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Semimartingales from the Fokker–Planck Equation*

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Abstract. We show the existence of a semimartingale of which one-dimensional marginal distributions are given by the solution of the Fokker–Planck equation with the pth integrable drift vector ($p > 1$).

Key Words. Marginals, Fokker–Planck equation.

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1. Introduction

Let $\mathcal{M}_1(\mathbf{R}^d)$ denote the complete separable metric space, with a weak topology, of Borel probability measures on $\mathbf{R}^d$ ($d \geq 1$).

Let b: $[0,1] \times \mathbf{R}^d \mapsto \mathbf{R}^d$ be measurable and let $\{P_t(dx)\}_{0 \leq t \leq 1} \subset \mathcal{M}_1(\mathbf{R}^d)$ satisfy the following Fokker–Planck equation: for $f \in C_b^{1,2}([0,1] \times \mathbf{R}^d)$ and $t \in [0,1]$,

$$\int_{\mathbf{R}^d} f(t,x)P_t(dx) - \int_{\mathbf{R}^d} f(0,x)P_0(dx) = \int_0^t ds \int_{\mathbf{R}^d} \left(\frac{\partial f(s,x)}{\partial s} + \frac{1}{2} \triangle f(s,x) + \langle b(s,x), D_x f(s,x) \rangle \right) P_s(dx), \tag{1.1}$$

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where $\triangle := \sum_{i=1}^{d} \partial^2/\partial x_i^2$, $D_x := (\partial/\partial x_i)_{i=1}^{d}$, and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbf{R}^d$.

Inspired by Born's probabilistic interpretation of a solution to Schrödinger's equation, Nelson proposed the problem of the construction of a diffusion process $\{X(t)\}_{0 \le t \le 1}$ for which the following hold (see [20]):

$$X(t) = X(0) + \int_0^t b(s, X(s))\, ds + W(t) \qquad (t \in [0, 1]), \tag{1.2}$$

$$P(X(t) \in dx) = P_t(dx) \qquad (t \in [0, 1]), \tag{1.3}$$

where $\{W(t)\}_{0 \le t \le 1}$ is a $\sigma[X(s)\colon 0 \le s \le t]$-Wiener process.

The first result was given by Carlen [2] (see also [23]). It was generalized, by Mikami [12], to the case where the second-order differential operator has a variable coefficient. Further generalization and almost complete resolution was made by Cattiaux and Léonard [3]–[6] (see also [1] and [13]–[15] for the related topics). However, in these papers, they assumed that

$$\int_0^1 dt \int_{\mathbf{R}^d} |b(t, x)|^2 P_t(dx) < \infty \tag{1.4}$$

for some b for which (1.1) holds. This is called the **finite energy condition** for $\{P_t(dx)\}_{0 \le t \le 1}$.

Remark 1.1. It is known that b is not unique for $\{P_t(dx)\}_{0 \le t \le 1}$ in (1.1) (see [12] or [3]–[6]).

In this paper we consider Nelson's problem under a weaker assumption than (1.4): there exists $p > 1$ such that

$$\int_0^1 dt \int_{\mathbf{R}^d} |b(t, x)|^p P_t(dx) < \infty \tag{1.5}$$

for some b for which (1.1) holds. We call (1.5) the **generalized finite energy condition** for $\{P_t(dx)\}_{0 \le t \le 1}$.

Let $L(t, x; u)\colon [0, 1] \times \mathbf{R}^d \times \mathbf{R}^d \mapsto [0, \infty)$ be continuous and be convex in u. Let $\mathcal{A}$ denote the set of all $\mathbf{R}^d$-valued, continuous semimartingales $\{X(t)\}_{0 \le t \le 1}$ on a complete filtered probability space such that there exists a Borel measurable $\beta_X\colon [0, 1] \times C([0, 1]) \mapsto \mathbf{R}^d$ for which

(i) $\omega \mapsto \beta_X(t, \omega)$ is $\mathcal{B}(C([0, t]))_+$-measurable for all $t \in [0, 1]$, where $\mathcal{B}(C([0, t]))$ denotes the Borel σ-field of $C([0, t])$ and $\mathcal{B}(C([0, t]))_+$ denotes the left-hand limit of $t \mapsto \mathcal{B}(C([0, t]))$, and

(ii) $\{W_X(t) := X(t) - X(0) - \int_0^t \beta_X(s, X)\, ds\}_{0 \le t \le 1}$ is a $\sigma[X(s)\colon 0 \le s \le t]$-Wiener process.

