi)$ is lower semicontinuous in $\mathbf{R}^N$;
(b) a *Carathéodory supremand* if:
  (i) for every $\xi \in \mathbf{R}^N$ the function $x \mapsto f(x, \xi)$ is measurable in $\Omega$;
  (ii) for a. a. $x \in \Omega$ the function $\xi \mapsto f(x, \xi)$ is continuous in $\mathbf{R}^N$;
(c) a *level convex normal* (respectively, a *level convex Carathéodory*) *supremand* if $f$ is a normal (respectively, a Carathéodory) supremand and $f(x, \cdot)$ is level convex on $\mathbf{R}^N$ for almost every $x \in \Omega$.

**Definition 2.2** A functional $F : X \to \overline{\mathbf{R}}$ defined on a topological vector space $X$ is said to be *level convex* if for every $t \in \mathbf{R}$ the level set $\{u \in X \ : \ F(u) \le t\}$ is convex.

Now we are in a position to state the main theorems of this paper. In Theorems 2.3–2.5 we deal with functionals $F : W^{1,\infty}(\Omega) \to \mathbf{R}$ of the form

$$F(u) = \operatorname*{ess\,sup}_{\Omega} f(x, Du(x)) \tag{2.1}$$

where $\Omega$ is an open subset of $\mathbf{R}^N$.

First we show that in many situations the level convexity of the functional $F$ is a consequence of its weak* lower semicontinuity.

**Theorem 2.3** *Let $\Omega \subset \mathbf{R}^N$ be a connected open set with Lipschitz continuous boundary. Let $f : \Omega \times \mathbf{R}^N \to \mathbf{R}$ be a Carathéodory supremand satisfying the following assumption: for any $M > 0$ there exists a modulus of continuity $\omega_M$ such that*

$$|f(x, \xi) - f(x, \eta)| \le \omega_M(|\xi - \eta|) \tag{2.2}$$

*for a.e. $x \in \Omega$ and for every $\xi, \eta \in B_M(0)$. If the functional $F$ defined by (2.1) is weakly* l.s.c. on $W^{1,\infty}(\Omega)$ then $F$ is a level convex functional.*

As shown in Remark 3.1 of [14], in the general case the above result does not imply as consequence the level convexity of $f(\cdot, \xi)$. However, we can prove that there exists at least a level convex supremand $\varphi$ for a level convex supremal functional $F$. We notice, as shown in Sect. 8, that $\varphi$ may not coincide with the level convex envelope of $f$.

**Theorem 2.4** *Let $\Omega \subset \mathbf{R}^N$ be an open set. Let $f : \Omega \times \mathbf{R}^N \to \mathbf{R}$ be a normal supremand and let $F$ be the functional defined by (2.1). Then $F$ is level convex if and only if there exists a level convex normal supremand $\varphi : \Omega \times \mathbf{R}^N \to \mathbf{R}$ such that*

$$F(u) = \operatorname*{ess\,sup}_{\Omega} \varphi(x, Du(x))$$

*for all $u \in W^{1,\infty}(\Omega)$. In particular if $F$ is level convex then $F$ is weakly* l.s.c. on $W^{1,\infty}(\Omega)$.*

In the previous theorem, if $f$ is globally Lipschitz continuous then it is possible to show that the function $\varphi$ is Lipschitz continuous as well (see Proposition 5.2). But when $f$ is a Carathéodory supremand it is not clear if $\varphi$ is a Carathéodory supremand too. However if $f$ satisfies (2.2) and a further coercivity condition, we may put together the previous results and obtain the following characterization.

**Theorem 2.5** *Let $\Omega \subset \mathbf{R}^N$ be a connected open set with Lipschitz continuous boundary. Let $f : \Omega \times \mathbf{R}^N \to \mathbf{R}$ be a Carathéodory supremand satisfying (2.2) and the following assumption: there exists an increasing continuous function $\alpha : \mathbf{R}^+ \to \mathbf{R}^+$ such that $\lim_{t \to +\infty} \alpha(t) = +\infty$ and*

$$f(x, \xi) \ge \alpha(|\xi|) \quad \textit{for a.e } x \in \Omega, \textit{ for every } \xi \in \mathbf{R}^N. \tag{2.3}$$

*Let $F$ be the functional defined by (2.1). The following facts are equivalent:*

1. *$F$ is weakly* l.s.c. on $W^{1,\infty}(\Omega)$*;
2. *$F$ is a level convex functional*;
3. *there exists a level convex Carathéodory supremand $\varphi : \Omega \times \mathbf{R}^N \to \mathbf{R}$ given by*

$$\varphi(x,\xi) := \inf\Big\{\operatorname*{ess\,sup}_{\Omega} f(y, Du(y)) \mid u \in W^{1,\infty}(\Omega) \text{ s.t. } x \in \widehat{u}, \text{ with } Du(x) = \xi\Big\} \tag{2.4}$$

*such that*

$$F(u) = \operatorname*{ess\,sup}_{\Omega} \varphi(x, Du(x))$$

*for all $u \in W^{1,\infty}(\Omega)$. Moreover $\varphi$ satisfies (2.3, 2.2) (for a suitable family $(\omega'_M)_M$ of moduli of continuity) and for a.e. $x \in \Omega$ $\varphi(x,\cdot) \geq f(x,\cdot)$.*

The following result concerns the class of 1-homogeneous supremal functional studied in [14]. Note that, compared with the assumptions in Theorem 2.5, in the next result we do not require that $f$ satisfies assumption (2.2).

**Theorem 2.6** *Let $\Omega \subset \mathbf{R}^N$ be a connected open set with Lipschitz continuous boundary. Let $f : \Omega \times \mathbf{R}^N \to \mathbf{R}$ be a Carathéodory supremand satisfying the following assumptions*

$$\alpha|\xi| \leq f(x,\xi) \leq \beta|\xi| \tag{2.5}$$

*and*

$$f(x, t\xi) = |t| f(x,\xi) \tag{2.6}$$

*for every $\xi \in \mathbf{R}^N$, for a.e. $x \in \Omega$ and for every $t \in \mathbf{R}$ and for some positive constants $\alpha, \beta > 0$. Let $F$ be the functional defined by (2.1), let $d : \Omega \times \Omega \to \mathbf{R}$ be the distance defined by*

$$d(x,y) = \sup\big\{u(x) - u(y), u \in W^{1,\infty}(\Omega) : F(u) \leq 1\big\}. \tag{2.7}$$

*and let $\varphi_d$ be the metric derivative of $d$ defined as*

$$\varphi_d(x,\eta) := \limsup_{t\to 0^+} \frac{d(x, x+t\eta)}{t}. \tag{2.8}$$

*Then the following facts are equivalent*:

1. *$F$ is weakly* l.s.c. on $W^{1,\infty}(\Omega)$*;
2. *$F$ is a convex functional*;
3. *for all $u \in W^{1,\infty}(\Omega)$*

$$F(u) = \operatorname*{ess\,sup}_{\Omega} \varphi_d^0(x, Du(x))$$

*where*

$$\varphi_d^0(x,\xi) := \sup\big\{\xi\cdot\eta : \varphi_d(x,\eta) \leq 1\big\}.$$

*Moreover if $\varphi$ is given by (2.4) there exists a negligible set $H \subset \Omega$ such that*

$$\varphi_d^0(x,\xi) = \varphi(x,\xi)$$

*for every $x \in \Omega \setminus H$ and for every $\xi \in \mathbf{R}^N$.*

Finally, through a localization method and an appropriate choice of the supremand, the results above can be extended to the class of supremal functionals $F : W^{1,\infty}(\Omega) \times \mathcal{A} \to \mathbf{R}$ of the form

$$F(u, A) = \operatorname*{ess\,sup}_A f(x, Du(x)). \tag{2.9}$$

In particular we give the following result:

**Theorem 2.7** *Let $\Omega$ be an open subset of $\mathbf{R}^N$. Let $f : \Omega \times \mathbf{R}^N \to \mathbf{R}$ be a Carathéodory supremand satisfying (2.2) and (2.3). Let $F(\cdot, A)$ be the functional defined by (2.9). The following facts are equivalent:*

1. *$F(\cdot, A)$ is weakly* l.s.c. on $W^{1,\infty}(\Omega)$ for every $A \in \mathcal{A}$;*
2. *$F(\cdot, A)$ is a level convex functional for every $A \in \mathcal{A}$;*
3. *there exists a level convex normal supremand $\tilde{f} : \Omega \times \mathbf{R}^N \to \mathbf{R}$ given by*

$$\tilde{f}(x,\xi) := \inf\Big\{ \operatorname*{ess\,sup}_{B_r(x)} f(y, Du(y)) \mid r > 0, \\ u \in W^{1,\infty}(\Omega) \text{ s.t. } x \in \widehat{u}, \text{ with } Du(x) = \xi \Big\} \tag{2.10}$$

*such that*

$$F(u, A) = \operatorname*{ess\,sup}_A \tilde{f}(x, Du(x))$$

*for all $u \in W^{1,\infty}(\Omega)$ and for all $A \in \mathcal{A}$. Moreover $\tilde{f}$ satisfies (2.3, 2.2) (for a suitable family $(\omega'_M)_M$ of moduli of continuity) and for a.e. $x \in \Omega$ $\tilde{f}(x,\cdot) \geq f(x,\cdot)$.*

Notice that in the previous theorem when $f(\cdot,\xi)$ is continuous for every $\xi \in \mathbf{R}^N$ then for a.e. $x \in \Omega$ $\tilde{f}(x,\cdot) = f(x,\cdot)$ (see Theorem 5.4) and therefore we obtain that if $F$ is weakly* l.s.c. then $f$ is a level convex supremand.

In order to show all the results above, we introduce some tools, recall some known facts and prove further preliminary results. For these reasons, the proofs of the previous theorems are postponed until Sect. 6.

## 3 The Class of Difference Quotients

In order to show that a weakly* lower semicontinuous supremal functional is level convex, we recall some results and some tools given for the 1-homogeneous supremal functionals in [14] with the aim to extend them to more general supremal functionals.

We consider a supremal functional (2.1) represented through a Carathéodory supremand $f : \Omega \times \mathbf{R}^N \to \mathbf{R}^+$ satisfying (2.2) and (2.3). Inspired by the distances introduced in [12], with every $\lambda \in \mathbf{R}$ such that the sub-level set $E_\lambda := \{u \in W^{1,\infty}(\Omega) : F(u) < \lambda\}$ is nonempty, we can associate a distance $d_\lambda$ in the following way:

$$d_\lambda(x, y) := \sup\left\{|u(x) - u(y)| : u \in W^{1,\infty}(\Omega) : F(u) \leq \lambda\right\}. \tag{3.1}$$

Notice that if $B_r(x) \subset \Omega$ then, from (2.3), we have that $d_\lambda(x, y) \leq \alpha^{-1}(\lambda)r$ for every $y \in B_r(x)$. In general if $\Omega$ is a connected open set, then for every $x, y \in \Omega$ the inequality

$$d_\lambda(x, y) \leq \alpha^{-1}(\lambda)|x - y|_\Omega \tag{3.2}$$

holds, where

$$|x - y|_\Omega = \inf\{\mathcal{L}(\gamma) : \gamma \in \Gamma_{x,y}(\Omega)\},$$

$\Gamma_{x,y}(\Omega)$ being the set of Lipschitz curves in $\Omega$ with end-points $x$ and $y$, and $\mathcal{L}(\gamma)$ the Euclidean length of $\gamma$. In particular if $\partial\Omega$ is Lipschitz continuous then there exists a constant $C > 0$ such that

$$d_\lambda(x, y) \leq |x - y|_\Omega \leq C|x - y|. \tag{3.3}$$

Moreover for every $\lambda \in \mathbf{R}$ there exists $\delta = \delta(\lambda)$ such that for every $x, y \in \Omega$

$$d_\lambda(x, y) \geq \delta|x - y|. \tag{3.4}$$

In fact since $E_\lambda$ is nonempty, then there exists $\varepsilon > 0$ and $u \in W^{1,\infty}(\Omega)$ such that $F(u) < \lambda - \varepsilon$. Now fix $x, y \in \Omega$ and, without loss of generality, assume $u(x) \geq u(y)$. Chosen $M > ||u||_{1,\infty}$ there exists $0 < \sigma < 1$ such that $w_{M+2}(t) \leq \varepsilon$ for every $0 < t \leq \sigma$. Thus if $0 < \delta < \min\{\frac{1}{\mathrm{diam}\Omega+1}, \sigma\}$ then the function

$$v(z) := u(z) + \delta\frac{(x - y)}{|x - y|} \cdot z$$

is such that

$$\|v\|_{1,\infty} \leq M + \delta(\mathrm{diam}\Omega) + \sigma \leq M + 2$$

and

$$F(v) \leq F(u) + w_{M+2}(\delta) < \lambda - \varepsilon + \varepsilon = \lambda.$$

This implies

$$\begin{aligned} d_\lambda(x, y) &\geq |v(x) - v(y)| = |\delta|x - y| + u(x) - u(y)| \\ &= \delta|x - y| + u(x) - u(y) \geq \delta|x - y|. \end{aligned}$$

Now, for every $\lambda$ such that $E_\lambda$ is nonempty, we consider the functional $R_\lambda : W^{1,\infty}(\Omega) \to \bar{\mathbf{R}}$ given by

$$R_\lambda(u) := \sup_{x,y\in\Omega,\, x\neq y} \frac{|u(x) - u(y)|}{d_\lambda(x, y)}. \tag{3.5}$$

The functional $R_\lambda$ is referred to as the *difference quotient* associated with the sub-level set $E_\lambda$ of $F$.

