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The Limits of Porous Materials in the Topology Optimization of Stokes Flows*

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Abstract. We consider a problem concerning the distribution of a solid material in a given bounded control volume with the goal to minimize the potential power of the Stokes flow with given velocities at the boundary through the material-free part of the domain. We also study the relaxed problem of the optimal distribution of the porous material with a spatially varying Darcy permeability tensor, where the governing equations are known as the Darcy–Stokes, or Brinkman, equations. We show that the introduction of the requirement of zero power dissipation due to the flow through the porous material into the relaxed problem results in it becoming a well-posed mathematical problem, which admits optimal solutions that have extreme permeability properties (i.e., assume only zero or infinite permeability); thus, they are also optimal in the original (non-relaxed) problem.

Two numerical techniques are presented for the solution of the constrained problem. One is based on a sequence of optimal Brinkman flows with increasing viscosities, from the mathematical point of view nothing but the exterior penalty approach applied to the problem. Another technique is more special, and is based on the "sizing" approximation of the problem using a mix of two different porous materials with high and low permeabilities, respectively.

This paper thus complements the study of Borrvall and Petersson (*Internat. J. Numer. Methods Fluids*, vol. 41, no. 1, pp. 77–107, 2003), where only sizing optimization problems are treated.

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1. Introduction

While topology optimization of structures in (very) rough terms can be described as the science of introducing holes in the structure to improve the structural performance, in the vast majority of the literature on the subject, especially computationally oriented ones, the appearence of holes is *precluded* from the very beginning by the requirement that the minimal structural dimension is positive at every point.

The reason for introducing such a constraint is twofold. From the numerical point of view, the FEM-stiffness matrix of the governing differential equation is guaranteed to be positive definite in this case, resulting in stable numerical procedures. However, more importantly, allowing some structural parts to disappear we often end up with an optimization problem having a non-closed feasible set and, as a result, lacking optimal solutions.

In topology optimization of solids and structures the classic problem of minimizing the structural compliance is known to possess optimal solutions, if we allow microstructures to be used in the optimal structure (see Appendix 5.2 in [BS3]). At the same time, if we are interested in a pure solid–void design, free of microstructures, the same problem lacks optimal solutions. Since the "grey" optimal solutions (the ones involving microstructures, as opposed to "black–white" pure solid–void solutions) are usually difficult to interpret and to manufacture, various restriction or regularization methods are considered in order to reduce the amount of "microstructural material" in the optimal structure; see the bibliographical notes (8) in [BS3]. The pure void parts, the very heart of topology optimization, are not allowed to appear in such methods and are usually modelled by a very compliant material. However, the limits of optimal designs as the properties of the compliant substitute approach those of void are not investigated.

In the case of topology optimization of truss structures, the question of the continuity of the optimal solutions with respect to the lower bound on the minimal structural dimension has received significant attention in the literature (see, e.g., the bibliographical notes (16) in [BS3] on the "stress singularity phenomenon"). Despite the abundant amount of literature on the topology optimization of linearly elastic continuous systems, similar studies have not been conducted for fluid mechanics.

Recently, topology optimization techniques have been applied to optimization problems in flow mechanics [BP], where traditionally shape optimization methods were prevailing (see the pioneering works of Pironneau [Pi1], [Pi2] on the optimality conditions for shape optimization in fluid mechanics; see also the bibliographical notes (2) in [BS3] for classical references, and [GS] for some recent advances in this area). The benefits of using topology optimization (or control in coefficients) over shape optimization include easier implementation and sensitivity analysis, and better integration with existing FEM codes (the monographs [BS3] and [Al] constitute excellent references for using control in coefficients in structural optimization). Borrvall and Petersson [BP] considered the optimal design of flow domains for minimizing the total power of the Stokes flows. The set of admissible designs is a set of porous materials with a spatially varying Darcy permeability tensor, under a constraint on the total volume of fluid in the control region. The appearence of internal walls in the domain (regions with pure solid material, not permitting flow; these can be interpreted as "holes in the flow") is not permitted. Thus, the *topology*, i.e., connectivity of the flow region does not change, and, carrying over the terminology from optimization in solid mechanics, we refer to this case as that of a "sizing" optimization.

In the present paper we study the "real" topology optimization case of the Stokes flow, i.e., pure solid and pure flow regions are allowed. We show that the relaxed problem of distributing porous material, as well as the pure solid–void (zero–one) problem, possesses optimal solutions. Furthermore, we show that the sizing optimal solutions have limits as the permeability of the porous material is allowed to vanish (i.e., converge to the permeability of solid material).

The outline of the present paper is as follows. In the next section we describe the necessary notation and state precisely the weak formulation of the governing equations, its interpretation, and the objective functional. Section 3 is dedicated to the proof of the existence of the optimal solutions to the relaxed problem, while in Section 4 we introduce a well-posed formulation of the zero–one optimal problem and establish the well-posedness of the latter. Two numerical approaches for the solution of the zero–one control problem are the topics of Sections 5 and 6. In Section 7 we show that for functionals other than the total power of the flow, the control problem might be ill-posed, even if rather strong continuity requirements are imposed on the objective functional. We end the paper with a brief discussion of further research topics.

2. Prerequisites

2.1. *Notation*

We follow standard engineering practice and denote vector quantities, such as vectors and vector-valued functions, using the **bold** font. However, for functional spaces of both scalar- and vector-valued functions we use regular font.

Let Ω be a connected bounded domain of $\mathbb{R}^d$, $d \in \{2, 3\}$, with a Lipschitz continuous boundary Γ. In this domain we would like to control the Darcy–Stokes, or Brinkman, equations [NB] with the prescribed flow velocities $\boldsymbol{g}$ on the boundary, and forces $\boldsymbol{f}$ acting in the domain by adjusting the inverse permeability α of the medium occupying Ω, which depends on the control function ρ:

$$\begin{cases} \left.\begin{aligned} &-\nu\boldsymbol{\Delta u} + \alpha(\rho)\boldsymbol{u} + \boldsymbol{\nabla} p = \boldsymbol{f} \\ &\operatorname{div}\boldsymbol{u} = 0 \end{aligned}\right\}, & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g}, & \text{on } \Gamma. \end{cases} \tag{1}$$

In system (1), $\boldsymbol{u}$ is the flow velocity, p is the pressure, and ν is the kinematic viscosity. Of course, the function $\boldsymbol{g}$ must satisfy the compatibility condition

$$\int_\Gamma \boldsymbol{g}\cdot\boldsymbol{n} = 0,$$

where $\boldsymbol{n}$ denotes the outward unit normal. Roughly speaking, if $\alpha(\rho(\boldsymbol{x})) = +\infty$ for some $\boldsymbol{x} \in \Omega$, we simply require $\boldsymbol{u}(\boldsymbol{x}) = 0$ in the first equation of (1) (the formalization and well-posedness of this requirement is discussed later).