For P_0 and $P_1 \in \mathcal{M}_1(\mathbf{R}^d)$, put

$$V(P_0, P_1) := \inf\left\{ E\left[\int_0^1 L(t, X(t); \beta_X(t, X))\, dt\right] \middle| \; PX(t)^{-1} = P_t (t = 0, 1),\, X \in \mathcal{A} \right\}, \tag{1.6}$$

$$v(P_0, P_1) := \inf\left\{ \int_0^1 \int_{\mathbf{R}^d} L(t, x; b(t, x)) P(t, dx) dt \,\middle|\, P(t, dx) = P_t(dx)(t = 0, 1), \{P(t, dx)\}_{0 \le t \le 1} \subset \mathcal{M}_1(\mathbf{R}^d), (b(t, x), P(t, dx)) \text{ satisfies } (1.1) \right\}. \quad (1.7)$$

In [12] where $u \mapsto L$ is quadratic, we proved and used the following:

$$V(P_0, P_1) = v(P_0, P_1). \quad (1.8)$$

Remark 1.2. As a typical case, when $L = |u|^2$, the minimizer of $V(P_0, P_1)$ is known to be the h-path process for the space–time Brownian motion (see [7], [18] and the references therein). It is known that its zero-noise limit exists and is the unique minimizer of Monge's problem (see [16] and [19]).

In this paper we prove (1.8) for a more general function L by the duality theorem for V. To make the point clearer, we describe [18] briefly. For P_0 and $P_1 \in \mathcal{M}_1(\mathbf{R}^d)$, put

$$\mathcal{V}(P_0, P_1) := \sup\left\{ \int_{\mathbf{R}^d} \varphi(1, y) P_1(dy) - \int_{\mathbf{R}^d} \varphi(0, x) P_0(dx) \right\}, \quad (1.9)$$

where the supremum is taken over all classical solutions φ to the following Hamilton–Jacobi–Bellman equation:

$$\frac{\partial \varphi(t, x)}{\partial t} + \frac{1}{2} \triangle \varphi(t, x) + H(t, x; D_x \varphi(t, x)) = 0((t, x) \in (0, 1) \times \mathbf{R}^d), \quad (1.10)$$
$$\varphi(1, \cdot) \in C_b^\infty(\mathbf{R}^d)$$

(see Lemma 3.1). Here for $(t, x, z) \in [0, 1] \times \mathbf{R}^d \times \mathbf{R}^d$,

$$H(t, x; z) := \sup_{u \in \mathbf{R}^d} \{\langle z, u \rangle - L(t, x; u)\}. \quad (1.11)$$

The following was proved in [18] and is called the duality theorem for the stochastic optimal control problem (1.6).

Theorem 1.1 (Duality Theorem). *Suppose that* (A.1)–(A.4) *in Section* 2 *hold. Then for any* P_0 *and* $P_1 \in \mathcal{M}_1(\mathbf{R}^d)$,

$$V(P_0, P_1) = \mathcal{V}(P_0, P_1)(\in [0, \infty]). \quad (1.12)$$

Suppose in addition that $V(P_0, P_1)$ *is finite. Then* $V(P_0, P_1)$ *has a minimizer and for any minimizer* $\{X(t)\}_{0 \le t \le 1}$ *of* $V(P_0, P_1)$,

$$\beta_X(t, X) = b_X(t, X(t)) := E[\beta_X(t, X) | (t, X(t))]. \quad (1.13)$$

Remark 1.3. Equation (1.12) can be considered as a counterpart in the stochastic optimal control theory of the duality theorem in the Monge–Kantorovich problem (see [10], [17], [21], [22] and the references therein).

