Appl Math Optim 50:209–228 (2004)
DOI: 10.1007/s00245-004-0800-3

Applied Mathematics & Optimization

A New Notion of Conjugacy for Isoperimetric Problems*

Javier F. Rosenblueth

IIMAS–UNAM, Apartado Postal 20-726,
México DF 01000, México
jfrl@servidor.unam.mx

Abstract. For problems in the calculus of variations with isoperimetric side constraints, we provide in this paper a set of points whose emptiness, independently of nonsingularity assumptions, is equivalent to the nonnegativity of the second variation along admissible variations. The main objective of introducing a characterization of this condition should be, of course, to obtain a simpler way of verifying it. There are two other sets of points available in the literature, introduced by Loewen and Zheng (1994) and Zeidan (1996), for which this necessary condition implies their emptiness. However, we show that verifying membership of these sets may be more difficult than checking directly if that condition holds. Contrary to this behavior, we prove that the desired objective of characterizing that condition is achieved by means of the set introduced in this paper.

Key Words. Isoperimetric problems, Calculus of variations, Conjugate points, Nonsingular extremals.

AMS Classification. 49K15.

1. Introduction

In order to illustrate the main objective of this paper, we start by briefly considering the following example. Suppose that we are interested in minimizing

$$I(x) = \int_0^{\pi} t\{\dot{x}^2(t) - 4x^2(t)\}\, dt$$

* This work was done during a sabbatical stay at The Weizmann Institute of Science, Rehovot, Israel. The support given by Consejo Nacional de Ciencia y Tecnología and Dirección General de Asuntos del Personal Académico, UNAM, is gratefully appreciated.

over all piecewise C^1 functions $x\colon [0,\pi] \to \mathbf{R}$ satisfying $x(0) = x(\pi) = 0$ and $\int_0^\pi x(t)\,dt = 0$. For such a calculus of variations problem, involving an isoperimetric constraint, necessary and sufficient conditions are well established. In particular, one readily verifies that $x_0 \equiv 0$ is a normal extremal for which the corresponding conditions of Legendre and Weierstrass hold. These conditions, therefore, leave x_0 as a candidate for optimality. On the other hand, the condition stating that the second variation with respect to x_0 is nonnegative along admissible variations corresponds to the inequality

$$\int_0^\pi t\{\dot{y}^2(t) - 4y^2(t)\}\,dt \geq 0$$

for all piecewise C^1 functions $y\colon [0,\pi] \to \mathbf{R}$ satisfying $y(0) = y(\pi) = 0$ and $\int_0^\pi y(t)\,dt = 0$. The verification of this condition is thus equivalent to the question of optimality of x_0.

One could try to apply the classical theory of "conjugate points" which, for the simple fixed-endpoint problem in the calculus of variations, plays a fundamental role in establishing both necessary and sufficient conditions for optimality. In particular, if H denotes the set of trajectories for which the second variation is nonnegative along admissible variations, Jacobi's necessary condition states that if x is nonsingular and $x \in H$, then there are no conjugate points with respect to x in the underlying open time interval. One can extend this notion to isoperimetric problems. Also, one can transform the original calculus of variations problem into one involving a system of differential equations, and try to apply results such as those of [25] which generalize in optimal control, from a classical point of view, the notion of conjugate points. However, for both cases, the nonexistence of conjugate points is implied only if the trajectory under consideration is *nonsingular*. For the above example, x_0 is singular, and none of these theories can be applied.

The question posed is essentially to find a characterization of the nonnegativity of a quadratic form. Several attempts in this direction have been made and, in particular, those by Bernhard [4], Breakwell and Ho [5], Caroff and Frankowska [6], Dmitruk [7], [8], Hestenes [9], [10], [12]–[15], Loewen and Zheng [16], Popescu [17], Stefani and Zezza [21], [22], Zeidan [23], [24], and Zeidan and Zezza [25]–[28] deserve special attenion.

It might be extremely complicated to compare these, or more, approaches to conjugacy, in particular when dealing with the isoperimetric problem. In this paper we concentrate on one line of research which has been widely quoted. It corresponds to the one initiated in 1994 by Loewen and Zheng [16] and extended to more general problems in 1996 by Zeidan [24].

For certain classes of optimal control problems, Loewen and Zheng [16] introduced a set $\mathcal{G}(x)$ whose emptiness in the open time interval, without nonsingularity assumptions, becomes in the normal case a necessary condition for optimality. This follows by showing that $x \in H \Rightarrow \mathcal{G}(x) \cap (t_0, t_1) = \emptyset$. When reduced to the fixed-endpoint problem in the calculus of variations, if the trajectory x is nonsingular, $\mathcal{G}(x)$ contains the set of usual conjugate points in the interior of the time interval. This new condition is thus a generalization of that of Jacobi both for more general problems as well as for singular extremals. For the problems considered in [16], the initial endpoint is fixed and a convexity assumption on the control set is required. For problems where both endpoints vary and the control set is not necessarily convex, Zeidan [24] introduced a set $\mathcal{Z}(x)$, containing that of Loewen and Zheng, and shows that a necessary condition for a normal extremal is again the nonexistence of such points in the open time interval. This condition is implied by H in the normal case, that is, if x is a normal extremal and $x \in H$, then $\mathcal{Z}(x) \cap (t_0, t_1) = \emptyset$. Problems in the calculus of variations with isoperimetric constraints can be seen to be a particular case of the optimal control problems considered in those papers when one adds a certain system of differential equations.

The main objective of introducing a characterization of the nonnegativity of a quadratic form should be, in general, to obtain a simpler way of verifying it. We point out that this is successfully achieved by means of the theory of Jacobi and the classical notion of conjugate points since the original question is reduced to solving Jacobi's differential equation (Euler's equation for the secondary problem). However, as mentioned before, this theory excludes the singular case.

With respect to the sets introduced in [16] and [24], as we shall see in this paper, one can easily find examples for which solving the question of their nonemptiness may be more difficult than verifying directly the existence of negative second variations. In those examples, one can exhibit an admissible variation y for which the second variation at x is negative, showing that $x \notin H$, but y does not satisfy the conditions defining membership of these sets. In other words, by using the theories of [16] or [24], one may fail to achieve the main objective of introducing a characterization of the second-order necessary condition.

The main purpose of this paper is to introduce a new set of points $\mathcal{R}(x)$, applicable to the fixed-endpoint problem in the calculus of variations involving isoperimetric side constraints, for which $x \in H \Leftrightarrow \mathcal{R}(x) = \emptyset$, and the objective of simplifying the conditions defining membership of H is achieved. This set corresponds to a generalization of a set of points introduced in [1] which is applicable to the fixed-endpoint problem without isoperimetric conditions. The idea underlying the definition of $\mathcal{R}(x)$ is simple. Given a trajectory x, a point s belongs to $\mathcal{R}(x)$ if there exist two functions y, u, depending on s, satisfying certain conditions. If $\mathcal{R}(x) \neq \emptyset$, then (extending y to the whole interval by zero) the choice of an admissible variation $u + \varepsilon y$ makes the second variation along x strictly negative for sufficiently small ε of appropriate sign. Conversely, if the second variation along x is negative for some variation y, then, by choosing $u \equiv y$, one of the endpoints belongs to $\mathcal{R}(x)$.

A further property of $\mathcal{R}(x)$ should be mentioned. We shall prove that $\mathcal{R}(x)$ contains $\mathcal{G}(x)$ for any trajectory x and, therefore, it also generalizes (both for isoperimetric problems as well as for singular trajectories) the usual notion of conjugate points and Jacobi's necessary condition. We emphasize that the classical conjugate point theory concerns the solution of a two points boundary value problem for a linear ordinary differential equation. The notion which is proposed here consists in finding a trajectory with special properties which assure the possibility of making the quadratic form negative.

Since we are dealing with a problem concerning a quadratic form, and the question of characterizing it can be seen independently of the variational problem, the set introduced in this paper can certainly be extended to more general optimal control problems such as those treated in [16] and [24]. We have chosen the case of isoperimetric constraints not only because of its importance in itself, but to be able to concentrate mainly on the unique aspects of the approach initiated in [16] and improved in several respects in the theory developed in this paper.

For the example we are dealing with, observe that the function $y(t) = \sin 2t$ ($t \in [0,\pi]$) satisfies the required conditions $y(0) = y(\pi) = 0$ and $\int_0^\pi y(t)\,dt = 0$, but $\int_0^\pi t\{\dot{y}^2(t) - 4y^2(t)\}\,dt = 0$. The condition that the second variation along x_0, evaluated at y, should be nonnegative, is not violated. However, as is easily proved, this function can be used to show that $0 \in \mathcal{R}(x)$ for any trajectory x, and hence one concludes that the problem has no solution. On the other hand, this particular function cannot be used in trying to prove that the point $s = 0$ belongs to $\mathcal{G}(x_0)$ or $\mathcal{Z}(x_0)$.