Our control set $\mathcal{H}$ is defined as follows:

$$\mathcal{H} = \left\{\rho \in L^\infty(\Omega) \mid 0 \le \rho \le 1,\ \text{a.e. in } \Omega,\ \int_\Omega \rho \le \gamma|\Omega|\right\},$$

where $0 < \gamma < 1$ is the maximal volume fraction that can be occupied by the fluid, and $|\cdot|$ denotes the Lebesgue measure of $\cdot$. Every element $\rho \in \mathcal{H}$ describes the scaled Darcy permeability tensor of the medium at a given point $\boldsymbol{x} \in \Omega$ in the following (informal) way: $\rho(\boldsymbol{x}) = 0$ corresponds to zero permeability at $\boldsymbol{x}$ (i.e., solid, which does not permit any flow at a given point), while $\rho(\boldsymbol{x}) = 1$ corresponds to infinite permeability (i.e., 100% flow region; no structural material is present). Formally, we relate the permeability α^{-1} to ρ using a convex, decreasing, and non-negative function $\alpha\colon\ [0,1] \to \mathbb{R}_+ \cup \{+\infty\}$, defined as

$$\alpha(\rho) = \rho^{-1} - 1.$$

Modelling the Stokes flow, we are interested only in the two extreme values of permeability, $\alpha^{-1} = 0$ or $\alpha^{-1} = +\infty$. For this purpose, we introduce the following subset of extreme points of a convex set $\mathcal{H}$:

$$\widetilde{\mathcal{H}} = \{\rho \in \mathcal{H} \mid \rho \in \{0,1\},\ \text{a.e. in } \Omega\}.$$

However, both from the analytical and the computational points of view, it is impossible to state the control problem in the set $\widetilde{\mathcal{H}}$, because it is non-convex, and not weakly* closed. Therefore, we first study the properties of the relaxed control problem posed over the convex, whence weakly* closed, set $\mathcal{H} = \operatorname{hull}(\widetilde{\mathcal{H}})$ (where $\operatorname{hull}(\cdot)$ denotes the operation of taking the convex hull of $\cdot$).

In the rest of the paper we use the symbol χ_A for $A \subset \Omega$ to denote the characteristic function of A: $\chi_A(\boldsymbol{x}) = 1$ for $\boldsymbol{x} \in A$; $\chi_A(\boldsymbol{x}) = 0$ otherwise.

2.2. *Weak Formulation*

To state the problem in a more analytically suitable way, and to incorporate the special case $\alpha = +\infty$ into the first equation of system (1), we introduce a weak formulation of the equations. We consider the set of admissible flow velocities and test functions

$$\mathcal{U} = \{\boldsymbol{v} \in H^1(\Omega) \mid \operatorname{tr} \boldsymbol{v} = \boldsymbol{g} \quad \text{on } \Gamma\},$$
$$\mathcal{V} = \{\boldsymbol{v} \in H^1(\Omega) \mid \operatorname{tr} \boldsymbol{v} = 0 \quad \text{on } \Gamma\},$$

and pressures

$$L_0^2(\Omega) = \left\{ q \in L^2(\Omega) \mid \int_\Omega q = 0 \right\}.$$

Then the weak formulation of (1) reads as follows: for $\boldsymbol{f} \in L^2(\Omega)$, compatible $\boldsymbol{g} \in H^{1/2}(\Gamma)$, and $\rho \in \mathcal{H}$ find $(\boldsymbol{u}, p) \in \mathcal{U} \times L_0^2(\Omega)$ such that

$$\begin{aligned} &\nu \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} + \int_\Omega \alpha(\rho) \boldsymbol{u} \cdot \boldsymbol{v} - \int_\Omega p \operatorname{div} \boldsymbol{v} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}, \qquad \forall \boldsymbol{v} \in \mathcal{V}, \\ &\int_\Omega q \operatorname{div} \boldsymbol{u} = 0, \qquad \forall q \in L_0^2(\Omega). \end{aligned} \tag{2}$$

(In the system above we use the usual convention $\infty \cdot 0 = 0$.)

Allowing designs with zero permeability significantly increases the complexity of the control problem. From the purely technical side, the inverse permeability α may be infinite on sets of positive measure, and thus does not belong to any of the common functional spaces. Even worse, internal walls that do not permit flows with the given boundary conditions might appear as limits of designs that admit flow, making the space of admissible designs not closed. The latter difficulty is demonstrated in the following example.

Example 2.1 (Diminishing Permeability). Let $\boldsymbol{g}$ be some compatible non-zero boundary condition, let $\boldsymbol{f}$ be arbitrary in $L^2(\Omega)$. Let $\rho_k \equiv 1/k$ in Ω, $\rho \equiv 0$ in Ω, so that $\rho_k \to \rho$ in $L^\infty(\Omega)$ as $k \to \infty$. It is not difficult to check (this follows from the standard theory for the Stokes equations as well as from the results in [BP]) that for each $k = 1, 2, \dots,$ there is a solution $(\boldsymbol{u}_k, p_k)$ to (2). However, since $\alpha(\rho) \equiv +\infty$ in Ω, from the first equation in (2) it follows that $\boldsymbol{u} \equiv 0$ in Ω, which is clearly not compatible with the boundary conditions. In other words, there is no solution $(\boldsymbol{u}, p)$ to (2) corresponding to the limiting control ρ, which means that the set of admissible controls is not closed even in the strong topology of $L^\infty(\Omega)$!

This is in vast contrast with the sizing case, which can be modelled by requiring $\underline{\rho} \le \rho \le \overline{\rho}$, a.e. in Ω, for some constants $0 < \underline{\rho} \le \overline{\rho} \le 1$. Under these conditions, Borrvall and Petersson [BP] show that the set of admissible controls is closed in the weak* topology of $L^\infty(\Omega)$. (In fact, the case $\rho = 1$ or $\alpha = 0$ is not allowed in the cited work; however, the arguments used there work for this case as well because, owing to Fredrichs' inequality, the semi-norm $|\cdot|_1$ is equivalent to the norm of $H^1(\Omega)$ in the problem we consider; see also Theorem 3.3).

Example 2.1 demonstrates that the lower semicontinuity of the objective functional alone is not sufficient for the topology optimization of the Darcy–Stokes flow to possess optimal solutions; e.g., take the problem of minimizing the "volume of the flow" $\int_\Omega \rho$ to recover a situation similar to that of Example 2.1. However, if the objective functional also enjoys an inf-compactness property with respect to the set of admissible controls, every minimizing sequence converges, thus making the control problem well-posed. In what follows we establish that the power functional, introduced below, for the Darcy–Stokes flow is both lower semicontinuous and inf-compact, thus extending the results of [BP] from sizing to topology optimization.