**Proposition 3.1** *For every $\lambda$ s.t. $E_\lambda \neq \emptyset$ the difference quotient $R_\lambda$ is a convex lower semicontinuous functional with respect to the strong convergence in $L^\infty$. Moreover $R_\lambda(u+v) \leq R_\lambda(u) + R_\lambda(v)$ for every $u, v \in W^{1,\infty}(\Omega)$.*

*Proof* Let $u \in W^{1,\infty}(\Omega)$ and let $\{u_n\} \subset W^{1,\infty}(\Omega)$ be a sequence converging to $u$ in $L^\infty(\Omega)$. We have that for every $x,\ y \in \Omega$ such that $0 < d_\lambda(x,y) < +\infty$

$$\frac{|u(x)-u(y)|}{d_\lambda(x,y)} = \lim_n \frac{|u_n(x)-u_n(y)|}{d_\lambda(x,y)} \leq \liminf_n R_\lambda(u_n).$$

Taking the supremum for $x, y \in \Omega, x \neq y$ we get the thesis. The convexity and the sublinearity of $R_\lambda$ are trivial. □

The key tool we will use in the sequel is the following lemma. It is an adaptation of Lemma 3.4 in [14]. We report its revised proof for the sake of completeness.

**Lemma 3.2** *Let $\Omega \subset \mathbf{R}^N$ be a connected open set with Lipschitz continuous boundary. Let $F$ be a supremal functional on $W^{1,\infty}(\Omega)$ represented by a Carathéodory supremand $f : \Omega \times \mathbf{R}^N \to \mathbf{R}$ satisfying (2.2) and (2.3). Let $v \in W^{1,\infty}(\Omega)$ be such that $R_\lambda(v) < 1$. Then there exists a sequence $\{v_n\} \subset W^{1,\infty}(\Omega)$ converging to $v$ in $L^\infty(\Omega)$ with $F(v_n) \leq \lambda$ for $n \in \mathbf{N}$.*

*Proof* Let us fix $r > 0$. By the fact that $R_\lambda(v) < 1$ and thanks to (3.4), for every $x$, $y \in \Omega$ with $|x-y| = r$

$$|v(y) - v(x)| < d_\lambda(x,y) - \gamma, \tag{3.6}$$

for a positive constant $\gamma$ depending on $r$. Let us fix $0 < \varepsilon < \frac{\gamma}{3}$. For every $x \in \Omega$ and for every $y \in \partial B_r(x) \cap \Omega$, by the definition of $d_\lambda$ there exists a function $w_r^{x,y} \in W^{1,\infty}(\Omega)$ such that:

1. $F(w_r^{x,y}) \leq \lambda$;
2. $|w_r^{x,y}(y) - w_r^{x,y}(x)| \geq d_\lambda(x,y) - \varepsilon$;
3. $w_r^{x,y}(x) = v(x)$;

the third property it is possible to fulfill thanks to the translation invariance of the first two. By properties 2, 3 and by (3.6), for every $y \in \partial B_r(x) \cap \Omega$

$$|w_r^{x,y}(y) - v(x)| \geq d_\lambda(x,y) - \varepsilon > |v(y) - v(x)| + \gamma - \varepsilon. \tag{3.7}$$

Note that by (2.3) we have that $\operatorname{ess\,sup}_\Omega |Dw_r^{x,y}| < \alpha^{-1}(\lambda)$, and hence there exists $\delta > 0$ (depending only on $\varepsilon$) such that

$$|w_r^{x,y}(z) - v(x)| > |v(z) - v(x)| + \gamma - 2\varepsilon > |v(z) - v(x)| + \varepsilon$$
$$\text{for every } z \in \partial B_r(x) \cap \Omega : |z - y| \leq \delta. \tag{3.8}$$

Moreover, since $w_r^{x,y}(x) = v(x)$, there exists $0 < r' < r$ (depending only on $\varepsilon$) such that

$$|w_r^{x,y}(z) - v(x)| < |v(z) - v(x)| + \varepsilon \quad \text{for every } z \in B_{r'}(x) \cap \Omega. \tag{3.9}$$

For every $x \in \Omega$, let us fix a finite set of points $\{y_1, \ldots, y_N\}$ on $\partial B_r(x) \cap \Omega$ such that

$$\partial B_r(x) \cap \Omega \subset \bigcup_{i=1}^{N} B_\delta(y_i),$$

and let us set the function $w_r^x : B_r(x) \cap \Omega \to \mathbf{R}$ defined by

$$w_r^x(z) := \max_i w_r^{x,y_i}(z) \qquad \text{for every } z \in B_r(x) \cap \Omega. \tag{3.10}$$

By construction and by (3.8, 3.9), we have:

1. $\operatorname{ess\,sup}_{B_r(x)\cap\Omega} f(z, Dw_r^x) \le \lambda$;
2. $|w_r^x(z) - v(x)| > |v(z) - v(x)| + \varepsilon$ for every $z \in \partial B_r(x) \cap \Omega$;
3. $|w_r^x(z) - v(x)| < |v(z) - v(x)| + \varepsilon$ for every $z \in B_{r'}(x) \cap \Omega$.

Now let $Z_r$ be a finite set of points of $\Omega$ such that

$$\Omega \subset \bigcup_{z \in Z_r} B_{r'}(z),$$

and consider the function $w_r : \Omega \to \mathbf{R}$ defined by

$$w_r(x) := \min_{z \in Z_r \cap B_r(x)} w_r^z(x). \tag{3.11}$$

From properties 2 and 3 above it follows that $w_r$ is continuous. Moreover, for almost every $x$ in $\Omega$, $Dw_r(x)$ coincides with $Dw_r^z(x)$ for some $z \in Z_r$ and this implies that $w_r \in W^{1,\infty}(\Omega)$ and $F(w_r) \le \lambda$.

Now let us prove that $\|w_r - v\|_\infty \to 0$ as $r \to 0^+$. To this aim, let us fix $x \in \Omega$, and let $z \in B_r(x)$ be such that $w_r(x) = w_r^z(x)$. Recalling that by construction $w_r^z(z) = v(z)$, and using (3.2) and (3.3), we conclude

$$\begin{aligned}
|w_r(x) - v(x)| &\le |w_r^z(x) - w_r^z(z)| + |w_r^z(z) - v(x)| \\
&= |w_r^z(x) - w_r^z(z)| + |v(z) - v(x)| \\
&\le \|Dw_r^z\|_\infty |x - z|_\Omega + d_\lambda(x, z) \\
&\le 2\alpha^{-1}(\lambda)|x - z|_\Omega \le 2\alpha^{-1}(\lambda) C r.
\end{aligned}$$

Therefore, for every $\{r_n\} \to 0$, the sequence $v_n := w_{r_n}$ does the job. □

The following result is a variant of the Lemma 3.2: we remove the assumption that $\Omega$ has Lipschitz continuous boundary but we require that the supremal functional is weakly* l.s.c. on every open subset of $\Omega$.

**Lemma 3.3** *Let $\Omega \subset \mathbf{R}^N$ be an open set. Let $F$ be a supremal functional on $W^{1,\infty}(\Omega)$ of the form (2.9) represented by a Carathéodory supremand $f : \Omega \times \mathbf{R}^N \to \mathbf{R}$ satisfying (2.2) and (2.3). Assume that $F(\cdot, A)$ is weakly* l.s.c. on $W^{1,\infty}(A)$ for every $A \in \mathcal{A}$. If $v \in W^{1,\infty}(\Omega)$ is such that $R_\lambda(v) < 1$ then $F(v,\Omega) \leq \lambda$.*

*Proof* Fix $\bar{r} \in \mathbf{R}^+$. If $x \in \Omega_{\bar{r}} = \{x \in \Omega : d(x, \partial\Omega) > \bar{r}\}$ then $B_r(x) \subset B_{\bar{r}}(x) \subset \Omega$ for every $0 < r < \bar{r}$. In particular for every $x, y \in \Omega_{\bar{r}}$ with $|x-y| = r$ we have that $d_\lambda(x,y) \leq 2\alpha^{-1}(\lambda) r$. By repeating the proof of the previous lemma, for every $r \in \mathbf{R}^+$ we can construct a function $w_r \in W^{1,\infty}(\Omega)$ such that $F(w_r, \Omega_{\bar{r}}) \leq \lambda$ and

$$|w_r(x) - v(x)| \leq 2\alpha^{-1}(\lambda)$$

for every $x \in \Omega_{\bar{r}}$. If $\{r_n\} \to 0$, the sequence $v_n := w_{r_n}$ weakly* converges to $v$ in $\Omega_{\bar{r}}$ and thanks to the weak* lower semicontinuity of $F(\cdot, \Omega_{\bar{r}})$ we have that

$$F(v, \Omega_{\bar{r}}) \leq \liminf_n F(w_{r_n}, \Omega_{\bar{r}}) \leq \liminf_n F(w_{r_n}, \Omega) \leq \lambda.$$

This easily implies that $F(v,\Omega) \leq \lambda$. □

Finally, we cite the following result obtained in [14] as a corollary of Lemma 3.2. This will be useful when we will be interested to the 1-homogeneous supremal functionals represented by a Carathéodory function.

**Proposition 3.4** (Proposition 3.5 in [14]) *Let $f : \Omega \times \mathbf{R}^N \to \mathbf{R}$ be a Carathéodory supremand satisfying (2.5) and (2.6). Let $d : \Omega \times \Omega \to \mathbf{R}^+$ be the distance defined by*

$$d(x,y) := \sup\left\{u(x) - u(y), u \in W^{1,\infty}(\Omega) : \operatorname*{ess\,sup}_\Omega f(x, Du(x)) \leq 1\right\}. \tag{3.12}$$

*If the functional $F(u) = \operatorname{ess\,sup}_\Omega f(x, Du(x))$ is a weakly* l.s.c. on $W^{1,\infty}(\Omega)$ then*

$$F(u) = \sup_{x,y \in \Omega, x \neq y} \frac{u(x) - u(y)}{d(x,y)}$$

*for every $u \in W^{1,\infty}(\Omega)$.*

## 4 Approximation of Supremal Functionals

### 4.1 Moreau–Yosida Transform

A key tool we shall use is a modification of the Moreau–Yosida Transform first introduced in [1] by Alvarez, Barron and Ishii for functions $f : \mathbf{R}^N \to \mathbf{R} \cup \{+\infty\}$. This modified infimal convolution is compatible with the max operator $\vee$ just as the classical convolution is compatible with the $+$ operator.

**Proposition 4.1** (Theorem 3.1 in [1]) *Let $f : \Omega \times \mathbf{R}^N \to [0, +\infty)$ be a normal supremand. If we set for every $\lambda > 0$*

$$f_\lambda(x,\xi) = \inf\left\{f(x,\eta) \vee \lambda|\xi - \eta| : \xi \in \mathbf{R}^N\right\} \tag{4.1}$$

*then we have that* $f_\lambda$ *is a normal supremand such that*

$$|f_\lambda(x,\xi) - f_\lambda(x,\eta)| \le \lambda|\xi - \eta|$$

*and*

$$f(x,\xi) = \sup\left\{f_\lambda(x,\xi) : \lambda > 0\right\}$$

*for every* $x \in \Omega$ *and for every* $\xi, \eta \in \mathbf{R}^N$.

In this section we apply this modified infimal convolution to a functional $F : W^{1,\infty}(\Omega) \to [0,+\infty]$ and we show an analogous approximation result.