Using a similar result to (1.8) on small time intervals $\subset [0, 1]$, we prove that for $\mathbf{P} := \{P_t(dx)\}_{0\le t\le 1} \subset \mathcal{M}_1(\mathbf{R}^d)$,

$$\mathbf{V}(\mathbf{P}) = \mathbf{v}(\mathbf{P}), \tag{1.14}$$

where

$$\mathbf{V}(\mathbf{P}) := \inf\left\{ E\left[\int_0^1 L(t, X(t); \beta_X(t, X))\, dt\right] \middle| \; PX(t)^{-1} = P_t (0 \le t \le 1), X \in \mathcal{A} \right\}, \tag{1.15}$$

$$\mathbf{v}(\mathbf{P}) := \inf\left\{ \int_0^1 dt \int_{\mathbf{R}^d} L(t, x; b(t, x)) P_t(dx) | b \text{ satisfies (1.1)} \right\}. \tag{1.16}$$

In particular, the existence of a minimizer of $\mathbf{V}(\mathbf{P})$ implies that of a semimartingale for which (1.2)–(1.3) hold. When $p = 2$ in (1.5), this semimartingale is Markovian. However, we do not know if it is also true even when $1 < p < 2$. This is our future problem.

In Section 2 we state our results which is proved in Section 4. Technical lemmas are given in Section 3.

2. Main Result

In this section we state our result. We state assumptions on L.

(A.1) There exists $p > 1$ such that

$$\liminf_{|u|\to\infty} \frac{\inf\{L(t, x; u) \colon (t, x) \in [0, 1] \times \mathbf{R}^d\}}{|u|^p} > 0.$$

(A.2)

$$\Delta L(\varepsilon_1, \varepsilon_2) := \sup \frac{L(t, x; u) - L(s, y; u)}{1 + L(s, y; u)} \to 0 \qquad \text{as} \quad \varepsilon_1, \varepsilon_2 \to 0,$$

where the supremum is taken over all (t, x) and $(s, y) \in [0, 1]\times\mathbf{R}^d$, for which $|t-s| \le \varepsilon_1$, $|x - y| < \varepsilon_2$ and all $u \in \mathbf{R}^d$.

(A.3)

(i) $L(t, x; u) \in C^3([0, 1] \times \mathbf{R}^d \times \mathbf{R}^d \colon [0, \infty))$,
(ii) $D_u^2 L(t, x; u)$ is positive definite for all $(t, x, u) \in [0, 1] \times \mathbf{R}^d \times \mathbf{R}^d$,
(iii) $\sup\{L(t, x; o) \colon (t, x) \in [0, 1] \times \mathbf{R}^d\}$ is finite,
(iv) $|D_x L(t, x; u)|/(1 + L(t, x; u))$ is bounded and
(v) $\sup\{|D_u L(t, x; u)| \colon (t, x) \in [0, 1] \times \mathbf{R}^d, |u| \le R\}$ is finite for all $R > 0$.

(A.4)

(i) $\Delta L(0, \infty)$ is finite, or
(ii) $p = 2$ in (A.1).

Remark 2.1. (i) Assumption (A.3(ii)) implies that $L(t, x; u)$ is strictly convex in u. (ii) $(1 + |u|^2)^{p/2}$ $(p > 1)$ satisfies (A.1)–(A.3) and (A.4(i)).

We state that (1.8) holds.

Theorem 2.1. *Suppose that* (A.1)–(A.4) *hold. Then for any* P_0 *and* $P_1 \in \mathcal{M}_1(\mathbf{R}^d)$,

$$V(P_0, P_1) = v(P_0, P_1)(\in [0, \infty]). \tag{2.1}$$

The following is our main result (see (1.15)–(1.16) for notations).

Theorem 2.2. *Suppose that* (A.1)–(A.4) *hold. Then*:

(i) *For any* $\mathbf{P} := \{P_t(dx)\}_{0 \le t \le 1} \subset \mathcal{M}_1(\mathbf{R}^d)$,

$$\mathbf{V}(\mathbf{P}) = \mathbf{v}(\mathbf{P})(\in [0, \infty]). \tag{2.2}$$

(ii) *For any* $\mathbf{P} := \{P_t(dx)\}_{0 \le t \le 1}, \subset \mathcal{M}_1(\mathbf{R}^d)$, *for which* $\mathbf{v}(\mathbf{P})$ *is finite, there exist a unique minimizer* $b_o(t, x)$ *of* $\mathbf{v}(\mathbf{P})$ *and a minimizer* $X \in \mathcal{A}$, *of* $\mathbf{V}(\mathbf{P})$. *In particular, for any minimizer* $X \in \mathcal{A}$, *of* $\mathbf{V}(\mathbf{P})$,

$$\beta_X(t, X) = b_o(t, X(t)) \tag{2.3}$$

and (1.2)–(1.3) *with* $b = b_o$ *hold.*

Remark 2.2. If $\mathbf{v}(\mathbf{P})$ is finite, then the generalized finite energy condition (1.5) holds from (A.1).