Let $\mathcal{J}^{\mathcal{S}}\colon \mathcal{U} \to \mathbb{R}$ denote the potential power of the Stokes flow:

$$\mathcal{J}^{\mathcal{S}}(\boldsymbol{u}) = \frac{1}{2}\int_\Omega \boldsymbol{\nabla u} \cdot \boldsymbol{\nabla u} - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u}.$$

Let us further define the additional power dissipation $\mathcal{J}^{\mathcal{D}}\colon \mathcal{H} \times \mathcal{U} \to \mathbb{R} \cup \{+\infty\}$, due to the presence of the porous medium:

$$\mathcal{J}^{\mathcal{D}}(\rho, \boldsymbol{u}) = \frac{1}{2}\int_\Omega \alpha(\rho)\boldsymbol{u} \cdot \boldsymbol{u}.$$

Finally, let $\mathcal{J}(\rho, \boldsymbol{u}) = \mathcal{J}^{\mathcal{S}}(\boldsymbol{u}) + \mathcal{J}^{\mathcal{D}}(\rho, \boldsymbol{u})$ denote the total power of the Darcy–Stokes flow.

Assuming $\alpha(\rho) < +\infty$, one can derive the variational formulation of system (1) (see [BP]):

$$\begin{aligned} \varphi(\rho) = {} & \min_{\boldsymbol{u} \in \mathcal{U}} \mathcal{J}(\rho, \boldsymbol{u}), \\ & \text{s.t. } \operatorname{div} \boldsymbol{u} = 0, \qquad \text{weakly in } \Omega, \end{aligned} \tag{3}$$

system (2) being the first-order necessary optimality conditions for (3). In particular, the pressure $p \in L^2_0(\Omega)$ is defined as a Lagrange multiplier for the constraint $\operatorname{div} \boldsymbol{u} = 0$. In what follows, we denote the feasible set of problem (3) by $\mathcal{U}_{\text{div}}$.

Now, assume that for a given $\rho \in \mathcal{H}$ there exists a solution $\boldsymbol{u} \in H^1(\Omega)$ to the variational problem (3). Let us consider a new domain $\check{\Omega} := \operatorname{int}[\operatorname{supp}(\boldsymbol{u})] \subseteq \Omega$. Clearly, $\alpha < +\infty$, a.e. in $\check{\Omega}$. Furthermore, if $\check{\Omega}$ is regular enough (e.g., with Lipschitz continuous boundary), then $\boldsymbol{u}$ solves the variational problem (3) in the domain $\check{\Omega}$ with the boundary conditions $\operatorname{tr} \boldsymbol{u} = \boldsymbol{g}$ on $\partial\check{\Omega} \cap \Gamma$, $\operatorname{tr} \boldsymbol{u} = 0$ on $\partial\check{\Omega} \setminus \Gamma$, and there exists an associated pressure p: $\check{\Omega} \to \mathbb{R}$ such that the pair $(\boldsymbol{u}, p)$ solves the weak formulation of the Darcy–Stokes equation in the domain $\check{\Omega}$ with the already described boundary condition. (In particular, if $\alpha = 0$ a.e. in $\check{\Omega}$, then $(\boldsymbol{u}, p)$ is a weak solution to the Stokes equation in the domain $\check{\Omega}$.) With this interpretation, we use the variational formulation (3) of the problem instead of (2) in the development that follows.

2.3. *Objective Functional*

The objective functional in our problem is to minimize the total potential power of the flow, which in the case of $\boldsymbol{f} = 0$ amounts to minimizing the power dissipated by the flow. (The same problem can be interpreted as a minimization of the average pressure drop, provided $\boldsymbol{f} = 0$ and $\boldsymbol{g} = g\boldsymbol{n}$ [BP].)

Therefore, the optimization problem we consider can be written as follows:

$$\min_{\rho \in \mathcal{H}} \varphi(\rho), \tag{4}$$

where φ: $\mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is defined in (3).

As has been announced above, with this functional the control problem (4) possesses optimal solutions *despite* the fact that the set of admissible controls is not closed (see Corollary 3.4). Furthermore, in contrast to the situation in the case of linear elasticity, the "discrete" problem of minimizing the total power of the Stokes flow with controls in $\widetilde{\mathcal{H}}$ possesses optimal solutions. However, special approximation techniques are necessary to find them (see Sections 5 and 6).

3. Existence of Optimal Solutions

In this section we prove that problem (4) admits optimal solutions; see Theorem 3.3 and its corollary. However, we need a few auxiliary results first.

Proposition 3.1. *The function* h: $[0, 1] \times \mathbb{R}^d \to \mathbb{R}_+ \cup \{+\infty\}$ *defined as* $h(x, \boldsymbol{y}) = x^{-1}\boldsymbol{y} \cdot \boldsymbol{y}$ *with the conventions* $0^{-1} = +\infty$ *and* $+\infty \cdot 0 = 0$ *is convex and lower semicontinuous*.

Proof. The proof is elementary and can be found on p. 83 in [Ro]. □

Lemma 3.2. *Let* $\{(\rho_k, \boldsymbol{u}_k)\} \subset \mathcal{H} \times \mathcal{U}_{\text{div}}$ *be such that*:

- $\liminf_{k\to+\infty} \mathcal{J}^{\mathcal{D}}(\rho_k, \boldsymbol{u}_k) = C$, *for some constant* $C < +\infty$;

- ∘ $w^*\text{-}\lim_{k\to+\infty}\rho_k = \rho$ *in* $L^\infty(\Omega)$;
- ∘ $w\text{-}\lim_{k\to\infty}\boldsymbol{u}_k = \boldsymbol{u}$ *in* $H^1(\Omega)$.

Then the pair $(\rho, \boldsymbol{u}) \in \mathcal{H} \times \mathcal{U}_{\text{div}}$, *and* $\mathcal{J}^{\mathcal{D}}(\rho, \boldsymbol{u}) \le C$.

Proof. The first claim is obvious.

Since $\int_\Omega \alpha(\rho_k)\boldsymbol{u}_k \cdot \boldsymbol{u}_k = \int_\Omega h(\rho_k, \boldsymbol{u}_k) - \int_\Omega \boldsymbol{u}_k \cdot \boldsymbol{u}_k$, where h is defined in Proposition 3.1, and the last integral converges to $\int_\Omega \boldsymbol{u} \cdot \boldsymbol{u}$, it remains to estimate $\liminf_{k\to+\infty} \int_\Omega h(\rho_k, \boldsymbol{u}_k)$. The weak lower semicontinuity of $(\rho, \boldsymbol{u}) \mapsto \int_\Omega h(\rho, \boldsymbol{u})$ follows from the (pointwise) convexity and lower semicontinuity of h (Proposition 3.1), Fatou's lemma, and Corollary 2.2 in [ET]. □

Now we are ready to establish the existence result.

Theorem 3.3 (Existence of Optimal Solutions). *The optimization problem*

$$\min_{(\rho,\boldsymbol{u})\in\mathcal{H}\times\mathcal{U}_{\text{div}}} \mathcal{J}(\rho, \boldsymbol{u}) \tag{5}$$

possesses at least one optimal solution $(\rho^*, \boldsymbol{u}^*)$.