The paper is organized as follows. In Section 2 we state the isoperimetric problem we deal with, and give a summary of well-known necessary conditions for optimality. In Section 3, by transforming the original problem into one involving a system of differential equations, we derive the corresponding sets of "generalized conjugate points" $\mathcal{G}(x)$ and "generalized coupled points" $\mathcal{Z}(x)$ introduced in [16] and [24], respectively. These sets are originally defined in the underlying open time interval. For completness we prove that, even if we include the (corresponding) endpoint in the definition, we have $\mathcal{G}(x) \subset \mathcal{Z}(x)$ and, if x is a normal trajectory, then $x \in H \Rightarrow \mathcal{Z}(x) = \emptyset$. Section 4 is devoted to two simple examples which illustrate serious difficulties in trying to prove nonemptiness of these sets. These examples motivate the need for introducing a new set of points for which these difficulties do not occur. In Section 5 we introduce such a set $\mathcal{R}(x)$, prove that its emptiness is equivalent to the necessary condition $x \in H$, and show, by an application of this result, that the problems of Section 4 have no solution.

2. Isoperimetric Problems

This paper concerns a characterization of the nonnegativity, along a set W, of a quadratic form given by

$$\int_{t_0}^{t_1} \{\langle y(t), F_{xx}(\tilde{x}(t))y(t)\rangle + 2\langle y(t), F_{x\dot{x}}(\tilde{x}(t))\dot{y}(t)\rangle + \langle \dot{y}(t), F_{\dot{x}\dot{x}}(\tilde{x}(t))\dot{y}(t)\rangle\}\,dt,$$

where $(\tilde{x}(t))$ is short for $(t, x(t), \dot{x}(t))$,

$$F(t,x,\dot{x}) = L(t,x,\dot{x}) + \sum_1^m \lambda_i L_i(t,x,\dot{x}),$$

and W is the set of all piecewise C^1 functions y: $[t_0,t_1] \to \mathbf{R}^n$ satisfying $y(t_0) = y(t_1) = 0$ and

$$\int_{t_0}^{t_1} \{\langle L_{ix}(\tilde{x}(t)), y(t)\rangle + \langle L_{i\dot{x}}(\tilde{x}(t)), \dot{y}(t)\rangle\}\,dt = 0 \qquad (i = 1,\ldots,m).$$

This specific quadratic form corresponds to the second variation of a calculus of variations problem involving isoperimetric side constraints. We state the problem together with well-known necessary conditions for a normal solution. A full account of these results can be found in [11] and [20].

Suppose we are given an interval $T := [t_0,t_1]$ in $\mathbf{R}$, two points ξ_0, ξ_1 in $\mathbf{R}^n$, an open set A in $T \times \mathbf{R}^n \times \mathbf{R}^n$, constants $\alpha_1,\ldots,\alpha_m$ in $\mathbf{R}$, and functions $L, L_1,\ldots,L_m$ mapping

$T \times \mathbf{R}^n \times \mathbf{R}^n$ to $\mathbf{R}$. Denote by X the vector space of all piecewise C^1 functions mapping T to $\mathbf{R}^n$, set

$$X(A) := \{x \in X \mid (t, x(t), \dot{x}(t)) \in A \ (t \in T)\},$$
$$X_e(A) := \{x \in X(A) \mid x(t_0) = \xi_0,\ x(t_1) = \xi_1\},$$

consider the functionals $I, I_1, \ldots, I_m\colon X \to \mathbf{R}$ given by

$$I(x) := \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))\, dt \qquad (x \in X),$$
$$I_i(x) := \alpha_i + \int_{t_0}^{t_1} L_i(t, x(t), \dot{x}(t))\, dt \qquad (x \in X,\ i = 1, \ldots, m),$$

and let

$$Z_e(A) := \{x \in X_e(A) \mid I_i(x) = 0 \ (i = 1, \ldots, m)\}.$$

The problem we deal with, which we label (P), is that of minimizing I over $Z_e(A)$.

Elements of X are called *arcs* or *trajectories* and they are *admissible* if they belong to $Z_e(A)$. An admissible trajectory x is said to *solve* (P) if $I(x) \leq I(y)$ for all $y \in Z_e(A)$. For any $x \in X$ we use the notation $(\tilde{x}(t))$ to represent $(t, x(t), \dot{x}(t))$ $(t \in T)$, and we assume throughout that $L, L_i \in C^2(A)$ $(i = 1, \ldots, m)$.

Definition 2.1. If F is any function mapping $T \times \mathbf{R}^n \times \mathbf{R}^n$ to $\mathbf{R}$ and

$$J(x) = \int_{t_0}^{t_1} F(t, x(t), \dot{x}(t))\, dt \qquad (x \in X),$$

then, for all $x \in X$, we define (whenever the derivatives involved exist) the *first variation of J at x* by

$$J'(x; y) := \int_{t_0}^{t_1} \{\langle F_x(\tilde{x}(t)), y(t)\rangle + \langle F_{\dot{x}}(\tilde{x}(t)), \dot{y}(t)\rangle\}\, dt \qquad (y \in X),$$

and the *second variation of J at x* by

$$J''(x; y) := \int_{t_0}^{t_1} 2\Omega(t, y(t), \dot{y}(t))\, dt \qquad (y \in X)$$

where, for all $(t, y, \dot{y}) \in T \times \mathbf{R}^n \times \mathbf{R}^n$,

$$2\Omega(t, y, \dot{y}) := \langle y, F_{xx}(\tilde{x}(t))y\rangle + 2\langle y, F_{x\dot{x}}(\tilde{x}(t))\dot{y}\rangle + \langle \dot{y}, F_{\dot{x}\dot{x}}(\tilde{x}(t))\dot{y}\rangle.$$

Denote by Y the space of trajectories $y \in X$ satisfying $y(t_0) = y(t_1) = 0$.

Definition 2.2. An admissible trajectory x is said to be *normal* to (P) if $\{I_i'(x; \cdot)\}_1^m$ is linearly independent on Y.

Note 2.3. Following the definition given in [11], if $J(x) = \int_{t_0}^{t_1} F(t, x(t), \dot{x}(t))\, dt$ $(x \in X)$, a trajectory x_0 is called an *extremaloid* for J if $J'(x_0; y) = 0$ for all $y \in Y$. Clearly, x_0 is a normal arc to (P) if it is not an extremaloid for an integral of the form $J(x) = \sum \lambda_i I_i(x)$ where $\lambda_1, \dots, \lambda_m$ are constants, not all zero. It is also a simple fact (see [11]) to show that x_0 is normal to (P) $\Leftrightarrow$ there exist $y_1, \dots, y_m \in Y$ such that $|I_i'(x_0; y_j)| \neq 0$ $(i, j = 1, \dots, m)$.

Let $C(x) := \{y \in X \mid I_i'(x; y) = 0\ (i = 1, \dots, m)\}$, and for all $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbf{R}^m$ define

$$F(t, x, \dot{x}; \lambda) := L(t, x, \dot{x}) + \sum_1^m \lambda_i L_i(t, x, \dot{x}) \qquad ((t, x, \dot{x}) \in T \times \mathbf{R}^n \times \mathbf{R}^n),$$

$$J_\lambda(x) := I(x) + \sum_1^m \lambda_i I_i(x) = \sum_1^m \lambda_i \alpha_i + \int_{t_0}^{t_1} F(t, x(t), \dot{x}(t); \lambda)\, dt \qquad (x \in X),$$

and consider the following sets:

$$E_\lambda := \{x \in X \mid J_\lambda'(x; y) = 0 \text{ for all } y \in Y\},$$
$$H_\lambda := \{x \in X \mid J_\lambda''(x; y) \geq 0 \text{ for all } y \in Y \cap C(x)\},$$
$$L_\lambda := \{x \in X \mid F_{\dot{x}\dot{x}}(\tilde{x}(t); \lambda) \geq 0 \text{ for all } t \in T\},$$
$$W_\lambda(A) := \{x \in X(A) \mid \mathcal{E}_\lambda(t, x(t), \dot{x}(t), u) \geq 0 \text{ for all } (t, u) \in T \times \mathbf{R}^n \text{ with } (t, x(t), u) \in A\},$$

where

$$\mathcal{E}_\lambda(t, x, \dot{x}, u) = F(t, x, u; \lambda) - F(t, x, \dot{x}; \lambda) - \langle u - \dot{x}, F_{\dot{x}}(t, x, \dot{x}; \lambda)\rangle.$$

The following theorem corresponds to a set of necessary conditions for a normal trajectory solving (P).