3. Lemmas

In this section we give technical lemmas.

In the same way as for $\mathcal{A}$, we define the set of semimartingales $\mathcal{A}_t$ in $C([t, 1])$. We recall the following result.

Lemma 3.1 [8, p. 210, Remark 11.2]. *Suppose that* (A.1) *and* (A.3) *hold. Then for any* $f \in C_b^\infty(\mathbf{R}^d)$, *the HJB equation* (1.10) *with* $\varphi(1, \cdot) = f$ *has a unique solution* $\varphi \in C^{1,2}([0, 1] \times \mathbf{R}^d) \cap C_b^{0,1}([0, 1] \times \mathbf{R}^d)$, *which can be written as follows*:

$$\varphi(t, x) = \sup_{X \in \mathcal{A}_t} \left\{ E[f(X(1)) | X(t) = x] - E\left[\int_t^1 L(s, X(s); \beta_X(s, X))\, ds \,\middle|\, X(t) = x \right] \right\}, \tag{3.1}$$

where for the maximizer $X \in \mathcal{A}_t$, *the following holds*:

$$\beta_X(s, X) = D_z H(s, X(s); D_x\varphi(s, X(s))).$$

Fix $P_0 \in \mathcal{M}_1(\mathbf{R}^d)$. For $f \in C_b(\mathbf{R}^d)$, put

$$V^*(f) := \sup_{P \in \mathcal{M}_1(\mathbf{R}^d)} \left\{ \int_{\mathbf{R}^d} f(x) P(dx) - V(P_0, P) \right\}, \tag{3.2}$$

$$v^*(f) := \sup_{P \in \mathcal{M}_1(\mathbf{R}^d)} \left\{ \int_{\mathbf{R}^d} f(x) P(dx) - v(P_0, P) \right\}. \tag{3.3}$$

The following lemma plays a crucial role in the proof of Theorem 2.1.

Lemma 3.2.

(i) *Suppose that* (A.3(i),(ii)) *hold. Then for any* Q_0 *and* $Q_1 \in \mathcal{M}_1(\mathbf{R}^d)$,

$$V(Q_0, Q_1) \geq v(Q_0, Q_1). \tag{3.4}$$

(ii) *Suppose in addition that* (A.1) *and* (A.3) *hold. Then for any* $f \in C_b^\infty(\mathbf{R}^d)$,

$$V^*(f) \geq v^*(f). \tag{3.5}$$

Proof. We first prove (i). For $X \in \mathcal{A}$ for which $E[\int_0^1 L(t, X(t); \beta_X(t, X))\, dt]$ is finite and for which $PX(t)^{-1} = Q_t$ $(t = 0, 1)$, $(b_X(t, x), P(X(t) \in dx))$ satisfies (1.1) with $(b(t, x), P_t(dx)) = (b_X(t, x), P(X(t) \in dx))$ (see (1.13) for notation). Indeed, for any $f \in C_b^{1,2}([0, 1] \times \mathbf{R}^d)$ and $t \in [0, 1]$, by Itô's formula,

$$\begin{aligned}
&\int_{\mathbf{R}^d} f(t, x) P(X(t) \in dx) - \int_{\mathbf{R}^d} f(0, x) P(X(0) \in dx) \\
&\quad = E[f(t, X(t)) - f(0, X(0))] \\
&\quad = \int_0^t ds E\left[\frac{\partial f(s, X(s))}{\partial s} + \frac{1}{2} \triangle f(s, X(s)) + \langle \beta_X(s, X), D_x f(s, X(s)) \rangle \right] \\
&\quad = \int_0^t ds E\left[\frac{\partial f(s, X(s))}{\partial s} + \frac{1}{2} \triangle f(s, X(s)) + \langle b_X(s, X(s)), D_x f(s, X(s)) \rangle \right] \\
&\quad = \int_0^t ds \int_{\mathbf{R}^d} \left(\frac{\partial f(s, x)}{\partial s} + \frac{1}{2} \triangle f(s, x) + \langle b_X(s, x), D_x f(s, x) \rangle \right) P(X(s) \in dx).
\end{aligned} \tag{3.6}$$

