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Optimal Control Problem Associated with Jump Processes

Yasushi Ishikawa

Department of Mathematics,
Faculty of Science, Ehime University,
Matsuyama Ehime 7908577, Japan
slishi@math.sci.ehime-u.ac.jp

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Abstract. An optimal portfolio/control problem is considered for a two-dimensional model in finance. A pair consisting of the wealth process and cumulutative consumption process driven by a geometric Lévy process is controlled by adapted processes. The value function appears and turns out to be a viscosity solution to some integro-differential equation, by using the Bellman principle.

Key Words. Stochastic control of jump type, Mathematical finance, Viscosity solution of PDE.

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Introduction

There are several articles which suggest that some so-called heavy tail distributions fit real data in finance well, and the importance of jump-type price processes which are driven by Lévy processes.

For example, Eberlein [12] suggests the importance of generalized hyperbolic (GH) distribution in empirical date. Barndorff-Nielsen [2] has pointed out the importance of heavy tail distributions such as (generalized) normal inverse Gaussian (GIG) distributions, in the form of exponential stochastic models driven by Lévy processes whose marginal distributions are heavy tail. Several models in mathematical finance based on this distribution or on GH distributions have been proposed with a considerable fit. See [13] for example.

In this paper by the Lévy process we mean that it is a process with stationary, independent increments which are continuous in probability. In this paper we do not treat Brownian motion (Wiener process) nor the Brownian motion part in this class. Hence the process admits and moves only by drifts and jumps, and can be regarded as a typical case in the jump-type setting.

We study a stochastic model in mathematical finance based on so-called geometric Lévy processes $S_t = S_0 e^{Z_t}$, which can be regarded as being driven by *canonical* (Marcus type) processes $\hat{Z}_t$ through the expression

$$S_t = S_0 + \int_0^t S_{u-}\, d\hat{Z}_u,$$

where

$$\hat{Z}_t = Z_t + \sum_{u \in [0,t)} (e^{\Delta Z_u} - 1 - \Delta Z_u).$$

We study the wealth process X_t which is based on S_t. We control X_t and the average consumption process Y_t with some decay rate using some parameter process, corresponding to portfolio, consumption rate or disposal of the asset, so that the average utility function (value function) attains its maximum. See Section 1 for more details, see also [11].

This type of problem is called Merton's optimal investment problem [22]. It is also called the consumption/investment problem. As for stochastic models, there are extensive works by Hindy and Huang [17], Pham [29], Benth et al. [4]–[6], and Framstad et al. [15]. Our setting is different from those in that we choose a pair consisting of wealth process and the cumultative consumption process, in that the utility depends only on the temporal consumption (consumption rate), or in that our utility functions can be quite general as long as they satisfy the so-called Gossen's law.

We can solve the Hamilton–Jacobi–Bellman (HJB) equation of integro-differential type associated to the stochastic model. By using stochastic analysis and the Bellman principle (Lemma 1.7), we show the existence and the uniqueness of viscosity solutions when the domain is bounded. By this uniqueness the viscosity solution proves to be the value function.

In some literature the Bellman principle for jump processes has been just stated to hold, or expected to hold, in order to show the existence and the smoothness of the solution to the HJB equation, see [30, Lemma 3.5], [28, Proposition 3.1], [5, (2.9)], [4, (2.10)], and [32, Theorem 3.4]. This principle first appeared in [20, Theorems 3.1.10, 3.1.11], in the diffusion case. However, in the jump case, as far as the author knows, it has not been explicitly shown in published form. To this end we follow an unpublished master thesis by Takanobu [34], which seems to have been partly influenced by Nisio's suggestive lecture note [25] and by Pragarauskas [30], whereas the notion of a viscosity solution was not known then (see [10]). We cite a good guide to this topic by the founders of the theory [9]. This principle will also be used for other stochastic models of jump type.

On the other hand, we could not afford to give the concrete form of the optimal policy. For an example of the explicit construction of the optimal control process associated to X_t, Y_t, refer to [5] and [19]. There is also a possibility of extending our model to the case where the wealth process is driven both by Wiener and Lévy processes (see [4]).

For an analytic approach to the HJB equation (see (1.10) below), refer to [8] and [18]. Roughly speaking, under the assumption that the Lévy measure satisfies $\nu(dz) = \nu(z)\, dz$

such that $|\nu(z)| \le C/|z|^{3-k}$ for some $k \in (0, 2)$ for $|z| \le 1$, the integro-differential equation with the Dirichlet boundary condition, which is similar to the HJB equation, admits a classical solution. The solution is weakly unique, and is more regular as the dumping factor $\alpha > 0$ gets bigger.

The composition of this paper is as follows. In Section 1 we prepare the various notions and state our main results, Theorems 1 (existence) and 2 (uniqueness). These theorems are proved in Sections 2 and 3, respectively, following mainly the arguments in [4] and [5]. As for the uniqueness, while general tools from the comparison theorem [9] may be used alternatively, we have tried to use more concrete ways of constructing functions. No mention has been made with respect to the regularity of v. In Section 4 we prove the Bellman principle (Lemma 1.7), which is necessary in the proof of Theorem 1, and its proof is the main feature of this paper. In Section 5 we provide small materials which are used in the text.

1. Preliminaries

Let $\tilde{N}(dt\,dz) = N(dt\,dz) - \nu(dz)\,dt$ be a compensated Poisson random measure, whose mean measure (Lévy measure) satisfies $\int_{\mathbf{R}\setminus\{0\}} \min(z2, 1)\nu(dz) < +\infty$.

Let Z_t be a Lévy process given by

$$Z_t = bt + \int_0^t \int_{|z|<1} z\tilde{N}(ds\,dz) + \int_0^t \int_{|z|\ge 1} zN(ds\,dz). \tag{1.1}$$

Here we do not admit the Gaussian part, and trajectories are chosen from the right continuous version. We put $S_t = S_0 e^{Z_t}$ with $S_0 > 0$ being a constant. The process (S_t) is called a *geometric Lévy process*.

Then S_t satisfies, by the Itô formula, the SDE

$$\begin{aligned} dS_t &= bS_t\,dt + S_t \int_{|z|<1} (e^z - 1 - z)\nu(dz)\,dt \\ &\quad + S_{t-}\left(\int_{|z|<1} (e^z - 1)\tilde{N}(dt\,dz) + \int_{|z|\ge 1} (e^z - 1)N(dt\,dz)\right). \end{aligned} \tag{1.2}$$

We assume

$$\int_{|z|\ge 1} (e^z - 1)\nu(dz) < \infty. \tag{1.3}$$

Then (1.2) can be rewritten as

$$\begin{aligned} dS_t &= bS_t\,dt + S_t \int_{\mathbf{R}\setminus\{0\}} (e^z - 1 - z1_{\{|z|<1\}})\nu(dz)\,dt \\ &\quad + S_{t-} \int_{\mathbf{R}\setminus\{0\}} (e^z - 1)\tilde{N}(dt\,dz). \end{aligned}$$

We put

$$\hat{b} = b + \int_{\mathbf{R}\setminus\{0\}} (e^z - 1 - z1_{\{|z|<1\}})\nu(dz),$$

which is finite due to (1.3). Then

$$dS_t = \hat{b}S_t\,dt + S_{t-}\int_{\mathbf{R}\setminus\{0\}}(e^z - 1)\tilde{N}(dt\,dz).$$

The probability space on which these processes are defined is denoted by $(\Omega, \mathcal{F}, P)$. Here the filtration is given by $\mathcal{F} = \sigma(\bigcup_{t\geq 0}\mathcal{F}_t)$ and $\mathcal{F}_t = \sigma((X_s, Y_s); s \leq t)$. The expectation with respect to P is denoted by E.

Let $\mathcal{S}$ be

$$\mathcal{S} = \{(x, y); y > 0, y + \beta x > 0\}.$$

The lines $y+\beta x = 0$ and $y = 0$ constitute lower boundaries of $\mathcal{S}$, whose normal vectors are $(\beta, 1)$ and $(0, 1)$, respectively. Here $\beta > 0$ is a weight factor which describes the dumping rate of the average past consumption (e.g., buying durable goods). This means that the bigger $\beta > 0$ corresponds to a preferance to more recent past consumption by the investor.

Based on the driving processes (Z_t), (S_t), we construct the processes $X = X_t^x$, $Y = Y_t^y$ depending on the parameter process (π_t, C_t, L_t) by

$$\begin{aligned} X_t &= x - C_t + \int_0^t (r + (\hat{b} - r)\pi_s)X_s\,ds + L_t \\ &\quad + \int_0^t \pi_{s-}X_{s-}\int_{\mathbf{R}\setminus\{0\}}(e^z - 1)\tilde{N}(ds\,dz), \\ X_0 &= x, \\ Y_t &= ye^{-\beta t} + \beta\int_0^t e^{-\beta(t-s)}dC_s, \qquad Y_0 = y. \end{aligned} \tag{1.4}$$

The background to defining X_t is the self-financing investment policy according to the portfolio π_t:

$$\frac{dX_t}{X_{t-}} = (1 - \pi_t)\frac{dB_t}{B_t} + \pi_t\frac{dS_t}{S_{t-}}, \tag{$*$}$$

where B_t denotes the riskless bond given by $dB_t = rB_t\,dt$. The second equation in (1.4) means $dY_t = -\beta Y_t\,dt + \beta\,dC_t$.

Here (π_t, C_t, L_t) denotes a control which satisfies the following conditions:

- **(A-i)** $C_t = \int_0^t c_s\,ds$, and $t \mapsto c_t$ is a non-decreasing adapted càdlàg process of finite variation such that $0 \leq c_t \leq M_1$ for all $t \geq 0$, for some $M_1 > 0$, and that $c_t > 0$ only for such t that $X_t \geq 0$.
- **(A-ii)** L_t is a non-decreasing adapted càdlàg process such that $L_{0-} = 0$, $L_t \geq 0$ a.s., $E[L_t] < \infty$ for all $t \geq 0$, $\Delta L_t > 0$ only for such t that $X_{t-} \in \mathcal{S}$ and $X_{t-} + \Delta X_t \notin \mathcal{S}$, and $L_t^c > 0$ only for such t that $X_t \leq 0$. Here L_t^c denotes the continuous part of L_t.
- **(A-iii)** π_t is an adapted cádlàg process with values in $[0, 1]$.
- **(A-iv)** π_t, C_t, L_t are processes such that

  $$\text{if} \quad (x, y) \in \mathcal{S}, \quad \text{then} \quad (X_t, Y_t) \in \bar{\mathcal{S}} \quad \text{a.s.} \tag{$**$}$$

  holds for $t \geq 0$.

Those controls (π_t, c_t, L_t) which satisfy (A-i)–(A-iv) are called *admissible*, and the set of admissible controls for (X_t, Y_t) starting from (x, y) will be denoted by $\mathcal{A}_{(x,y)}$ which may often be written simply be $\mathcal{A}$.

An example of such a control is $(\pi_t, c_t, L_t) \equiv (0, 0, 0)$ with $r = 0$. Another (non-trivial) example of reflection type is stated in Theorem 4 of [15]. Hence $\mathcal{A}$ is not empty. See the Remark below.

Intuitively, X_t denotes the wealth process, C_t the cumultative consumption up to time t, and $\pi_t \in [0, 1]$ the fraction of wealth invested in the risky asset (e.g., stock) subject to S_t, whereas the constant $r \geq 0$ is the interest rate of the safe asset (e.g., bond). The control C_t is used only when the investor has non-negative wealth ($X_t \geq 0$). On the other hand, the process L_t is a control to adjust the wealth (X_t may take a negative value), which may correspond to some sporadic additional income (e.g., selling some asset, recieving aid). This control is used only when the investor has a debt ($X_t \leq 0$), and its jump part is used only when (X_t, Y_t) may exit $\mathcal{S}$.

The process Y_t models the average past consumption process, which must be non-negative (otherwise the investor abandons her living). The intuition behind (1.4) is that if a jump brings the process (X_t, Y_t) out of $\bar{\mathcal{S}}$, then an admissible control will bring it back to $\mathcal{S}$ immediately. We consider no transaction costs with respect to those controls.

We remark $(\partial X/\partial C, \partial Y/\partial C) = (-1, \beta)$, $(\Delta_L X, \Delta_L Y) = (1, 0)$. We also remark that $\Delta_N X_t \geq -\pi_{t-} X_{t-} \geq -X_{t-}$ if $X_{t-} \geq 0$, since $e^z - 1 \geq -1$.

In case that the value of (π_t, c_t, L_t) is determined by the value of (X_t, Y_t), this control is called a Markov control.

Viewing $(\pi_., c_., L_.)$ as a fixed parameter, we put $v^{(\pi_., c_., L_.)}$ by

$$v^{(\pi_., c_., L_.)}(t; x, y) = E^{(X_{t\wedge.}^{(\pi_., c_., L_.)}, Y_{t\wedge.}^{(\pi_., c_., L_.)})} \left[\int_0^t e^{-\alpha s} U(c_s)\, ds \right],$$

where $X_t^{(\pi_., c_., L_.)}$, $Y_t^{(\pi_., c_., L_.)}$ are processes X_t, Y_t given $(\pi_., c_., L_.)$. Also we put the value functions

$$v(t; x, y) = \sup_{(\pi, c, L) \in \mathcal{A}} E^{(X_{t\wedge.}^{(\pi_., c_., L_.)}, Y_{t\wedge.}^{(\pi_., c_., L_.)})} \left[\int_0^t e^{-\alpha s} U(c_s)\, ds \right], \tag{1.5}$$

$$v(x, y) = \sup_{(\pi, c, L) \in \mathcal{A}} E^{(X^{(\pi_., c_., L_.)}, Y^{(\pi_., c_., L_.)})} \left[\int_0^\infty e^{-\alpha s} U(c_s)\, ds \right], \tag{1.6}$$

where $\alpha > 0$ is the dumping rate of the utility, the supremum is taken over admissible controls $(\pi_., c_., L_.)$, and the expectation is taken with respect to the law of (X_t, Y_t) due to $N(dt\, dz)$. Our goal is to characterize v as a viscosity solution to the HJB equation stated below.