Theorem 2.4. *If x solves* (P) *and is normal to* (P), *then there exists a unique $\lambda \in \mathbf{R}^m$ such that $x \in E_\lambda$. Moreover, $x \in H_\lambda \cap L_\lambda \cap W_\lambda(A)$.*

The sets E_λ and H_λ depend explicitly on the first and second variations of J_λ, respectively. The first one is usually characterized as follows:

Proposition 2.5. *Let $x \in X(A)$ and $\lambda \in \mathbf{R}^m$. Then $x \in E_\lambda \Leftrightarrow$ there exists $c \in \mathbf{R}^n$ such that*

$$F_{\dot{x}}(\tilde{x}(t); \lambda) = \int_{t_0}^{t} F_x(\tilde{x}(s); \lambda)\, ds + c \qquad (t \in T).$$

The theory of "conjugate points" in the calculus of variations, leading to Jacobi's necessary and sufficient conditions, characterizes the conditions that are expressed in terms of the second variation. This theory depends upon the hypothesis that the trajectory under consideration is *nonsingular* which, for the problem we are dealing with, corresponds to the assumption that $|F_{\dot{x}\dot{x}}(\tilde{x}(t);\lambda)| \neq 0$ for all $t \in T$. As explained in the introduction, for isoperimetric problems such as the one we are considering, one can find in the literature two sets of points for which the nonnegativity of the second variation implies their emptiness independently of the nonsingularity of the trajectory. We devote the next section to a study of these two sets.

3. Generalized Conjugate and Coupled Points

The problem we are dealing with can be transformed into one involving a system of differential equations for which the notions of "generalized conjugate points" and "generalized coupled points," introduced in [16] and [24], respectively, can be applied.

To do so, let $\alpha := (\alpha_1, \ldots, \alpha_m)$ and, for any $(t, x, z, u) \in T \times \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^n$, define

$$\tilde{L}(t, x, z, u) := L(t, x, u), \qquad f(t, x, z, u) := (u, L_1(t, x, u), \ldots, L_m(t, x, u)).$$

Denote by $\tilde{X}$ the space of all piecewise C^1 functions (x, z) mapping T to $\mathbf{R}^n \times \mathbf{R}^m$, and by U the space of all piecewise continuous functions u mapping T to $\mathbf{R}^n$. As one readily verifies, our original problem (P) is equivalent to the problem, which we label ($\tilde{\mathrm{P}}$), of minimizing

$$\tilde{I}(x, z, u) := \int_{t_0}^{t_1} \tilde{L}(t, x(t), z(t), u(t))\, dt$$

over all $(x, z, u) \in \tilde{X} \times U$ satisfying

$$\begin{cases} (\dot{x}(t), \dot{z}(t)) = f(t, x(t), z(t), u(t)) \qquad (t \in T), \\ (t, x(t), u(t)) \in A, \\ (x(t_0), z(t_0), x(t_1), z(t_1)) = (\xi_0, 0, \xi_1, -\alpha). \end{cases}$$

An element of $\tilde{X} \times U$ is called a *process* and it is *admissible* if it satisfies the above three conditions. For this class of problems define, for all $(t, x, z, u, p, q) \in T \times \mathbf{R}^{n+m} \times \mathbf{R}^n \times \mathbf{R}^{n+m}$,

$$\mathcal{H}(t, x, z, u, p, q) := \langle (p, q), f(t, x, z, u) \rangle - \tilde{L}(t, x, z, u).$$

According to the definition given in [16], an admissible process (x, z, u) is called *extremal* if there exists $(p, q) \in \tilde{X}$ such that

$$(\dot{p}(t), \dot{q}(t)) = (-\mathcal{H}_x, -\mathcal{H}_z) \quad \text{and} \quad 0 = \mathcal{H}_u, \tag{1}$$

where the partial derivatives of $\mathcal{H}$ are evaluated at $(t, x(t), z(t), u(t), p(t), q(t))$.

Now, for any $s \in [t_0, t_1)$, set $T_s := [s, t_1]$, let $\tilde{X}_s$ be the space of piecewise C^1 functions mapping T_s to $\mathbf{R}^n \times \mathbf{R}^m$, let U_s be the space of piecewise continuous functions mapping T_s to $\mathbf{R}^n$, and let $\tilde{Y}_s(x, z, u)$ be the set of functions $(y, w, v) \in \tilde{X}_s \times U_s$ such that $(y(s), w(s)) = (y(t_1), w(t_1)) = (0, 0)$ and

$$\begin{pmatrix} \dot{y}(t) \\ \dot{w}(t) \end{pmatrix} = A(t) \begin{pmatrix} y(t) \\ w(t) \end{pmatrix} + B(t)v(t) \qquad (t \in T_s),$$

where $A(t) = (f_x, f_z)$, $B(t) = f_u$, and the partial derivatives are evaluated at $(t, x(t), z(t), u(t))$. Denote by $\tilde{Y}(x, z, u)$ the set $\tilde{Y}_{t_0}(x, z, u)$.

We are now in a position to define, for the problem $(\tilde{\mathrm{P}})$, the set of "generalized conjugate points" introduced in [16].

Definition 3.1. Let $(x, z, u) \in \tilde{X} \times U$ be an extremal and let $(p, q) \in \tilde{X}$ satisfy (1). A point $s \in [t_0, t_1)$ is called a *generalized conjugate point* (to t_1 with respect to (x, z, u)) if there exist $(y, w, v) \in \tilde{Y}_s(x, z, u)$ and $(q_1, q_2) \in \tilde{X}_s$ such that if

$$\mu(t) := B^*(t)\begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} + (\mathcal{H}_{ux}, \mathcal{H}_{uz})\begin{pmatrix} y(t) \\ w(t) \end{pmatrix} + \mathcal{H}_{uu}v(t) \qquad (t \in T_s),$$

then:

(i) $\begin{pmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{pmatrix} + A^*(t)\begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} = -\begin{pmatrix} \mathcal{H}_{xx} & \mathcal{H}_{xz} \\ \mathcal{H}_{zx} & \mathcal{H}_{zz} \end{pmatrix}\begin{pmatrix} y(t) \\ w(t) \end{pmatrix} - \begin{pmatrix} \mathcal{H}_{xu} \\ \mathcal{H}_{zu} \end{pmatrix} v(t)$ $(t \in T_s)$.

(ii) $(q_1(s), q_2(s)) \neq (0, 0)$.

(iii) $\langle v(t), \mu(t)\rangle \geq 0$ $(t \in T_s)$ and either (a) or (b) holds:
  - (a) $\langle v(t), \mu(t)\rangle > 0$ on a set of positive measure.
  - (b) There exists $(y_1, w_1, v_1) \in \tilde{Y}(x, z, u)$ such that
    - (i) $\langle y_1(s), q_1(s)\rangle + \langle w_1(s), q_2(s)\rangle > 0$,
    - (ii) $\langle v_1(t), \mu(t)\rangle \geq 0$ $(t \in T_s)$.

For the specific functions we are dealing with, this definition can be simplified as follows. Observe first that the function $\mathcal{H}$ corresponds to

$$\mathcal{H}(t, x, z, u, p, q) = \langle p, u\rangle + \sum_1^m q_i L_i(t, x, u) - L(t, x, u).$$

Hence (x, z, u) in $\tilde{X} \times U$ is an extremal if and only if it is admissible and there exists $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbf{R}^m$ such that

$$p(t) = L_{\dot{x}}(\tilde{x}(t)) + \sum_1^m \lambda_i L_{i\dot{x}}(\tilde{x}(t))$$

$$\Rightarrow \quad \dot{p}(t) = L_x(\tilde{x}(t)) + \sum_1^m \lambda_i L_{ix}(\tilde{x}(t)) \qquad (t \in T).$$

Observe that $A(t)$ and $B(t)$ are $(n+m) \times (n+m)$ and $(n+m) \times n$ matrices, respectively, given by

$$A(t) = \begin{pmatrix} 0_{n\times n} & 0_{n\times m} \\ L_{1x}(\tilde{x}(t)) & 0_{1\times m} \\ \vdots & \vdots \\ L_{mx}(\tilde{x}(t)) & 0_{1\times m} \end{pmatrix}, \qquad B(t) = \begin{pmatrix} I_{n\times n} \\ L_{1u}(\tilde{x}(t)) \\ \vdots \\ L_{mu}(\tilde{x}(t)) \end{pmatrix}$$

and so the differential equation appearing in the definition of $\tilde{Y}_s(x, z, u)$ is equivalent to

$$\dot{y}(t) = v(t) \quad \text{and} \quad \dot{w}_i(t) = L_{ix}(\tilde{x}(t))y(t) + L_{iu}(\tilde{x}(t))v(t) \qquad (i = 1, \ldots, m).$$

For any $s \in [t_0, t_1)$ let X_s be the space of piecewise C^1 functions mapping T_s to $\mathbf{R}^n$, let Y_s be the set of functions $y \in X_s$ for which $y(s) = y(t_1) = 0$ and, for all $x \in X_s$, let

$$C_s(x) := \left\{ y \in X_s \;\middle|\; \int_s^{t_1} \{\langle L_{ix}(\tilde{x}(t)), y(t)\rangle + \langle L_{i\dot{x}}(\tilde{x}(t)), \dot{y}(t)\rangle\}\, dt = 0 \quad (i = 1, \ldots, m) \right\}.$$

It follows that if (x, z, u) is an admissible process and $(y, w, v) \in \tilde{Y}_s(x, z, u)$, then $y \in Y_s \cap C_s(x)$.