Hence, from Remark 2.1(i), by Jensen's inequality,

$$\begin{aligned}
&E\left[\int_0^1 L(t, X(t); \beta_X(t, X))\, dt \right] \\
&\quad \geq E\left[\int_0^1 L(t, X(t); b_X(t, X(t)))\, dt \right] \\
&\quad = \int_0^1 dt \int_{\mathbf{R}^d} L(t, x; b_X(t, x)) P(X(t) \in dx) \geq v(Q_0, Q_1).
\end{aligned} \tag{3.7}$$

Next we prove (ii). For φ in (3.1) and $\{(b(t, x), P(t, dx))\}_{0 \leq t \leq 1}$ for which $\{P(t, dx)\}_{0 \leq t \leq 1} \subset \mathcal{M}_1(\mathbf{R}^d)$ and (1.1) with $P(0, dx) = P_0$ holds,

$$\int_{\mathbf{R}^d} f(x) P(1, dx) - \int_{\mathbf{R}^d} \varphi(0, x) P_0(dx) \leq \int_0^1 dt \int_{\mathbf{R}^d} L(t, x; b(t, x)) P(t, dx). \tag{3.8}$$

Indeed, take $\psi \in C_o^\infty(\mathbf{R}^d\colon [0,\infty))$ for which $\psi(x) = 1$ ($|x| \le 1$) and $\psi(x) = 0$ ($|x| \ge 2$), and put $\psi_R(x) := \psi(x/R)$ for $R > 0$. Then from (1.1),

$$\begin{aligned}
&\int_{\mathbf{R}^d} \psi_R(x) f(x) P(1,dx) - \int_{\mathbf{R}^d} \psi_R(x)\varphi(0,x)P(0,dx)\\
&\quad = \int_0^1 dt \int_{\mathbf{R}^d} \psi_R(x)\left[\frac{\partial\varphi(t,x)}{\partial t} + \frac{1}{2}\triangle\varphi(t,x) + \langle b(t,x), D_x\varphi(t,x)\rangle\right]P(t,dx)\\
&\qquad + \int_0^1 dt \int_{\mathbf{R}^d}\Big[\langle D_x\psi_R(x), D_x\varphi(t,x)\rangle + \tfrac{1}{2}\triangle\psi_R(x)\varphi(t,x)\\
&\qquad\qquad + \langle b(t,x), D_x\psi_R(x)\rangle\varphi(t,x)\Big]P(t,dx). \qquad (3.9)
\end{aligned}$$

Let $R \to \infty$. Then we obtain (3.8) from (1.10), (A.1) and Lemma 3.1.

Lemma 3.1 and (3.8) implies (ii). Indeed,

$$\begin{aligned}
v^*(f) &= \sup\Big\{\int_{\mathbf{R}^d} f(x)P(1,dx) - \int_0^1 dt\int_{\mathbf{R}^d} L(t,x;b(t,x))P(t,dx) \,\Big|\\
&\qquad P(0,dx) = P_0(dx), \{P(t,dx)\}_{0\le t\le 1} \subset \mathcal{M}_1(\mathbf{R}^d),\\
&\qquad (b(t,x), P(t,dx)) \text{ satisfies (1.1)}\Big\}\\
&\le \int_{\mathbf{R}^d}\varphi(0,x)P_0(dx) \qquad \text{(from (3.8))}\\
&= \sup\Big\{E\Big[f(X(1)) - \int_0^1 L(t,X(t);\beta_X(t,X))\,dt\Big]\,\Big|\\
&\qquad PX(0)^{-1} = P_0, X \in \mathcal{A}\Big\} \qquad \text{(from Lemma 3.1)}\\
&= V^*(f). \quad \square \qquad (3.10)
\end{aligned}$$