**Theorem 4.2** *Let* $F : W^{1,\infty}(\Omega) \to [0,+\infty]$ *be a (strongly) l.s.c. functional such that* $F(u+c) = F(u)$ *for every* $u \in W^{1,\infty}(\Omega)$ *and for every* $c \in \mathbf{R}$. *If we set*

$$F_\lambda(u) := \inf\left\{F(v) \vee \lambda\|Du - Dv\|_\infty : v \in W^{1,\infty}(\Omega)\right\}$$

*for every* $\lambda > 0$ *then we have*

$$F(u) = \sup_\lambda F_\lambda(u).$$

*Moreover, the functional* $F_\lambda$ *satisfies the condition*

$$F_\lambda(u) \le F_\lambda(v) + \lambda\|Du - Dv\|_\infty \quad \text{for every } u, v \in W^{1,\infty}(\Omega). \tag{4.2}$$

*Therefore if* $F$ *is finite in at least one point then*

$$|F_\lambda(u) - F_\lambda(v)| \le \lambda\|Du - Dv\|_\infty \quad \text{for every } u, v \in W^{1,\infty}(\Omega).$$

*Proof* Fix $u \in W^{1,\infty}(\Omega)$. By taking $v = u$ in the definition of $F_\lambda(u)$ we obtain the inequality

$$F_\lambda(u) \le F(u).$$

Take now $t < F(u)$; since $F$ is lower semicontinuous there exists $\delta > 0$ such that

$$t < \inf\left\{F(w) : w \in W^{1,\infty}(\Omega),\ \|u - w\|_{W^{1,\infty}(\Omega)} < \delta\right\}.$$

Let $0 < \delta' < \frac{\delta}{1+\operatorname{diam}\Omega}$. Then

$$t < \inf\left\{F(v) : v \in W^{1,\infty}(\Omega),\ \|Du - Dv\|_\infty < \delta'\right\}. \tag{4.3}$$

In fact, let $v \in W^{1,\infty}(\Omega)$ be such that

$$\|Du - Dv\|_\infty < \delta'.$$

Fix $x_0 \in \Omega$ and define $w(x) := v(x) + u(x_0) - v(x_0)$. Then

$$\|Du - Dw\|_\infty = \|Du - Dv\|_\infty \le \delta'$$

and for a.e. $x \in \Omega$

$$\begin{aligned} |u(x) - w(x)| &\le |(u(x) - u(x_0)) - (w(x) - w(x_0)| \\ &\le \|Du - Dw\|_\infty |x - x_0| \le \delta' \cdot \mathrm{diam}\Omega. \end{aligned}$$

Therefore

$$\|u - v\|_{W^{1,\infty}(\Omega)} \le \delta' + \delta' \cdot \mathrm{diam}\Omega < \delta.$$

This implies $F(v) = F(w) \ge \inf\{F(w) : w \in W^{1,\infty}(\Omega),\ \|u - w\|_{W^{1,\infty}(\Omega)} < \delta\}$ and thus (4.3) follows. Now let $\lambda > 0$ be such that $\lambda \cdot \delta' > t$. For every $v \in W^{1,\infty}(\Omega)$ with $\|Du - Dv\|_\infty < \delta'$ we have

$$F(v) \vee \lambda \|Du - Dv\|_\infty \ge F(v) > t$$

whereas for every $v \in W^{1,\infty}(\Omega)$ with $\|Du - Dv\|_\infty \ge \delta$ we have

$$F(v) \vee \lambda \|Du - Dv\|_\infty \ge \lambda \delta' > t.$$

Hence $F_\lambda(u) \ge t$ and, since $t$ was arbitrary, this proves the inequality

$$F(u) \le \sup\left\{F_\lambda(u) : \lambda > 0\right\}.$$

Finally let $u, v \in W^{1,\infty}(\Omega)$. For fixed $\varepsilon > 0$, let $w \in W^{1,\infty}(\Omega)$ be such that

$$F_\lambda(v) \ge F(w) \vee \lambda \|Dv - Dw\|_\infty - \varepsilon.$$

Taking into account that for every $a, b, c \in \mathbf{R}$ we have

$$a \vee b \le a \vee c + |b - c|$$

we obtain

$$\begin{aligned} F_\lambda(u) &\le F(w) \vee \lambda \|Du - Dw\|_\infty \\ &\le F(w) \vee \lambda \|Dv - Dw\|_\infty + \lambda \Big| \|Du - Dw\|_\infty - \|Dv - Dw\|_\infty \Big| \\ &\le F_\lambda(v) + \varepsilon + \lambda \|Du - Dv\|_\infty. \end{aligned}$$

Since $\varepsilon$ was arbitrary, inequality (4.2) follows and the proof is achieved. □

### 4.2 An Approximation through Coercive Functionals

Here we approximate a non-negative supremal functional through a sequence of coercive supremal functionals.

**Proposition 4.3** *Let $\Omega$ be an open subset of $\mathbf{R}^N$. Let $g : \Omega \times \mathbf{R}^N \to \mathbf{R}^+$ be a normal supremand. Let*

$$g_n(x, \xi) := g(x, \xi) \vee \frac{1}{n} |\xi| \tag{4.4}$$

*and for every* $n \in \mathbf{N}$ *let* $G, G_n : W^{1,\infty}(\Omega) \to \mathbf{R}$ *be the functionals defined by*

$$G(u) = \operatorname*{ess\,sup}_{\Omega} g(x, Du(x)) \tag{4.5}$$

*and by*

$$G_n(u) = \operatorname*{ess\,sup}_{\Omega} g_n(x, Du(x)) \tag{4.6}$$

*respectively. Then*:

(i) *for every* $n \in \mathbf{N}$ *and for every* $u \in W^{1,\infty}(\Omega)$

$$G_n(u) = G(u) \vee \frac{1}{n}\|Du\|_{L^\infty(\Omega)}; \tag{4.7}$$

(ii) *the sequence* $(G_n)_n$ *pointwise converges to* $G$;

(iii) *if* $G_n$ *is a level convex functional for every* $n \in \mathbf{N}$, *then* $G$ *is a level convex functional.*

*Proof* In order to show (i), fix $n \in \mathbf{N}$ and $u \in W^{1,\infty}(\Omega)$. The inequality $G_n(u) \leq G(u) \vee \frac{1}{n}\|Du\|_{L^\infty(\Omega)}$ is trivial. For the converse inequality, for fixed $\delta$ there exists $B_\delta \subset \Omega$ with $|B_\delta| > 0$ such that $g_n(x, Du(x)) \geq G_n(u) - \delta$ for every $x \in B_\delta$. Set $B_\delta^+ = \{x \in B_\delta : g_n(x,\xi) = g(x,\xi)\}$ and $B_\delta^- = \{x \in B_\delta : g_n(x,\xi) = \frac{1}{n}|\xi|\}$. If $|B_\delta^+| > 0$ then $G(u) \geq G_n(u) - \delta$ while if $|B_\delta^-| > 0$ then $\frac{1}{n}\|Du\|_{L^\infty(\Omega)} \geq G_n(u) - \delta$. In both cases $\frac{1}{n}\|Du\|_{L^\infty(\Omega)} \vee G(u) \geq G_n(u) - \delta$ for every $\delta > 0$. Then $\frac{1}{n}\|Du\|_{L^\infty(\Omega)} \vee G(u) \geq G_n(u)$. As a consequence, for every $u \in W^{1,\infty}(\Omega)$

$$\lim_n G_n(u) = \lim_n G(u) \vee \frac{1}{n}\|Du\|_{L^\infty(\Omega)} = G(u) \vee 0 = G(u).$$

Finally, concerning (iii), fix $u, v \in X$ and $\lambda \in (0, 1)$. Then

$$G(\lambda u + (1-\lambda)v) = \lim_n G_n(\lambda u + (1-\lambda)v) \leq \lim_n G_n(u) \vee G_n(v) = G(u) \vee G(v)$$

i.e. $G$ is a level convex functional. □

## 5 Some Representation Results

This section is devoted to the construction of admissible supremands for supremal functionals of the form (2.1) and (2.9). Note that, as shown in the example given in Sect. 1 of [8], if $F$ is a supremal functional of the form (2.9), it is not possible to represent $F$ through the function $h$ defined by

$$h(x,\xi) := \inf_{r>0} F(w_{x,\xi}, B_r(x))$$

where $w_{x,\xi}(y) := u + \xi \cdot (y - x)$.

The following proof is inspired by the proof of Lemma 3.3 in [8].

**Proposition 5.1** *Let $\Omega$ be an open subset of $\mathbf{R}^N$. Let $F : W^{1,\infty}(\Omega) \to \overline{\mathbf{R}}$ be a functional such that for any $M > 0$ there exists a modulus of continuity $\omega_M$ such that*

$$|F(u) - F(v)| \leq \omega_M(\|Du - Dv\|_\infty)$$

*for every $u, v \in W^{1,\infty}(\Omega)$ s.t. $\|Du\|_\infty, \|Dv\|_\infty \leq M$. Let $\varphi : \Omega \times \mathbf{R}^N \to \mathbf{R}$ be defined by*

$$\varphi(x,\xi) := \inf\big\{F(u) \mid u \in W^{1,\infty}(\Omega) \text{ s.t. } x \in \widehat{u}, \text{ with } Du(x) = \xi\big\}. \tag{5.1}$$

*Then for every $\xi \in \mathbf{R}^N$ the function $x \mapsto \varphi(x,\xi)$ is measurable in $\Omega$.*

*Moreover if there exists an increasing continuous function $\alpha : \mathbf{R}^+ \to \mathbf{R}^+$ such that* $\lim_{t\to+\infty} \alpha(t) = +\infty$ *and $F(u) \geq \alpha(\|Du\|_\infty)$ then*:

(i) *$\varphi$ is a Carathéodory supremand satisfying* (2.3) *and* (2.2) *for a suitable family $(\omega'_M)_M$ of moduli of continuity*;
(ii) *for any $u \in W^{1,\infty}(\Omega)$*

$$F(u) \geq \operatorname*{ess\,sup}_{\Omega} \varphi(x, Du(x)).$$

*Proof* Let $\xi \in \mathbf{R}^N$ and $\lambda \in \mathbf{R}$ be fixed. Define the sets

$$A(x) := \{u \in W^{1,\infty}(\Omega) : x \in \widehat{u} \text{ with } Du(x) = \xi\}, \tag{5.2}$$

and

$$K_\lambda := \{x \in \Omega : \forall u \in A(x)\ F(u) \geq \lambda\} = \{x \in \Omega : \varphi(x,\xi) \geq \lambda\}. \tag{5.3}$$

If we prove that $K_\lambda$ is measurable for every $\lambda \in \mathbf{R}$, then $\varphi(\cdot,\xi)$ is measurable. Suppose that $K_\lambda$ is not measurable. Then there is a set $C$ with $K_\lambda \subset C$ s.t. $C$ is measurable and of minimal measure. Let $x_0 \in \widehat{C} \setminus K_\lambda$ where $\widehat{C}$ the set of the points of density 1 of $C$. From the definition of $K_\lambda$, there is some $u \in A(x_0)$ such that $F(u) < \lambda$. Now, fix $\varepsilon > 0$ such that $F(u) < \lambda - \varepsilon$. Since the functional $F$ is strongly continuous in $W^{1,\infty}(\Omega)$, then there exists $\delta > 0$ such that $F(v) < \lambda$ for every $v \in W^{1,\infty}(\Omega)$ such that $\|v - u\|_{W^{1,\infty}(\Omega)} \leq \delta$. Set

$$A_1 = \Big\{x \in \Omega \mid x \in \widehat{u}, |u(x) - u(x_0)| \leq \delta/2, |Du(x) - \xi| \leq \tfrac{\delta}{2\mathrm{diam}(\Omega)}\Big\}. \tag{5.4}$$

Note that $A_1$ is measurable and since $x_0$ is a Lebesgue point of $Du$, then $|A_1| > 0$. We claim that $A_1 \cap K_\lambda = \emptyset$. In fact, if $x \in A_1$, then the function $v_x \in W^{1,\infty}(\Omega)$ defined by

$$v_x(y) := u(y) + (u(x_0) - u(x)) + \langle Du(x_0) - Du(x), y - x\rangle$$

belongs to $A(x)$ and $\|v_x - u\|_{W^{1,\infty}(\Omega)} \leq \delta$. Thus it easily follows that $F(v_x) < \lambda$. So $x \notin K_\lambda$ which implies that $K_\lambda \subset C \setminus A_1$. Moreover the set $C \setminus A_1$ is still measurable. If we show that $x_0$ is a point of density 1 of $A_1$ then there exists $r > 0$ such that

$$|A_1 \cap B_r(x_0)| \geq \frac{1}{2}|B_r(x_0)| \quad \text{and}$$

$$|C \cap B_r(x_0)| \geq \frac{3}{4}|B_r(x_0)|.$$

Therefore

$$\begin{aligned}|A_1 \cap C| &\geq |A_1 \cap B_r(x_0) \cap C| = |A_1 \cap B_r(x_0)| - |A_1 \cap B_r(x_0,) \setminus C| \\ &\geq |A_1 \cap B_r(x_0)| - |B_r(x_0) \setminus C| \\ &\geq |A_1 \cap B_r(x_0)| - |B_r(x_0)| + |B_r(x_0) \cap C| \\ &\geq \frac{3}{4}|B_r(x_0)| - |B_r(x_0)| + \frac{1}{2}|B_r(x_0)| \\ &= \frac{1}{2}|B_r(x_0)|.\end{aligned}$$