Proof. Let $\boldsymbol{u}_0$ be the solution to the Stokes problem in Ω (i.e., the solution to (3) corresponding to $\rho \equiv 1$ in Ω); set $\rho_0 \equiv \gamma/|\Omega|$. Then $(\rho_0, \boldsymbol{u}_0) \in \mathcal{H} \times \mathcal{U}_{\text{div}}$, and $\mathcal{J}(\rho_0, \boldsymbol{u}_0) < +\infty$. Furthermore, for all $(\rho, \boldsymbol{u}) \in \mathcal{H} \times \mathcal{U}_{\text{div}}$ it holds that $\mathcal{J}(\rho, \boldsymbol{u}) \ge \mathcal{J}(1, \boldsymbol{u}_0) > -\infty$, i.e., problem (5) is feasible and $\mathcal{J}$ is proper with respect to its feasible set.

The set $\mathcal{H}$ is weakly* compact in $L^\infty(\Omega)$, and the set $\mathcal{U}_{\text{div}}$ is weakly closed in $H^1(\Omega)$.

Owing to the weak lower semicontinuity of $\mathcal{J}^{\mathcal{S}}$ in $H^1(\Omega)$ (see Theorem 2.3 in [Da]), and lower semicontinuity of $\mathcal{J}^{\mathcal{D}}$ in the weak*×weak topology of $L^\infty(\Omega) \times H^1(\Omega)$ (see Lemma 3.2), it remains to show that every minimizing sequence $\{(\rho_k, \boldsymbol{u}_k)\}$ of (5) has bounded second components.

The valid inequality $+\infty > \limsup_{k\to+\infty} \mathcal{J}(\rho_k, \boldsymbol{u}_k) \ge \limsup_{k\to+\infty} \mathcal{J}(1, \boldsymbol{u}_k) = \limsup_{k\to+\infty} \mathcal{J}^{\mathcal{S}}(\boldsymbol{u}_k)$ implies that $\{|\boldsymbol{u}_k|_1\}$ is bounded. Since Ω is bounded, and $\operatorname{tr} \boldsymbol{u}_k = \boldsymbol{g}$, Fredrichs' inequality implies that $\{\|\boldsymbol{u}_k\|\}$ is bounded. □

Corollary 3.4. *The optimization problem* (4) *possesses at least one optimal solution.*

Proof. Let $(\rho^*, \boldsymbol{u}^*)$ be optimal solution to (5); then ρ^* is optimal in (4). □

4. Existence of Black–White Solutions

From the engineering point of view, it is important to find optimal solutions to problem (4) that also lie in $\widetilde{\mathcal{H}}$. Such optimal solutions are traditionally called *zero–one*, or *black–white*, solutions in the topology optimization literature. Zero–one optimal solutions are easy to interpret and to manufacture (e.g., one does not need to include microstructures into the final design in linear elasticity, or materials with varying porosity in Darcy–Stokes flow mechanics).

Let $\boldsymbol{u}^*$ be a flow that is optimal in problem (4). We can always obtain an optimal control ρ^* for this flow as a solution to the following opimization problem:

$$\min_{\rho\in\mathcal{H}} \mathcal{J}(\rho,\boldsymbol{u}^*). \tag{6}$$

For problem (6) to admit optimal solutions at the extreme points of the control set $\mathcal{H}$, i.e., in $\widetilde{\mathcal{H}}$, it is necessary for the inverse permeability α to depend on ρ in a *concave* way. At the same time, the lower semicontinuity of the objective functional $\mathcal{J}$ depends on the fact that α (in fact, h, see Lemma 3.2) depends on its arguments in a *convex* manner. Clearly, there is no function mapping [0, 1] onto $[0,+\infty]$ satisfying both requirements. Therefore, we need to specify the requirement that there must be at least one solution to (4) in $\widetilde{\mathcal{H}}$ as an additional constraint. As will be shown in Theorem 4.1, this can be achieved by adding a requirement of zero energy dissipation due to the flow through the porous material, i.e., $\mathcal{J}^{\mathcal{D}}(\rho,\boldsymbol{v}) = 0$.

On the other hand, in the case of the sizing optimization problems considered in [BP], the design space $\mathcal{H}$ describes inverse permeabilities α which belong to the bounded subset $\{0 < \underline{\alpha} \le \alpha \le \overline{\alpha} < +\infty\}$ of $L^\infty(\Omega)$. Therefore, one has a freedom to choose an affine mapping (that is, both convex and concave) $\alpha^{(\ell)}(\rho) = \overline{\alpha} + (\underline{\alpha} - \overline{\alpha})\rho$ to describe the dependence of the inverse permeability on the design; with such a choice, there is always an optimal solution $\rho^* \in \widetilde{\mathcal{H}}$ to the sizing optimization problem (see Corollary 3.1 in [BP]). However, the zero–one optimal solutions obtained in [BP] are not black–white in the traditional interpretation (i.e., black denotes solid material, and white is its opposite: void in linear elasticity, or flow region in flow mechanics), but rather "dark-grey–light-grey"! Namely, they are composed of two porous materials with high and low permeabilities, respectively. A priori, it is not clear how close they are to the real black–white solutions (if any of the latter exist).

Therefore, our further goals are as follows. In this section we show how to set up, in an analytically suitable manner, an optimization problem for minimizing the potential power of the Stokes flow that possesses black–white solutions. This problem is not suitable for numerical computations though, because the zero–one solution requirement is posed as a complementarity condition between the inverse permeability and the velocity of the flow. (Complementarity conditions are known to generate highly non-convex feasible sets, which often violate standard constraint qualifications [LPR] and are therefore extremely hard to solve to global or even local optimality.) As a remedy, in the two subsequent sections we propose two computational approaches to the zero–one problem: one is based on a penalty function, with the viscosity of the flow playing the role of a penalty parameter; the other one is based on the aforementioned "dark-grey—light-grey" approximations.

Theorem 4.1 (Existence of 0–1 Solutions). *The optimization problem*

$$\begin{cases} \min_{(\rho,\boldsymbol{u})\in\mathcal{H}\times\mathcal{U}_{\text{div}}} \mathcal{J}^{\mathcal{S}}(\boldsymbol{u}), \\ s.t. \quad \mathcal{J}^{\mathcal{D}}(\rho,\boldsymbol{u}) = 0, \end{cases} \tag{7}$$

possesses at least one optimal solution $(\widetilde{\rho},\boldsymbol{u}^*) \in \widetilde{\mathcal{H}} \times \mathcal{U}_{\text{div}}$.

Proof. The constraint of problem (7) can be equivalently written as $\mathcal{J}^{\mathcal{D}}(\rho,\boldsymbol{u}) \le 0$, which, together with Lemma 3.2, implies the closedness of the feasible set of problem (7) in the weak* $\times$ weak topology of $L^\infty(\Omega) \times H^1(\Omega)$. Therefore, following the proof of Theorem 3.3, we can establish existence of the optimal solution $(\rho^*, \boldsymbol{u}^*) \in \mathcal{H} \times \mathcal{U}_{\text{div}}$, provided there is at least one solution that is feasible in (7).