Let $(\Omega, \mathbf{B}, \{\mathbf{B}_t\}_{t\ge 0}, P)$ be a complete filtered probability space, let X_o be a $(\mathbf{B}_0)$-adapted random variable, and let $\{W(t)\}_{t\ge 0}$ denote a d-dimensional $(\mathbf{B}_t)$-Wiener process for which $W(0) = o$ (see, e.g., [11]). For an $\mathbf{R}^d$-valued, $(\mathbf{B}_t)$-progressively measurable stochastic process $\{u(t)\}_{0\le t\le 1}$, put

$$X^u(t) = X_o + \int_0^t u(s)\,ds + W(t) \qquad (t \in [0,1]). \qquad (3.11)$$

Then the following is known.

Lemma 3.3. *Suppose that $E[\int_0^1 |u(t)|\,dt]$ is finite. Then $\{X^u(t)\}_{0\le t\le 1} \in \mathcal{A}$ and*

$$\beta_{X^u}(t, X^u) = E[u(t)|X^u(s), 0 \le s \le t] \qquad (3.12)$$

(*see p.* 270 *of* [11]). *Besides, by Jensen's inequality,*

$$E\left[\int_0^1 L(t, X^u(t); u(t))\,dt\right] \ge E\left[\int_0^1 L(t, X^u(t); \beta_{X^u}(t, X^u))\,dt\right]. \qquad (3.13)$$

For $\mathbf{P} := \{P_t(dx)\}_{0\le t\le 1} \subset \mathcal{M}_1(\mathbf{R}^d)$ and $n \ge 1$, put

$$V_n(\mathbf{P}) := \inf\left\{ E\left[\int_0^1 L(t, X(t); \beta_X(t, X))\, dt\right] \right| \\ PX(t)^{-1} = P_t \left(t = \frac{i}{2^n}, i = 0, \dots, 2^n\right), X \in \mathcal{A}\right\}, \tag{3.14}$$

$$v_n(\mathbf{P}) := \inf\left\{ \int_0^1 dt \int_{\mathbf{R}^d} L(t, x; b(t, x)) P(t, dx) \right| \\ P(t, dx) = P_t(dx) \left(t = \frac{i}{2^n}, i = 0, \dots, 2^n\right), \\ \{P(t, dx)\}_{0\le t\le 1} \subset \mathcal{M}(\mathbf{R}^d), \\ (b(t, x), P(t, dx)) \text{ satisfies (1.1)}\right\}. \tag{3.15}$$

Then we have

Lemma 3.4. *Suppose that (A.1)–(A.4) hold. Then for any* $\mathbf{P} := \{P_t(dx)\}_{0\le t\le 1} \subset \mathcal{M}_1(\mathbf{R}^d)$ *and* $n \ge 1$,

$$v_n(\mathbf{P}) = V_n(\mathbf{P}). \tag{3.16}$$

Proof. For $i = 0, \dots, 2^n - 1$, put

$$V_{n,i}(\mathbf{P}) := \inf\left\{ E\left[\int_0^{1/2^n} L(t, X(t); \beta_X(t, X))\, dt\right] \right| \\ PX(t)^{-1} = P_{t+i/2^n} \left(t = 0, \frac{1}{2^n}\right), X \in \mathcal{A}\right\}, \tag{3.17}$$

$$v_{n,i}(\mathbf{P}) := \inf\left\{ \int_0^{1/2^n} dt \int_{\mathbf{R}^d} L(t, x; b(t, x)) P(t, dx) \right| \\ P(t, dx) = P_{t+i/2^n}(dx) \left(t = 0, \frac{1}{2^n}\right), \\ \{P(t, dx)\}_{0\le t\le 1/2^n} \subset \mathcal{M}(\mathbf{R}^d), \\ (b(t, x), P(t, dx)) \text{ satisfies (1.1) on } [0, 1/2^n]\right\}. \tag{3.18}$$

Then, from Theorem 2.1,

$$v_n(\mathbf{P}) = \sum_{i=0}^{2^n-1} v_{n,i}(\mathbf{P}) = \sum_{i=0}^{2^n-1} V_{n,i}(\mathbf{P}). \tag{3.19}$$