The function $U(\cdot)$ is a utility function following the so-called Gossen's law, which depends on the consumption rate, and the hasty investor would like to maximize the utility. We assume U is strictly increasing, differentiable, and concave on $[0, \infty)$ such that $U(0) = 0$, $U(\infty) = \infty$, $U'(0) = \infty$, $U'(\infty) = 0$. Since $U(\cdot)$ is differentiable, it is continuous and locally bounded. Since $c \mapsto U(c)$ is concave, it is bounded by $K(1 + c)$ for some $K > 0$.

Examples of such functions are the power function $c \mapsto (1/\gamma)c^\gamma$ $(0 < \gamma < 1)$ and the logarithmic $c \mapsto \log(c + 1)$. For a more realistic meaning of those types of utilities, see [3] and [19].

Remark. We can show the existence of the optimal control to (1.6). That is, there exists $(\pi^*, c^*, L^*) \in \mathcal{A}$ such that

$$v(x, y) = E^{(X^{(\pi^*,c^*,L^*)}, Y^{(\pi^*,c^*,L^*)})}\left[\int_0^\infty e^{-\alpha s} U(c_s^*)\, ds\right] \tag{1.7}$$

holds.

We denote the trajectory associated to this optimal control by (X_t^*, Y_t^*). We give the sketch of the proof for this assertion in Section 5.

Associated to (X_t, Y_t), the generator (integro-differential operator) $A = A^{\pi,c}$ is given by

$$\begin{aligned} Av(x, y) = -\alpha v - \beta y v_y \\ + \Big\{(r + \pi(\hat{b} - r))x v_x + \int (v(x + \pi x(e^z - 1), y) - v(x, y) \\ - \pi x v_x(e^z - 1))\nu(dz)\Big\} \\ + U(c) - c(v_x - \beta v_y), \qquad \pi \in [0, 1], \quad c \in [0, M_1]. \end{aligned} \tag{1.8}$$

Further, we put

$$\begin{aligned} Nv &= v_x \cdot 1_{\{x \le 0\}}, \\ Mv &= (\beta v_y - v_x) \cdot 1_{\{x \ge 0\}}. \end{aligned} \tag{1.9}$$

Here v_x (resp. v_y) denotes the partial derivative with respect to x (resp. y). We note that the operators M, N correspond in (1.4) to the continuous parts of the controls C_t, L_t, respectively (see Section 2).

The HJB equation (integro-variational inequality) is then

$$\begin{aligned} &\max\Big\{Nv, \sup_{\pi,c}\{Av\}, Mv\Big\} = 0 \qquad \text{in } \mathcal{S}, \\ &v = 0 \qquad \text{outside of } \mathcal{S}. \end{aligned} \tag{1.10}$$

Conditions $Nv \le 0$, $Mv \le 0$ in $\mathcal{S}$ may be viewed as a Neumann-type condition. We seek solutions to (1.10) in a weaker sense. For classical studies for this type of operator, see [8] and the reference therein.

We write the function space

$$C_l(\bar{\mathcal{S}}) = \left\{\varphi \in C(\bar{\mathcal{S}});\ \sup_{(x,y)\in\bar{\mathcal{S}}} \left|\frac{\varphi(x, y)}{(1 + |x| + |y|)^l}\right| < \infty\right\} \tag{1.11}$$

for $l \ge 0$. This is a space of functions having the constraint on the asymptotic order at infinity. We remark $C_l(\bar{\mathcal{S}}) \subset C_{l'}(\bar{\mathcal{S}})$ if $l \le l'$.

If it holds for $\tilde{v} \in C^2(\bar{\mathcal{S}}) \cap C_l(\bar{\mathcal{S}})$ for some $l > 0$ that

$$N\tilde{v} \le 0, \qquad M\tilde{v} \le 0, \quad \text{and} \quad \sup_{\pi,c} A\tilde{v} \le 0 \quad \text{in } \mathcal{S},$$

then it is well known that, for the value function v,

$$v \le \tilde{v} \qquad \text{in } \mathcal{S}. \tag{1.12}$$

The proof is given in Section 5.

We put the constant $k(\gamma)$ by

$$k(\gamma) = \max_{\pi} \Bigg[\gamma(r + \pi(\hat{b} - r)) + \int_{\mathbf{R}\setminus\{0\}} ((1 + \pi(e^z - 1))^\gamma - 1 - \gamma\pi(e^z - 1))\nu(dz) \Bigg],$$

which is finite for each $\gamma > 0$ due to (1.3), and $k(\gamma) \to 0$ as $\gamma \to 0$.

We write

$$B^\pi((x, y), v) = \int (v(x + \pi x(e^z - 1), y) - v(x, y) - \pi x v_x(e^z - 1))\nu(dz),$$

and for $\delta > 0$, $p \in \mathbf{R}$,

$$\begin{aligned} B^{\pi,\delta}((x, y), \varphi, p) &= \int_{|z|>\delta} (\varphi(x + \pi x(e^z - 1), y) - \varphi(x, y) \\ &\qquad - \pi x p(e^z - 1))\nu(dz), \\ B^\pi_\delta((x, y), \varphi, p) &= \int_{|z|\le\delta} (\varphi(x + \pi x(e^z - 1), y) - \varphi(x, y) \\ &\qquad - \pi x p(e^z - 1))\nu(dz); \end{aligned}$$

so that

$$B^\pi((x, y), v) = B^{\pi,\delta}((x, y), v, v_x) + B^\pi_\delta((x, y), v, v_x), \qquad \delta > 0.$$

Further, we use the notation $F = F^{\delta,c}$ given by

$$\begin{aligned} &F((x, y), w, s, t; \varphi, p, \psi, q) \\ &\quad = -\alpha w - \beta y t + \max_{0\le\pi\le1} \{(r + \pi(\hat{b} - r))xs \\ &\qquad\qquad + B^{\pi,\delta}((x, y), \varphi, p) + B^\pi_\delta((x, y), \psi, q)\} \\ &\qquad + U(c) - c(s - \beta t) \end{aligned} \tag{1.13}$$

when it is necessary. Here s, t, p, q are scalars. We note that

$$Av(x, y) = F((x, y), v, v_x, v_y; v, v_x, v, v_x).$$

Definition 1.1 (see [4] and [5]). Let $\mathrm{E} \subset \bar{\mathcal{S}}$.

(1) Any $v \in C(\bar{\mathcal{S}})$ is a viscosity subsolution (resp. supersolution) of (1.10) in E iff for all $(x, y) \in \mathrm{E}$ all $\delta > 0$ and all $\varphi \in C^2(\bar{\mathcal{S}}) \cap C_1(\bar{\mathcal{S}})$ such that (x, y) is a global maximizer (resp. minimizer) of $v - \varphi$ relative to E, it holds that

$$\max\left(N\varphi, \sup_c (F(\cdot, v, \varphi_x, \varphi_y; \varphi, \varphi_x, \varphi, \varphi_x)), M\varphi\right)(x, y) \geq 0 \tag{1.14}$$

$$\left(\text{resp. } \max\left(N\varphi, \sup_c (F(\cdot, v, \varphi_x, \varphi_y; \varphi, \varphi_x, \varphi, \varphi_x)), M\varphi\right)(x, y) \leq 0\right).$$

(2) $v \in C(\bar{\mathcal{S}})$ is a constrained viscosity solution of (1.10) iff v is a viscosity subsolution of (1.10) in $\bar{\mathcal{S}}$ and a supersolution of (1.10) in $\mathcal{S}$.

Definition 1.2. Let $\mathrm{E} \subset \bar{\mathcal{S}}$. Any $v \in C(\bar{\mathcal{S}})$ is a strict supersolution in E iff for every $(x, y) \in \mathrm{E}$, $\varphi \in C^2(\bar{\mathcal{S}}) \cap C_1(\bar{\mathcal{S}})$ such that (x, y) is a global minimizer of $v - \varphi$ relative to E, there exists $\nu > 0$ such that

$$\max\left(N\varphi, \sup_c (F(\cdot, v, \varphi_x, \varphi_y; \varphi, \varphi_x, \varphi, \varphi_x)), M\varphi\right)(x, y) \leq -\nu.$$

Showing that the solution to the HJB equation exists in the sense of viscosity solutions was studied, initially, in terms of analytical methods (e.g., elliptic regularization), see [33] and [23]. Here we use a stochastic method. To this end we define the following functions:

We put, for $(\pi_\cdot, c_\cdot, L_\cdot) \in \mathcal{A}$, $t \geq 0$ and $u \in C_b(\mathcal{S})$,

$$\bar{G}_t^{(\pi_\cdot, c_\cdot, L_\cdot)}(x, y) = E^{(X^x, Y^y)}\left[\int_0^t e^{-\alpha s} U(c_s)\, ds\right],$$

$$G_t^{(\pi_\cdot, c_\cdot, L_\cdot)} u(x, y) = E^{(X^x, Y^y)}\left[\int_0^t e^{-\alpha s} U(c_s)\, ds + e^{-\alpha t} u(X_t^x, Y_t^y)\right],$$

and let

$$\bar{G}_t(x, y) \equiv \sup_{(\pi_\cdot, c_\cdot, L_\cdot) \in \mathcal{A}} \bar{G}_t^{(\pi_\cdot, c_\cdot, L_\cdot)}(x, y),$$

$$G_t u(x, y) \equiv \sup_{(\pi_\cdot, c_\cdot, L_\cdot) \in \mathcal{A}} G_t^{(\pi_\cdot, c_\cdot, L_\cdot)} u(x, y)$$

for $u \in C_b(\mathbf{R})$.

Proposition 1.3.

(1) *For each bounded $E \subset \mathcal{S}$,*

$$\lim_{\delta \to 0} \sup_{t, s \in [0, T], |t-s| \leq \delta} \sup_{(x, y) \in E} |v(t; x, y) - v(s; x, y)| = 0. \tag{1.15}$$

(2) *The value function* $v(t; x, y)$ *is continuous in* (x, y) *uniformly to* $t \in [0, T]$.
(3) *The function* $G_t u(x, y)$ *is continuous in* (x, y) *uniformly in* $t \in [0, T]$.

The proof of this proposition is given in Section 5.
We now have our first main result.

Theorem 1. *The value function* $v(x, y)$ *is well defined, and it is a constrained viscosity solution of* (1.10). *Further, it is bounded.*

The proof of this theorem is given in Section 2. For the proof of this theorem we prepare several lemmas.

Lemma 1.4 [15, Lemma 2.4]. *Let* $(x', y') = (x, y) + m(-1, \beta) + l(1, 0)$. *Then, for all* $l \geq 0, m \geq 0$,

$$v(x, y) \geq v(x', y'). \tag{1.16}$$

Lemma 1.5 (Bellman Principle). *For any stopping time* τ *and any* $t \geq 0$,

$$v(x, y) = \sup_{(\pi,c,L)\in\mathcal{A}} E\left[\int_0^{\tau\wedge t} e^{-\alpha s} U(c_s)\, ds + e^{-\alpha(\tau\wedge t)} v(X^x_{\tau\wedge t}, Y^y_{\tau\wedge t})\right],$$
$$(x, y) \in \mathcal{S}, \tag{1.17}$$

where $(\pi_., c_., L_.)$ *is taken over admissible controls.*

This principle is mentioned in a simple form in Theorem 3.19 of [16]. The exact expression is obtained by approxmating τ by a sequence (s_n) such that $s_n \to \tau$. An unpublished article by Takanobu [34] proves it more precisely. In the case where (X_t), (Y_t) are diffusion processes, this result is well known (see, e.g., Chapter XI of [26] and see also p. 134 of [20]). We give a proof of the above in Section 4.

By considering $e^{-\alpha t} \cdot 1_{\{[0,T]\}}(t)$ in place of $e^{-\alpha t}$ and by restricting t to $[0, T]$, we have the following lemma.

Lemma 1.6 (Bellman Principle (2)). *For any stopping time* τ *in* $[0, T]$ *and any* $0 \leq t \leq T$,

$$v(t; x, y) = \sup_{(\pi,c,L)\in\mathcal{A}|_{[0,T]}} E\left[\int_0^{\tau\wedge t} e^{-\alpha u} U(c_u)\, du + 1_{\{\tau\leq t\}} \cdot e^{-\alpha(\tau\wedge t)} v(t-\tau; X^x_\tau, Y^y_\tau)\right],$$
$$(x, y) \in \mathcal{S}. \tag{1.18}$$

The proof of this lemma follows that of Lemma 1.5.
A slight modification of the definition of super- and subsolutions is the following.