To construct a feasible solution, we choose a closed set $\Omega_0 \Subset \Omega$, such that $|\Omega_0| = (1-\gamma)|\Omega|$ and $\Omega\backslash\Omega_0$ is connected and has a Lipschitz continuous boundary. Let $\boldsymbol{u}^{\mathcal{S}}$ be the Stokes flow in $\Omega\backslash\Omega_0$ with boundary conditions $\boldsymbol{u}^{\mathcal{S}} = \boldsymbol{g}$ on Γ and $\boldsymbol{u}^{\mathcal{S}} = 0$ on $\partial(\Omega\backslash\Omega_0)\backslash\Gamma$; set $\rho^{\mathcal{S}} = \chi_{\Omega\backslash\Omega_0}$. When $\mathcal{J}^{\mathcal{S}}(\boldsymbol{u}^{\mathcal{S}}) < +\infty$ and $\mathcal{J}^{\mathcal{D}}(\rho^{\mathcal{S}}, \boldsymbol{u}^{\mathcal{S}}) = 0$.

Now, let $\widetilde{\rho} = \chi_{\Omega_{\text{nz}}(\boldsymbol{u}^*)}$, where $\Omega_{\text{nz}}(\boldsymbol{u}^*)$ is defined up to the sets of measure zero as $\{\boldsymbol{x} \in \Omega \mid \|\hat{\boldsymbol{u}}^*(\boldsymbol{x})\| \neq 0\}$, and $\hat{\boldsymbol{u}}^*\colon \Omega \to \mathbb{R}^d$ is an arbitrary representative of $\boldsymbol{u}^*$. Then $\int_\Omega \widetilde{\rho} \leq \int_\Omega \rho^*$ and $\mathcal{J}^{\mathcal{D}}(\widetilde{\rho}, \boldsymbol{u}^*) = 0$, yielding an optimal solution $(\widetilde{\rho}, \boldsymbol{u}^*) \in \widetilde{\mathcal{H}} \times \mathcal{U}_{\text{div}}$. ☐

We stress the fact that, owing to Theorem 4.1, for every optimal solution to (7), there is an optimal solution to the following zero–one problem,

$$\begin{cases} \min_{(\rho,\boldsymbol{u})\in L^\infty(\Omega)\times\mathcal{U}_{\text{div}}} \mathcal{J}^{\mathcal{S}}(\boldsymbol{u}), \\ \text{s.t.} \begin{cases} \rho(\boldsymbol{x}) = 0 \Longrightarrow \boldsymbol{u}(\boldsymbol{x}) = 0, & \text{a.e. in } \Omega, \\ \boldsymbol{u}(\boldsymbol{x}) \neq 0 \Longrightarrow \rho(\boldsymbol{x}) = 1, & \text{a.e. in } \Omega, \\ \displaystyle\int_\Omega \rho \leq \gamma|\Omega|, \end{cases} \end{cases} \tag{8}$$

having the same objective value. Therefore, every optimal solution to (8) is also optimal in (7). In this sense, problems (8) and (7) are *equivalent*, i.e., neither one is a relaxation nor a restriction of the other. Such an equivalence is a very important and unique fact about the topology optimization of Stokes flows. We recall that the zero–one problem “as is” in linear elasticity is ill-posed, and either relaxation or restriction is *necessary* to guarantee the existence of optimal solutions (see the bibliographical notes (8) in [BS3] for an extensive account of relaxation and restriction methods in topology optimization in solid mechanics).

5. Black–White Solutions Via Increasing the Viscosity

There is a school of thought arguing that under some circumstances the viscosity ν and permeability α^{-1} in system (1) alone do not adequately describe the Stokes flow in porous media. An additional parameter μ is introduced into the first PDE as follows [NB]:

$$-\nu\boldsymbol{\Delta u} + \mu\alpha(\rho)\boldsymbol{u} + \boldsymbol{\nabla} p = \boldsymbol{f}.$$

Now, the parameter μ is the viscosity of the flow, while ν is an “effective viscosity.” Repeating the arguments of Section 1, we then arrive at the following formulation of the optimization problem (4):

$$\min_{(\varrho,\boldsymbol{v})\in\mathcal{H}\times\mathcal{U}_{\text{div}}} \mathcal{J}^{\mathcal{S}}(\boldsymbol{v}) + \mu\mathcal{J}^{\mathcal{D}}(\varrho, \boldsymbol{v}). \tag{9}$$

Clearly, this is nothing but the exterior penalty reformulation of problem (7), with the viscosity μ playing the role of a penalty parameter. The arguments of Theorem 3.3 are applicable to problem (9) as well, so that there exists a family of optimal solutions

$\{\rho^*_\mu, \boldsymbol{u}^*_\mu\}$, $\mu > 0$, to (9). From the standard theory for non-linear programs (see Theorem 9.2.2 in [BSS]), it follows that every weak* $\times$ weak limit point of this sequence as $\mu \to +\infty$ (and there is at least one) is an optimal solution to (7).

We note that problem (9) does not contain any complicating state constraints, and thus is much easier to solve than (7). While the penalty method might converge quite slowly, and the approximating designs might contain quite a large amount of porous material with intermediate values of permeability, we think it is instructive to mention this approach, owing to its clear mathematical and physical interpretations (compare with, e.g., the most popular "SIMP" approach [BS1] in the topology optimization of elastic materials, or the more material science-compatible "RAMP" method [SS]; see also the discussion on p. 64 in [BS3].

6. Black–White Solutions as Limits of "Dark-Grey–Light-Grey" Solutions

In this section we approximate the zero–one problem (7) using the aforementioned two-value "dark-grey–light-grey" optimal controls obtained in [BP]. To perform such an approximation, we introduce two sequences, $\{\underline{\alpha}_k\} \downarrow 0$ and $\{\overline{\alpha}_k\} \uparrow +\infty$, of extreme inverse permeabilities. Further, we let $\underline{\rho}_k = (\overline{\alpha}_k + 1)^{-1}$, $\overline{\rho}_k = (\underline{\alpha}_k + 1)^{-1}$, and define an affine function $\alpha^{(\ell,k)}\colon [\underline{\rho}_k, \overline{\rho}_k] \to \mathbb{R}_+$ so that $\alpha^{(\ell,k)}(\underline{\rho}_k) = \overline{\alpha}_k$, $\alpha^{(\ell,k)}(\overline{\rho}_k) = \underline{\alpha}_k$. To simplify the discussion somewhat, we assume that the sequence $\{(\underline{\alpha}_k, \overline{\alpha}_k)\}$ is chosen so that the inequality $\overline{\rho}_k\gamma + \underline{\rho}_k(1-\gamma) \le \gamma$ is satisfied. Then we can also define the approximating control sets $\mathcal{H}_k = \{\varrho \in \mathcal{H} \mid \underline{\rho}_k \le \varrho \le \overline{\rho}_k,\ \text{a.e. in } \Omega\}$ and $\widetilde{\mathcal{H}}_k = \{\varrho \in \mathcal{H} \mid \varrho \in \{\underline{\rho}_k, \overline{\rho}_k\},\ \text{a.e. in } \Omega\}$. Finally, we define $\mathcal{J}^{\mathcal{D}}_k(\varrho, \boldsymbol{v}) = \frac{1}{2}\int_\Omega \alpha^{(\ell,k)}(\varrho)\boldsymbol{v}\cdot\boldsymbol{v}$ and $\mathcal{J}_k(\varrho, \boldsymbol{v}) = \mathcal{J}^{\mathcal{S}}(\boldsymbol{v}) + \mathcal{J}^{\mathcal{D}}_k(\varrho, \boldsymbol{v})$.