Since $V_n(\mathbf{P}) \ge v_n(\mathbf{P})$ from (3.6)–(3.7), we only have to prove the following:

$$\sum_{i=0}^{2^n-1} V_{n,i}(\mathbf{P}) \ge V_n(\mathbf{P}). \tag{3.20}$$

Suppose that the left-hand side of (3.20) is finite. For $i = 0, \dots, 2^n - 1$, take a minimizer $X_{n,i}$ of $V_{n,i}(\mathbf{P})$ (see Theorem 1.1), and put

$$P_{n,i} := PX_{n,i}\left(\cdot - \frac{i}{2^n}\right)^{-1}$$
$$\text{on} \quad \left(C\left(\left[\frac{i}{2^n}, \frac{i+1}{2^n}\right]: \mathbf{R}^d\right), \mathcal{B}\left(C\left(\left[\frac{i}{2^n}, \frac{i+1}{2^n}\right]: \mathbf{R}^d\right)\right)\right), \tag{3.21}$$

$$P_n\left(dX|_{C([0,1]:\ \mathbf{R}^d)}\right) := P_{n,0}\left(dX|_{C([0,1/2^n]:\ \mathbf{R}^d)}\right) \times \Pi_{i=1}^{2^n-1} P_{n,i}\left(dX|_{C([i/2^n,(i+1)/2^n]:\ \mathbf{R}^d)} \middle| X_{n,i}\left(\frac{i}{2^n}\right) = X\left(\frac{i}{2^n}\right)\right) \tag{3.22}$$

on $(C([0,1]:\ \mathbf{R}^d), \mathcal{B}(C([0,1]:\ \mathbf{R}^d)))$. Under the completion of this measure, the coordinate process $\{X_n(t)\}_{0\le t\le 1}$ satisfies the following:

$$X_n(t) = X_n(0) + \sum_{i=0}^{2^n-1} \int_{\min(i/2^n,t)}^{\min((i+1)/2^n,t)} b_{n,i}\left(s - \frac{i}{2^n}, X_n(s)\right) ds + W_{X_n}(t)$$
$$(0 \le t \le 1), \tag{3.23}$$

where $b_{n,i}$ denotes the drift vector of $X_{n,i}$ (see Theorem 1.1). In particular, $PX_n(t)^{-1} = P_t$ $(t = i/2^n, i = 0, \dots, 2^n)$, which implies (3.20). □

4. Proofs

In this section we prove our results given in Section 2.

When $L = |u|^2$, the following proof extremely simplifies that of Lemma 2.5 of [12].

Proof of Theorem 2.1. Lemma 3.2(i) and the following complete the proof:

$$\begin{aligned} v(P_0, P_1) &\ge \sup_{f\in C_b^\infty(\mathbf{R}^d)} \left\{\int_{\mathbf{R}^d} f(x)P_1(dx) - v^*(f)\right\} \qquad \text{(from (3.3))} \\ &\ge \sup_{f\in C_b^\infty(\mathbf{R}^d)} \left\{\int_{\mathbf{R}^d} f(x)P_1(dx) - V^*(f)\right\} \qquad \text{(from Lemma 3.2(ii))} \\ &= V(P_0, P_1) \qquad \text{(from Theorem 1.1 (see (3.10)))}. \quad \square \end{aligned} \tag{4.1}$$

Proof of Theorem 2.2. We first prove (i). From (3.6)–(3.7), $\mathbf{V}(\mathbf{P}) \ge \mathbf{v}(\mathbf{P})$. Therefore we only have to show that

$$\mathbf{v}(\mathbf{P}) \ge \mathbf{V}(\mathbf{P}). \tag{4.2}$$

Suppose that $\mathbf{v}(\mathbf{P})$ is finite. Then, from Lemma 3.4,

$$\mathbf{v}(\mathbf{P}) \ge v_n(\mathbf{P}) = V_n(\mathbf{P}) \tag{4.3}$$

and X_n constructed in (3.23) is a minimizer of $V_n(\mathbf{P})$.