Lemma 1.7 [7, Lemma 4.1]. *Let* $v \in C_1(\bar{\mathcal{S}})$ *and* $E \subset \bar{\mathcal{S}}$. *Then* v *is a viscosity subsolution of* (1.10) *in* $E = \bar{\mathcal{S}}$ (*resp. supersolution of* (1.10) *in* $E = \mathcal{S}$) *if and only if, for every* $\varphi \in C^2(\bar{\mathcal{S}})$ *and* $\delta > 0$,

$$\max\left(N\varphi, \sup_c (F(\cdot, v, \varphi_x, \varphi_y; v, \varphi_x, \varphi, \varphi_x)), M\varphi\right)(x, y) \geq 0 \tag{1.19}$$

whenever $(x, y) \in E$ *is a global maximizer of* $v - \varphi$ *relative to* E

$$(\textit{resp.}\quad \max\left(N\varphi, \sup_c(F(\cdot, v, \varphi_x, \varphi_y; v, \varphi_x, \varphi, \varphi_x)), M\varphi\right)(x, y) \le 0$$

whenever $(x, y) \in E$ *is a global minimizer of* $v - \varphi$ *relative to* E).

With respect to the uniqueness of the viscosity solution, we have the following result.

Theorem 2. *For each* $\bar{\gamma} > 0$ *choose* $\alpha > 0$ *so that* $\alpha > k(\bar{\gamma})$. *Assume* $v_0 \in C_{\bar{\gamma}}(\bar{\mathcal{S}})$ *is a subsolution of* (1.10) *in* $\bar{\mathcal{S}}$ *and* $\bar{v} \in C_{\bar{\gamma}}(\bar{\mathcal{S}})$ *is a supersolution of* (1.10) *in* $\mathcal{S}$. *Then*

$$v_0 \le \bar{v} \qquad \text{on } \bar{\mathcal{S}}. \tag{1.20}$$

Consequently, the HJB equation admits at most one constrained viscosity solution in $C_{\bar{\gamma}}(\bar{\mathcal{S}})$.

This implies that the solution must coincide with the value function, since it is bounded and hence belongs to $C_{\bar{\gamma}}(\bar{\mathcal{S}})$ for all $\bar{\gamma} > 0$. The proof of this theorem is given in Section 3.

2. Proof of Theorem 1

We divide the proof into three steps.

(0) *Property of* v. The value function $v(x, y)$ is well defined as a non-negative function on $\bar{\mathcal{S}}$, by the boundedness of $t \mapsto c_t$ and the local boundedness of $U(\cdot)$. The continuity of v follows from Proposition 1.3 and the local boundedness of $U(\cdot)$.

For the boundedness, since $U(\cdot)$ is concave we have

$$U(c) \le K(1 + c)$$

for some large $K > 0$. Hence

$$E\left[\int_0^t e^{-\alpha s} U(c_s)\, ds\right] \le E\left[K \int_0^t e^{-\alpha s}(1 + c_s)\, ds\right] \le K \int_0^t e^{-\alpha s}(1 + M_1)\, ds \le \frac{1}{\alpha}(1 + M_1)K(1 - e^{-\alpha t}).$$

We reach the conclusion by letting $t \to \infty$.

(1) v *is a Subsolution.* Let $\varphi \in C^2(\bar{\mathcal{S}}) \cap C_1(\bar{\mathcal{S}})$ and let (x, y) be the global maximizer of $v - \varphi$ in $\bar{\mathcal{S}}$. We assume $(v - \varphi)(x, y) = 0$.

We would prove

$$0 \le \max\left[Nv, \sup_{\pi, c} Av, Mv\right]. \tag{2.1}$$

Assume on the contrary

$$0 > \max \left[Nv, \sup_{\pi,c} Av, Mv \right] (x, y). \tag{2.2}$$

By the continuity, there exist an open ball $B_r = B_r((x, y))$ with center (x, y) and radius $r > 0$, $\varepsilon > 0$, and $\hat{c}$ such that

$$0 \geq M\varphi, \qquad 0 \geq N\varphi, \tag{2.3}$$

$$U(\hat{c}) - \hat{c}(\varphi_x - \beta\varphi_y) - \alpha v - \beta y \varphi_y \\ + \max_{\pi}\{(r + \pi(\hat{b} - r))xv_x + B^{\pi}((x, y), \varphi, \varphi_x)\} \leq -\varepsilon\alpha, \tag{2.4}$$

on $\overline{B_r \cap \mathcal{S}}$, and that

$$v \leq \varphi - \varepsilon \qquad \text{on} \quad \partial B_r \cap \bar{\mathcal{S}}. \tag{2.5}$$

Let $(X_0, Y_0) = (x, y)$, and put

$$\tau^* = \inf\{t \geq 0; (X_t, Y_t) \notin B_r\}, \qquad \tau_L = \inf\{t \geq 0; \Delta_L X_t \neq 0\}.$$

Further, put $\tau = \min(\tau_L, \tau^*)$.

We first remark $\tau_L > 0$ a.s. Indeed, let (π^*, c^*, L^*) denote an optimal control given in (1.7), and let (X^*, Y^*) denote the corresponding optimal trajectory. Then we have the following lemma, according to Lemma 5.3 of [6].

Lemma 2.1 (see Lemma 5.3 of [6]). *Let A_L denote the event that the optimal trajectory (X_t^*, Y_t^*) starting from (x, y) has an initial jump of size > 0 at $t = 0$ due to the control L. Suppose that for each φ which appears in* (1.14)

$$M\varphi \leq 0, \qquad N\varphi \leq 0, \qquad \sup_{\pi,c} A\varphi \leq -\varepsilon\alpha$$

holds. Then we have $P(A_L) = 0$.

Proof. We denote the position of (X_t^*, Y_t^*) after the initial jump from (x, y) caused by $(\Delta L; \Delta L > 0)$ by

$$(\hat{X}, \hat{Y}) = (\hat{X}(\Delta L), \hat{Y}(\Delta L)) = (x + \Delta L, y).$$

By the Bellman principle, we may assume without loss of generality that $(\hat{X}, \hat{Y}) \in \overline{B_r \cap \mathcal{S}}$ for some $r > 0$.

We have by the Bellman principle

$$v(x, y) = E[v(\hat{X}, \hat{Y})] = \int_{A_L} v(\hat{X}, \hat{Y})\, dP + \int_{\Omega \setminus A_L} v(x, y)\, dP.$$

Hence

$$\int_{A_L} (v(\hat{X}, \hat{Y}) - v(x, y))\, dP = 0.$$

Since $v \leq \varphi$ and $(v - \varphi)(x, y) = 0$,

$$\int_{A_L} (\varphi(\hat{X}, \hat{Y}) - \varphi(x, y))\, dP \geq 0. \tag{2.6}$$

We denote ΔL_t at $t = 0$ by ε_L. By the assumption,

$$\varphi(\hat{X}(\varepsilon_L), \hat{Y}(\varepsilon_L)) \leq \varphi(\hat{X}(\varepsilon), \hat{Y}(\varepsilon)) \tag{2.7}$$

for $0 < \varepsilon \leq \varepsilon_L$.

Suppose first that $\varepsilon_L > 0$. Then

$$\varphi_x(x, y) \cdot P(A_L) \geq 0 \qquad \text{for} \quad \varepsilon \leq \varepsilon_L. \tag{2.8}$$

Indeed, by (2.6) and (2.7)

$$\int_{A_L} (\varphi(\hat{X}(\varepsilon), \hat{Y}(\varepsilon)) - \varphi(x, y))\, dP \geq 0$$

for $\varepsilon \leq \varepsilon_L$. Hence by Fatou's lemma,

$$\int_{A_L} \limsup_{\varepsilon \to 0} \left(\frac{1}{\varepsilon} (\varphi(x + \varepsilon, y) - \varphi(x, y)) \right) dP \geq 0.$$

Hence (2.8) follows.

Hence, in view of the assumption of Lemma 2.1 and (2.8), we have

$$P(A_L) = 0$$

as long as $\varepsilon_L > 0$. □

We now return to the proof of Theorem 1.

(i) On $\{\tau^* < \tau_L\}$, any one of the terms C_t, L^c, or N will make the process (X_t, Y_t) move out of B_r. Let (x', y') be on the intersection of ∂B_r and the line connecting $(X_{\tau^*-}, Y_{\tau^*-})$ to (X_{τ^*}, Y_{τ^*}). The slope vector of this line is $(-1, \beta)$ or $(1, 0)$, and v is decreasing along this line by Lemma 1.4.

Hence we have from the above, for some $\varepsilon > 0$,

$$v(X_{\tau^*}, Y_{\tau^*}) \leq v(x', y') \leq \varphi(x', y') - \varepsilon \leq \varphi(X_{\tau^*}, Y_{\tau^*}) - \varepsilon.$$

Then

$$v(x, y) = \int_0^{\tau^*} e^{-\alpha s} U(c_s)\, ds + e^{-\alpha \tau^*} v(X_{\tau^*}, Y_{\tau^*})$$

$$
\begin{aligned}
&\le \int_0^{\tau^*} e^{-\alpha s} U(c_s)\,ds + e^{-\alpha\tau^*} v(X_{\tau^*}, Y_{\tau^*}) - \varepsilon e^{-\alpha\tau^*} \\
&\quad + e^{-\alpha\tau^*} v(X_{\tau^*-}, Y_{\tau^*-}) - \varepsilon e^{-\alpha\tau^*} \\
&= \int_0^{\tau^*} e^{-\alpha s} U(c_s)\,ds \\
&\quad + \Biggl\{ \varphi(x, y) + \int_0^{\tau^*} (-\alpha) e^{-\alpha s} \varphi(X_s, Y_s)\,ds + \int_0^{\tau^*} e^{-\alpha s} \varphi_x(X_s, Y_s)\,dX_s \\
&\qquad + \int_0^{\tau^*} e^{-\alpha s} \varphi_y(X_s, Y_s)\,dY_s + \int_0^{\tau^*} e^{-\alpha s} \varphi_{xy}(X_s, Y_s)\,d[X, Y]^{\mathrm{c}}_s \\
&\qquad + \frac{1}{2}\int_0^{\tau^*} e^{-\alpha s} \varphi_{xx}(X_s, Y_s)\,d[X, X]^{\mathrm{c}}_s \\
&\qquad + \frac{1}{2}\int_0^{\tau^*} e^{-\alpha s} \varphi_{yy}(X_s, Y_s)\,d[Y, Y]^{\mathrm{c}}_s \\
&\qquad + \sum_{s\in[0,\tau^*)} e^{-\alpha s} \{\varphi(X_s, Y_s) - \varphi(X_{s-}, Y_{s-}) - (\varphi_x(X_{s-}, Y_{s-})\Delta X_s \\
&\qquad\qquad + \varphi_y(X_{s-}, Y_{s-})\Delta Y_s)\} \Biggr\} - \varepsilon e^{-\alpha\tau^*} \\
&= -\varepsilon e^{-\alpha\tau^*} + \varphi(x, y) \\
&\quad + \int_0^{\tau^*} e^{-\alpha s} \{U(c_s) - \alpha e^{-\alpha s} \varphi(X_s, Y_s) + (r + \pi(\hat{b} - r)) X_s \varphi_x \\
&\qquad\qquad - \beta Y_s \varphi_y + B^{\pi}((X_s, Y_s), \varphi, \varphi_x)\}\,ds \\
&\quad + \int_0^{\tau^*} e^{-\alpha s} (-\varphi_x + \beta\varphi_y)(X_s, Y_s) c_s\,ds + \int_0^{\tau^*} e^{-\alpha s} \varphi_x(X_s, Y_s)\,dL^{\mathrm{c}}_t \\
&\quad + \sum_{s\in[0,\tau^*)} e^{-\alpha s} \{\varphi(X_{s-} + \Delta L_s, Y_{s-} - \gamma\Delta L_s) - \varphi(X_{s-}, Y_{s-})\} \\
&\quad + \int_0^{\tau^*}\!\!\int e^{-\alpha s} (\varphi(X_{s-} + \pi_{s-} X_{s-}(e^z - 1), Y_{s-}) \\
&\quad - \varphi(X_{s-}, Y_{s-}))\tilde{N}(ds\,dz).
\end{aligned} \tag{2.9}
$$

Since $-\varphi_x + \beta\varphi_y \le 0$ on $\{x \ge 0\}$, and $\varphi_x \le 0$ on $\{x \le 0\}$, $\varphi(X_{s-} + \Delta L_s, Y_{s-}) - \varphi(X_{s-}, Y_{s-}) \le 0$, and since $-\alpha\varphi \le -\alpha v - \varepsilon\alpha \le -\alpha v$,

$$
\begin{aligned}
\text{R.H.S.} &\le \varphi(x, y) - \varepsilon e^{-\alpha\tau^*} + \int_0^{\tau^*} e^{-\alpha s}(-\varepsilon\alpha)\,ds \\
&\quad + \int_0^{\tau^*}\!\!\int e^{-\alpha s} (\varphi(X_{s-} + \pi_{s-} X_{s-}(e^z - 1), Y_{s-}) \\
&\qquad - \varphi(X_{s-}, Y_{s-}))\tilde{N}(ds\,dz) \\
&\le \varphi(x, y) - \varepsilon + \int_0^{\tau^*}\!\!\int e^{-\alpha s} (\varphi(X_{s-} + \pi_{s-} X_{s-}(e^z - 1), Y_{s-}) \\
&\qquad - \varphi(X_{s-}, Y_{s-}))\tilde{N}(ds\,dz).
\end{aligned}
$$