In view of these remarks, we can redefine the set introduced in [16] as follows:

Definition 3.2. For all $x \in X$ and $\lambda \in \mathbf{R}^m$ let $\mathcal{G}_\lambda(x)$ be the set of points $s \in [t_0, t_1)$ for which there exist $y \in Y_s \cap C_s(x)$, $q \in X_s$, and $k = (k_1, \ldots, k_m) \in \mathbf{R}^m$ such that if

$$\mu(t) := q(t) + \sum_1^m k_i L_{i\dot{x}}(\tilde{x}(t)) - F_{\dot{x}x}(\tilde{x}(t); \lambda) y(t) - F_{\dot{x}\dot{x}}(\tilde{x}(t); \lambda)\dot{y}(t) \qquad (t \in T_s),$$

then:

(i) $\dot{q}(t) + \sum_1^m k_i L_{ix}(\tilde{x}(t)) = F_{xx}(\tilde{x}(t); \lambda) y(t) + F_{x\dot{x}}(\tilde{x}(t); \lambda)\dot{y}(t)$ $(t \in T_s)$.
(ii) $(q(s), k) \neq (0, 0)$.
(iii) $\langle \dot{y}(t), \mu(t)\rangle \geq 0$ $(t \in T_s)$
and either (a) or (b) holds:
 (a) $\langle \dot{y}(t), \mu(t)\rangle > 0$ on a set of positive measure.
 (b) There exists $u \in Y \cap C(x)$ such that if
 $$\rho_i(t) := \int_{t_0}^t \{\langle L_{ix}(\tilde{x}(\tau)), u(\tau)\rangle + \langle L_{i\dot{x}}(\tilde{x}(\tau)), \dot{u}(\tau)\rangle\}\, d\tau \qquad (t \in T),$$
 then
  (i) $\langle u(s), q(s)\rangle + \sum_1^m k_i \rho_i(s) > 0$,
  (ii) $\langle \dot{u}(t), \mu(t)\rangle \geq 0$ $(t \in T_s)$.

The main result in [16] relating this set to the condition that the second variation is nonnegative along admissible variations states that, for any $x \in X$ and $\lambda \in \mathbf{R}^m$, $x \in H_\lambda \Rightarrow \mathcal{G}_\lambda(x) \cap (t_0, t_1) = \emptyset$ [16, Theorem 4.3]. Combining this result with Theorem 2.4 we obtain that if x is a normal solution of P(A), then there exists $\lambda \in \mathbf{R}^m$ such that $\mathcal{G}_\lambda(x) \cap (t_0, t_1) = \emptyset$ [16, Theorem 4.6].

We turn now to the set of points defined in [24]. Consider the system

$$\begin{pmatrix} \dot{y}(t) \\ \dot{w}(t) \end{pmatrix} = A(t) \begin{pmatrix} y(t) \\ w(t) \end{pmatrix} + B(t)v(t) \quad (t \in T), \qquad \begin{pmatrix} y(t_0) \\ w(t_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for all $(y, w, v) \in \tilde{X} \times U$. Let Φ: $T \to \mathbf{R}^{n+m} \times \mathbf{R}^{n+m}$ satisfy

$$\dot{\Phi}(t) = -\Phi(t)A(t) \quad (t \in T), \qquad \Phi(t_1) = I$$

so that any solution (y, w, v) of the above system can be expressed as

$$\begin{pmatrix} y(t) \\ w(t) \end{pmatrix} = \Phi^{-1}(t) \int_{t_0}^t \Phi(\tau)B(\tau)v(\tau)\, d\tau \qquad (t \in T).$$

The definition of "generalized coupled points" given in [24], applied to problem $(\tilde{\mathrm{P}})$, is the following:

Definition 3.3. Let $(x, z, u) \in \tilde{X} \times U$ be an extremal and let $(p, q) \in \tilde{X}$ satisfy (1). A point $s \in [t_0, t_1)$ is called a *generalized coupled point* (to t_1 with respect to (x, z, u)) if there exist $(y, w, v) \in \tilde{Y}_s(x, z, u)$ and $(q_1, q_2) \in \tilde{X}_s$ such that if

$$\mu(t) := B^*(t)\begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} + (\mathcal{H}_{ux}, \mathcal{H}_{uz})\begin{pmatrix} y(t) \\ w(t) \end{pmatrix} + \mathcal{H}_{uu}v(t) \qquad (t \in T_s),$$

then:

(i) $$\begin{pmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{pmatrix} + A^*(t)\begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} = -\begin{pmatrix} \mathcal{H}_{xx} & \mathcal{H}_{xz} \\ \mathcal{H}_{zx} & \mathcal{H}_{zz} \end{pmatrix}\begin{pmatrix} y(t) \\ w(t) \end{pmatrix} - \begin{pmatrix} \mathcal{H}_{xu} \\ \mathcal{H}_{zu} \end{pmatrix} v(t) \qquad (t \in T_s).$$

(ii) $\langle v(t), \mu(t)\rangle \geq 0$ $(t \in T_s)$.

(iii) If the inequality in (ii) is equality for all $t \in T_s$, then, for any $\alpha \in \mathbf{R}^n \times \mathbf{R}^m$ satisfying

$$\langle r(t), \mu(t) - B^*(t)\Phi^*(t)\alpha\rangle \leq 0 \qquad \text{for all} \quad t \in T_s$$
$$\text{and} \quad r\colon T_s \to \mathbf{R}^n \quad \text{piecewise continuous,}$$

there exists $v_1\colon [t_0, s] \to \mathbf{R}^n$ piecewise continuous with

$$\left\langle \begin{pmatrix} y_1(s) \\ w_1(s) \end{pmatrix}, \Phi^*(s)\alpha - \begin{pmatrix} q_1(s) \\ q_2(s) \end{pmatrix} \right\rangle < 0,$$

where (y_1, w_1) is the solution of

$$\begin{pmatrix} \dot{y}_1(t) \\ \dot{w}_1(t) \end{pmatrix} = A(t)\begin{pmatrix} y_1(t) \\ w_1(t) \end{pmatrix} + B(t)v_1(t) \quad (t \in [t_0, s]),$$
$$\begin{pmatrix} y_1(t_0) \\ w_1(t_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One can easily verify that the matrix $\Phi(t)$, for the specific functions delimiting the problem, is given by

$$\Phi(t) = \begin{pmatrix} I_{n\times n} & 0_{n\times m} \\ \int_t^{t_1} L_{1x}(\tilde{x}(\tau))\,d\tau & \\ \vdots & I_{m\times m} \\ \int_t^{t_1} L_{mx}(\tilde{x}(\tau))\,d\tau & \end{pmatrix} = I_{(n+m)\times(n+m)} + \int_t^{t_1} A(\tau)\,d\tau.$$

In view of this fact and the remarks following Definition 3.1, we can redefine the set of points defined in [24] as follows:

Definition 3.4. For all $x \in X$ and $\lambda \in \mathbf{R}^m$ let $\mathcal{Z}_\lambda(x)$ be the set of points $s \in [t_0, t_1)$ for which there exist $y \in Y_s \cap C_s(x)$, $q \in X_s$ and $k = (k_1, \ldots, k_m) \in \mathbf{R}^m$ such that if

$$\mu(t) := q(t) + \sum_1^m k_i L_{i\dot{x}}(\tilde{x}(t)) - F_{\dot{x}x}(\tilde{x}(t); \lambda) y(t) - F_{\dot{x}\dot{x}}(\tilde{x}(t); \lambda)\dot{y}(t) \quad (t \in T_s),$$

then:

(i) $\dot{q}(t) + \sum_1^m k_i L_{ix}(\tilde{x}(t)) = F_{xx}(\tilde{x}(t); \lambda) y(t) + F_{x\dot{x}}(\tilde{x}(t); \lambda)\dot{y}(t)$ $(t \in T_s)$.
(ii) $\langle \dot{y}(t), \mu(t)\rangle \geq 0$ $(t \in T_s)$.
(iii) If the inequality in (ii) is equality for all $t \in T_s$, then, for any $(\alpha, \beta) \in \mathbf{R}^n \times \mathbf{R}^m$ satisfying

$$\mu(t) = \alpha + \left(L_{1\dot{x}}(\tilde{x}(t)) + \int_t^{t_1} L_{1x}(\tilde{x}(\tau))\, d\tau, \ldots, L_{m\dot{x}}(\tilde{x}(t)) + \int_t^{t_1} L_{mx}(\tilde{x}(\tau))\, d\tau\right)\beta \qquad (t \in T_s),$$

there exists $u\colon [t_0, s] \to \mathbf{R}^n$ piecewise C^1 with $u(t_0) = 0$ such that if

$$\rho_i(t) := \int_{t_0}^t \{\langle L_{ix}(\tilde{x}(\tau)), u(\tau)\rangle + \langle L_{i\dot{x}}(\tilde{x}(\tau)), \dot{u}(\tau)\rangle\}\, d\tau \qquad (t \in [t_0, s]),$$

then

$$\left\langle u(s), \alpha + \left(\int_s^{t_1} L_{1x}(\tilde{x}(t))\, dt, \ldots, \int_s^{t_1} L_{mx}(\tilde{x}(t))\, dt\right)\beta\right\rangle + \sum_1^m \beta_i \rho_i(s) < \langle u(s), q(s)\rangle + \sum_1^m k_i \rho_i(s).$$

In [24] it is proved (by applying a weak version of Pontryagin's maximum principle) that if x is a normal solution to (P), then there exists $\lambda \in \mathbf{R}^m$ such that $\mathcal{Z}_\lambda(x) \cap (t_0, t_1) = \emptyset$ [24, Theorem 5.1].