In particular $|C \setminus A_1| < |C|$ and since $K \subset (C \setminus A_1)$, we have contradicted the minimality of $C$. In order to show that $x_0$ is a point of density 1 of $A_1$ note that there exists $r_0 = r_0(\delta) > 0$ such that for every $r < r_0$

$$B_r(x_0) \cap A_1 = \left\{x \in B_r(x_0); x \in \widehat{u}, |Du(x) - \xi_0| \leq \tfrac{\delta}{2\text{diam}(\Omega)}\right\}. \tag{5.5}$$

Now

$$\begin{aligned}\frac{|B_r(x_0) \cap A_1|}{|B_r(x_0)|} &= 1 - \frac{|B_r(x_0) \setminus A_1|}{|B_r(x_0)|} = 1 - \frac{\int_{B_r(x_0)\setminus A_1} dx}{|B_r(x_0)|} \\ &\geq 1 - \frac{2\text{diam}(\Omega)}{\delta |B_r(x_0)|}\int_{B_r(x_0)} |Du(x) - \xi_0| dx\end{aligned}$$

and since $x_0$ is a Lebesgue point of $Du$ with $Du(x_0) = \xi_0$ we obtain

$$\lim_{r\to 0^+} \frac{|B_r(x_0) \cap A_1|}{|B_r(x_0)|} = 1$$

i.e. $x_0$ is a point of density 1 of $A_1$. Now we assume that $F$ satisfies also the coercivity assumption $F(u) \geq \alpha(\|Du\|_\infty)$ for every $u \in W^{1,\infty}(\Omega)$. We show that $\varphi$ satisfies assumption (2.2) for a suitable family of moduli of continuity. Let $M > 0$ be fixed. Then there is some constant $K = K(M)$ such that, for any $(x, \xi) \in \Omega \times \mathbf{R}^N$ with $|\xi| \leq M$ and for any $v \in W^{1,\infty}(\Omega)$,

$$[F(v) \leq \varphi(x, \xi) + 1] \quad \Rightarrow \quad \|Dv\|_\infty \leq M'.$$

In fact, from the continuity assumption on $F$,

$$F(v) \leq \varphi(x, \xi) + 1 \leq F(\varphi_\xi) + 1 \leq F(0) + \omega_M(|\xi|) + 1$$

where $\varphi_\xi(x) := \xi \cdot x$ and, from the coercivity condition on $F$, we have

$$\|Dv\|_\infty \leq \alpha^{-1}(F(0) + \omega_M(M) + 1).$$

In particular if $\xi, \eta \in B_M(0)$ and $u_\xi \in W^{1,\infty}(\Omega)$ is such that $F(u_\xi) \leq \varphi(x, \xi) + \varepsilon$ where $0 < \varepsilon < 1$ then

$$\varphi(x, \eta) \leq F(u_\xi + \varphi_{\eta-\xi}) \leq F(u_\xi) + \omega_{M'}(|\xi - \eta|) \leq \varphi(x, \xi) + \omega_{M'}(|\xi - \eta|) + \varepsilon$$

where $M' = \alpha^{-1}(\omega_M(M) + F(0) + 1) + 2M$. As $\varepsilon \to 0^+$ the last inequality implies

$$\varphi(x,\eta) \le \varphi(x,\xi) + \omega_{M'}(|\xi - \eta|)$$

and, by changing the roles of $\xi$ and $\eta$, it follows

$$|\varphi(x,\eta) - \varphi(x,\xi)| \le \omega_{M'}(|\xi - \eta|).$$

Then it is sufficient to define $\omega'_M := \omega_{M'}$. Finally by the definition of $\varphi$ it follows that

$$\operatorname*{ess\,sup}_{\Omega} \varphi(x, Du(x)) \le F(u)$$

for every $u \in W^{1,\infty}(\Omega)$. □

Under the stronger assumption that $F$ is a Lipschitz continuous functional, it is not necessary to require a coercivity assumption in order to show that $\varphi$ is a Carathéodory supremand.

**Proposition 5.2** *Let $\Omega$ be an open subset of $\mathbf{R}^N$. Let $F : W^{1,\infty}(\Omega) \to \overline{\mathbf{R}}$ be a functional. Assume that there exists $L > 0$ such that $F$ is $L$-Lipschitz continuous functional i.e.*

$$|F(u) - F(v)| \le L\|u - v\|_{W^{1,\infty}(\Omega)} \quad \textit{for every } u, v \in W^{1,\infty}(\Omega).$$

*Then*:

(i) *the function $\varphi$ defined by* (5.1) *is a Carathéodory supremand such that*

$$|\varphi(x,\xi) - \varphi(x,\eta)| \le L|\xi - \eta| \tag{5.6}$$

*for every $x \in \Omega$ and for every $\xi, \eta \in \Omega$;*

(ii) *for any $u \in W^{1,\infty}(\Omega)$*

$$F(u) \ge \operatorname*{ess\,sup}_{\Omega} \varphi(x, Du(x)).$$

*Proof* Thanks to Proposition 5.1, for every $\xi \in \mathbf{R}^N$ the function $x \mapsto \varphi(x,\xi)$ is measurable in $\Omega$. In order to show that for every $x \in \Omega$ $\varphi(x,\cdot)$ is $L$-Lipschitz continuous, fix $x \in \Omega$ and $\xi, \eta \in \mathbf{R}^N$. Let $u_\xi \in W^{1,\infty}(\Omega)$ be such that $F(u_\xi) \le \varphi(x,\xi) + \varepsilon$. Then

$$\varphi(x,\eta) \le F(u_\xi + \varphi_{\eta-\xi}) \le F(u_\xi) + L|\xi - \eta| \le \varphi(x,\xi) + L|\xi - \eta| + \varepsilon.$$

As $\varepsilon \to 0^+$ the last inequality implies

$$\varphi(x,\eta) \le \varphi(x,\xi) + L|\xi - \eta|$$

and, by changing the roles of $\xi$ and $\eta$, it follows

$$|\varphi(x,\eta) - \varphi(x,\xi)| \le L|\xi - \eta|.$$ □

**Lemma 5.3** *Let $\Omega$ be an open subset of $\mathbf{R}^N$. Let $f : \Omega \times \mathbf{R}^N \to \mathbf{R}^+$ be a Carathéodory supremand satisfying assumption (2.2) and let $\varphi$ and $\tilde{f}$ be the functions given respectively by (2.4) and (2.10). Then there exists a negligible set $H \subset \Omega$ such that*

$$\varphi \geq \tilde{f} \geq f \text{ on } (\Omega \setminus H) \times \mathbf{R}^N.$$

*Proof* By definition, it easily follows that $\varphi \geq \tilde{f}$. Now, for every $\xi \in \mathbf{Q}^N$ let $N_\xi$ be the negligible set such that $\Omega \setminus N_\xi$ is the set of the Lebesgue points of $f(\cdot, \xi)$. Then $\bigcup_{\xi \in \mathbf{Q}^N} N_\xi$ is a negligible set and if $x \in \Omega' = \Omega \setminus \big(\bigcup_{\xi \in \mathbf{Q}^N} N_\xi\big)$ then $x$ is a Lebesgue point of $f(\cdot, \eta)$. In fact, let $(\xi_n)_n \subset \mathbf{Q}^N$ be such that $\xi_n \to \eta$. Fix $\varepsilon > 0$ and $n_0 \in \mathbf{N}$ such that $|\xi_n - \eta| \leq \varepsilon$ for every $n \geq n_0$. Then for every $n \geq n_0$ and for every $\rho > 0$

$$\begin{aligned}
&\int_{B_\rho(x)} |f(y,\eta) - f(x,\eta)| dy \\
&\quad \leq \int_{B_\rho(x)} |f(y,\eta) - f(y,\xi_n)| dy + \int_{B_\rho(x)} |f(y,\xi_n) - f(x,\xi_n)| dy \\
&\qquad + \int_{B_\rho(x)} |f(x,\xi_n) - f(x,\eta)| dy \\
&\quad \leq \int_{B_\rho(x)} 2w_M(|\eta - \xi_n|) dy + \int_{B_\rho(x)} |f(y,\xi_n) - f(x,\xi_n)| dy \\
&\quad \leq 2|B_\rho| w_M(\varepsilon) + \int_{B_\rho(x)} |f(y,\xi_n) - f(x,\xi_n)| dy.
\end{aligned}$$

Letting $\rho \to 0$ we obtain

$$\lim_{\rho \to 0} \frac{1}{|B_\rho|} \int_{B_\rho(x)} |f(y,\eta) - f(x,\eta)| dy \leq 2w_M(\varepsilon)$$

and from the arbitrariness of $\varepsilon$ we can conclude. Now we show that for every $x \in \Omega'$ and for every $\xi \in \mathbf{R}^N$ it holds

$$\tilde{f}(x,\xi) \geq f(x,\xi). \tag{5.7}$$

Note that $\Omega \setminus \Omega'$ is a negligible set. Fix $x \in \Omega'$ and $\xi \in \mathbf{R}^N$. Then there exists $B_r(x)$ and $u_x \in W^{1,\infty}(\Omega)$ such that $x \in \widehat{u_x}$, $Du_x(x) = \xi$ and $\tilde{f}(x,\xi) \geq \operatorname{ess\,sup}_{B_r(x)} f(y, Du_x(y)) - \varepsilon$. If we show that $x$ is a Lebesgue point of $h = f(\cdot, Du_x(\cdot))$ then

$$\tilde{f}(x,\xi) \geq \operatorname*{ess\,sup}_{B_r(x)} f(y, Du_x(y)) - \varepsilon \geq f(x,\xi) - \varepsilon$$

for every $\varepsilon > 0$ and thus we can conclude. Let us show that $x$ is a Lebesgue point of the function $h(y) := f(y, Du_x(y))$. For $M = \|Du_x\|_\infty$ we have

$$\int_{B_\rho(x)} |f(y, Du_x(y)) - f(x, Du_x(x))| dy$$

$$\leq \int_{B_\rho(x)} |f(y, Du_x(y)) - f(y, Du_x(x)|dy$$
$$+ \int_{B_\rho(x)} |f(y, Du_x(x)) - f(x, Du_x(x))|dy$$
$$\leq \int_{B_\rho(x)} \omega_M(|Du_x(y) - Du_x(x)|)dy$$
$$+ \int_{B_\rho(x)} |f(y, Du_x(x)) - f(x, Du_x(x))|dy.$$

Since $x \in \Omega'$ we have that

$$\frac{1}{|B_\rho|} \int_{B_\rho(x)} |f(y, Du_x(x)) - f(x, Du_x(x))|dy \to 0 \tag{5.8}$$

when $\rho \to 0$. Then for fixed $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon)$ such that for every $\rho \leq r_0$

$$\int_{B_\rho(x)} |Du_x(y) - Du_x(x)|dy \leq \varepsilon |B_\rho|.$$

By Chebishev Theorem we have that

$$\left|\{y \in B_\rho(x) : |Du(y) - Du(x)| \geq \sqrt{\varepsilon}\}\right| \leq \frac{1}{\sqrt{\varepsilon}} \int_{B_\rho(x)} |Du_x(y) - Du_x(x)|dy$$
$$\leq \sqrt{\varepsilon} |B_\rho|.$$

Thus for every $\rho \leq r_0$, we have

$$\int_{B_\rho(x)} \omega_M(|Du_x(y) - Du_x(x)|)dy$$
$$= \int_{\{y \in B_\rho(x) : |D(y) - Du(x)| \geq \sqrt{\varepsilon}\}} \omega_M(|Du_x(y) - Du_x(x)|)dy$$
$$+ \int_{\{y \in B_\rho(x) : |Du(y) - Du(x)| \leq \sqrt{\varepsilon}\}} \omega_M(|Du_x(y) - Du_x(x)|)dy$$
$$\leq C\sqrt{\varepsilon}|B_\rho| + \omega_M(\sqrt{\varepsilon})|B_\rho|$$

where $C = \max\{\omega_M(\xi) : |\xi| \leq 2M\}$. Thus

$$\lim_{\rho \to 0^+} \frac{1}{|B_\rho|} \int_{B_\rho(x)} \omega_M(|Du_x(y) - Du_x(x)|)dy \leq C\sqrt{\varepsilon} + \omega_M(\sqrt{\varepsilon})$$

for every $\varepsilon > 0$ and thus

$$\lim_{\rho \to 0^+} \frac{1}{|B_\rho|} \int_{B_\rho(x)} \omega_M(|Du_x(y) - Du_x(x)|)dy = 0. \tag{5.9}$$

In particular from (5.8) and (5.9) it follows that

$$\frac{1}{|B_\rho|}\int_{B_\rho(x)} |f(y, Du_x(y)) - f(x, Du_x(x))|dy \to 0$$

when $\rho \to 0$. This completes the proof of (5.7). □

We underline the difference between the following theorem and the representation result shown in Theorem 2.2 in [8]. We prove that the function $\tilde{f}$ is an admissible supremand for a localized supremal functional $F(u, A) = \operatorname{ess\,sup}_A f(x, Du(x))$ without requiring that for every open set $A \subset \Omega$ $F(\cdot, A)$ is weakly* lower semicontinuous.