(ii) On $\{\tau^* \geq \tau_L\}$, $\tau = \tau_L > 0$ a.s. Then

$$\begin{aligned}
&\int_0^{\tau_L} e^{-\alpha s} U(c_s)\, ds + e^{-\alpha \tau_L} v(X_{\tau_L}, Y_{\tau_L}) \\
&\quad \leq \int_0^{\tau_L} e^{-\alpha s} U(c_s)\, ds + e^{-\alpha \tau_L} \varphi(X_{\tau_L}, Y_{\tau_L}) \\
&\quad \leq \varphi(x, y) + \int_0^{\tau_L} ds e^{-\alpha s} \{U(c_s) - \alpha e^{-\alpha s} \varphi(X_s, Y_s) + (r + \pi(\hat{b} - r)) X_s \varphi_x \\
&\qquad\qquad - \beta Y_s \varphi_y + B^\pi((X_s, Y_s), \varphi, \varphi_x)\} \\
&\qquad + \sum_{s \in [0, \tau_L)} e^{-\alpha s} \{\varphi(X_{s-} + \Delta L_s, Y_{s-}) - \varphi(X_{s-}, Y_{s-})\} \\
&\qquad + \int_0^{\tau_L} \int e^{-\alpha s} (\varphi(X_{s-} + \pi_{s-} X_{s-}(e^z - 1), Y_{s-}) - \varphi(X_{s-}, Y_{s-})) \tilde{N}(ds\, dz) \\
&\quad \leq \varphi(x, y) - \varepsilon(1 - e^{-\alpha \tau_L}) \\
&\qquad + \int_0^{\tau_L} \int e^{-\alpha s} (\varphi(X_{s-} + \pi_{s-} X_{s-}(e^z - 1), Y_{s-}) - \varphi(X_{s-}, Y_{s-})) \tilde{N}(ds\, dz).
\end{aligned}$$

From cases (i) and (ii), we have

$$\begin{aligned}
&E\left[\int_0^{\tau} e^{-\alpha s} U(c_s)\, ds + e^{-\alpha \tau} v(X_\tau, Y_\tau)\right] \\
&\quad \leq E\left[1_{\{\tau^* < \tau_L\}} \cdot \left(\int_0^{\tau^*} e^{-\alpha s} U(c_s)\, ds + e^{-\alpha \tau^{**}} v(X_{\tau^{**}}, Y_{\tau^{**}})\right)\right] \\
&\qquad + E\left[1_{\{\tau^* \geq \tau_L\}} \cdot \left(\int_0^{\tau_L} e^{-\alpha s} U(c_s)\, ds + e^{-\alpha \tau_L} v(X_{\tau_L}, Y_{\tau_L})\right)\right] \\
&\quad \leq \varphi(x, y) - \varepsilon E[1 - 1_{\{\tau^* \geq \tau_L\}}.e^{-\alpha \tau_L}] \leq \varphi(x, y) - \varepsilon E[1 - e^{-\alpha \tau_L}]. \qquad (2.10)
\end{aligned}$$

By the *Bellman Principle* (Lemma 1.5),

$$v(x, y) = \sup_{(\pi, c, L) \in \mathcal{A}} E\left[\int_0^{\tau \wedge t} e^{-\alpha s} U(c_s)\, ds + e^{-\alpha(\tau \wedge t)} v(X_{\tau \wedge t}, Y_{\tau \wedge t})\right], \qquad (2.11)$$

where $v(x, y) = \varphi(x, y)$, we have a contradiction by letting $t \to \infty$ in view of Lemma 2.1.

(2) *v is a Supersolution.* Let $\varphi \in C^2(\bar{\mathcal{S}}) \cap C_1(\bar{\mathcal{S}})$, and let $(x, y) \in \mathcal{S}$ be the global minimizer of $v - \varphi$ in $\bar{\mathcal{S}}$. We assume $(v - \varphi)(x, y) = 0$. Then by Lemma 1.4

$$\varphi(x, y) = v(x, y) \geq v(x - m + l, y + \beta m) \geq \varphi(x - m + l, y + \beta m). \qquad (2.12)$$

Hence

$$0 \geq \varphi((x, y) + m(-1, \beta) + l(1, 0)) - \varphi(x, y).$$

Dividing by m (resp. l) and letting $m \to 0$ (resp. $l \to 0$), we get

$$0 \geq -\varphi_x + \beta\varphi_y, \qquad 0 \geq \varphi_x. \tag{2.13}$$

Let τ_r be the exit time from $B_r = B_r((x, y))$. We apply Lemma 1.5 with $\pi_t = \pi$, $c_t = 0$, $\tau = \tau_r \wedge h$. Further, by the assumption $v(x, y) = \varphi(x, y)$, we obtain

$$\begin{aligned} 0 &\geq E\left[\int_0^{\tau\wedge t} e^{-\alpha s} U(c_s)\, ds + e^{-\alpha(\tau\wedge t)}\varphi(X_{\tau\wedge t}, Y_{\tau\wedge t})\right] - \varphi(x, y) \\ &\geq E\left[\int_0^{\tau\wedge t} e^{-\alpha s}\{U(c_s) - \alpha\varphi - \beta Y_s\varphi_y + (r + (\hat{b} - r)\pi)X_s\varphi_x \right. \\ &\qquad\qquad \left. + B^\pi((X_s, Y_s), \varphi, \varphi_x)\}\, ds\right] \\ &\geq E[(1/\alpha)(1 - e^{-\alpha(h\wedge\tau_r)})] \cdot \inf_{(x,y)\in B_r}[U(c) - \alpha\varphi - \beta y\varphi_y + (r + (\hat{b} - r)\pi)x\varphi_x \\ &\qquad\qquad + B^\pi((x, y), \varphi, \varphi_x)]. \end{aligned}$$

By the right continuity of the paths, $\tau_r > 0$ a.s. Hence $\lim_{h\to 0} E[(1/h)(1 - e^{-\alpha(h\wedge\tau_r)})] = \alpha$. Dividing the above inequality by h, and then letting $h \to 0$ and $r \to 0$, we obtain

$$U(c) - \alpha\varphi - \beta y\varphi_y + (r + (\hat{b} - r)\pi)x\varphi_x + B^\pi((x, y), \varphi, \varphi_x) \leq 0 \tag{2.14}$$

for every $\pi \in [0, 1]$. This implies, in view of (2.13), that v is a viscosity supersolution.

This proves the assertions. □

3. Uniqueness of the Viscosity Solution

The proof goes almost parallel to Section 4 of [4], hence we give a sketch of the proof. We prove the uniqueness of the viscosity solution under this assumption. For the sake of notational convenience, we write (only in this section) the coordinate by (x_1, x_2) instead of the previous (x, y). For example, a point in $\mathcal{S}$ will be written $\mathbf{X}$ or $\mathbf{X}_m$, and their coordinates will be written by (x_1, x_2) or (x_{m1}, x_{m2}), respectively. We believe that the reader will not confuse these symbols with the original processes.

First we begin with Theorem 2 in Section 1:

Theorem 3.1. *For each $\bar{\gamma} > 0$ choose $\alpha > 0$ so that $\alpha > k(\bar{\gamma})$. Assume $v_0 \in C_{\bar{\gamma}}(\bar{\mathcal{S}})$ is a subsolution of* (1.10) *in $\bar{\mathcal{S}}$ and $\bar{v} \in C_{\bar{\gamma}}(\bar{\mathcal{S}})$ is a supersolution of* (1.10) *in $\mathcal{S}$. Then*

$$v_0 \leq \bar{v} \qquad \text{on } \bar{\mathcal{S}}. \tag{3.1}$$

Consequently, the HJB equation admits at most one constrained viscosity solution in $C_{\bar{\gamma}}(\bar{\mathcal{S}})$.

Choose $\gamma' > \bar{\gamma}$ such that $\alpha > k(\gamma')$. To show the assertion above we shall prove that

(1) there exists $K > 0$ such that, for any $\theta \in (0, 1]$,

$$\bar{v}^\theta \equiv (1-\theta)\bar{v} + \theta\left(K + \left(1 + |x_1| + \frac{|x_2|}{\beta}\right)^{\gamma'}\right) \in C_{\gamma'}(\bar{\mathcal{S}})$$

is a strict supersolution; and that

(2) $v_0 \le \bar{v}^\theta$ in any $\mathrm{E} \subset \bar{\mathcal{S}}$.

Then (1) and (2) imply (3.1) by letting $\theta \to 0$.

Proof of (1). We put

$$w = K + \left(1 + |x_1| + \frac{|x_2|}{\beta}\right)^{\gamma'} \tag{3.2}$$

for some $K > 0$. Here $K > 0$ is a large constant. We note that $w_{x_1} = \gamma'(1 + |x_1| + |x_2|/\beta)^{\gamma'-1}\,\mathrm{sgn}(x_1)$ and $w_{x_2} = (\gamma'/\beta)(1 + |x_1| + |x_2|/\beta)^{\gamma'-1}\,\mathrm{sgn}(x_2)$.

We first claim for some $f \in C(\bar{\mathcal{S}})$, strictly positive in any $\mathrm{E} \subset \mathcal{S}$, that

$$\max\left(Nw, \sup_c(F(\cdot, w, w_{x_1}, w_{x_2}; w, w_{x_1}, w, w_{x_1})), Mw\right)(x_1, x_2) \le -f(x_1, x_2) \qquad \text{on E.} \tag{3.3}$$

This implies that w is a strict supersolution in any $\mathrm{E} \subset \mathcal{S}$.

Then we claim that $\bar{v}^\theta$ is a strict supersolution of (1.10) in $\mathcal{S}$.

For any $\varphi \in C^2(\bar{\mathcal{S}})$, $(x_1, x_2) \in \mathcal{S}$ is a global minimizer of $\bar{v} - \varphi$ iff (x_1, x_2) is a global minimizer of $\bar{v}^\theta - \varphi^\theta$, where

$$\varphi^\theta = (1-\theta)\varphi + \theta w, \qquad \bar{v}^\theta = (1-\theta)\bar{v} + \theta w.$$

Since $\bar{v}$ is a supersolution of (1.10) in $\mathcal{S}$,

$$M\varphi = \beta\varphi_{x_2} - \varphi_{x_1} \le 0 \qquad \text{for} \quad x_1 \ge 0,$$
$$N\varphi = \varphi_{x_1} \le 0 \qquad \text{for} \quad x_1 \le 0,$$

and hence

$$M\varphi^\theta = (1-\theta)(\beta\varphi_{x_2} - \varphi_{x_1}) + \theta(\beta w_{x_2} - w_{x_1}) = 0 \qquad \text{for} \quad x_1 \ge 0, \tag{3.4}$$
$$N\varphi^\theta = (1-\theta)\varphi_{x_1} + \theta w_{x_1} \le -\theta\gamma'\chi^{\gamma'-1} \qquad \text{for} \quad x_1 \le 0. \tag{3.5}$$

Here we put $\chi(x_1, x_2) = 1 + |x_1| + |x_2|/\beta$.

We denote by $\pi^* \in [0, 1]$ the maximizer of

$$\varphi^\theta_{x_1}\pi x_1(\hat{b} - r) + \int (\varphi^\theta(x_1 + \pi x_1(e^z - 1), x_2) - \varphi^\theta(x_1, x_2) - \varphi^\theta_{x_1}(x_1, x_2)\pi x_1(e^z - 1))\, d\nu(z). \tag{3.6}$$

Then

$$\begin{aligned}
&F((x_1,x_2),\bar v^\theta,\varphi^\theta_{x_1},\varphi^\theta_{x_2};\varphi^\theta,\varphi^\theta_{x_1},\varphi^\theta,\varphi^\theta_{x_1})\\
&\quad=(1-\theta)U(c)-c((1-\theta)(\varphi_{x_1}-\beta\varphi_{x_2}))-\alpha(1-\theta)\bar v-\beta x_2(1-\theta)\varphi_{x_2}\\
&\qquad+(r+(\hat b-r)\pi^*)x_1(1-\theta)\varphi_{x_1}+(1-\theta)B^{\pi^*}((x_1,x_2),\varphi)\\
&\qquad+\theta U(c)-c\theta(w_{x_1}-\beta w_{x_1})-\alpha\theta w-\beta x_2\theta w_{x_1}\\
&\qquad+(r+(\hat b-r)\pi^*)x_1\theta w_{x_1}+\theta B^{\pi^*}((x_1,x_2),w)\\
&\quad\le(1-\theta)F((x_1,x_2),\bar v,\varphi_{x_1},\varphi_{x_2};\varphi,\varphi_{x_1},\varphi,\varphi_{x_1})\\
&\qquad+\theta F((x_1,x_2),w,w_{x_1},w_{x_2};w,w_{x_1},w,w_{x_1})\\
&\quad\le-((1-\theta)\nu+\theta f)\le-\theta f. \qquad (3.7)
\end{aligned}$$

This implies the assertion.

Assertion (3.3) follows almost similarly to (4.11) in Section 4 of [4], where we use the condition for $k(\gamma')$.

Proof of (2). We observe

$$v_0(x_1,x_2)-\bar v^\theta(x_1,x_2)\le C(1+|x_1|+|x_2|)^{\bar\gamma}-\theta\left(1+|x_1|+\frac{|x_2|}{\beta}\right)^{\gamma'}. \qquad (3.8)$$

The right-hand side tends to $-\infty$ as $|(x_1,x_2)|\to\infty$ in $\mathcal{S}$.

Hence we may assume $\mathrm{E}=\{(x_1,x_2);-(1/\beta)x_2<x_1<R,0<x_2<R\}$ for some large $R>0$ without loss of generality. Assume on the contrary that

$$M\equiv\max_{\bar{\mathrm{E}}}(v_0-\bar v^\theta)>0. \qquad (3.9)$$

Since $v_0,\bar v^\theta$ are continuous, there exists some $\mathbf{Z}\in\bar{\mathrm{E}}$ so that the maximum is attained. Then either $\mathbf{Z}\in\mathrm{E}$ or $\mathbf{Z}\in\partial\mathrm{E}$. First we consider the case $\mathbf{Z}\in\partial\mathrm{E}$.