Now, it is a simple fact to show that the set of generalized coupled points contains that of generalized conjugate points (see Lemma 5.2 in [24]). Moreover, if x is normal to P(A), then $x \in H_\lambda \Rightarrow \mathcal{Z}_\lambda(x) = \emptyset$. For completness we prove these results when they are reduced to the problem we are considering in this paper, and including the endpoint $t = t_0$.

Note 3.5. For any $(x, \lambda) \in X \times \mathbf{R}^m$, $\mathcal{G}_\lambda(x) \subset \mathcal{Z}_\lambda(x)$.

Proof. Let $s \in \mathcal{G}_\lambda(x)$ and let $y \in Y_s \cap C_s(x)$, $q \in X$, and $k = (k_1, \ldots, k_m) \in \mathbf{R}^m$ be as in Definition 3.2. If (a) holds, then $s \in \mathcal{Z}_\lambda(x)$. If (a) does not hold, then the inequality in (ii) of Definition 3.4 is equality for all $t \in T_s$ and there exists $u \in Y \cap C(x)$ such that if

$$\rho_i(t) := \int_{t_0}^t \{\langle L_{ix}(\tilde{x}(\tau)), u(\tau)\rangle + \langle L_{i\dot{x}}(\tilde{x}(\tau)), \dot{u}(\tau)\rangle\}\, d\tau \qquad (t \in T),$$

then $\langle u(s), q(s)\rangle + \sum_1^m k_i\rho_i(s) > 0$ and $\langle \dot u(t), \mu(t)\rangle \geq 0$ $(t \in T_s)$. Assume, without loss of generality, that there exists $(\alpha, \beta) \in \mathbf{R}^n \times \mathbf{R}^m$ such that, for all $t \in T_s$,

$$\mu(t) = \alpha + \left(L_{1\dot x}(\tilde x(t)) + \int_t^{t_1} L_{1x}(\tilde x(\tau))\, d\tau, \ldots, L_{m\dot x}(\tilde x(t)) \right.$$
$$\left. + \int_t^{t_1} L_{mx}(\tilde x(\tau))\, d\tau\right)\beta.$$

Since $u \in C(x)$, we have

$$\int_{t_0}^{s} \{\langle L_{ix}(\tilde x(t)), u(t)\rangle + \langle L_{i\dot x}(\tilde x(t)), \dot u(t)\rangle\}\, dt$$
$$= -\int_s^{t_1} \{\langle L_{ix}(\tilde x(t)), u(t)\rangle + \langle L_{i\dot x}(\tilde x(t)), \dot u(t)\rangle\}\, dt$$

and therefore, as one readily verifies,

$$\left\langle u(s), \alpha + \left(\int_s^{t_1} L_{1x}(\tilde x(t))\, dt, \ldots, \int_s^{t_1} L_{mx}(\tilde x(t)) dt\right)\beta\right\rangle + \sum_1^m \beta_i\rho_i(s)$$
$$= -\int_s^{t_1} \langle \dot u(t), \mu(t)\rangle\, dt \leq 0 < \langle u(s), q(s)\rangle + \sum_1^m k_i\rho_i(s). \qquad \square$$

Theorem 3.6. *Suppose x is normal to* (P). *Then, for any $\lambda \in \mathbf{R}^m$, $x \in H_\lambda \Rightarrow \mathcal{Z}_\lambda(x) = \emptyset$.*

Proof. Suppose there exists $s \in \mathcal{Z}_\lambda(x)$. Let y, q, k, μ be as in Definition 3.4, and define $z(t) := 0$ for $t \in [t_0, s]$ and $z(t) := y(t)$ for $t \in [s, t_1]$. Then $z \in Y \cap C(x)$ and

$$J_\lambda''(x; z) = \int_{t_0}^{t_1} 2\Omega_\lambda(t, z(t), \dot z(t))\, dt = -\int_s^{t_1} \langle \dot y(t), \mu(t)\rangle\, dt \ \leq\ 0,$$

where $2\Omega_\lambda$ is the integrand of the second variation of J_λ at x. If condition (ii) of Definition 3.4 holds strictly on a set of positive measure, then $J_\lambda''(x; z) < 0$, contradicting that $x \in H_\lambda$. Therefore, the inequality in (ii) of Definition 3.4 is equality for all $t \in T_s$ and so $J_\lambda''(x; z) = 0$. In this event, z is a (normal) minimum of the isoperimetric problem of minimizing $J_\lambda''(x; \cdot)$ over $Y \cap C(x)$. Hence (applying Proposition 2.5 to this so-called accesory problem) there exists a unique $\nu = (\nu_1, \ldots, \nu_m) \in \mathbf{R}^m$ such that if

$$p(t) = F_{\dot x x}(\tilde x(t); \lambda)z(t) + F_{\dot x\dot x}(\tilde x(t); \lambda)\dot z(t) + \sum_1^m \nu_i L_{i\dot x}(\tilde x(t)) \qquad (t \in T),$$

then

$$\dot p(t) = F_{xx}(\tilde x(t); \lambda)z(t) + F_{x\dot x}(\tilde x(t); \lambda)\dot z(t) + \sum_1^m \nu_i L_{ix}(\tilde x(t)) \qquad (t \in T).$$

Let $\alpha := q(t_1) - p(t_1)$ and $\beta := k + \nu$. By (i) of Definition 3.4 we have

$$\mu(t) = q(t) - p(t) + \sum_1^m \beta_i L_{i\dot{x}}(\tilde{x}(t)) = \alpha + \sum_1^m \int_t^{t_1} \beta_i L_{ix}(\tilde{x}(\tau))\, d\tau$$
$$+ \sum_1^m \beta_i L_{i\dot{x}}(\tilde{x}(t)) \qquad (t \in T_s).$$

Observe that, up to this point, s can take any value in the half-open interval $[t_0, t_1)$. Now, by condition (iii) of Definition 3.4, there exists $u\colon\ [t_0, s] \to \mathbf{R}^n$ piecewise C^1 with $u(t_0) = 0$ such that if

$$\rho_i(t) := \int_{t_0}^t \{\langle L_{ix}(\tilde{x}(\tau)), u(\tau)\rangle + \langle L_{i\dot{x}}(\tilde{x}(\tau)), \dot{u}(\tau)\rangle\}\, d\tau \qquad (t \in [t_0, s]),$$

then

$$0 < \left\langle u(s), q(s) - \alpha - \sum_1^m \beta_i \int_s^{t_1} L_{ix}(\tilde{x}(t)); dt \right\rangle - \sum_1^m \nu_i \rho_i(s).$$

This implies, in particular, that the assumption $s = t_0$ yields the desired contradiction. For the case $s > t_0$, observe that the right-hand side of the last expression is equal to

$$\langle u(s), p(s)\rangle - \int_{t_0}^s \{\langle \dot{p}(t), u(t)\rangle + \langle p(t), \dot{u}(t)\rangle\}\, dt = 0$$

and so, in all cases, we reach a contradiction. □

4. Examples

As mentioned in the Introduction, there are examples where one can exhibit a function y for which the second variation along a normal extremal $x \in E_\lambda$ is negative, showing that $x \notin H_\lambda$, but y does not satisfy the conditions defining membership of $\mathcal{G}_\lambda(x)$ or $\mathcal{Z}_\lambda(x)$. The first example we consider in this section illustrates this fact.

Example 4.1. Minimize $I(x) = \int_0^7 t\{\dot{x}^2(t) - x^2(t)\}\, dt$ subject to $x(0) = x(7) = 0$ and $\int_0^7 x(t)\, dt = 0$.