**Theorem 5.4** *Let $\Omega$ be an open subset of $\mathbf{R}^N$. Let $f : \Omega \times \mathbf{R}^N \to \mathbf{R}^+$ be a Carathéodory supremand satisfying assumptions (2.2) and (2.3). Then*

(i) *the functions $\varphi$ and $\tilde{f}$ given respectively by (2.4) and (2.10) are Carathéodory supremands satisfying (2.3) and (2.2) for a suitable family $(\omega'_M)_M$ of moduli of continuity*;
(ii) *for every $u \in W^{1,\infty}(\Omega)$ and for every $A \in \mathcal{A}$*

$$\operatorname*{ess\,sup}_{\Omega} f(x, Du(x)) = \operatorname*{ess\,sup}_{\Omega} \varphi(x, Du(x)) \quad \textit{and}$$

$$\operatorname*{ess\,sup}_{A} f(x, Du(x)) = \operatorname*{ess\,sup}_{A} \tilde{f}(x, Du(x));$$

(iii) *if $f(\cdot, \xi)$ is continuous on $\Omega$ for every $\xi \in \mathbf{R}^N$ then there exists a negligible set $H$ such that $f = \tilde{f}$ on $(\Omega \setminus H) \times \mathbf{R}^N$.*

*Proof* By applying Proposition 5.1 to the functional $F : W^{1,\infty}(\Omega) \to \overline{\mathbf{R}}$ defined by

$$F(u) = \operatorname*{ess\,sup}_{\Omega} f(x, Du(x))$$

we obtain that the function $\varphi$ is a Carathéodory supremand satisfying (2.3) and (2.2) for a suitable family $(\omega'_M)_M$ of moduli of continuity and such that

$$\operatorname*{ess\,sup}_{\Omega} f(x, Du(x)) \geq \operatorname*{ess\,sup}_{\Omega} \varphi(x, Du(x))$$

for any $u \in W^{1,\infty}(\Omega)$. The converse inequality follows from Lemma 5.3.

Now let us choose a countable base $(A_n)_{n\in\mathbf{N}}$ of open subsets of $\Omega$ and for every $n \in \mathbf{N}$ and for every $u \in W^{1,\infty}(\Omega)$ define $F_n(u) := \operatorname{ess\,sup}_{A_n} f(x, Du(x))$. By applying Proposition 5.1 to the functional $F_n$ we obtain that for every $n \in \mathbf{N}$

$$\varphi_n(x, \xi) := \inf\left\{\operatorname*{ess\,sup}_{A_n} f(x, Du(x)) \mid u \in W^{1,\infty}(A_n) \text{ s.t. } x \in \widehat{u}, \text{ with } Du(x) = \xi\right\}$$

is a Carathéodory supremand such that

$$\operatorname*{ess\,sup}_{A_n} f(x, Du(x)) \geq \operatorname*{ess\,sup}_{A_n} \varphi_n(x, Du(x)) \tag{5.10}$$

for any $u \in W^{1,\infty}(A_n)$. Since

$$\tilde{f} = \inf \varphi_n,$$

we have that $\tilde{f}$ is a Borel function. Moreover by applying Proposition 5.1 for any $M > 0$ there exists a modulus of continuity $\omega'_M$ such that

$$|\varphi_n(x,\xi) - \varphi_n(x,\eta)| \leq \omega'_M(|\xi - \eta|)$$

for a.e. $x \in \Omega$ and for every $\xi, \eta \in B_M(0)$ and for every $n \in \mathbf{N}$. This implies that

$$|\tilde{f}(x,\xi) - \tilde{f}(x,\eta)| \leq \omega'_M(|\xi - \eta|)$$

for a.e. $x \in \Omega$ and for every $\xi, \eta \in B_M(0)$. Therefore $\tilde{f}$ is a Carathéodory supremand. Finally thanks to (5.10) we have that for every $n \in \mathbf{N}$

$$\operatorname*{ess\,sup}_{A_n} f(x, Du(x)) \geq \operatorname*{ess\,sup}_{A_n} \tilde{f}(x, Du(x))$$

for any $u \in W^{1,\infty}(A_n)$. Thanks to Lemma 5.3 the converse inequalities follow and since $(A_n)_{n\in\mathbf{N}}$ is a countable base of open subsets of $\Omega$ it follows that

$$\operatorname*{ess\,sup}_{A} f(x, Du(x)) = \operatorname*{ess\,sup}_{A} \tilde{f}(x, Du(x))$$

for any $u \in W^{1,\infty}(\Omega)$ and for every $A \in \mathcal{A}$.

Finally in order to show (iii) thanks to (5.7) it is sufficient to show that $f(x_0,\xi) \geq \tilde{f}(x_0,\xi)$ for every $x_0 \in \Omega'$. Fix $x_0 \in \Omega'$. By the definition of $\tilde{f}$

$$\operatorname*{ess\,sup}_{B_r(x_0)} f(x,\xi) \geq \tilde{f}(x_0,\xi).$$

By letting $r \to 0$ and by using the continuity of $f(\cdot,\xi)$ it easily follows $f(x_0,\xi) \geq \tilde{f}(x_0,\xi)$. □

Finally we consider the particular case in which $f(x,\cdot)$ is a 1-homogeneous function. Note that inequality (5.11) cannot be proved directly by applying Lemma 5.3 since $f$ does not satisfy (2.2).

**Theorem 5.5** *Let $\Omega$ be an open subset of $\mathbf{R}^N$. Let $f : \Omega \times \mathbf{R}^N \to \mathbf{R}^+$ be a Carathéodory supremand satisfying assumption* (2.5) *and* (2.6). *Then the function $\varphi$ given by* (2.4) *is a Carathéodory supremand satisfying assumption* (2.5) *and* (2.6) *and such that*:

(i) *for a.e. $x \in \Omega$*

$$\varphi(x,\xi) \geq f(x,\xi) \quad \forall \xi \in \mathbf{R}^N; \tag{5.11}$$

(ii) *for every $u \in W^{1,\infty}(\Omega)$*

$$\operatorname*{ess\,sup}_{\Omega} f(x, Du(x)) = \operatorname*{ess\,sup}_{\Omega} \varphi(x, Du(x)).$$

*Proof* Let $F$ be the functional defined by

$$F(u) = \operatorname*{ess\,sup}_{\Omega} f(x, Du(x))$$

for all $u \in W^{1,\infty}(\Omega)$. In order to show (i) of Definition 2.1(b), let $\xi \in \mathbf{R}^N$ and $\lambda \in \mathbf{R}$ be fixed. Consider the sets $A(x)$ and $K_\lambda$ defined respectively by (5.2, 5.3). If we prove that $K_\lambda$ is measurable for every $\lambda \in \mathbf{R}$, then $\varphi(\cdot, \xi)$ is measurable. Suppose that $K_\lambda$ is not measurable. Then there is a set $C$ with $K_\lambda \subset C$ s.t. $C$ is measurable and of minimal measure. In fact, let $x_0 \in \widehat{C} \setminus K_\lambda$ where $\widehat{C}$ the set of the points of density 1 of $C$. From the definition of $K_\lambda$, there is some $u \in A(x_0)$ such that $F(u) < \lambda$. Now, fix $\varepsilon > 0$ such that $F(u) < \lambda - \varepsilon$ and set $\delta := \frac{\varepsilon}{\beta}$. By (2.5) we have that $F(v) \le \varepsilon$ for every $v \in W^{1,\infty}(\Omega)$ such that $\|v\|_{W^{1,\infty}(\Omega)} \le \delta$. Now consider the set $A_1$ defined by (5.4). We claim that $A_1 \cap K_\lambda = \emptyset$. Now, if $x \in A_1$, then the function $v_x \in W^{1,\infty}(\Omega)$ defined by

$$v_x(y) := u(y) + (u(x_0) - u(x)) + \langle Du(x_0) - Du(x), y - x\rangle$$

belongs to $A(x)$ and $\|v_x - u\|_{W^{1,\infty}(\Omega)} \le \delta$. By Proposition 3.4 we can write $F$ as

$$F(u) = \sup_{x,y\in\Omega,\, x\neq y} \frac{u(x) - u(y)}{d_1(x, y)}$$

where $d_1$ is given by (3.12). In particular, by Proposition 3.1

$$F(u + v) \le F(u) + F(v) \tag{5.12}$$

for every $u, v \in W^{1,\infty}(\Omega)$. This implies

$$F(v_x) \le F(v_x - u) + F(u) < \varepsilon + \lambda - \varepsilon = \lambda.$$

So $x \notin K_\lambda$. This implies that $K_\lambda \subset C \setminus A_1$. Moreover the set $C\setminus A_1$ is still measurable. Repeating the proof of Proposition 5.1 one can show that

$$|A_1 \cap C| \ge \frac{1}{2}|B_r(x_0)|$$

which implies that $|C \setminus A_1| < |C|$ and since $K \subset (C\setminus A_1)$, we have contradicted the minimality of $C$. Now we show (ii) of Definition 2.1(b). Let us fix $x \in \Omega$, $\varepsilon > 0$ and $\xi_1, \xi_2 \in \mathbf{R}^N$ such that $|\xi_1 - \xi_2| \le \min\{\frac{\varepsilon}{2\beta \mathrm{diam}(\Omega)}, \frac{\varepsilon}{2\beta}\}$. From the definition of $\varphi$ we can find some $u \in W^{1,\infty}(\Omega)$ such that $Du(x) = \xi_1$ and $\varphi(x, \xi_1) \ge F(u) - \epsilon$. Then, defined $w_{x,\xi_i}(y) := \xi_i \cdot (y - x)$, we have

$$\|w_{x,\xi_1} - w_{x,\xi_2}\|_{W^{1,\infty}(\Omega))} \le \frac{\varepsilon}{\beta}$$

and thus

$$\varphi(x, \xi_2) \le F(u + w_{x,\xi_2} - w_{x,\xi_1}) \le F(u) + F(w_{x,\xi_2} - w_{x,\xi_1}) \le \varphi(x, \xi_1) + \varepsilon.$$

By changing the roles of $\xi_1$ and $\xi_2$ we obtain that $\varphi(x,\cdot)$ is uniformly continuous on $\mathbf{R}^N$. Concerning (5.11), for every $n \in \mathbf{N}$ one can define

$$f_n(x,\xi) = \inf_{\eta \in \mathbf{R}^N} \{f(x,\eta) \vee n|\xi - \eta|\}. \tag{5.13}$$

Then, thanks to Proposition 4.1, for every $n \in \mathbf{N}$ the function $f_n$ is Lipschitz continuous with Lipschitz constant equal to $n$, $f_n \le f_m$ if $n \le m$ and since $f(x,\cdot)$ is continuous, we have

$$f(x,\xi) = \sup\left\{f_n(x,\xi) : n \in \mathbf{N}\right\}.$$

Now for every $n \in \mathbf{N}$ let $\psi_n$ be defined by

$$\psi_n(x,\xi) := \inf\left\{\operatorname*{ess\,sup}_{\Omega} f_n(x, Du(x)) \mid u \in W^{1,\infty}(\Omega) \text{ s.t. } x \in \widehat{u}, \text{ with } Du(x) = \xi\right\}. \tag{5.14}$$

By definition, we have that $\psi_n \le \varphi$ for every $n \in \mathbf{N}$ and, thanks to Theorem 5.4, $\psi_n$ is a Carathéodory supremand such that there exists a negligible set $H_n \subset \Omega$ such that

$$\psi_n \ge f_n \quad \text{on } (\Omega \setminus H_n) \times \mathbf{R}^N.$$

In particular, by defining $H = \bigcup_{n=1}^{\infty} H_n$ we obtain that

$$\varphi(x,\xi) \ge \bigvee_{n\in\mathbf{N}} \psi_n(x,\xi) \ge \bigvee_{n\in\mathbf{N}} f_n(x,\xi) = f(x,\xi) \text{ on } (\Omega \setminus H) \times \mathbf{R}^N$$

which implies that for every $u \in W^{1,\infty}(\Omega)$ the inequality

$$\operatorname*{ess\,sup}_{\Omega} \varphi(x, Du(x)) \ge \operatorname*{ess\,sup}_{\Omega} f(x, Du(x))$$

holds. By the definition of $\varphi$ the converse inequality is also true and thus (ii) follows. □

## 6 The Proofs

Now we are in a position to show the main theorems of this paper. In the proofs it is fundamental the application of Lemma 3.2 in order to deduce the convexity of the sub-level sets of a weakly* lower semicontinuous functional.