Since $\partial\mathrm{E}$ is piecewise C^2, there exist $h_0>0,k>0$, a uniformly continuous map $\eta\colon\ \bar{\mathrm{E}}\to\mathbf{R}^2$ such that

$$B_{hk}(\mathbf{X}+h\eta(\mathbf{X}))\subset\mathrm{E}\qquad\text{for all}\quad\mathbf{X}\in\bar{\mathrm{E}},\quad h\in(0,h_0] \qquad (3.10)$$

(see p. 1111 of [32]). We write it by $\eta(\mathbf{X})=(\eta(\mathbf{X})_1,\eta(\mathbf{X})_2)$.

For $m>1$ and $\varepsilon\in(0,1)$ we define the function $\Phi=\Phi^{(m,\varepsilon)}:\bar{\mathrm{E}}\times\bar{\mathrm{E}}\to\mathbf{R}$ given by

$$\Phi(\mathbf{X},\mathbf{Y})=v_0(\mathbf{X})-\bar v^\theta(\mathbf{Y})-|m(\mathbf{X}-\mathbf{Y})+\varepsilon\eta(\mathbf{Z})|^2-\varepsilon|\mathbf{X}-\mathbf{Z}|^2.$$

Let $M_m\equiv\max_{\bar{\mathrm{E}}\times\bar{\mathrm{E}}}\Phi(\mathbf{X},\mathbf{Y})$, and let $(\mathbf{X}_m,\mathbf{Y}_m)$ be the maximizer of $\Phi^{(m,\varepsilon)}(\cdot,\cdot)$. We observe $\mathbf{Y}_m\in\mathrm{E}$ and $\mathbf{X}_m\in\bar{\mathrm{E}}$. We put

$$\begin{aligned}
\varphi(\mathbf{X})&=\bar v^\theta(\mathbf{Y}_m)+|m(\mathbf{X}-\mathbf{Y}_m)+\varepsilon\eta(\mathbf{Z})|^2+\varepsilon|\mathbf{X}_m-\mathbf{Z}|^2,\\
\psi(\mathbf{Y})&=v_0(\mathbf{X}_m)+|m(\mathbf{X}_m-\mathbf{Y})+\varepsilon\eta(\mathbf{Z})|^2+\varepsilon|\mathbf{X}_m-\mathbf{Z}|^2.
\end{aligned}$$

Since $\bar v^\theta$ is a strict supersolution in $\mathcal{S}$, due to the maximum principle (Lemma 1.7), for every $\psi\in C^2(\bar{\mathcal{S}})$, every $\delta>0$, and every admissible control $c_.$ we have, for some $\nu>0$,

$$F(\mathbf{Y}_m,\bar v^\theta,\psi_{y_1},\psi_{y_2};\bar v^\theta,\psi_{y_1},\psi,\psi_{y_1})<-\nu. \qquad (3.11)$$

Similarly, since v_0 is a subsolution in $\bar{\mathcal{S}}$ we have, due to Lemma 1.7, for every $\varphi \in C^2(\bar{\mathcal{S}})$, every $\delta > 0$, and every admissible control $c_.$,

$$F(\mathbf{X}_m, v_0, \varphi_{x_1}, \varphi_{x_2}; v_0, \varphi_{x_1}, \varphi, \varphi_{x_1}) \geq 0. \tag{3.12}$$

We subtract (3.11) from (3.12):

$$\begin{aligned} 0 < {} & F(\mathbf{X}_m, v_0, \varphi_{x_1}, \varphi_{x_2}; v_0, \varphi_{x_1}, \varphi, \varphi_{x_1}) - F(\mathbf{Y}_m, \bar{v}^\theta, \psi_{y_1}, \psi_{y_2}; \bar{v}^\theta, \psi_{y_1}, \psi, \psi_{y_1}) \\ \leq {} & [(U(c) - c(v_{0,x_1} - \beta v_{0,x_2})(\mathbf{X}_m)) - (U(c) - c(\bar{v}^\theta_{x_1} - \beta \bar{v}^\theta_{0,x_2})(\mathbf{Y}_m))] \\ & - \alpha[v_0(\mathbf{X}_m) - \bar{v}^\theta(\mathbf{Y}_m)] - \beta[x_{m2}\varphi_{x_2}(\mathbf{X}_m) - y_{m2}\psi_{y_2}(\mathbf{Y}_m)] \\ & + \max_{\pi_. \in [0,1]} [(r + (\hat{b} - r)\pi)[x_{m1}\varphi_{x_1}(\mathbf{X}_m) - y_{m1}\psi_{y_1}(\mathbf{Y}_m)] \\ & \qquad + I_1 + I_2 + [B^\pi_\delta(\mathbf{X}_m, \varphi) - B^\pi_\delta(\mathbf{Y}_m, \psi)]]. \end{aligned} \tag{3.13}$$

Here we put

$$\begin{aligned} I_1 = \int_{\{\delta < |z| < 1\}} & \{[v_0(T^\pi(z, \mathbf{X}_m)) - \bar{v}^\theta(T^\pi(z, \mathbf{Y}_m))] - [v_0(\mathbf{X}_m) - \bar{v}^\theta(\mathbf{Y}_m)] \\ & - \pi[x_{m1}\varphi_{x_1}(\mathbf{X}_m) - y_{m1}\psi_{y_1}(\mathbf{Y}_m)](e^z - 1)\}\nu(dz), \end{aligned} \tag{3.14}$$

$$\begin{aligned} I_2 = \int_{\{|z| \geq 1\}} & \{[v_0(T^\pi(z, \mathbf{X}_m)) - \bar{v}^\theta(T^\pi(z, \mathbf{Y}_m))] - [v_0(\mathbf{X}_\alpha) - \bar{v}^\theta(\mathbf{Y}_m)] \\ & - \pi[x_{m1}\varphi_{x_1}(\mathbf{X}_m) - y_{m1}\psi_{y_1}(\mathbf{Y}_m)](e^z - 1)\}\nu(dz), \end{aligned} \tag{3.15}$$

where we put

$$T^\pi(z, \mathbf{X}) = (x_1 + \pi x_1(e^z - 1), x_2).$$

Note first that

$$B^\pi_\delta(\mathbf{X}_m, \varphi) \to 0 \quad \text{and} \quad B^\pi_\delta(\mathbf{Y}_m, \psi) \to 0$$

as $\delta \to 0$, since φ and ψ are C^2 on $\bar{\mathrm{E}}$.

Next, we have by the (Lipschitz) continuity of v_0, $\bar{v}^\theta$ and by direct calculation (see (4.19) of [4]) that

$$\mathbf{Y}_m - \mathbf{X}_m = (\varepsilon/m)\eta(\mathbf{Z}) + o(1/m) \qquad \text{as} \quad m \to +\infty,$$

and hence

$$\begin{aligned} [x_{mi}\varphi_{x_i}(\mathbf{X}_m) - y_{mi}\psi_{y_i}(\mathbf{Y}_m)] &= 2m(x_{mi} - y_{mi})[m(x_{mi} - y_{mi}) + \varepsilon\eta(\mathbf{Z})_i] \\ &= o(1) \qquad \text{as} \quad m \to +\infty, \quad i = 1, 2. \end{aligned}$$

We obtain, since $M = \max_{\bar{\mathcal{S}}}(v_0 - \bar{v}^\theta) = (v_0 - \bar{v}^\theta)(\mathbf{Z})$ for some $\mathbf{Z} \in \partial\mathrm{E}$,

$$\begin{aligned} I_2 \leq \int_{\{|z| \geq 1\}} & [\{(M + \bar{v}^\theta(T^\pi(z, \mathbf{X}_m)) - \bar{v}^\theta(T^\pi(z, \mathbf{Y}_m))) - M_m\} \\ & - [x_{m1}\varphi_{x_1}(\mathbf{X}_m) - y_{m1}\psi_{y_1}(\mathbf{Y}_m)](e^z - 1)]\nu(dz) \end{aligned}$$

$$\le (M - M_m)\nu(\{|z| \ge 1\}) + \int_{\{|z|\ge 1\}} (\bar{v}^\theta(T^\pi(z, \mathbf{X}_m)) - \bar{v}^\theta(T^\pi(z, \mathbf{Y}_m)))\nu(dz) + \left(\int_{\{|z|\ge 1\}} |e^z - 1|\nu(dz)\right) \cdot o(1). \tag{3.16}$$

The third term tends to 0 as $m \to \infty$, the second also tends to 0 since $\bar{v}^\theta$ is continuous and since $\mathbf{X}_m \to \mathbf{Z}$, $\mathbf{Y}_m \to \mathbf{Z}$, and the first one tends to 0 since $M_m \to M$ as $m \to \infty$.

Next we give the estimate of I_1. Since $T^\pi(z, \mathbf{X}_m), T^\pi(z, \mathbf{Y}_m) \in \mathrm{E}$ for $z \in (-1, 1)$, we have by an easy calculation that

$$\begin{aligned}
&\Phi(T^\pi(z, \mathbf{X}_m), T^\pi(z, \mathbf{Y}_m)) - \Phi(\mathbf{X}_m, \mathbf{Y}_m) \\
&\quad = [v_0(T^\pi(z, \mathbf{X}_m)) - \bar{v}^\theta(T^\pi(z, \mathbf{Y}_m)) - (v_0(\mathbf{X}_m) - \bar{v}^\theta(\mathbf{Y}_m))] \\
&\quad - \varepsilon(\pi x_{m1}(e^z - 1)(\pi x_{m1}(e^z - 1) - 2z_1)) \\
&\quad - \{m^2(x_{m1} - y_{m1})^2\pi^2(e^z - 1)^2 \cdot (x_{m1} - y_{m1})(\pi(e^z - 1))\}.
\end{aligned}$$

Hence the *integrand of* I_1 is equal to

$$\begin{aligned}
&\Phi(T^\pi(z, \mathbf{X}_m), T^\pi(z, \mathbf{Y}_m)) - \Phi(\mathbf{X}_m, \mathbf{Y}_m) + \varepsilon(\pi x_{m1}(e^z - 1)(\pi x_{m1}(e^z - 1) - 2z_1)) \\
&\quad + m^2(x_{m1} - y_{m1})^2(e^z - 1)(\pi^2(e^z - 1) - 2\pi),
\end{aligned}$$

and

$$I_1 \le (\varepsilon(x_{m1}^2 + c_1) + m^2(x_{m1} - y_{m1})^2) \int_{\{\delta<|z|<1\}} (e^z - 1)^2 \nu(dz). \tag{3.17}$$

Since $\alpha(x_{m1} - y_{m1}) \to -\varepsilon\eta(\mathbf{Z})_1$ as $\alpha \to \infty$ and since the integral is convergent, we have

$$\lim_{\varepsilon\to 0} \lim_{m\to\infty} I_1 \le 0.$$

Returning to (3.13), since

$$x_{mi}\varphi_{x_i}(\mathbf{X}_m) - y_{mi}\psi_{y_i}(\mathbf{Y}_m) \to 0$$

as $m \to \infty$, $i = 1, 2$, we have, by letting $m \to \infty$,

$$0 < c_\cdot((\bar{v}^\theta_{x_1} - \beta\bar{v}^\theta_{x_2})(\mathbf{Z}) - (v_{0,x_1} - \beta v_{0,x_2})(\mathbf{Z})) - m[v_0(\mathbf{Z}) - \bar{v}^\theta(\mathbf{Z})].$$

Since $c_\cdot$ is arbitrary in the above, we have

$$-m[v_0(\mathbf{Z}) - \bar{v}^\theta(\mathbf{Z})] > 0, \tag{3.18}$$

which is a contradiction since $m > 0$.

The case for $\mathbf{Z} \in \mathrm{E}$ follows without much difficulity, and we omit the details. See Case II in Section 4 of [4]. □

4. Bellman Principle

In this section we prove the Bellman principle (the dynamic programing principle, Lemma 1.5) for jump processes. As stated above, this principle makes it possible to show the existence of a (sub)solution for (1.8). In previous articles, this principle in the framework of jump processes has been either just assumed (e.g., p. 282 of [4], p. 452 of [5], p. 84 of [6], and p. 9 of [28]) or stated with a very rough sketch of the proof (e.g., the Remark in Section 3 of [28]).

The proof is rather long. The idea is that we seek a representation of $v(t; x, y) = \sup_{\mathbf{a}\in\mathcal{A}} v^{\mathbf{a}}(t; x, y)$ in terms of approximating step controls instead of general adapted control $\mathbf{a} \in \mathcal{A}$. This idea and the proof are due to the Master's thesis of Takanobu [34], which seems to have been partly influenced by Nisio [25] and Pragarauskas [30]. See also Section 3.4 of [16]. Since [34] is not published, we repeat his argument in a somewhat proper form.

Let $\mathbf{A} = [0, 1] \times [0, M_1] \times [0, \infty)$. In this section we denote by $\mathbf{a}_t = (\pi_t, c_t, L_t) : [0, \infty) \times \Omega \to \mathbf{A}$ any control satisfying conditions (A-i)–(A-iii) in Section 1. The set of such controls is denoted by $\mathcal{A}^0$. Note that $\mathcal{A}^0 \supset \mathcal{A}$.

In this section we fix the starting point (x, y) of (X_t, Y_t), and denote $\mathcal{A}_{(x,y)}$ (resp. $\mathcal{A}^0_{(x,y)}$) by $\mathcal{A}$ (resp. $\mathcal{A}^0$).