In this case $n = m = 1$, $T = [0, 7]$, $\xi_0 = \xi_1 = \alpha_1 = 0$, $A = T \times \mathbf{R}^2$, $L(t, x, \dot{x}) = t(\dot{x}^2 - x^2)$, and $L_1(t, x, \dot{x}) = x$.

We want to see if $x_0 \equiv 0$ is a solution of the problem. First observe that, since $I_1(x) = \int_0^7 x(t)\ dt$, we have $I_1'(x; y) = \int_0^7 y(t)\ dt$, and $x \in Z_e(A)$ is normal if there exists $y \in X$ with $y(0) = y(7) = 0$ such that $|\int_0^7 y(t)\ dt| \neq 0$. Thus, any trajectory $x \in Z_e(A)$ is normal. Also, $x \in L_\lambda \cap W_\lambda(A)$ for any $x \in X$ and $\lambda \in \mathbf{R}$, since $F_{\dot{x}\dot{x}}(\tilde{x}(t); \lambda) = 2t \geq 0$ $(t \in T)$ and the "excess" function $\mathcal{E}_\lambda$ is given by $t(u - \dot{x})^2$. Now, for all $\lambda \in \mathbf{R}$,

$$J_\lambda(x) = \int_0^7 F(\tilde{x}(t); \lambda)\, dt = \int_0^7 \{L(\tilde{x}(t)) + \lambda x(t)\}\, dt$$

and so

$$J'_\lambda(x; y) = \int_0^7 \{2t(\dot{x}(t)\dot{y}(t) - x(t)y(t)) + \lambda y(t)\}\, dt.$$

Thus x_0 belongs to E_0. One necessary condition remains to be verified, namely, that x_0 belongs to H_0. By definition, this set corresponds to those $x \in X$ for which

$$\int_0^7 t\{\dot{y}^2(t) - y^2(t)\}\, dt \geq 0$$

for all $y \in X$ satisfying $y(0) = y(7) = 0$ and $\int_0^7 y(t)\, dt = 0$. Let $a := \frac{7}{4}$ and define

$$y(t) := \begin{cases} t & \text{if} \quad t \in [0, a], \\ 2a - t & \text{if} \quad t \in [a, 3a], \\ t - 4a & \text{if} \quad t \in [3a, 4a]. \end{cases}$$

By construction, $y(0) = y(7) = 0$ and $\int_0^7 y(t)\, dt = 0$. Moreover, as one readily verifies,

$$\begin{aligned} &\int_0^7 t\{\dot{y}^2(t) - y^2(t)\}\, dt \\ &\quad = \int_0^a t\{1 - t^2\}\, dt + \int_a^{3a} t\{1 - (2a - t)^2\}\, dt + \int_{3a}^{4a} t\{1 - (t - 4a)^2\}\, dt \\ &\quad = a^2\left(8 - 3a^2 + \frac{a^2}{3}\right) = -\frac{49}{96} < 0 \end{aligned}$$

and therefore $x_0 \notin H_0$. Thus x_0 is not a solution of the problem. Also, since H_λ is independent of λ, the same proof shows that the problem has no solution at all.

We turn now to the sets $\mathcal{G}_\lambda(x)$ and $\mathcal{Z}_\lambda(x)$. By definition, s belongs to $\mathcal{G}_\lambda(x)$ if $s \in [0, 7)$ and there exist $y \in Y_s$ with $\int_s^7 y(t)\, dt = 0$, $q \in X_s$, and $k \in \mathbf{R}$ such that if $\mu(t) := q(t) - 2t\dot{y}(t)$ $(t \in T_s)$, then:

(i) $\dot{q}(t) + k = -2ty(t)$ $(t \in T_s)$.

(ii) $(q(s), k) \neq (0, 0)$.

(iii) $\dot{y}(t)\mu(t) \geq 0$ $(t \in T_s)$
and either (a) or (b) holds:

- (a) $\dot{y}(t)\mu(t) > 0$ on a set of positive measure.
- (b) There exists $u \in Y$ with $\int_0^7 u(t)\, dt = 0$ such that if $\rho(t) := \int_0^t u(\tau)\, d\tau$ $(t \in T)$, then
  - (i) $u(s)q(s) + k\rho(s) > 0$,
  - (ii) $\dot{u}(t)\mu(t) \geq 0$ $(t \in T_s)$.

On the other hand, $\mathcal{Z}_\lambda(x)$ is given by those points $s \in [0, 7)$ for which there exist $y \in Y_s$ with $\int_s^7 y(t)\, dt = 0$, $q \in X_s$, and $k \in \mathbf{R}$ such that if $\mu(t) := q(t) - 2t\dot{y}(t)$ $(t \in T_s)$, then:

(i) $\dot{q}(t) + k = -2ty(t)$ $(t \in T_s)$.
(ii) $\dot{y}(t)\mu(t) \geq 0$ $(t \in T_s)$.
(iii) If the inequality in (ii) is equality for all $t \in T_s$ then, for any $(\alpha, \beta) \in \mathbf{R}^2$ satisfying $\mu(t) = \alpha + \beta(7 - t)$ $(t \in T_s)$, there exists $u\colon [0, s] \to \mathbf{R}^n$ piecewise C^1 with $u(0) = 0$ such that if $\rho(t) := \int_0^t u(\tau)\, d\tau$ $(t \in [0, s])$, then

$$u(s)(\alpha + \beta(7 - s)) + \beta\rho(s) < u(s)q(s) + k\rho(s).$$

It follows that if a point $s \in [0, 7)$ belongs to $\mathcal{G}_\lambda(x)$ or $\mathcal{Z}_\lambda(x)$, then, necessarily, there exist $c, k \in \mathbf{R}$ and $y \in Y_s$ with $y \not\equiv 0$ and $\int_s^7 y(t)\, dt = 0$, such that

$$\dot{y}(t)\left(c - kt - 2\int_s^t \tau y(\tau)\, d\tau - 2t\dot{y}(t)\right) \geq 0 \qquad \text{for all} \quad t \in [s, 7].$$

Consider now the function y defined above satisfying $y(0) = y(7) = 0$ and $\int_0^7 y(t)\, dt = 0$, and for which the second variation along any trajectory is strictly negative. Observe that, in view of the above inequality, we require the constants $c, k \in \mathbf{R}$ to satisfy

$$c - kt - 2\int_0^t \tau^2\, d\tau \geq 2t \qquad \text{for all} \quad t \in [0, \tfrac{7}{4}]$$

and

$$c - kt - 2\int_0^{7/4} \tau^2\, d\tau - \int_{7/4}^t \tau(7 - 2\tau)\, d\tau \leq -2t \qquad \text{for all} \quad t \in [\tfrac{7}{4}, \tfrac{7}{2}].$$

However, there are no constants $c, k \in \mathbf{R}$ for which both relations hold, and so y fails to satisfy the conditions defining membership of these two sets.

This fact can be easily generalized to any function y whose derivative does not vanish and it changes sign in an interval (for such functions the condition $y \in C_s(x)$ is not even required). These functions, just like the one defined above, are natural to be considered in trying to prove nonemptiness of $\mathcal{G}_\lambda(x)$ or $\mathcal{Z}_\lambda(x)$ since the conditions $y \in Y_s$ and $y \not\equiv 0$ imply that $\dot{y}$ changes sign along the interval $[s, 7]$. Suppose then that, as in the previous case, for some $b, \varepsilon > 0$ with $[b-\varepsilon, b+\varepsilon]$ contained in $(s, 7)$, $\dot{y}(t) > 0$ for $t \in [b - \varepsilon, b]$ and $\dot{y}(t) < 0$ for $t \in [b, b + \varepsilon]$. In this event the constants $c, k \in \mathbf{R}$ must satisfy

$$c - kt - 2\int_s^t \tau y(\tau)\, d\tau \geq 2t\dot{y}(t) > 0 \qquad \text{for all} \quad t \in [b - \varepsilon, b]$$

and

$$c - kt - 2\int_s^t \tau y(\tau)\, d\tau \le 2t\dot{y}(t) < 0 \qquad \text{for all} \quad t \in [b, b+\varepsilon],$$

which is not possible.

In the next example, posed in the Introduction, the verification that a certain trajectory does not belong to H_λ is not as simple as in the previous one. One can exhibit, however, an admissible variation for which the second variation vanishes and this function fails to satisfy the conditions defining membership of $\mathcal{G}_\lambda(x)$ and $\mathcal{Z}_\lambda(x)$.

Example 4.2. Minimize $I(x) = \frac{1}{2}\int_0^\pi t\{\dot{x}^2(t) - 4x^2(t)\}\, dt$ subject to $x(0) = x(\pi) = 0$ and $\int_0^\pi x(t)\, dt = 0$.

In this case $n = m = 1$, $T = [0, \pi]$, $\xi_0 = \xi_1 = \alpha_1 = 0$, $A = T \times \mathbf{R}^2$, $L(t, x, \dot{x}) = t(\dot{x}^2 - 4x^2)/2$, and $L_1(t, x, \dot{x}) = x$.