*Proof of Theorem 2.3* Suppose that $f$ is a Carathéodory supremand satisfying also the coercivity assumption (2.3). Let $\lambda \in \mathbf{R}$ be such that the sub-level set $E_\lambda := \{u \in W^{1,\infty}(\Omega) : F(u) \le \lambda\}$ is nonempty. If $K_\lambda := \{u \in W^{1,\infty}(\Omega) : F(u) < \lambda\}$ is nonempty too, then let $R_\lambda$ be the corresponding difference quotient defined by (3.5). Now we show that

$$E_\lambda = \{u \in W^{1,\infty}(\Omega) : R_\lambda(u) \le 1\}. \tag{6.1}$$

In fact, if $R_\lambda(u) \le 1$ then for every $0 < \delta < 1$ we have that $R_\lambda(\delta u) \le \delta < 1$. By Lemma 3.2 there exists a sequence $\{u_n\} \subset W^{1,\infty}(\Omega)$ converging to $\delta u$ in $L^\infty(\Omega)$

with $F(u_n) \le \lambda$. Since $F$ is weakly* lower semicontinuous it follows $F(\delta u) \le \lambda$ and then, by the same reason, letting $\delta \to 1$ we can conclude $F(u) \le \lambda$. Vice versa, if $F(u) \le \lambda$ then for every $x, y \in \Omega$ we have $d_\lambda(x, y) \ge |u(x) - u(y)|$ by definition of $d_\lambda$. This implies $R_\lambda(u) \le 1$. By (6.1), by applying Proposition 3.1, it follows that $E_\lambda$ is a convex set. Finally, if $K_\lambda$ is empty, note that $K_{\lambda+\varepsilon}$ is nonempty for every $\varepsilon > 0$ since $E_\lambda \subset K_{\lambda+\varepsilon}$ and from the first part of this proof it follows that $E_{\lambda+\varepsilon}$ is a convex set. Since

$$E_\lambda = \bigcap_{\varepsilon>0} E_{\lambda+\varepsilon}$$

then $E_\lambda$ is a convex set, too. Now we remove the previous coercivity assumption on $f$. Define

$$g(x,\xi) := \arctan(f(x,\xi)) + \frac{\pi}{2}$$

and for every $n \in \mathbf{N}$ consider the function $g_n$ given by (4.4) and the functionals $G$ and $G_n$ given by (4.5) and (4.6) respectively. Then it holds:

(i) $g_n(x,\xi) \ge \frac{1}{n}|\xi|$ for every $n \in \mathbf{N}$, for a.e. $x \in \Omega$ and for every $\xi \in \mathbf{R}^N$;
(ii) for every $n \in \mathbf{N}$, for a.e. $x \in \Omega$ and for every $\xi, \eta \in B_M(0)$

$$|g_n(x,\xi) - g_n(x,\eta)| \le \omega'_{M,n}(|\xi - \eta|) \tag{6.2}$$

where $\omega'_{M,n}(s) := \omega_M(s) \vee \frac{1}{n}|s|$.

Property (i) is trivial; concerning (ii), fix $n \in \mathbf{N}$ and let $x \in \Omega$, $\xi, \eta \in B_M(0)$. If $g_n(x,\xi) \ge \frac{1}{n}|\xi|$ and $g_n(x,\eta) \ge \frac{1}{n}|\eta|$ then (6.2) is trivial. If $g_n(x,\xi) \le \frac{1}{n}|\xi|$ and $g_n(x,\eta) \le \frac{1}{n}|\eta|$ then (6.2) is trivial. It remains to show (6.2) when $g_n(x,\xi) < \frac{1}{n}|\xi|$ and $g_n(x,\eta) > \frac{1}{n}|\eta|$. In this case

$$\begin{aligned} g_n(x,\xi) - g_n(x,\eta) &= \frac{1}{n}|\xi| \vee g(x,\xi) - g(x,\eta) \\ &\le \left(\frac{1}{n}|\xi - \eta| + \frac{1}{n}|\eta|\right) \vee \big(g(x,\eta) + \omega_M(|\xi - \eta|)\big) - g(x,\eta) \\ &\le \frac{1}{n}|\xi - \eta| \vee \omega_M(|\xi - \eta|) \end{aligned}$$

and

$$\begin{aligned} g_n(x,\xi) - g_n(x,\eta) &= \frac{1}{n}|\xi| - g(x,\eta) \vee \frac{1}{n}|\eta| \\ &\ge \frac{1}{n}|\xi| - \big(g(x,\xi) + \omega_M(|\xi - \eta|)\big) \vee \left(\frac{1}{n}|\xi - \eta| + \frac{1}{n}|\xi|\right) \\ &\ge -\left(\frac{1}{n}|\xi - \eta| \vee \omega_M(|\xi - \eta|)\right). \end{aligned}$$

Now, since the function $h(s) = \arctan s$ is uniformly continuous and increasing on $\mathbf{R}$, then the supremal functional $G$ is weakly* lower semicontinuous on $W^{1,\infty}(\Omega)$. Thus, thanks to (4.7), for every $n \in \mathbf{N}$ the supremal functional $G_n$ is a weakly* lower semicontinuous on $W^{1,\infty}(\Omega)$. Since $G_n$ is represented through a coercive function, from the first part of this proof it follows that $(G_n)_n$ is a sequence of level convex functionals. Since $(G_n)_n$ pointwise converges to $G$, then $G$ is a level convex functional, too. As an easy consequence, $F$ is a level convex functional. □

*Proof of Theorem 2.4* One implication is trivial. Now assume that $F$ is a level convex functional. For every $n \in \mathbf{N}$, let $F_n : W^{1,\infty}(\Omega) \to \mathbf{R}$ be the functional defined by

$$F_n(u) := \inf\big\{F(v) \vee n\|Dv - Du\|_\infty : v \in W^{1,\infty}(\Omega)\big\}.$$

Then, thanks to Proposition 4.2, for every $n \in \mathbf{N}$ $F_n$ is an $n$-Lipschitz continuous functional such that

$$F = \bigvee_n F_n.$$

Moreover, for every $n \in \mathbf{N}$, the functional $F_n$ is level convex. In fact let $u_1, u_2 \in W^{1,\infty}(\Omega)$ and $\theta \in (0,1)$. For fixed $\varepsilon > 0$ there exist $v_1, v_2 \in W^{1,\infty}(\Omega)$ such that

$$F_n(u_i) \geq F(v_i) \vee n\|Dv_i - Du_i\|_\infty - \varepsilon$$

for every $i \in \{1,2\}$. Then

$$\begin{aligned}
&F_n(\theta u_1 + (1-\theta)u_2)\\
&\quad \leq F(\theta v_1 + (1-\theta)v_2) \vee n\big\|D(\theta u_1 + (1-\theta)u_2) - D(\theta v_1 + (1-\theta)v_2)\big\|_\infty\\
&\quad \leq F(v_1) \vee F(v_2) \vee n\|Du_1 - Dv_1\|_\infty \vee n\|Du_2 - Dv_2)\|_\infty\\
&\quad \leq F_n(u_1) \vee F_n(u_2) + \varepsilon.
\end{aligned}$$

Since $\varepsilon$ is arbitrary, it follows that

$$F_n\big(\theta u_1 + (1-\theta)\big)u_2 \leq F_n(u_1) \vee F_n(u_2),$$

i.e. $F_n$ is a level convex functional. Now define

$$\varphi_n(x,\xi) := \inf\big\{F_n(u) \mid u \in W^{1,\infty}(\Omega) \text{ s.t. } x \in \widehat{u}, \text{ with } Du(x) = \xi\big\}. \tag{6.3}$$

Thanks to Proposition 5.2, for every $n \in \mathbf{N}$ $\varphi_n$ is a Carathéodory supremand, $n$-Lipschitz continuous w.r.t. $\xi$ and such that

$$\operatorname*{ess\,sup}_\Omega \varphi_n(x, Du(x)) \leq F_n(u). \tag{6.4}$$

Moreover, for every $n \in \mathbf{N}$ $\varphi_n$ is a level convex function. Fix $x \in \Omega$, $\xi, \eta \in \mathbf{R}^N$ and $\lambda \in (0,1)$. By the definition of $\varphi_n$ there exist $B_r(x)$, $u_\varepsilon, v_\varepsilon \in W^{1,\infty}(\Omega)$, differentiable at $x$ such that $Du_\varepsilon(x) = \xi$, $Dv_\varepsilon(x) = \eta$ and $\varphi_n(x,\xi) \geq F_n(u_\varepsilon) - \varepsilon$ and $\varphi_n(x,\eta) \geq F_n(v_\varepsilon) - \varepsilon$. Since $F_n$ is a level convex functional then

$$\begin{aligned}
\varphi_n(x, \lambda\xi + (1-\lambda)\eta) &\leq F_n(\lambda u_\varepsilon + (1-\lambda)v_\varepsilon)\\
&\leq F_n(u_\varepsilon) \vee F_n(v_\varepsilon) \leq (\varphi_n(x,\xi) - \varepsilon) \vee (\varphi_n(x,\eta) - \varepsilon).
\end{aligned}$$

Letting $\varepsilon \to 0$ we obtain the thesis. The sequence $\{\varphi_n\}_n$ is non decreasing and thus

$$\exists \lim_n \varphi_n(x,\xi) = \bigvee_n \varphi_n(x,\xi) =: \varphi(x,\xi).$$

It is easy to verify that $\varphi$ is a level convex normal supremand and letting $n \to \infty$ in (6.4) we obtain

$$\operatorname*{ess\,sup}_\Omega \varphi(x, Du(x)) \le F(u).$$

In order to show the converse inequality, for every $n \in \mathbf{N}$ we use again the functions $f_n, \psi_n$ defined by (5.13) and (5.14). Now for every $n \in \mathbf{N}$ and for every $u \in W^{1,\infty}(\Omega)$ it holds

$$\operatorname*{ess\,sup}_\Omega f_n(x, Du(x)) \le F_n(u). \tag{6.5}$$

In fact, by definition, for fixed $\varepsilon > 0$ there exists $u_\varepsilon \in W^{1,\infty}(\Omega)$ such that

$$F_n(u) \ge \operatorname*{ess\,sup}_\Omega f(x, Du_\varepsilon(x)) \vee n\|Du_\varepsilon - Du\|_\infty - \varepsilon.$$

In particular for a.e. $x \in \Omega$

$$F_n(u) \ge f(x, Du_\varepsilon(x)) \vee n|Du_\varepsilon(x) - Du(x)| - \varepsilon \ge f_n(x, Du(x)) - \varepsilon$$

which implies

$$F_n(u) \ge \operatorname*{ess\,sup}_\Omega f_n(x, Du(x)) - \varepsilon.$$

Since $\varepsilon$ is arbitrary, then (6.5) follows. From (6.5) and the definitions of $f_n, \psi_n, \varphi_n$, we deduce

$$f_n(x,\xi) \le \psi_n(x,\xi) \le \varphi_n(x,\xi) \text{ on } (\Omega \setminus H) \times \mathbf{R}^N$$

where $H \subset \Omega$ is a negligible set. By passing to the limit as $n \to \infty$ in the previous inequality and by applying the fact that

$$f(x,\xi) = \bigvee_{n\in\mathbf{N}} f_n(x,\xi)$$

we obtain that

$$f(x,\xi) \le \varphi(x,\xi) \quad \text{on } (\Omega \setminus H) \times \mathbf{R}^N$$

which implies that

$$F(u) \le \operatorname*{ess\,sup}_\Omega \varphi(x, Du(x)). \qquad \square$$

*Proof of Theorem 2.5* 1 $\Longrightarrow$ 2. It follows from Theorem 2.3.