Let d be a given metric on the parameter space $\mathbf{A}$ such that $d(p, q) \le 1$, $p, q \in \mathbf{A}$. We introduce a metric ρ on $\mathcal{A}^0$ by

$$\rho(\mathbf{a}^1, \mathbf{a}^2) \equiv \sum_{m=1}^{\infty} 2^{-m} \left(1 \wedge E\left[\int_0^m d(\mathbf{a}^1_t, \mathbf{a}^2_t)\, dt\right]\right), \qquad \mathbf{a}^1, \mathbf{a}^2 \in \mathcal{A}^0.$$

Here the expectation is taken with respect to the controls. We define the topology on $\mathcal{A}^0$ in terms of the convergence with respect to ρ, that is,

$$\mathbf{a}^m \to \mathbf{a} \text{ in } \mathcal{A}^0 \quad \Longleftrightarrow \quad \rho(\mathbf{a}^m, \mathbf{a}) \to 0 \ \text{ as } \ m \to \infty. \tag{4.1}$$

In the case where $\mathbf{a} \in \mathcal{A}$ and $\mathbf{a}^m \to \mathbf{a}$ in $\mathcal{A}^0$, we say $\mathbf{a}^m \to \mathbf{a}$ in $\mathcal{A}$.

We sometimes denote, for the sake of simplicity, by $x^{\mathbf{a}}_t = x^{\mathbf{a},(x,y)}_t = (X^{\mathbf{a},x}_t, Y^{\mathbf{a},y}_t)$ the process with the control $\mathbf{a}$ defined in Section 1.

We begin with the following simple assertion.

Proposition 4.1. *For any random variable τ_1, τ_2 such that $0 \le \tau_1 \le \tau_2 \le T$, the mapping $\mathbf{a} = (\pi, c, L) \in \mathcal{A}^0 \to \mathbf{R}$,*

$$\mathbf{a}_{\cdot} \mapsto \int_{\tau_1}^{\tau_2} e^{-\alpha s} U(c_s)\, ds \tag{4.2}$$

is L^2-continuous.

Proof. Let $(\mathbf{a}^m)_{m=1}^{\infty}$, $\mathbf{a}^m = (\pi^m, c^m, L^m)$, be a sequence of the controls in $\mathcal{A}^0$ such that $\mathbf{a}^m \to \mathbf{a}$ in $\mathcal{A}^0$. Since $|U(c^m_s) - U(c_s)| \le 2U'(c_s)|c^m_s - c_s|$ as $m \to \infty$, $E[\int_0^T |U(c^m_s) -$

$U(c_s)|^2\,ds] \to 0$ as $m \to \infty$. Hence

$$E\left[\left|\int_{\tau_1}^{\tau_2} e^{-\alpha s} U(c_s^m)\,ds - \int_{\tau_1}^{\tau_2} e^{-\alpha s} U(c_s)\,ds\right|^2\right]$$
$$\leq E\left[\left(\int_{\tau_1}^{\tau_2} |e^{-\alpha s} U(c_s^m) - e^{-\alpha s} U(c_s)|\,ds\right)^2\right]$$
$$\leq T\,E\left[\int_0^T \{e^{-\alpha s}|U(c_s^m) - U(c_s)|\}^2\,ds\right] \to 0$$

as $m \to \infty$. □

Corollary to Proposition 4.1.

(1) *$v^{\mathbf{a}}(t; x, y)$ is continuous in $(\mathbf{a}, (x, y)) \in \mathcal{A}^0 \times \mathcal{S}$ for each t.*
(2) *$G_t u(x, y)$ is continuous in $(x, y) \in \mathcal{S}$ for each t, $u \in C_b(\mathcal{S})$.*

Proof. The first assertion follows from the expression

$$v^{\mathbf{a}}(t; x, y) = E^{(X^{\mathbf{a},x}, Y^{\mathbf{a},y})}\left[\int_0^t e^{-\alpha s} U(c_s^{(\mathbf{a})})\,ds\right]$$

and Proposition 4.1. Here $c^{(\mathbf{a})}$ is such that $\mathbf{a} = (\pi, c^{(\mathbf{a})}, L)$. The second assertion follows from the definition of G_t. □

Next we consider a series of step controls approximating arbitrarily given control $\mathbf{a} \in \mathcal{A}$.

Let Δ: $0 = t_0 < t_1 < t_2 < \cdots < t_m < \cdots \to \infty$ be any partition of $[0, \infty)$. Let

$$\{\beta(i);\ i = 1, 2, 3, \ldots\} \tag{4.3}$$

be any *given* countable dense subset of $\mathbf{A}$. For $N = 1, 2, 3, \ldots$, we define a set of *step controls* $\mathcal{A}_N(\Delta) \subset \mathcal{A}^0$ by

$$\begin{aligned}\mathcal{A}_N(\Delta) = \{\mathbf{a}_t;\ &\text{there exist } \mathcal{F}_{t_i}\text{-measurable functions } \mathbf{a}_i\colon\ \Omega \to \{\beta(1), \ldots, \beta(N)\}\\ &\text{such that } \mathbf{a}_t|_{t=0} = \mathbf{a}_0 \text{ and } \mathbf{a}_t = \mathbf{a}_i \text{ for}\\ &t_i \leq t < t_{i+1};\ i = 0, 1, 2, \ldots\},\end{aligned} \tag{4.4}$$

and put

$$\mathcal{A}(\Delta) \equiv \bigcup_{N=1}^{\infty} \mathcal{A}_N(\Delta). \tag{4.5}$$

The topology of $\mathcal{A}(\Delta)$ is the weak topology induced by that of each $\mathcal{A}_N(\Delta)$.

Proposition 4.2. *For any series of partitions $(\Delta_m)_{m=1}^{\infty}$ of $[0, \infty)$ such that $\lim_{m\to\infty} |\Delta_m| = 0$, the set $\bigcup_{m=1}^{\infty} \mathcal{A}(\Delta_m)$ is dense in $\mathcal{A}^0$.*

Proof. We divide the proof into four steps.

For $N = 1, 2, 3, \ldots$, we define another set of step controls $\mathcal{A}_N \subset \mathcal{A}^0$ by

$$\mathcal{A}_N = \{\mathbf{a}_t \in \mathcal{A}^0;\ \text{for each } t \in [0,\infty) \text{ and } \omega,\ \mathbf{a}_t(\omega) \text{ takes values in } \{\beta(1), \ldots, \beta(N)\}\}. \tag{4.6}$$

We put

$$\mathcal{A}^* \equiv \bigcup_{N=1}^{\infty} \mathcal{A}_N. \tag{4.7}$$

Step 1. *$\mathcal{A}^*$ is dense in $\mathcal{A}^0$*. That is, for any given $\mathbf{a} \in \mathcal{A}^0$ there exists a series $(\mathbf{a}^m)_{m=1}^{\infty} \subset \mathcal{A}^*$ such that $\mathbf{a}^m \to \mathbf{a}$ in $\mathcal{A}^0$.

Indeed, for each $\varepsilon > 0$ and $\alpha \in \mathbf{A}$, we put

$$i_\varepsilon(\alpha) \equiv \min\{i;\, d(\alpha, \beta(i)) \le \varepsilon\}. \tag{4.8}$$

Then $i_\varepsilon(\alpha)$ is Borel measurable with respect to α, since

$$\{\alpha \in \mathbf{A};\, i_\varepsilon(\alpha) = i\} = \{\alpha \in \mathbf{A};\, d(\alpha, \beta(1)) > \varepsilon\} \cap \cdots \cap \{\alpha \in \mathbf{A};\, d(\alpha, \beta(i-1)) > \varepsilon\}.$$

We next put $k_{\varepsilon,N}(\alpha) \equiv \beta(N \wedge i_\varepsilon(\alpha))$. Then $k_{\varepsilon,N}(\alpha)$ is Borel measurable with respect to α, since

$$\{\alpha \in \mathbf{A};\, k_{\varepsilon,N}(\alpha) = \beta(i)\} = \{\alpha \in \mathbf{A};\, N \wedge i_\varepsilon(\alpha) = i\}. \tag{4.9}$$

For each $\mathbf{a} \in \mathcal{A}^0$ we put $\mathbf{a}^{m,N} = (\mathbf{a}_t^{m,N})$ by $\mathbf{a}_t^{m,N} \equiv k_{1/m,N}(\mathbf{a}_t)$. Then $\mathbf{a}^{m,N} \in \mathcal{A}_N$, and we have, for any $T > 0$,

$$\limsup_{N\to\infty} E\left[\int_0^T d(\mathbf{a}_t^{m,N}, \mathbf{a}_t)\, dt\right] \le \frac{T}{m}. \tag{4.10}$$

In effect, it follows from (4.8), (4.9) that $\mathbf{a}^{m,N} \in \mathcal{A}_N$. On the other hand, since $d(\mathbf{a}_t^{m,N}, \mathbf{a}_t) = d(\beta(N \wedge i_{1/m}(\mathbf{a}_t)), \mathbf{a}_t)$,

$$\lim_{N\to\infty} d(\mathbf{a}_t^{m,N}, \mathbf{a}_t) = d(\beta(i_{1/m}(\mathbf{a}_t)), \mathbf{a}_t) \le 1/m.$$

Since $d(\mathbf{a}^{m,N}, \mathbf{a}_t) \le 1$, we have

$$\limsup_{N\to\infty} E\left[\int_0^T d(\mathbf{a}_t^{m,N}, \mathbf{a}_t)\, dt\right] \le E\left[\int_0^T \lim_{N\to\infty} d(\mathbf{a}_t^{m,N}, \mathbf{a}_t)\, dt\right] \le \frac{T}{m}.$$

Next we show that, for any $\mathbf{a} \in \mathcal{A}^0$, there exists a series $(\mathbf{a}^m)_{m=1}^{\infty} \subset \mathcal{A}^*$ such that $\mathbf{a}^m \to \mathbf{a}$ in $\mathcal{A}^0$.

Indeed, for $m = 1$, $T = 1$ we can find in (4.10) some $N_1 \ge 1$ such that $E[\int_0^1 d(\mathbf{a}_t^{1,N}, \mathbf{a}_t)\, dt] \le 2(1/1)$, for $N \ge N_1$. For $m = 2$, $T = \sqrt{2}$ we can find in (4.10) some $N_2 > N_1$ such that $E[\int_0^{\sqrt{2}} d(\mathbf{a}_t^{2,N}, \mathbf{a}_t)\, dt] \le 2(1/\sqrt{2})$, for $N \ge N_2$. For $m = 3$, $T = \sqrt{3}$ we can find in (4.10) some $N_3 > N_2$ such that $E[\int_0^{\sqrt{3}} d(\mathbf{a}_t^{3,N}, \mathbf{a}_t)\, dt] \le 2(1/\sqrt{3})$, for $N \ge N_3$. We repeat this procedure, to obtain the sequence $\{N_m\}_{m=1}^{\infty}$.

We now have

$$N_{m+1} > N_m,$$

$$E\left[\int_0^{\sqrt{m}} d(\mathbf{a}_t^{m,N}, \mathbf{a}_t)\, dt\right] \le 2\left(\frac{1}{\sqrt{m}}\right) \qquad \text{for} \quad N \ge N_m.$$

Hence $E[\int_0^{\sqrt{m}} d(\mathbf{a}_t^{m,N_m}, \mathbf{a}_t)\, dt] \le 2(1/\sqrt{m})$. We choose $\mathbf{a}^m = (\mathbf{a}_t^m)$, by $\mathbf{a}_t^m \equiv \mathbf{a}_t^{m,N_m}$. Then $\mathbf{a}^m \in \mathcal{A}^*$ and $\mathbf{a}^m \to \mathbf{a}$ in $\mathcal{A}0$. This follows since

$$\limsup_{m\to\infty} E\left[\int_0^T d(\mathbf{a}_t^m, \mathbf{a}_t)\, dt\right] \le \limsup_{m\to\infty} E\left[\int_0^{\sqrt{m}} d(\mathbf{a}_t^m, \mathbf{a}_t)\, dt\right]$$
$$\le \lim_{m\to\infty} 2\left(\frac{1}{\sqrt{m}}\right) = 0.$$

This proves the assertion of Step 1.

Step 2. *For each* $\mathbf{a} \in \mathcal{A}_N$ *there exists a sequence* $(\mathbf{a}^m)_{m=1}^{\infty} \subset ((\bigcup_{\Delta} \mathcal{A}(\Delta)) \cap \mathcal{A}_N)$ *such that* $\mathbf{a}^m \to \mathbf{a}$ *in* $\mathcal{A}^0$. Let $x_1, \ldots, x_N \in [0,1] \times [0,\infty) \times [0,\infty)$ be the points such that $|x_i - x_j| \ge 1$ $(i \ne j)$. For $(t,\omega) \in [0,\infty) \times \Omega$ we put $\mathbf{b}_t(\omega) = \sum_{i=1}^N 1_{\{\mathbf{a}_.=\beta(i)\}}(t,\omega) \cdot x_i$. Then $\mathbf{b}_.$ is an adapted process which satisfies $d(\mathbf{a}_t, \mathbf{a}_s) \le |\mathbf{b}_t - \mathbf{b}_s|^2$. This is since $|\mathbf{b}_t - \mathbf{b}_s|^2 = |x_i - x_j|^2 = 0$ $(i = j)$, $|\mathbf{b}_t - \mathbf{b}_s|^2 \ge 1$ $(i \ne j)$, both of which are greater than or equal to $d(\mathbf{a}_t, \mathbf{a}_s)$ by the definition of d.