As in Example 4.1, any trajectory $x \in Z_e(A)$ is normal and $x \in L_\lambda \cap W_\lambda(A)$ for any $x \in X$ and $\lambda \in \mathbf{R}$. Also, $x_0 \equiv 0$ belongs to E_0, and we want to see if it is a solution of the problem.

By definition, H_λ is given by those $x \in X$ for which

$$\int_0^\pi t\{\dot{y}^2(t) - 4y^2(t)\}\, dt \ge 0$$

for all $y \in X$ satisfying $y(0) = y(\pi) = 0$ and $\int_0^\pi y(t)\, dt = 0$. To begin with, a function like the one defined in the previous example (setting $a := \pi/4$, so that it vanishes at 0 and π and satisfies $\int_0^\pi y(t)\, dt = 0$), yields a positive value to the above integral. On the other hand, for the function $y(t) = \sin 2t$ $(t \in [0, \pi])$, the required conditions hold but the above integral vanishes. It is in fact not clear (at this point) how to exhibit a function y satisfying the endpoint conditions and for which the second variation is strictly negative.

Though the function $\sin 2t$ does not yield a negative value to the second variation, let us see if it can be used to prove nonemptiness of the sets defined in [16] and [24]. Similar arguments to the ones given in Example 4.1 show that if a point s belongs to $\mathcal{G}_\lambda(x)$ or $\mathcal{Z}_\lambda(x)$, then, necessarily, there exist $c, k \in \mathbf{R}$ and $y \in Y_s$ with $y \not\equiv 0$ and $\int_s^\pi y(t)\, dt = 0$ such that

$$\dot{y}(t)\left(c - kt - 4\int_s^t \tau y(\tau)\, d\tau - t\dot{y}(t)\right) \ge 0 \qquad \text{for all} \quad t \in [s, \pi].$$

Note that, for the function $y(t) = \sin 2t$ $(t \in [0, \pi])$, we require that

$$\begin{aligned} 2\cos 2t\left(c - kt - 4\int_0^t \tau \sin 2\tau\, d\tau - 2t\cos 2t\right) &= 2\cos 2t(c - kt - \sin 2t) \\ &\ge 0 \qquad \text{for all} \quad t \in [0, \pi]. \end{aligned}$$

However, as before, there are no constants c and k in $\mathbf{R}$ for which this relation holds. Indeed, observe that $t = \pi/2 \Rightarrow 2c \le k\pi$, and $t = \pi \Rightarrow c \ge k\pi$, implying that $k \le 0$.

If $k = 0$, then $c = 0$ which is clearly not possible. Thus $k \neq 0$ and so $c < 0$. However, $t = 0 \Rightarrow c \geq 0$ which is a contradiction.

5. A New Notion of Conjugacy

In this section we introduce a new set of points whose emptiness is equivalent to the nonnegativity of the second variation, and for which the difficulties that appear in trying to apply the theories of [16] and [24] do not occur. Moreover, as mentioned in the Introduction, this set achieves (contrary to the sets defined in [16] and [24]) the main objective of introducing a characterization of this second-order necessary condition, namely, to obtain a simpler way of verifying it.

This new set of points corresponds to a generalization of a set first introduced in [1] for problems without isoperimetric constraints. We refer the reader to [2], [3], [18], and [19] for further properties of that set.

Definition 5.1. For any $x \in X$ and $\lambda \in \mathbf{R}^m$ let $\mathcal{R}_\lambda(x)$ be the set of points $s \in [t_0, t_1)$ for which there exists $y \in Y_s \cap C_s(x)$ such that if

$$\begin{aligned} v(t) &:= F_{\dot{x}x}(\tilde{x}(t); \lambda) y(t) + F_{\dot{x}\dot{x}}(\tilde{x}(t); \lambda) \dot{y}(t), \\ w(t) &:= F_{xx}(\tilde{x}(t); \lambda) y(t) + F_{x\dot{x}}(\tilde{x}(t); \lambda) \dot{y}(t) \qquad (t \in T_s), \end{aligned}$$

then:

(i) $\int_s^{t_1} \{\langle \dot{y}(t), v(t)\rangle + \langle y(t), w(t)\rangle\}\, dt \leq 0$.

(ii) There exists $u \in Y \cap C(x)$ such that $\gamma := \int_s^{t_1} \{\langle \dot{u}(t), v(t)\rangle + \langle u(t), w(t)\rangle\}\, dt \neq 0$.

Theorem 5.2. *For all $x \in X(A)$ and $\lambda \in \mathbf{R}^m$, $x \in H_\lambda \Leftrightarrow \mathcal{R}_\lambda(x) = \emptyset$.*

Proof. ($\Rightarrow$) Suppose there exists $s \in \mathcal{R}_\lambda(x)$, and let y, u be as in Definition 5.1. Since $\gamma \neq 0$, we have $y \not\equiv 0$. Let $z(t) := 0$ for $t \in [t_0, s]$ and $z(t) := y(t)$ for $t \in [s, t_1]$. Note first that

$$J''_\lambda(x; z) = \int_{t_0}^{t_1} 2\Omega_\lambda(t, z(t), \dot{z}(t))\, dt = \int_s^{t_1} \{\langle y(t), w(t)\rangle + \langle \dot{y}(t), v(t)\rangle\}\, dt \leq 0.$$

Set $k := J''_\lambda(x; u)$, $\alpha := -(\gamma + k/2\gamma)$, and $y_\alpha := u + \alpha z$. Then y_α belongs to $Y \cap C(x)$ and

$$\begin{aligned} J''_\lambda(x; y_\alpha) &= \int_{t_0}^{t_1} 2\Omega_\lambda(t, y_\alpha(t), \dot{y}_\alpha(t))\, dt = k + \alpha^2 J''_\lambda(x; z) \\ &\quad + 2\alpha \int_s^{t_1} \{\langle u(t), w(t)\rangle + \langle \dot{u}(t), v(t)\rangle\}\, dt \\ &\leq k + 2\alpha\gamma = -2\gamma^2 < 0. \end{aligned}$$

($\Leftarrow$) Suppose $x \notin H_\lambda$. Let $y \in Y \cap C(x)$ be such that $J''_\lambda(x; y) < 0$ and let $u \equiv y$. Then $t_0 \in \mathcal{R}_\lambda(x)$. ☐

Combining Theorems 2.4 and 5.2 we obtain the following necessary condition:

Theorem 5.3. *If x is a normal solution of* (P), *then there exists $\lambda \in \mathbf{R}^m$ such that $\mathcal{R}_\lambda(x) = \emptyset$.*

Note 5.4. Denote by $\tilde{\mathcal{R}}_\lambda(x)$ the set of points $s \in [t_0, t_1)$ for which there exists $y \in Y_s \cap C_s(x)$ satisfying strictly the inequality in (i) of Definition 5.1. Then $\tilde{\mathcal{R}}_\lambda(x) \subset \mathcal{R}_\lambda(x)$. This follows simply by setting $u(t) := 0$ for $t \in [t_0, s]$ and $u(t) := y(t)$ for $t \in [s, t_1]$. Then $u \in Y \cap C(x)$ and $\gamma < 0$. A similar reasoning shows that $s \in \mathcal{R}_\lambda(x) \Leftrightarrow [t_0, s] \subset \mathcal{R}_\lambda(x)$, so that $\mathcal{R}_\lambda(x) \neq \emptyset$ is always an interval containing the point t_0.

We now return to the sets defined in [16] and [24]. A simple proof provided below shows that, for any $x \in X$ and $\lambda \in \mathbf{R}^m$, $\mathcal{R}_\lambda(x)$ contains all generalized conjugate points with respect to x and λ. With respect to the set of generalized coupled points observe that, by Theorems 3.6 and 5.2, if x is normal to (P), then $\mathcal{Z}_\lambda(x) \neq \emptyset \Rightarrow \mathcal{R}_\lambda(x) \neq \emptyset$. This implies, in particular, that if one detects nonoptimality of x by applying the theories of [16] or [24], proving nonemptiness of any of these sets, one also detects it by means of the set introduced in this paper.