2 $\Longrightarrow$ 3. Thanks to Lemma 5.3 and Theorem 5.4 it remains to show that $\varphi$ is level convex w.r.t. $\xi$. Fix $x \in \Omega$, $\xi, \eta \in \mathbf{R}^N$ and $\lambda \in (0,1)$. By definition of $\varphi$ there exist $u_\varepsilon, v_\varepsilon \in W^{1,\infty}(\Omega)$, differentiable at $x$, such that $Du_\varepsilon(x) = \xi$, $Dv_\varepsilon(x) = \eta$ and

$\varphi(x,\xi) \geq F(u_\varepsilon) - \varepsilon$ and $\varphi(x,\eta) \geq F(v_\varepsilon) - \varepsilon$. Since $F$ is a level convex functional, we have that

$$\begin{aligned}\varphi(x,\lambda\xi+(1-\lambda)\eta) &\leq F(\lambda u_\varepsilon+(1-\lambda)v_\varepsilon) \leq F(u_\varepsilon)\vee F(v_\varepsilon)\\ &\leq (\varphi(x,\xi)-\varepsilon)\vee(\varphi(x,\eta)-\varepsilon).\end{aligned}$$

Letting $\varepsilon \to 0$ we obtain the thesis.

$3 \Longrightarrow 1$ It follows by Theorem 1.1. □

*Proof of Theorem 2.6* $1 \Longrightarrow 2$. By applying Proposition 3.4 we have that

$$F(u) = \sup_{x,y\in\Omega,\ x\neq y} \frac{u(x)-u(y)}{d(x,y)}.$$

Then $F$ is convex (see for instance the proof of Proposition 3.1).

$2 \Longrightarrow 3$. Thanks to Theorem 5.5 we can represent $F$ through the function $\varphi$ given by (2.4), i.e. $F(u) = \operatorname{ess\,sup}_\Omega \varphi(x, Du(x))$ for every $u \in W^{1,\infty}(\Omega)$. If we show that $\varphi$ is convex w.r.t. $\xi$ then by applying Propositions 2.4 and 2.5 in [14] it follows that $F$ can be represented also through the function $\varphi_d^0$. Fix $x\in\Omega$, $\xi,\eta\in\mathbf{R}^N$ and $\lambda\in(0,1)$. By definition of $\varphi$, there exist $u_\varepsilon, v_\varepsilon \in W^{1,\infty}(\Omega)$, differentiable at $x$ such that $Du_\varepsilon(x)=\xi$, $Dv_\varepsilon(x)=\eta$ and $\varphi(x,\xi)\geq F(u_\varepsilon)-\varepsilon$ and $\varphi(x,\eta)\geq F(v_\varepsilon)-\varepsilon$. Since $F$ is a convex functional, we have that

$$\begin{aligned}\varphi(x,\lambda\xi+(1-\lambda)\eta) &\leq F(\lambda u_\varepsilon+(1-\lambda)v_\varepsilon) \leq \lambda F(u_\varepsilon)+(1-\lambda)F(v_\varepsilon)\\ &\leq \lambda(\varphi(x,\xi)-\varepsilon)+(1-\lambda)(\varphi(x,\eta)-\varepsilon),\end{aligned}$$

and letting $\varepsilon\to 0$ we obtain that $\varphi$ is convex w.r.t. $\xi$. Finally, thanks to Lemma 5.3, we have that there exists a negligible set $H\subset\Omega$ such that

$$\varphi_d^0(x,\xi)\leq\varphi(x,\xi) \quad \text{for every } x\in\Omega\setminus H,\ \text{for } \xi\in\mathbf{R}^N.$$

The converse inequality follows by applying Proposition 1.6 in [14].

$3 \Longrightarrow 1$. It follows by Theorem 1.1. □

*Proof of Theorem 2.7* $1 \Longrightarrow 2$. For every open subset $A\subset\Omega$ with Lipschitz continuous boundary, it is sufficient to apply Theorem 2.3 to the functional $F(u) := F(u,A)$. Now let $A\subset\Omega$ be a generic open set and $u_1,u_2\in W^{1,\infty}(\Omega)$ and $\theta\in(0,1)$. Then there exists a countable family of open sets $(A_n)_{n\in\mathbf{N}}$ with Lipschitz continuous boundaries such that $A=\bigcup_n A_n$. For every $n\in\mathbf{N}$ we have that

$$F(\theta u_1+(1-\theta)u_2, A_n) \leq F(v_1,A_n)\vee F(v_2,A_n).$$

This implies

$$\begin{aligned}F(\theta u_1+(1-\theta)u_2, A) &= \bigvee_{n\in\mathbf{N}} F(\theta u_1+(1-\theta)u_2, A_n)\\ &\leq \bigvee_{n\in\mathbf{N}} F(v_1,A_n)\vee\bigvee_{n\in\mathbf{N}} F(v_2,A_n)\\ &= F(v_1,A)\vee F(v_2,A).\end{aligned}$$

$2 \Longrightarrow 3$. Since $f$ satisfies all the assumptions of Theorem 5.4, $F$ can be represented also by the Carathéodory supremand $\tilde{f}$. It remains to show that $\tilde{f}$ is a level convex function. Fix $x \in \Omega$, $\xi, \eta \in \mathbf{R}^N$ and $\lambda \in (0,1)$. By the definition of $\tilde{f}$ there exist $B_r(x)$, $u_\varepsilon, v_\varepsilon \in W^{1,\infty}(\Omega)$, differentiable at $x$ such that $Du_\varepsilon(x) = \xi$, $Dv_\varepsilon(x) = \eta$ and $\tilde{f}(x,\xi) \geq F(u_\varepsilon, B_r(x)) - \varepsilon$ and $\tilde{f}(x,\eta) \geq F(v_\varepsilon, B_r(x)) - \varepsilon$. Since $F(\cdot, B_r(x))$ is a level convex functional then

$$\begin{aligned}\tilde{f}(x, \lambda\xi + (1-\lambda)\eta) &\leq F(\lambda u_\varepsilon + (1-\lambda)v_\varepsilon, B_r(x)) \leq F(u_\varepsilon, B_r(x)) \vee F(v_\varepsilon, B_r(x)) \\ &\leq (\tilde{f}(x,\xi) - \varepsilon) \vee (\tilde{f}(x,\eta) - \varepsilon).\end{aligned}$$

Letting $\varepsilon \to 0$ we obtain the thesis.

$3 \Longrightarrow 1$. It follows by Theorem 1.1. □

## 7 An Existence Theorem of AMLs and a Principle of Comparison

In general a minimizer for a supremal functional is not necessarily a local minimizer (see Example 1.2. in [9]). Then, by analogy with the case of integral functionals, G. Aronsson introduced the following notion of local minimizers for a supremal functional of the form

$$F(u, A) = \operatorname*{ess\,sup}_{A} f(x, Du(x)) \tag{7.1}$$

(for instance see [2–4]).

**Definition 7.1** Let $g$ be a Lipschitz function defined on $\partial\Omega$ and let us denote by $W_g^{1,\infty}(\Omega)$ the space of functions such that $(u-g) \in W^{1,\infty}(\Omega) \cap \mathcal{C}_0(\Omega)$. An absolute minimizer or an AML for the problem

$$\min_{v \in W_g^{1,\infty}(\Omega)} F(v, \Omega) \tag{7.2}$$

(where $F$ is given by (7.1)) is a minimizer $u$ such that for all open subset $V \subset\subset \Omega$ one has

$$F(u, V) \leq F(v, V)$$

for all $v$ in $W^{1,\infty}(V)$ such that $v = u$ on $\partial V$.

With different techniques Barron et al. in [6] and Champion et al. in [9] have proved an existence theorem of AML for a supremal functional $F$ by assuming that it is represented by a level convex function $f$. Now, thanks to Theorem 2.7, we can give the following:

**Theorem 7.2** *Let $f : \Omega \times \mathbf{R}^N \to \mathbf{R}$ be a Carathéodory supremand satisfying (2.2) and (2.3). Let $F(\cdot, A)$ be the functional defined by (7.1). Let $g$ be a Lipschitz function defined on $\partial\Omega$. If $F(\cdot, A)$ is lower semicontinuous with respect to the weak* convergence of $W^{1,\infty}(\Omega)$ for every $A \in \mathcal{A}$ then there exists at least one absolute minimizer $v \in W_g^{1,\infty}(\Omega)$.*

*Proof* By applying Theorem 2.7, we can represent $F$ through a level convex Carathéodory supremand satisfying (2.2) and (2.3). Then we can conclude by applying Theorem 4.1 in [9]. □

*Remark 7.3* In [9] (see Theorem 4.7) the authors give another existence theorem for AML based on a Perron-like method. They show that if the functional (7.1) is weakly* l.s.c. and coercive and satisfies the additional hypothesis that for any $A \in \mathcal{A}$, $w \in W^{1,\infty}(A) \cap \mathcal{C}(\bar{A})$ and $y \in A$, the image set

$$A_{y,w} = \{u(y) : u \in W^{1,\infty}_w(A),\ u \text{ is an AML}\}$$

is connected, then there exists at least one absolute minimizer $v \in W^{1,\infty}_g(\Omega)$. Now if the functional (7.1) is weakly* lower semicontinuous for every $A \in \mathcal{A}$ and $f$ satisfies (2.2) and (2.3), then the last assumption is trivially satisfied. In fact, thanks to Theorem 2.3, $F$ is a level convex functional and then the sets $A_{y,w}$ are convex.

In Theorem 3.5 of the paper [10], Champion and De Pascale characterize the absolute minimizers of a wide class of supremal functionals by extending the principle of comparison with cones introduced by Crandall et al. in [11] for the minimizing Lipschitz Extension Problem. Their characterization relies on the fact that when $f = f(x,\xi)$ is a lower semicontinuous function, satisfying (2.3) and level convex in the $\xi$-variable, then for every open set $V \subset \mathbf{R}^N$ the pseudo-distances

$$\rho^V_\lambda(x,y) = \sup\left\{u(x) - u(y), u \in W^{1,\infty}(V) : \operatorname*{ess\,sup}_V f(x, Du(x)) \le \lambda\right\} \tag{7.3}$$

coincide with the following distances:

$$\delta^V_\lambda(x,y) = \inf_{\gamma \in \Gamma_{x,y}(V)} \int_0^1 f^0(\gamma, \gamma', \lambda)\, dt$$

where $f^0$ is defined by

$$f^0(x,\xi,\lambda) := \sup\left\{\xi \cdot \eta : f(x,\eta) \le \lambda\right\}$$

(see Lemma B.3 and Proposition A.2 in [10]). Now if $f$ is not lower semicontinuous this equality could fail. In fact let $\Omega = (-2,2)^2$ and consider the segment $S = (-1,1) \times \{0\}$. Consider the Carathéodory supremand defined by

$$f(x,\xi) = \begin{cases} \beta|\xi| & \text{if } x \in \Omega \setminus S, \\ \alpha|\xi| & \text{if } x \in S, \end{cases}$$

with $0 < \beta < \alpha$. Then $f$ is not globally lower semicontinuous on $\Omega$ and for every $\lambda \ge 0$ and for every $x, y \in V = (-1,1)^2$ it holds

$$\delta^V_\lambda(x,y) = \frac{\lambda}{\beta}|x - y|.$$

If $\bar{x} = (-1, 0)$ and $\bar{y} = (0, 1)$ then

$$\delta_\lambda^V(\bar{x}, \bar{y}) = \frac{2\lambda}{\alpha} < \frac{2\lambda}{\beta} = \rho_\lambda^V(\bar{x}, \bar{y}).$$

This inequality is due to the fact that when one modifies the values of the supremand $f$ on a negligible subset of $\Omega$ the distances $\delta_\lambda^V$ can change while the distances $\rho_\lambda^V$ do not depend on the supremand $f$ chosen to represent $F$.