Let b_n denote the drift vector of $\{X_n(t)\}_{0\le t\le 1}$. It is easy to see that $\{(X_n(t), \int_0^t b_n(s, X_n(s))\,ds)\colon\ t \in [0,1]\}_{n\ge 1}$ is tight in $C([0,1]\colon\ \mathbf{R}^{2d})$ from (A.1) (see [23, Theorem 3] or [9]). Take a weakly convergent subsequence $\{(X_{n_k}(t), \int_0^t b_{n_k}(s, X_{n_k}(s))\,ds)\colon\ t \in [0,1]\}_{k\ge 1}$ such that

$$\begin{aligned}\liminf_{n\to\infty} E\left[\int_0^1 L(t, X_n(t); b_n(s, X_n(s)))\,dt\right]\\ = \lim_{k\to\infty} E\left[\int_0^1 L(t, X_{n_k}(t); b_{n_k}(s, X_{n_k}(s)))\,dt\right].\end{aligned} \tag{4.4}$$

Let $\{(X(t), A(t))\}_{t\in[0,1]}$ denote the limit of $\{(X_{n_k}(t), \int_0^t b_{n_k}(s, X_{n_k}(s))\,ds)\colon\ t\in[0,1]\}_{k\ge 1}$ as $k \to \infty$. Then $\{X(t) - X(0) - A(t)\}_{t\in[0,1]}$ is a $\sigma[X(s)\colon\ 0 \le s \le t]$-Wiener process and $\{A(t)\}_{t\in[0,1]}$ is absolutely continuous (see [23, Theorem 5] or [9]). We can also prove, in the same way as in the proof of (3.17) of [15], the following: from (4.3)–(4.4), (A.2) and (A.3(ii)) (see Remark 2.1(i)),

$$\begin{aligned}\mathbf{v}(\mathbf{P}) &\ge \liminf_{n\to\infty} E\left[\int_0^1 L(t, X_n(t); b_n(t, X_n(t)))\,dt\right]\\ &\ge E\left[\int_0^1 L\left(t, X(t); \frac{dA(t)}{dt}\right)dt\right]\\ &\ge \tilde{E}\left[\int_0^1 L\,(t, X(t); \beta_X(t, X))\,dt\right] \qquad \text{(from Lemma 3.3)}\\ &\ge \mathbf{V}(\mathbf{P}).\end{aligned} \tag{4.5}$$

Here $\tilde{E}$ denotes the mean value by the completion of $PX(\cdot)^{-1}$ and we used the fact that $P(X(t) \in dx) = P_t(dx)$ for all $t \in [0,1]$. Indeed,

$$P(X(t) \in dx) = \lim_{n\to\infty} P\left(X\left(\frac{[2^n t]}{2^n}\right)\in dx\right) \qquad \text{weakly,}$$

$$P\left(X\left(\frac{[2^n t]}{2^n}\right)\in dx\right) = P_{[2^n t]/2^n}(dx) \to P_t(dx) \qquad \text{as} \quad n\to\infty \quad \text{weakly.}$$

Next we prove (ii). Suppose that $\mathbf{v}(\mathbf{P})$ is finite. Then (2.2) and (4.5) show the existence of a minimizer X of $\mathbf{V}(\mathbf{P})$. In the same way as in (3.7), Theorem 2.2(i) and the strict convexity of $u \mapsto L(t, x; u)$ (see Remark 2.1(i)) imply that $\beta_X(t, X) = b_X(t, X(t))$ and $b_X(t, x)$ is a minimizer of $\mathbf{v}(\mathbf{P})$.

Let b_1 and b_2 be minimizers of $\mathbf{v}(\mathbf{P})$. Then for any $\lambda \in (0,1)$, $\lambda b_1(t,x) + (1 - \lambda)b_2(t,x)$ satisfies (1.1), and

$$\begin{aligned}\mathbf{v}(\mathbf{P}) &\le \int_0^1\int_{\mathbf{R}^d} L(t, x; \lambda b_1(t,x) + (1-\lambda)b_2(t,x))P_t(dx)\\ &\le \lambda\int_0^1\int_{\mathbf{R}^d} L(t, x; b_1(t,x))P_t(dx) + (1-\lambda)\int_0^1\int_{\mathbf{R}^d} L(t, x; b_2(t,x))P_t(dx)\\ &= \mathbf{v}(\mathbf{P}).\end{aligned} \tag{4.6}$$

The strict convexity of $u \mapsto L(t, x; u)$ implies the uniqueness of a minimizer of $\mathbf{v}(\mathbf{P})$. □

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