Theorem 5.5. *For any $(x, \lambda) \in X \times \mathbf{R}^m$, $\mathcal{G}_\lambda(x) \subset \mathcal{R}_\lambda(x)$.*

Proof. Let $s \in \mathcal{G}_\lambda(x)$ and let $y \in Y_s \cap C_s(x)$, $q \in X$ and $k = (k_1, \ldots, k_m) \in \mathbf{R}^m$ be as in Definition 3.2. Note that, in terms of these functions, v, w defined in Definition 5.1 satisfy

$$v(t) = q(t) + \sum_1^m k_i L_{i\dot{x}}(\tilde{x}(t)) - \mu(t) \quad \text{and}$$

$$w(t) = \dot{q}(t) + \sum_1^m k_i L_{ix}(\tilde{x}(t)) \qquad (t \in T_s).$$

Condition (i) of Definition 5.1 is a consequence of condition (iii) of Definition 3.2 since

$$\begin{aligned}
&\int_s^{t_1} \{\langle \dot{y}(t), v(t)\rangle + \langle y(t), w(t)\rangle\}\, dt \\
&\quad = \sum_1^m k_i \int_s^{t_1} \{\langle \dot{y}(t), L_{i\dot{x}}(\tilde{x}(t))\rangle + \langle y(t), L_{ix}(\tilde{x}(t))\rangle\}\, dt - \int_s^{t_1} \langle \dot{y}(t), \mu(t)\rangle\, dt \\
&\quad = -\int_s^{t_1} \langle \dot{y}(t), \mu(t)\rangle\, dt \leq 0.
\end{aligned}$$

If (a) holds, then $s \in \tilde{\mathcal{R}}_\lambda(x) \subset \mathcal{R}_\lambda(x)$. If (b) holds, then

$$\begin{aligned}
\gamma &= \int_s^{t_1} \{\langle \dot{u}(t), v(t)\rangle + \langle u(t), w(t)\rangle\}\, dt \\
&= -\langle u(s), q(s)\rangle - \sum_1^m k_i \rho_i(s) - \int_s^{t_1} \langle \dot{u}(t), \mu(t)\rangle\, dt < 0. \qquad \square
\end{aligned}$$

We now consider the examples given in Section 4.

For Example 4.1, the function y for which the second variation at x is negative shows that $0 \in \tilde{\mathcal{R}}_\lambda(x)$, and an application of Theorem 5.3 implies that the problem has no solution.

For Example 4.2, $\mathcal{R}_\lambda(x)$ is given by those points $s \in [0, \pi)$ for which there exists $y \in Y_s$ with $\int_s^\pi y(t)\,dt = 0$ such that:

(i) $\int_s^\pi t\{\dot{y}^2(t) - 4y^2(t)\}\,dt \le 0$.

(ii) There exists $u \in Y \cap C(x)$ such that $\gamma := \int_s^\pi t\{\dot{u}(t)\dot{y}(t) - 4u(t)y(t)\}\,dt \neq 0$.

We claim that $0 \in \mathcal{R}_\lambda(x)$. Let $y(t) := \sin 2t$ $(t \in [0, \pi])$. Since $y(0) = y(\pi) = 0$ and $\int_0^\pi y(t)\,dt = 0$, we have $y \in Y_0 \cap C_0(x) = Y \cap C(x)$, and condition (i) holds since

$$\int_0^\pi t\{\dot{y}^2(t) - 4y^2(t)\}\,dt = 4\int_0^\pi t\cos 4t\,dt = 0.$$

To show that (ii) also is satisfied observe first that if we set

$$v(t) := 2t\cos 2t \;(= t\dot{y}(t)) \quad \text{and} \quad w(t) := -4t\sin 2t \;(= -4ty(t)),$$

then $w(t) - \dot{v}(t) = -2\cos 2t$ and, therefore,

$$\begin{aligned}\gamma &= \int_0^\pi \{\dot{u}(t)v(t) + u(t)w(t)\}\,dt = u(\pi)v(\pi) + \int_0^\pi u(t)\{w(t) - \dot{v}(t)\}\,dt\\ &= u(\pi)v(\pi) - 2\int_0^\pi u(t)\cos 2t\,dt.\end{aligned}$$

Let, for example, $u(t) := \sin 8t$ for $t \in [0, \pi/4]$ and $u(t) := 0$ for $t \in [\pi/4, \pi]$. We have $u(0) = u(\pi) = 0$ and $\int_0^\pi u(t)\,dt = 0$, so that $u \in Y \cap C(x)$. Moreover, as one readily verifies,

$$\gamma = -2\int_0^{\pi/4} \sin 8t\cos 2t\,dt = -\tfrac{8}{15}.$$

Hence (ii) holds and so $0 \in \mathcal{R}_\lambda(x)$ for any $x \in X$ and $\lambda \in \mathbf{R}$. This proves the claim. By Theorem 5.3, (P) has no solution.

Acknowledgments

The author thanks The Weizmann Institute of Science for the facilities provided during this research, as well as Professor Zvi Artstein for all his support.

References

1. Berlanga R, Rosenblueth JF (2002) Jacobi's condition for singular extremals: an extended notion of conjugate points, Applied Mathematics Letters, 15:453–458
2. Berlanga R, Rosenblueth JF (2004) A Sturm-Liouville approach applicable to different notions of conjugacy, Applied Mathematics Letters, 17:467–472
3. Berlanga R, Rosenblueth JF (2004) Extended conjugate points in the calculus of variations, IMA Journal of Mathematical Control & Information, 21:159–173

4. Bernhard P (1983) La théorie de la seconde variation et le problème linéaire quadratique, in Advances in Hamiltonian Systems (Rome, 1981), Birkhäuser, Boston, MA, pp. 109-142
5. Breakwell JV, Ho YC (1965) On the conjugate point condition for the control problem, International Journal of Engineering Science, 2:565–579
6. Caroff N, Frankowska H (1996) Conjugate points and shocks in nonlinear optimal control, Transactions of the American Mathematical Society, 348:3133–3153
7. Dmitruk A (1976) The Euler–Jacobi equation in variational calculus, Matematicheskie Zametki, 20:847–858 (in Russian). English translation (1976), Mathematical Notes, 20:1032–1038
8. Dmitruk A (1984) Jacobi-type conditions for the problem of Bolza with inequalities, Matematicheskie Zametki, 35:813-827 (in Russian). English translation (1984), Mathematical Notes, 35:427-435
9. Hestenes MR (1951) Applications of the theory of quadratic forms in Hilbert space to the calculus of variations, Pacific Journal of Mathematics, 1:525-581
10. Hestenes MR (1961) Quadratic variational theory and linear elliptic partial differential equations, Transactions of the American Mathematical Society, 101:306-350
11. Hestenes MR (1966) Calculus of Variations and Optimal Control Theory, Wiley, New York
12. Hestenes MR (1969) Quadratic variational theory, in Control Theory and the Calculus of Variations (edited by AV Balakrishnan), Academic Press, New York, pp. 1–37
13. Hestenes MR (1975) Quadratic control problems, Journal of Optimization Theory & Applications, 17:1–42
14. Hestenes MR (1976) On quadratic control problems, in Calculus of Variations and Control Theory (edited by DL Russell), Academic Press, New York, pp. 289–304
15. Hestenes MR (1983) Singular quadratic variational problems, Journal of Optimization Theory & Applications, 41:123–137
16. Loewen PD, Zheng H (1994) Generalized conjugate points for optimal control problems, Nonlinear Analysis, Theory, Methods & Applications, 22:771–791
17. Popescu M (2002) Singular normal extremals and conjugate points for Bolza functionals, Journal of Optimization Theory & Applications, 115:267–282
18. Rosenblueth JF (2002) Conjugate intervals for singular trajectories in the calculus of variations, Proceedings of the XXXIV National Congress of the Mexican Mathematical Society, Aportaciones Matemáticas, Serie Comunicaciones, vol 30, pp. 81–98
19. Rosenblueth JF (2003) Conjugate intervals for the linear fixed-endpoint control problem, Journal of Optimization Theory & Applications, 116:393-420
20. Rosenblueth JF (2003) Necessity and sufficiency for isoperimetric problems, Proceedings of the XXXV National Congress of the Mexican Mathematical Society, Aportaciones Matemáticas, Serie Comunicaciones, vol 32, pp. 163–181
21. Stefani G, Zezza P (1996) Optimality conditions for a constrained optimal control problem, SIAM Journal on Control & Optimization, 34:635-659
22. Stefani G, Zezza P (1997) Constrained regular LQ-control problems, SIAM Journal on Control & Optimization, 35:876-900
23. Zeidan V (1994) The Riccati equation for optimal control problems with mixed state-control constraints: necessity and sufficiency, SIAM Journal on Control & Optimization, 32:1297–1321
24. Zeidan V (1996) Admissible directions and generalized coupled points for optimal control problems, Nonlinear Analysis, Theory, Methods & Applications, 26:479–507
25. Zeidan V, Zezza P (1988) The conjugate point condition for smooth control sets, Journal of Mathematical Analysis & Applications, 132:572–589
26. Zeidan V, Zezza P (1988) Necessary conditions for optimal control problems: conjugate points, SIAM Journal on Control & Optimization, 26:592–608
27. Zeidan V, Zezza P (1989) Conjugate points and optimal control: counterexamples, IEEE Transactions on Automatic Control, 34:254–255
28. Zeidan V, Zezza P (1991) Coupled points in optimal control theory, IEEE Transactions on Automatic Control, 36:1276–1281

Accepted 24 *May* 2004. *Online publication* 10 *September* 2004.