Now when $f$ is not globally l.s.c., it is an open problem if it is possible to characterize the AMLs of the problem (7.2) through a comparison principle with distance functions. However, thanks to the results shown in Sects. 3 and 6, we can give a partial result. We consider a supremal functional $F$ represented by a Carathéodory supremand $f : \Omega \times \mathbf{R}^N \to \mathbf{R}^+$, level convex in the $\xi$-variable, satisfying (2.2) and (2.3). For any open set $V \subset\subset \Omega$, for every $x, y \in V$ and for every $\lambda \in \mathbf{R}$ we define

$$d_\lambda^V(x, y) := \begin{cases} \sup\{|u(x) - u(y)| : u \in W^{1,\infty}(\Omega), F(u, V) \leq \lambda\} \\ \quad \text{if } \lambda > \inf_{v \in W^{1,\infty}(V)} F(v, V), \\ \inf_{\varepsilon > 0} d_{\lambda+\varepsilon}^V(x, y) \quad \text{if } \lambda = \inf_{v \in W^{1,\infty}(V)} F(v, V), \\ -\infty \qquad\qquad\qquad \text{if } \lambda < \inf_{v \in W^{1,\infty}(V)} F(v, V). \end{cases} \tag{7.4}$$

Moreover for every $x, y \in \bar{V}$ and for every $\lambda \in \mathbf{R}$ we define

$$d_\lambda^V(x, y) := \inf\Big\{\liminf_n d_\lambda^V(x_n, y_n) : (x_n)_n, (y_n)_n \subset V,\ x_n \to x,\ y_n \to y\Big\}. \tag{7.5}$$

We point out that, since the boundary of $V$ may be non regular, it may happen that $d_\lambda^V(\bar{x}, \bar{y}) = +\infty$ for some $\lambda \in \mathbf{R}$ and for some $\bar{x} \in \partial V$ and $\bar{y} \in V$. In this case it is easy to show that $d_\lambda^V(\bar{x}, y) = +\infty$ for every $y \in V$.

**Definition 7.4** We shall say that a continuous function $u : \bar{\Omega} \to \mathbf{R}$ satisfies the comparison with the distance functions $d_\lambda^V$ from above in $\Omega$ if and only if for any connected open subset $V \subset\subset \Omega$, any $x_0 \in \bar{V}$, any $\lambda \in \mathbf{R}$ and $\alpha \in \mathbf{R}$ the inequality

$$u \leq d_\lambda^V(x_0, .) + \alpha \quad \text{on } \partial(V \setminus \{x_0\})$$

implies

$$u \leq d_\lambda^V(x_0, .) + \alpha \quad \text{on } \bar{V}.$$

Now we can easily show the theorem.

**Theorem 7.5** *Under the assumption of Theorem 7.2 if $u \in W^{1,\infty}(\Omega)$ satisfies the Comparison with the Distance Functions $d_\lambda^V$ on $\Omega$ from above then $u$ is an absolute minimizer of $F$.*

*Proof* Assume that $u$ satisfies the comparison principle with the distance functions $d_\lambda^V$. Let $V \subset\subset \Omega$ be an open set. We will show that

$$F(u, V) = \inf\{F(v, V) : v \in W^{1,\infty}(V), v = u \text{ on } \partial V\}.$$

In fact, since $F(\cdot, V)$ is weakly* l.s.c. and coercive, there exists $w \in W^{1,\infty}(V)$ such that

$$F(w, V) = \lambda = \min\{F(v, V) : v \in W^{1,\infty}(V), v = u \text{ on } \partial V\}.$$

If $\lambda > \inf\{F(v, V) : \ v \in W^{1,\infty}(V)\}$ then

$$|w(x) - w(y)| \le d_\lambda^V(x, y) \tag{7.6}$$

for every $x, y \in V$. Now $w - u = 0$ on $\partial V$ means that there exists a sequence $w_n \in C_0^\infty(V)$ weakly* converging to $w - u$. In particular the sequence $(w_n + u)_n$ is bounded in $W^{1,\infty}(\Omega)$ and thus it admits a subsequence which weakly* converges to a function $v \in W^{1,\infty}(\Omega)$. In particular $v = w$ on $V$ and therefore

$$|v(x) - v(y)| \le d_\lambda^V(x, y)$$

for every $x, y \in V$. Since $v \in C(\bar{V})$ and $v = u$ on $\partial V$, by continuity we obtain that

$$|u(x) - u(y)| \le d_\lambda^V(x, y) \tag{7.7}$$

for every $x, y \in \partial V$. Since $u$ satisfies the principle of comparison, (7.7) yields to $u(x) \le u(y) + d_\lambda^V(x, y)$ for every $y \in \partial V$ and for every $x \in V$. By applying again the principle of comparison we obtain that $u(x) \le u(y) + d_\lambda^V(x, y)$ for every $y, x \in V$ i.e.

$$R_\lambda(u, V) := \sup_{x,y \in V, x \neq y} \frac{|u(x) - u(y)|}{d_\lambda^V(x, y)} \le 1$$

and thus for every $0 < \delta < 1$ $R_\lambda(\delta u, V) \le \delta < 1$. Now we can notice that thanks to Theorem 2.7 $F(\cdot, V)$ is weakly* l.s.c. in $W^{1,\infty}(V)$. Thus by applying Lemma 3.3 we have that $F(\delta u, V) \le \lambda$. By letting $\delta \to 1$ we can conclude that $F(u, V) \le \lambda$. If $\lambda = \min\{F(v, V) : \ v \in W^{1,\infty}(V)\}$ it is sufficient to note that

$$|w(x) - w(y)| \le d_{\lambda+\varepsilon}^V(x, y)$$

for every $x, y \in V$ and for every $\varepsilon > 0$. By repeating the first part of this proof, it is easy to show that $F(u, V) \le \lambda + \varepsilon$. By letting $\varepsilon \to 0^+$ we obtain the thesis. □

## 8 A Counterexample

In this section we will show that in general even if $F(u) = \operatorname{ess\,sup}_\Omega f(x, Du(xf))$ is weakly* lower semicontinuous on $W^{1,\infty}(\Omega)$ it cannot be represented by the level convex envelope $f^{lc}$ of $f$ i.e. in general $\operatorname{ess\,sup}_\Omega f^{lc}(x, Du(x)) < F(u)$.

In fact, a suitable modification of Example 3.2 in [14] gives the following:

*Example 8.1* Let us call $\mathcal{G}$ the set of all continuous functions $g : \mathbf{R}^N \to \mathbf{R}$, positively 1-homogeneous and satisfying $\alpha|\xi| \le g(\xi) \le \beta|\xi|$ for all $\xi \in \mathbf{R}^N$, and let

$$\mathcal{C} := \{C \subseteq \mathbf{R}^N : C = \{\xi \in \mathbf{R}^N : \ g(\xi) \le 1\} \text{ for some } g \in \mathcal{G}\}. \tag{8.1}$$

Note that the sets in $\mathcal{C}$ are closed, star-shaped (with respect to the origin), and that by definition with every $C \in \mathcal{C}$ is associated a function $g \in \mathcal{G}$, which we denote by $g_C$. Moreover $\mathcal{C}$ is closed under intersection and union.

Let now $B$ be the unit closed ball in $\mathbf{R}^N$ centered at 0. Then $B \in \mathcal{C}$ with $g_B(\xi) = |\xi|$. Let $H \in \mathcal{C}$ satisfy the following properties:

1. $H$ is not convex;
2. $H \cap B$ is convex;
3. $H \setminus B \neq \emptyset$ and $B \setminus H \neq \emptyset$;
4. $B$ is contained in the convex hull of $H$.

Finally let us construct an open and dense set $A \subset \Omega$ with $0 < |A| < |\Omega|$ as follows. Let $\{v_i\}_{i\in\mathbf{N}}$ be a dense subset of $\partial B$ and let $\{p_j\}_{j\in\mathbf{N}}$ be dense in $\Omega$. For a given positive constant $\delta > 0$ we define

$$A := \bigcup_{i,j\in\mathbf{N}} \left\{ x \in \Omega : \operatorname{dist}(x, \{p_i + s v_j, s \in \mathbf{R}\}) < \frac{\delta}{2^{ij}} \right\}.$$

Clearly, if $\delta$ is small enough, we have that $0 < |A| < |\Omega|$. Roughly speaking the set $A$ is given by a countable union of thin strips along a dense set of directions.

We consider the functions $f, f_+ : \Omega \times \mathbf{R}^N \to \mathbf{R}$ defined by

$$f(x,\xi) := \begin{cases} g_B(\xi) & \text{if } x \in A, \\ g_H(\xi) & \text{if } x \in \Omega \setminus A; \end{cases} \qquad f_+(x,\xi) := \begin{cases} g_B(\xi) & \text{if } x \in A, \\ g_{H\cap B}(\xi) & \text{if } x \in \Omega \setminus A. \end{cases}$$

The associated supremal functionals are

$$F(u) := \operatorname*{ess\,sup}_{\Omega} f(x, Du(x)) \quad \text{and} \quad F_+(u) := \operatorname*{ess\,sup}_{\Omega} f_+(x, Du(x)).$$

Now we show the following facts:

1. $F = F_+$ and therefore $F$ is weakly* lower semicontinuous (in fact $F_+$ is weakly* lower semicontinuous being represented by a (level) convex function);
2. $\exists u \in W^{1,\infty}(\Omega)$ such that $F(u) > \operatorname{ess\,sup}_{\Omega} f^{lc}(x, Du(x))$.

In fact, by construction we have that $F_+ \geq F$, and so let us assume by contradiction that for some $u \in W^{1,\infty}(\Omega)$ we have

$$F_+(u) > 1 \quad \text{while } F(u) < 1. \tag{8.2}$$

This will imply that $Du \in H \setminus \overline{B}$ on a set of positive measure. Therefore there exists a point $x \in \Omega$ of differentiability for $u$ with $|Du(x)| > 1$. To simplify the notation we can assume $x = 0$ and $u(0) = 0$. Let $\{\rho_n\}$ be a sequence converging to zero, and for every $n$ let us consider the function $u_n : B \to \mathbf{R}$ defined by

$$u_n(x) := \frac{1}{\rho_n} u(\rho_n x) \quad \text{for every } x \in B.$$

By the definition of $A$, for every $n$ and for every $\varepsilon > 0$ we can find an open strip $L_n^\varepsilon$ in $B$ such that $\rho_n L_n^\varepsilon \subset A$ and such that $L_n^\varepsilon$ contains two points $a_n^\varepsilon$ and $b_n^\varepsilon$ with

$$\left|a_n^\varepsilon - \frac{Du(0)}{|Du(0)|}\right| + \left|b_n^\varepsilon - \left(-\frac{Du(0)}{|Du(0)|}\right)\right| \leq \varepsilon. \tag{8.3}$$

By Proposition 3.4, we have that

$$\operatorname*{ess\,sup}_{L_n^\varepsilon} |Du_n(x)| = \sup_{x,y\in L_n^\varepsilon} \frac{u_n(x)-u_n(y)}{|x-y|} \geq \frac{u_n(a_n^\varepsilon)-u_n(b_n^\varepsilon)}{|a_n^\varepsilon - b_n^\varepsilon|}.$$

Using that, by the differentiability of $u$ at 0, $\{u_n\}$ converges to $Du(0)\cdot x$ uniformly, by (8.3) we deduce that, for $n$ big enough,

$$\operatorname*{ess\,sup}_{L_n^\varepsilon} |Du_n(x)| \geq |Du(0)| + o(\varepsilon),$$

where $o(\varepsilon) \to 0$ as $\varepsilon \to 0$. Therefore, recalling that $|Du(0)| > 1$, we can find $\varepsilon$ and $n$ such that $\operatorname{ess\,sup}_{L_n^\varepsilon} |Du_n(x)| > 1$. We conclude that

$$F(u) \geq \operatorname*{ess\,sup}_{A} |Du(x)| \geq \operatorname*{ess\,sup}_{\rho_n L_n^\varepsilon} |Du(x)| = \operatorname*{ess\,sup}_{L_n^\varepsilon} |Du_n(x)| > 1,$$

which is in contradiction with (8.2).

Finally let $\xi \in B$ such that $\xi$ is not in $H \cap B$. In particular $\xi$ belongs to the convex hull of $H$ and therefore

$$f^{lc}(x,\xi) \leq 1 \quad \text{a.e. on } \Omega.$$

On the other hand by the definition of $f_+$ and the choice of $\xi$ we have

$$f_+(x,\xi) > 1 \quad \text{a.e. on } \Omega \setminus A. \tag{8.4}$$

Therefore if we define $\bar{u}(x) = x\cdot\xi$ we obtain

$$F(\bar{u}) > 1 \geq \operatorname*{ess\,sup}_{\Omega} f^{lc}(x,\xi) = \operatorname*{ess\,sup}_{\Omega} f^{lc}(x, D\bar{u}(x)).$$

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