We extend the above $\mathbf{b}_.$ by putting $\mathbf{b}_t \equiv \mathbf{b}_0$ for $t < 0$. For $n = 1, 2, \ldots,$ put $k(n,t) \equiv j \cdot 2^{-n}$ if $j \cdot 2^{-n} \le t < (j+1) \cdot 2^{-n}$ for $j \in \mathbf{Z}$. Then there exists some $s \in [0,1]$ and a subsequence (n') of $\mathbf{N}$ such that, for any $T > 0$,

$$E\left[\int_0^T |\mathbf{b}_t - \mathbf{b}_{k(n',t-s)+s}|^2\, dt\right] \to 0 \tag{4.11}$$

as $n' \to \infty$.

To show (4.11), we first remark that $|\mathbf{b}_t| \le \max_{1\le i\le N} |x_i|$ and that $t - 2^{-n} \le k(n,t) < t$. This implies $\int_0^T |\mathbf{b}_{t+h} - \mathbf{b}_t|^2\, dt \to 0$ as $h \to 0$ by the definition of $\mathbf{b}$. Hence

$$\int_0^1 ds \int_0^T |\mathbf{b}_t - \mathbf{b}_{k(n,t-s)+s}|^2\, dt \to 0$$

as $n \to \infty$.

Indeed,

$$\int_0^1 ds \int_0^T |\mathbf{b}_t - \mathbf{b}_{k(n,t-s)+s}|^2\, dt$$
$$= \int_0^1 ds \int_{-s}^{T-s} |\mathbf{b}_{t+s} - \mathbf{b}_{k(n,t)+s}|^2\, dt$$

$$
\begin{aligned}
&\leq \int_0^1 ds \int_{-1}^T |\mathbf{b}_{t+s} - \mathbf{b}_{k(n,t)+s}|^2\, dt \\
&= \int_{-1}^T dt \int_0^1 |\mathbf{b}_{t+s} - \mathbf{b}_{k(n,t)+s}|^2\, ds \\
&= \int_{-1}^T dt \int_{k(n,t)}^{1+k(n,t)} |\mathbf{b}_{t+s-k(n,t)} - \mathbf{b}_s|^2\, ds \\
&\leq \int_{-1}^T dt \int_0 2|\mathbf{b}_{t+s-k(n,t)} - \mathbf{b}_s|^2\, ds \\
&\leq (T+1) \sup_{|h|\leq 2^{-n}} \int_{-1}^2 |\mathbf{b}_{s+h} - \mathbf{b}_s|^2\, ds \to 0
\end{aligned}
$$

as $n \to \infty$. By the dominated convergence theorem

$$E\left[\int_0^1 ds \int_0^T |\mathbf{b}_t - \mathbf{b}_{k(n,t-s)+s}|^2\, dt\right] \to 0$$

as $n \to \infty$, that is,

$$\int_0^1 ds E\left[\int_0^T |\mathbf{b}_t - \mathbf{b}_{k(n,t-s)+s}|^2\, dt\right] \to 0$$

as $n \to \infty$. This implies, by taking a subsequence, (4.11).

We now put $\mathbf{a}^{n'} = (\mathbf{a}_t^{n'})$, $\mathbf{a}_t^{n'} = \mathbf{a}_{(k(n',t-s)+s)\vee 0}$, where $s \in [0,1]$ is given in (4.11). Then $\mathbf{a}^{n'} \in (\bigcup_\Delta \mathcal{A}(\Delta)) \cap \mathcal{A}_N$. Further, we have $\mathbf{a}^{n'} \to \mathbf{a}$ in $\mathcal{A}^0$, since

$$d(\mathbf{a}_t^{n'}, \mathbf{a}_t) = d(\mathbf{a}_{(k(n',t-s)+s)\vee 0}, \mathbf{a}_t) \leq |\mathbf{b}_{(k(n',t-s)+s)\vee 0} - \mathbf{b}_t|^2 = |\mathbf{b}_{k(n',t-s)+s} - \mathbf{b}_t|^2,$$

and we can apply (4.11).

Step 3. *Let* $\mathbf{a} \in (\bigcup_\Delta \mathcal{A}(\Delta)) \cap \mathcal{A}_N$ *be given. Then there exists* $(\mathbf{a}^m) \subset \mathcal{A}_N(\Delta_m)$ *such that* $\mathbf{a}^m \to \mathbf{a}$ *in* $\mathcal{A}^0$. We now fix $\mathbf{a} \in (\bigcup_\Delta \mathcal{A}(\Delta)) \cap \mathcal{A}_N$ arbitrarily. Then by definition there exist $0 = s_0 < s_1 < \cdots < s_p < \cdots \to \infty$, and a set of controls $\mathbf{a}_i\colon \Omega \to \{\beta(i)\}_{i=1}^N$: $\mathcal{F}_{s_i}$-measurable such that $\mathbf{a}_t = \mathbf{a}_0$ $(t = 0)$, $\mathbf{a}_t = \mathbf{a}_i$ $(s_i \leq t < s_{i+1})$, $i = 0, 1, 2, \ldots$.

We denote the partition by $\Delta_m : 0 = t_0^m < t_1^m < \cdots < t_p^m \cdots \to \infty$. Since $\lim_{m\to\infty} |\Delta_m| = 0$, we have

$$|\Delta_m| < \inf\{(s_{i+1} - s_i); i \geq 0\}, \qquad m \geq m_0, \tag{4.12}$$

for some $m_0 > 0$.

For $t > 0$, $m \geq m_0$, we put

$$p(t) \equiv \max\{p; s_p < t\}, \qquad q_m(t) \equiv \max\{q; t_q^m < t\}.$$

We then have $s_{p(t_q^m)} < t_q^m \le s_{p(t_q^m)+1}$ by the definition. We also have due to the assumption (4.12) that

$$\begin{aligned}
&\#\{q; 0 \le q \le q_m(T), p(t_q^m) \ne p(t_{q+1}^m)\} \\
&\quad = \#\{q; 0 \le q \le q_m(T), p(t_q^m) < p(t_{q+1}^m)\} \\
&\quad \le p(t_{q_m(T)}^m) \le p(T).
\end{aligned}$$

We now put, for $m \ge m_0$, $\mathbf{a}_t^m \equiv \mathbf{a}_0$ $(t = 0)$, $\mathbf{a}_t^m \equiv \mathbf{a}_{p(t_q^m)}$ $(t_q^m \le t < t_{q+1}^m)$, $q = 0, 1, 2, \ldots$. Then $\mathbf{a}_{p(t_q^m)}$ takes values in $\{\beta(1), \ldots, \beta(N)\}$, $\mathcal{F}_{t_q^m}$-measurable. We have, further,

$$E\left[\int_0^T d(\mathbf{a}_t^m, \mathbf{a}_t)\,dt\right] \to 0 \tag{4.13}$$

as $m \to \infty$ for each $T > 0$.

Indeed,

$$\begin{aligned}
&\int_0^T d(\mathbf{a}_t^m, \mathbf{a}_t)\,dt \\
&\quad \le \int_0^{t_{q_m(T)+1}^m} d(\mathbf{a}_t^m, \mathbf{a}_t)\,dt \\
&\quad = \sum_{q=0}^{q_m(T)} \int_{t_q^m}^{t_{q+1}^m} d(\mathbf{a}_t^m, \mathbf{a}_t)\,dt = \sum_{q=0}^{q_m(T)} \int_{t_q^m}^{t_{q+1}^m} d(\mathbf{a}_{p(t_q^m)}, \mathbf{a}_t)\,dt \\
&\quad = \left(\sum_{q:0 \le q \le q_m(T),\, p(t_q^m) = p(t_{q+1}^m)} + \sum_{q:0 \le q \le q_m(T),\, p(t_q^m) \ne p(t_{q+1}^m)} \right) \\
&\qquad \times \int_{t_q^m}^{t_{q+1}^m} d(\mathbf{a}_{p(t_q^m)}, \mathbf{a}_t)\,dt \\
&\quad = \sum_{q:0 \le q \le q_m(T),\, p(t_q^m) \ne p(t_{q+1}^m)} \int_{t_q^m}^{t_{q+1}^m} d(\mathbf{a}_{p(t_q^m)}, \mathbf{a}_t)\,dt \le K|\Delta_m| p(T),
\end{aligned}$$

which tends to 0 as $m \to \infty$. This implies (4.13). Hence $\mathbf{a}_\cdot^m \to \mathbf{a}_\cdot$.

Step 4. *Finally we show that for each* $\mathbf{a} \in \mathcal{A}^0$ *there exists a subsequences* m' *and a sequence* $(\mathbf{a}^{m'}) \subset \mathcal{A}(\Delta^{m'})$ *such that* $\mathbf{a}^{m'} \to \mathbf{a}$ *in* $\mathcal{A}^0$. We fix $\mathbf{a} \in \mathcal{A}^0$. It is sufficient to show that for each $\varepsilon > 0$ there exists $\mathbf{a}^\varepsilon \in \bigcup_{m=1}^\infty \mathcal{A}(\Delta_m)$ such that $\rho(\mathbf{a}^\varepsilon, \mathbf{a}) < \varepsilon$.

By Step 1 there exist $N \ge 1$ and $\mathbf{a}' \in \mathcal{A}_N$ such that $\rho(\mathbf{a}', \mathbf{a}) < \varepsilon/3$. By Step 2 there exists $\mathbf{a}'' \in (\bigcup_\Delta \mathcal{A}(\Delta)) \cap \mathcal{A}_N$ such that $\rho(\mathbf{a}'', \mathbf{a}') < \varepsilon/3$. By Step 3 there exists $\mathbf{a}''' \in \bigcup_m \mathcal{A}(\Delta_m)$ such that $\rho(\mathbf{a}''', \mathbf{a}'') < \varepsilon/3$. These imply the first assertion. The second assertion follows immediately. □

Corollary to Proposition 4.2. *Let* (Δ_m) *be a series of partitions of* $[0, \infty)$ *such that* $\Delta_m \subset \Delta_{m+1}$, $m = 1, 2, \ldots$ *(that is,* Δ_{m+1} *is more fine than* Δ_m*). Then for any* $\mathbf{a} \in \mathcal{A}$ *there exists a sequence* (Δ_m) *of controls* $\mathbf{a}^m \in \mathcal{A}(\Delta_m)$ *such that* $\mathbf{a}^m \to \mathbf{a}$ *in* $\mathcal{A}$.

Indeed, there exist, by Proposition 4.2, a subsequence $(m_k)_{k=1}^{\infty}$ and controls $\mathbf{a}^{m_k} \in \mathcal{A}(\Delta_{m_k})$ such that $\mathbf{a}^{m_k} \to \mathbf{a}$ in $\mathcal{A}$. Since $\Delta_m \subset \Delta_{m+1}$, $\mathcal{A}(\Delta_m) \subset \mathcal{A}(\Delta_{m+1})$. Hence by putting $\mathbf{a}^m \equiv \mathbf{a}^{m_k}$, $m_k \le m < m_{k+1}$, this $(\mathbf{a}^m)$ satisfies that $\mathbf{a}^m \in \mathcal{A}(\Delta_m)$, $\mathbf{a}^m \to \mathbf{a}$ in $\mathcal{A}$.

Proposition 4.3. *For any series of partitions* (Δ_m) *of* $[0,\infty)$ *such that* $\Delta_m \subset \Delta_{m+1}$, $m = 1, 2, \ldots$ *and that* $\lim_{m\to\infty} |\Delta_m| = 0$, *we have*

$$v(t; x, y) = \lim_{m\to\infty} \sup_{\mathbf{a}\in\mathcal{A}(\Delta_m)} v^{\mathbf{a}}(t; x, y), \qquad t \ge 0. \tag{4.14}$$

Proof. We fix $\varepsilon > 0$. Since $v(t; x, y) = \sup_{\mathbf{a}\in\mathcal{A}} v^{\mathbf{a}}(t; x, y)$, there exists $\mathbf{a}^{\varepsilon} \in \mathcal{A}$ such that $v^{\mathbf{a}^{\varepsilon}}(t; x, y) > v(t; x, y) - \varepsilon$. By the corollary to Proposition 4.2 there exists $(\mathbf{a}^m)_{m=1}^{\infty}$, $\mathbf{a}^m \in \mathcal{A}(\Delta_m)$, such that $\mathbf{a}^m \to \mathbf{a}^{\varepsilon}$ in $\mathcal{A}$.

By Proposition 4.1, $v^{\mathbf{a}^m}(t; x, y) \to v^{\mathbf{a}^{\varepsilon}}(t; x, y)$, $t \ge 0$. Hence $v(t; x, y) - \varepsilon < v^{\mathbf{a}^{\varepsilon}}(t; x, y) = \lim_{m\to\infty} v^{\mathbf{a}^m}(t; x, y) \le \lim_{m\to\infty} \sup_{\mathbf{a}\in\mathcal{A}(\Delta_m)} v^{\mathbf{a}}(t; x, y)$.

We let $\varepsilon \to 0$ to obtain

$$v(t; x, y) \le \lim_{m\to\infty} \sup_{\mathbf{a}\in\mathcal{A}(\Delta_m)} v^{\mathbf{a}}(t; x, y).$$

On the other hand, we have

$$v(t; x, y) \ge \lim_{m\to\infty} \sup_{\mathbf{a}\in\mathcal{A}(\Delta_m)} v^{\mathbf{a}}(t; x, y)$$

by definition. Hence the equality holds. □

Next we seek a representation of $v(t; x, y)$ using the functions G_t and $\bar{G}_t$ of Section 1.