Throughout this section we also use the following notion. For $\boldsymbol{u} \in H^1(\Omega)$, we denote by $\Omega(\boldsymbol{u})$ a subset of Ω, which we define as follows. We choose and fix an arbitrary representative $\hat{\boldsymbol{u}}\colon \Omega \to \mathbb{R}^d$ of $\boldsymbol{u}$ (i.e., $\hat{\boldsymbol{u}} = \boldsymbol{u}$, a.e.), and define a constant C_γ by

$$C_\gamma := \inf\{C\colon |\{\hat{\boldsymbol{u}} > C\}| < \gamma|\Omega|\}.$$

We then choose an arbitrary set $\Omega(\boldsymbol{u})$ of measure $\gamma|\Omega|$ such that $\Omega(\boldsymbol{u}) \supseteq \{\hat{\boldsymbol{u}} > C_\gamma\}$ and $\Omega(\boldsymbol{u}) \subseteq \{\hat{\boldsymbol{u}} \ge C_\gamma\}$. The fact that $\Omega(\boldsymbol{u})$ is not uniquely defined (even up to the sets of measure zero) will not make any difference for the discussion that follows.

The main result of this section is Theorem 6.3, establishing the convergence (under some arguably mild conditions) of the "dark-grey–light-grey" approximations towards the black–white limits. We begin with some auxiliary results.

The following lemma allows us to define a "limiting" design $\widetilde{\rho} \in \widetilde{\mathcal{H}}$, corresponding to the limiting flow $\boldsymbol{u}$, even though the sequence of "dark-grey–light-grey" controls $\{\rho_k\}$ might have no limit points in $\widetilde{\mathcal{H}}$ in the usual weak* sense.

Lemma 6.1. *Let $\{\boldsymbol{u}_k\} \subset H^1(\Omega)$ weakly converge to $\boldsymbol{u} \in H^1(\Omega)$. Define $\rho_k = \overline{\rho}_k\chi_{\Omega(\boldsymbol{u}_k)} + \underline{\rho}_k\chi_{\Omega\setminus\Omega(\boldsymbol{u}_k)}$, and assume that $\rho_k \in \widetilde{\mathcal{H}}_k$ (i.e., $\int_\Omega \rho_k \le \gamma|\Omega|$), and that*

$$\liminf_{k\to+\infty} \mathcal{J}^{\mathcal{D}}(\rho_k, \boldsymbol{u}_k) = \liminf_{k\to+\infty} \frac{1}{2}\left[\underline{\alpha}_k \int_{\Omega(\boldsymbol{u}_k)} \boldsymbol{u}_k\cdot\boldsymbol{u}_k + \overline{\alpha}_k \int_{\Omega\setminus\Omega(\boldsymbol{u}_k)} \boldsymbol{u}_k\cdot\boldsymbol{u}_k\right] \le C,$$

for some constant $C < +\infty$. Then there is $\widetilde{\rho} \in \widetilde{\mathcal{H}}$ such that

$$\mathcal{J}^{\mathcal{D}}(\widetilde{\rho}, \boldsymbol{u}) = 0. \tag{10}$$

In particular, $|\Omega_{\mathrm{nz}}(\boldsymbol{u})| \le \gamma|\Omega|$, where $\Omega_{\mathrm{nz}}(\boldsymbol{u})$ is defined up to the sets of measure zero as $\{\boldsymbol{x} \in \Omega \mid \|\hat{\boldsymbol{u}}(\boldsymbol{x})\| \ne 0\}$, and $\hat{\boldsymbol{u}}\colon\ \Omega \to \mathbb{R}^d$ is an arbitrary representative of $\boldsymbol{u}$.

Proof. The existence of limit points follows from the inclusion $\widetilde{\mathcal{H}}_k \subset \mathcal{H}$, $k = 1, 2, \dots,$ and the weak*-compactness of the latter. Therefore, we assume that the original sequence $\{\rho_k\}$ weakly* converges to $\rho \in \mathcal{H}$.

The control function ρ_k is a solution to the following optimization problem with a linear objective functional and weak*-compact feasible set:

$$\max_{\varrho \in \mathcal{H}_k} \int_\Omega \varrho \boldsymbol{u}_k \cdot \boldsymbol{u}_k. \tag{11}$$

Since $\{\boldsymbol{u}_k \cdot \boldsymbol{u}_k\}$ converges strongly in $L^1(\Omega)$, from Proposition 4.4 in [BS2] it follows that ρ must solve the following optimization problem:

$$\max_{\varrho \in \mathcal{H}} \int_\Omega \varrho \boldsymbol{u} \cdot \boldsymbol{u}. \tag{12}$$

Further, since the objective functional of (12) is linear (in ϱ), the problem possesses a zero–one optimal solution $\widetilde{\rho} \in \widetilde{\mathcal{H}}$; we can always take $\widetilde{\rho} = \chi_{\Omega(\boldsymbol{u})}$.

Clearly,

$$2C \ge \liminf_{k\to+\infty} \int_\Omega \alpha(\rho_k)\boldsymbol{u}_k \cdot \boldsymbol{u}_k = \liminf_{k\to+\infty} \overline{\alpha}_k \int_{\Omega\setminus\Omega(\boldsymbol{u}_k)} \boldsymbol{u}_k \cdot \boldsymbol{u}_k,$$

which implies that

$$0 = \liminf_{k\to+\infty} \int_{\Omega\setminus\Omega(\boldsymbol{u}_k)} \boldsymbol{u}_k \cdot \boldsymbol{u}_k = \lim_{k\to+\infty} \int_\Omega \rho_k \boldsymbol{u}_k \cdot \boldsymbol{u}_k = \int_\Omega \widetilde{\rho}\boldsymbol{u} \cdot \boldsymbol{u} = \int_{\Omega\setminus\Omega(\boldsymbol{u})} \boldsymbol{u} \cdot \boldsymbol{u},$$

where we used the convergence of optimal values for problem (11) to the one of problem (12) as k goes to $+\infty$ (again, by Proposition 4.4 in [BS2]). We conclude that $\boldsymbol{u} \equiv 0$ on $\Omega\setminus\Omega(\boldsymbol{u})$, which implies (10). ☐

Corollary 6.2. *In addition to the assumptions of Lemma* 6.1, *assume that $|\Omega_{\mathrm{nz}}(\boldsymbol{u})| = \gamma|\Omega|$. Then the sequence $\{\rho_k\}$ converges to $\widetilde{\rho} \in \widetilde{\mathcal{H}}$ strongly in $L^1(\Omega)$.*

Proof. The additional assumption implies that problem (12) possesses the only optimal solution $\widetilde{\rho} = \chi_{\Omega(\boldsymbol{u})} = \chi_{\Omega_{\mathrm{nz}}(\boldsymbol{u})} \in \mathcal{H}$. This implies the weak* convergence of the sequence $\{\rho_k\}$ towards $\widetilde{\rho}$ in $L^\infty(\Omega)$. Strong convergence in $L^1(\Omega)$ then follows from Corollary 3.2 in [Pe]. ☐

Now, the main result of this section can be established.

Theorem 6.3 (Convergence of "Dark-Grey–Light-Grey" Approximations). *Consider the sequence of sizing optimization problems*:

$$\min_{(\varrho,\boldsymbol{v})\in\mathcal{H}_k\times\mathcal{U}_{\mathrm{div}}} \mathcal{J}_k(\varrho,\boldsymbol{v}), \qquad k=1,\dots. \tag{13}$$

Let $\{(\rho_k^*,\boldsymbol{u}_k^*)\}$ *be a sequence of "dark-grey–light-grey" optimal solutions to* (13) (*i.e.*, $(\rho_k^*,\boldsymbol{u}_k^*)\in\widetilde{\mathcal{H}}_k\times\mathcal{U}_{\mathrm{div}}$, $k=1,2,\dots$), *which exists by Corollary* 3.1 *in* [*BP*]. *Then an arbitrary weak limit point* $\boldsymbol{u}$ *of the sequence* $\{\boldsymbol{u}_k^*\}\subset H^1(\Omega)$ (*and there is at least one*) *defines a control* $\rho=\chi_{\Omega(\boldsymbol{u})}\in\widetilde{\mathcal{H}}$ *such that* $(\rho,\boldsymbol{u})$ *is an optimal solution to problem* (7).

If, in addition, $|\Omega_{\mathrm{nz}}(\boldsymbol{u})|=\gamma|\Omega|$, *then* $\{\rho_k\}$ *strongly converges to* ρ *in* $L^1(\Omega)$ ($\Omega_{\mathrm{nz}}(\cdot)$ *is defined in Lemma* 6.1).

Proof. Let $\boldsymbol{u}^{\mathcal{S}}$ be the Stokes flow constructed in the proof of Theorem 4.1; set $\rho_k=\overline{\rho}_k\chi_{\Omega_{\mathrm{nz}}(\boldsymbol{u}^{\mathcal{S}})}+\underline{\rho}_k\chi_{\Omega\setminus\Omega_{\mathrm{nz}}(\boldsymbol{u}^{\mathcal{S}})}$. Then $(\rho_k,\boldsymbol{u}^{\mathcal{S}})$ is feasible in (13), $k=1,2,\dots$. Therefore, the following inequalities hold:

$$\begin{aligned}\limsup_{k\to+\infty}\mathcal{J}_k(\rho_k^*,\boldsymbol{u}_k^*) &\le \limsup_{k\to+\infty}\mathcal{J}_k(\rho_k,\boldsymbol{u}^{\mathcal{S}})\\ &\le \mathcal{J}^{\mathcal{S}}(\boldsymbol{u}^{\mathcal{S}})+\lim_{k\to+\infty}1/2\underline{\alpha_k}\|\boldsymbol{u}^{\mathcal{S}}\|_{L^2(\Omega)}\\ &= \mathcal{J}^{\mathcal{S}}(\boldsymbol{u}^{\mathcal{S}})<+\infty.\end{aligned}$$

This directly implies the boundedness of the sequence $\{\boldsymbol{u}_k^*\}$; we therefore assume that the original sequence weakly converges to $\boldsymbol{u}$. Furthermore, owing to Lemma 6.1, the pair $(\rho,\boldsymbol{u})$, with $\rho=\chi_{\Omega(\underline{\boldsymbol{u}})}$, is feasible in (7).

Let $(\rho^*,\boldsymbol{u}^*)\in\mathcal{H}\times\mathcal{U}_{\mathrm{div}}$ be an arbitrary zero–one optimal solution to (7). By the weak lower semicontinuity of $\mathcal{J}^{\mathcal{S}}$, we have

$$\mathcal{J}^{\mathcal{S}}(\rho^*,\boldsymbol{u}^*)\le\mathcal{J}^{\mathcal{S}}(\rho,\boldsymbol{u})\le\liminf_{k\to+\infty}\mathcal{J}^{\mathcal{S}}(\boldsymbol{u}_k^*)\le\liminf_{k\to+\infty}\mathcal{J}_k(\rho_k^*,\boldsymbol{u}_k^*).$$

On the other hand, letting $\widetilde{\rho}_k=\overline{\rho}_k\chi_{\Omega(\boldsymbol{u}^*)}+\underline{\rho}_k\chi_{\Omega\setminus\Omega(\boldsymbol{u}^*)}$, we obtain the reverse inequality:

$$\mathcal{J}^{\mathcal{S}}(\rho^*,\boldsymbol{u}^*)=\mathcal{J}(\rho^*,\boldsymbol{u}^*)=\lim_{k\to+\infty}\mathcal{J}_k(\widetilde{\rho}_k,\boldsymbol{u}^*)\ge\limsup_{k\to+\infty}\mathcal{J}_k(\rho_k^*,\boldsymbol{u}_k^*),$$

owing to the feasibility of $(\widetilde{\rho}_k,\boldsymbol{u}^*)$ in (13), $k=1,2,\dots$. This establishes the optimality of $(\rho,\boldsymbol{u})$ in (7).

The last claim is a simple application of Corollary 6.2. □

Now we are ready to discuss the additional assumption of Theorem 6.3 (the assumption of Corollary 6.2), which guarantees the strong convergence of the optimal approximating controls. This condition necessarily holds if the flow volume constraint $\int_\Omega\rho\le\gamma|\Omega|$ is active (binding) at *every* control that is optimal in (7). While we do not know if this condition holds in every instance of problem (7), it can always be satisfied by decreasing the flow volume factor γ, if the convergence towards the flow $\boldsymbol{u}$ with $|\Omega_{\mathrm{nz}}(\boldsymbol{u})|<\gamma|\Omega|$ is observed, and resolving the problem.

There is an obstacle, however, which might prevent this from working in practice: each of the approximating problems (13) is non-convex, and, therefore, we cannot expect them to be solved to global optimality by numerical algorithms. (Many structural optimization problems are rather difficult to approximate due to the inherent non-convexity of the approximating problems; see [SS].) Despite this fact, in realistic instances of (7) we expect the flow volume constraint to be binding.

7. Bilevel Programming in Flow Mechanics: a Possible Generalization?

Assume that we are interested in the optimal control of the Darcy–Stokes equations with respect to an alternative objective functional $\mathcal{F}$: $\mathcal{H} \times H^1(\Omega) \to \mathbb{R} \cup \{\infty\}$, where $\mathcal{H}$ denotes the abstract control set. Formally, we would like to solve the following *bilevel* (see p. 10 in [LPR]) programming problem:

$$\begin{cases} \min_{(\rho,\boldsymbol{u})\in\mathcal{H}\times H^1(\Omega)} \mathcal{F}(\rho,\boldsymbol{u}), \\ \text{s.t. } \boldsymbol{u} \in \operatorname{argmin}_{\boldsymbol{v}\in\mathcal{U}_{\text{div}}} \mathcal{J}(\rho,\boldsymbol{v}). \end{cases} \tag{14}$$

Similarly, if we are interested only in pure Stokes flows, the optimization problem can be posed as follows:

$$\begin{cases} \min_{(\rho,\boldsymbol{u})\in\mathcal{H}\times H^1(\Omega)} \mathcal{F}(\rho,\boldsymbol{u}), \\ \text{s.t. } \begin{cases} \boldsymbol{u} \in \operatorname{argmin}_{\boldsymbol{v}\in\mathcal{U}_{\text{div}}} \mathcal{J}(\rho,\boldsymbol{v}), \\ \mathcal{J}^{\mathcal{D}}(\rho,\boldsymbol{u}) = 0. \end{cases} \end{cases} \tag{15}$$

Of course, the minimization of the power function is the simplest problem one can consider in flow topology optimization, owing to the fact that we can join the lower-level and upper-level optimization problems into one: then the bilevel program (14) reduces to (5). This fact allows us to minimize the objective functional $\mathcal{F} \equiv \mathcal{J}$ *simultaneously* with respect to $(\rho, \boldsymbol{u})$, resulting in a problem with an inf-compact, lower semicontinuous functional (with respect to the weak* $\times$ weak topology of $L^\infty(\Omega) \times H^1(\Omega)$) that, thus, possesses optimal solutions. In the bilevel case the mapping $\rho \to \operatorname{argmin}_{\boldsymbol{v}\in\mathcal{U}_{\text{div}}} \mathcal{J}(\rho,\boldsymbol{v})$ is not closed in the weakly* $\times$ weakly topology of $L^\infty(\Omega) \times H^1(\Omega)$. The next example shows that this mapping is not closed even in the strong topology of $L^1(\Omega) \times H^1(\Omega)$, which in particular prevents us from using the weak* topology of $BV(\Omega)$ (or even $SBV(\Omega)$, see [AFP]) for the design space of problems (14) and (15).

Example 7.1 (Disappearing Wall in the Driven Cavity Flow Problem). Let $\Omega = (0,1) \times (-1,1) \subset \mathbb{R}^2$, $\Omega_+ = (0,1) \times (0,1)$, $\Omega_- = \Omega \backslash \Omega_+$, $\boldsymbol{f} \equiv 0$ in Ω, $\boldsymbol{g} \equiv (1,0)$ on the "upper" boundary (the line connecting the points (0, 1) and (1, 1)), and $\boldsymbol{g} \equiv 0$ otherwise. Define $\boldsymbol{u}_+$ to be the solution to the "lid-driven cavity flow" problem (see, e.g., p. 146 in [Ji]) in Ω_+, $\boldsymbol{u}_+ = 0$ in Ω_-.

Consider a sequence $\{\rho_k\} \subset L^\infty(\Omega) \cap BV(\Omega)$, with $\rho_k \equiv 1 - \chi_{(1,0)\times(-1/k,0)}$ in Ω, $k = 1, 2, \ldots$. The solution to the Darcy–Stokes problem (2) in this case is $\boldsymbol{u}_k = \boldsymbol{u}_+$; thus $\{(\rho_k, \boldsymbol{u}_k)\} \to (1, \boldsymbol{u}_+)$ strongly in $L^1(\Omega) \times H^1(\Omega)$. At the same time, the flow corresponding to $\rho \equiv 1$ in Ω is the solution to the driven cavity flow problem in Ω, which is not equal to $\boldsymbol{u}_+$. Thus, the mapping $\rho \to \operatorname{argmin}_{\boldsymbol{v}\in\mathcal{U}_{\text{div}}} \mathcal{J}(\rho,\boldsymbol{v})$ is not closed even in the strong topology of $L^1(\Omega) \times H^1(\Omega)$, even though $\limsup_{k\to+\infty} \mathcal{J}(\rho_k, \boldsymbol{u}_k) < +\infty$.

Now, define $\mathcal{F}(\varrho, \boldsymbol{v}) = \|1-\varrho\|_{BV(\Omega)} + \|\boldsymbol{v}-\boldsymbol{u}_+\|_{H^1(\Omega)}$, $\mathcal{H} = \{\varrho \in BV(\Omega) \mid 0 \leq \varrho \leq 1, \text{ a.e. in } \Omega\}$. Then the sequence $\{(\rho_k, \boldsymbol{u}_k)\}$ is a minimizing sequence for both problems (14) and (15), which does not converge to a feasible point of either of the problems. Therefore, the classic "flow tracking problem" posed as a bilevel topology optimization problem of Darcy–Stokes flow has no solutions.

If we restrict the set of admissible controls so that $\rho \geq \underline{\rho} > 0$ in Ω, problem (14) becomes well-posed for every continuous enough objective functional; however, making such a restriction we arrive at a less interesting, for us, sizing case. Therefore, the problem of choosing practically interesting and well-posed formulations of the topology optimization of Stokes flows with objective functionals other than the total power $\mathcal{J}$ remains open.

8. Conclusions and Further Research

We have shown that the topology optimization problem of the Darcy–Stokes equations with respect to total power minimization admits optimal solutions, even if the limiting zero and infinite permeabilities are included in the design domain. We have further established that the problem of finding a zero–one optimal control, or optimal pure Stokes flow, can be set up in a well-posed way; no additional restriction techniques are necessary in contrast to the case of linear elasticity (see [BS3]). Two techniques were proposed for solving the zero–one optimal control problem. We have also shown that the topology optimization problem with respect to alternative functionals might be ill-posed, and might lack optimal solutions.

It would be particularly interesting to study the zero–one topology optimization problem of Navier–Stokes or Euler flows. For the Navier–Stokes flows, which are of much engineering interest, one can take the same design parametrization as for the Stokes flows [Ge]. The problematic part, as is typical in topology optimization, is to establish the inf-compactness property of the chosen objective functional on the set of admissible designs [Ev]. The theory for the sizing case is straightforward, and only the numerical part needs to be investigated. For the Euler flows, even the design parametrization is unclear, partly due to the fact that flows of inviscid fluids through porous media are not so well investigated in the literature.

As for the Stokes flow, further study of bilevel optimization problems might be interesting, as well as consideration of alternative flow boundary conditions (see [Section 8.2.2 in [Ji]).

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