Lemma 4.4. *Let* $t_1 < t_2$ *and let* $\gamma\colon \Omega \to \mathbf{A}$ *be a given* $\mathcal{F}_{t_1}$*-measurable function. Let* $\mathbf{a} \in \mathcal{A}$ *be a constant control such that* $\mathbf{a}_t \equiv \gamma = (\pi^0, c^0, L^0)$ *for* $t \in [t_1, t_2)$. *Then for* $u \in C_b(\mathcal{S})$,

$$E\left[\int_{t_1}^{t_2} e^{-\alpha s} U(c_s^0)\, ds + e^{-\alpha t_2} u(X_{t_2}^x, Y_{t_2}^y) | \mathcal{F}_{t_1}\right] = e^{-\alpha t_1} G_{t_2-t_1}^{\gamma} u(X_{t_1}^x, Y_{t_1}^y). \tag{4.15}$$

Proof.

Step 1. We denote by $p(t)$ the Poisson point process associated to the original Poisson random measure $N(ds\, dz)$. We put $\bar{p}(t)$ to be the shifted process $\bar{p}(t) \equiv p(t + t_1)$ of $p(t)$ by t_1, for $t \in D_{\bar{p}} \equiv \{t; t + t_1 \in D_p\}$, $\bar{\mathcal{F}}_t \equiv \mathcal{F}_{t+t_1}$.

For $\mathbf{a} = (\pi, c, L)$, $\mathbf{b} = (\pi', c', L')$, we denote by $y_t^{\mathbf{b}}$, z_t the processes given by $y_t^{\mathbf{b}} = (y_t^{\mathbf{b},x}, y_t^{\mathbf{b},y})$;

$$y_t^{\mathbf{b},x} = x + \int_0^t (r + (\hat{b} - r)\pi_s') y_s^{\mathbf{b},x}\, ds + \int_0^t \int \pi_{s-}' y_s^{\mathbf{b},x} (e^z - 1) \tilde{N}_{\bar{p}}(ds\, dz),$$

$$y_t^{\mathbf{b},y} = y e^{-\beta t} + \beta \int_0^t e^{-\beta(t-s)} c_s'\, ds,$$

and $z_t = (z_t^x, z_t^y)$;

$$z_t^x = X_{t_1}^{\mathbf{a},x} + \int_0^t (r + (\hat{b} - r)\pi_s^0) z_s\, ds + \int_0^t \int \pi_{s-}^0 z_s (e^z - 1) \tilde{N}_{\bar{p}}(ds\, dz),$$
$$z_t^y = Y_{t_1}^{\mathbf{a},y} e^{-\beta t} + \beta \int_0^t e^{-\beta(t-s)} c_s^0\, ds,$$

where $N_{\bar{p}}(ds\, dz)$ denotes the new random measure induced by $\bar{p}$.

Then by Theorem 5.1 in Section 5 we have for any measurable $F\colon \mathbf{A} \times D \to [0, \infty)$,

$$E[F(\gamma, z)|\bar{\mathcal{F}}_0] = E[F(\mathbf{b}, y^{\mathbf{b},x})]|_{\mathbf{b}=\gamma, x=x_{t_1}^{\mathbf{a},(x,y)}}.$$

Step 2. We have $z_t = x_{t+t_1}^{\mathbf{a}}$ for $0 \le t \le t_2 - t_1$.

Indeed, by the Burkholder–Davis–Gundy (BDG) inequality,

$$\begin{aligned} E\left[\sup_{0\le t\le t_2-t_1} |z_t^x - X_{t+t_1}^{\mathbf{a},x}|^2\right] &\le C \int_0^{t_2-t_1} \Bigg\{ E[|(r + (\hat{b} - r)\pi_s^0) z_s^x - (r + (\hat{b} - r)\pi_{s+t_1}) z_s^x|^2] \\ &\quad + E\left[\int_0^{t_2-t_1} |\pi_{s-}^0 z_s^x (e^z - 1) - \pi_{(s+t_1)-} z_s^x (e^z - 1)|^2 \nu(dz)\right]\Bigg\}\, ds. \end{aligned}$$

Since $\mathbf{a}_t = \gamma$ for $t \in [t_1, t_2)$,

$$E\left[\sup_{0\le t\le t_2-t_1} |z_t^x - X_{t+t_1}^{\mathbf{a},x}|^2\right] = 0.$$

The assertion that $z_t^y = Y_{t+t_1}^{\mathbf{a},y}$ follows similarly.

Step 3. We have

$$E\left[\int_{t_1}^{t_2} e^{-\alpha(s-t_1)} U(c_s)\, ds + e^{-\alpha(t_2-t_1)} u(x_{t_2}^{\mathbf{a}}) | \mathcal{F}_{t_1}\right] = G_{t_2-t_1}^{\gamma}(x_{t_1}^{\mathbf{a}}).$$

Indeed,

$$\begin{aligned} \int_{t_1}^{t_2} e^{-\alpha s} U(c_s)\, ds + e^{-(t_2-t_1)} u(x_{t_2}^{\mathbf{a}}) &= \int_0^{t_2-t_1} e^{-\alpha s} U(\gamma_s)\, ds + e^{-(t_2-t_1)} u(x_{t_2-t_1+t_1}^{\mathbf{a}}) \\ &\equiv F(\gamma, z) \qquad \text{(say)} \end{aligned}$$

by Step 2. Hence by Step 1

$$\begin{aligned} &E\left[\int_{t_1}^{t_2} e^{-\alpha(s-t_1)} U(c_s)\, ds + e^{-(t_2-t_1)} u(x_{t_2}^{\mathbf{a}}) | \mathcal{F}_{t_1}\right] \\ &\quad = E[F(\gamma, z)|\bar{\mathcal{F}}_0] = E[F(\mathbf{b}, y^{\mathbf{b}})]|_{\mathbf{b}=\gamma, (x,y)=x_{t_1}^{\mathbf{a}}} \end{aligned}$$

$$= E\left[\int_0^{t_2-t_1} e^{-\alpha s} U(c_s')\,ds + e^{-\alpha(t_2-t_1)} u(y^{\mathbf{b}}_{t_2-t_1})\right]\Bigg|_{\mathbf{b}=\gamma,(x,y)=x^{\mathbf{a}}_{t_1}}.$$

Since $x^{\mathbf{b}}_t =^d y^{\mathbf{b}}_t$, the right-hand side is equal to $G^{\mathbf{b}}_{t_2-t_1} u(x, y)|_{\mathbf{b}=\gamma,(x,y)=x^{\mathbf{a}}_{t_1}}$. This leads to the assertion.

Step 4. Finally we have

$$E\left[\int_{t_1}^{t_2} e^{-\alpha s} U(c_s)\,ds + e^{-\alpha t_2} u(x^{\mathbf{a}}_{t_2})|\mathcal{F}_{t_1}\right] = e^{-\alpha t} G^{\gamma}_{t_2-t_1} u(x^{\mathbf{a}}_{t_1}).$$

Indeed,

$$\begin{aligned}\text{L.H.S.} &= E\left[e^{-\alpha t_1}\left(\int_{t_1}^{t_2} U(c_s)\,ds + e^{-\alpha(t_2-t_1)} u(x^{\mathbf{a}}_{t_2})\right)\Bigg|\mathcal{F}_{t_1}\right]\\ &= e^{-\alpha t_1} E\left[\int_{t_1}^{t_2} U(c_s)\,ds + e^{-\alpha(t_2-t_1)} u(x^{\mathbf{a}}_{t_2})|\mathcal{F}_{t_1}\right] = e^{-\alpha t_1} G^{\gamma}_{t_2-t_1}(x^{\mathbf{a}}_{t_1}). \qquad \square\end{aligned}$$

We denote by $\Delta([0, t])$ a partition of $[0, t]$ such that $0 = s_0 < s_1 < \cdots < s_m = t$.

Lemma 4.5. *For any series of partitions* $(\Delta_i([0, t]))$ *of* $[0, t]$ *such that* $\Delta_i([0, t]) \subset \Delta_{i+1}([0, t])$, $i = 1, 2, \ldots$ *and that* $\lim_{i\to\infty} |\Delta_i([0, t])| = 0$, *we have*

$$v(t; x, y) \le \liminf_{i\to\infty} G_{s^i_1-s^i_0} \cdots \bar{G}_{s^i_{m(i)}-s^i_{m(i-1)}}(x, y). \tag{4.16}$$

Here $m(i)$ *is given by the relation* $\Delta_i([0, t]) : 0 < s^i_0 < s^i_1 < \cdots < s^i_{m(i)} = t$.

Proof. We extend the set of partitions $(\Delta_i([0, t]))$ on $[0, t]$ to the partitions (Δ_i) on $[0, \infty)$ as follows; we put

$$\Delta_i|_{[0,t]} = \Delta_i([0, t]),$$

$$\Delta_i \subset \Delta_{i+1}, \qquad i = 1, 2, \ldots, \lim_{i\to\infty} |\Delta_i| = 0.$$

By Proposition 4.3 we have

$$v(t; x, y) = \lim_{i\to\infty} \sup_{\mathbf{a}\in\mathcal{A}(\Delta_i)} v^{\mathbf{a}}(t; x, y).$$

Hence we can choose some series $(\mathbf{a}^i)$, $\mathbf{a}^i \in \mathcal{A}(\Delta_i)$, of controls such that $v^{\mathbf{a}^i}(t; x, y) \to v(t; x, y)$. Here we must have for each i, $\mathbf{a}^i_s = \mathbf{a}^i_j$ for $s \in [s^i_j, s^i_{j+1})$, for some $\{\mathbf{a}^i_j;\ j = 0, 1, 2, \ldots, m(i) - 1\}$. We write $\mathbf{a}^i = (\pi^i, c^i, L^i)$.

For each i we define a sequence $(u_j^i)_{j=0}^{m(i)}$, $u_j^i \in C_b(\mathcal{S})$, of functions by

$$u_{m(i)}^i \equiv 0, \qquad u_{m(i)-1}^i = \bar{G}_{s_{m(i)}^i - s_{m(i)-1}^i},$$

$$u_j^i = G_{s_{j+1}^i - s_j^i} u_{j+1}^i, \qquad j = 0, 1, 2, \ldots, m(i) - 2.$$

Then we have by Lemma 4.4 for $j = 0, 1, \ldots, m(i) - 2$,

$$\begin{aligned} E&\left[\int_{s_j}^{s_{j+1}} e^{-\alpha r} U(c_r^i)\, dr + e^{-\alpha s_{j+1}^i} u_{j+1}^i(X_{s_{j+1}^i}^{\mathbf{a}^i}, Y_{s_{j+1}^i}^{\mathbf{a}^i})\right] \\ &= E[e^{-\alpha s_{j+1}^i} G_{s_{j+1}^i - s_j^i}^{\mathbf{a}_j^i} u_{j+1}^i(X_{s_j^i}^{\mathbf{a}^i}, Y_{s_j^i}^{\mathbf{a}^i})] \\ &\leq E[e^{-\alpha s_{j+1}^i} G_{s_{j+1}^i - s_j^i} u_{j+1}^i(X_{s_j^i}^{\mathbf{a}^i}, Y_{s_j^i}^{\mathbf{a}^i})] = E[e^{-\alpha s_{j+1}^i} u_j^i(X_{s_j^i}^{\mathbf{a}^i}, Y_{s_j^i}^{\mathbf{a}^i})]. \end{aligned}$$

That is,

$$\begin{aligned} E&\left[\int_{s_j}^{s_{j+1}} e^{-\alpha r} U(c_r^i)\, dr\right] \\ &\leq -E[e^{-\alpha s_{j+1}^i} u_{j+1}^i(X_{s_{j+1}^i}^{\mathbf{a}^i}, Y_{s_{j+1}^i}^{\mathbf{a}^i})] + E[e^{-\alpha s_{j+1}^i} u_j^i(X_{s_j^i}^{\mathbf{a}^i}, Y_{s_j^i}^{\mathbf{a}^i})]. \end{aligned}$$

Hence

$$\begin{aligned} E\left[\int_0^t e^{-\alpha r} U(c_r^i)\, dr\right] &= \sum_{j=0}^{m(i)-1} E\left[\int_{s_j^i}^{s_{j+1}^i} e^{-\alpha r} U(c_r^i)\, dr\right] \\ &\leq E\left[\sum_{j=0}^{m(i)-1} (e^{-\alpha s_j} u_j^i(X_{s_j^i}^{\mathbf{a}^i}, Y_{s_j^i}^{\mathbf{a}^i}) - e^{-\alpha s_{j+1}} u_{j+1}^i(X_{s_{j+1}^i}^{\mathbf{a}^i}, Y_{s_{j+1}^i}^{\mathbf{a}^i}))\right] \\ &= u_0^i(x) - E[e^{-\alpha t} u_{m(i)}^i(X_t^{\mathbf{a}^i}, Y_t^{\mathbf{a}^i})]. \end{aligned}$$

Since $u_0^i(x, y) = G_{s_1^i - s_0^i} \cdots \bar{G}_{s_{m(i)}^i - s_{m(i)-1}^i}(x, y)$ and since $u_{m(i)} \equiv 0$, we have

$$v^{\mathbf{a}^i}(t; x, y) \leq G_{s_1^i - s_0^i} \cdots \bar{G}_{s_{m(i)}^i - s_{m(i)-1}^i}(x, y).$$

Letting $i \to \infty$,

$$\begin{aligned} v(t; x, y) &\leq \liminf_{i \to \infty} G_{s_1^i - s_0^i} \cdots \bar{G}_{s_{m(i)}^i - s_{m(i)-1}^i}(x, y) \\ &\leq \sup_i G_{s_1^i - s_0^i} \cdots \bar{G}_{s_{m(i)}^i - s_{m(i)-1}^i}(x, y). \end{aligned}$$ □

Lemma 4.6. *For any partition* $\Delta([0, t])$ *of* $[0, t]$ *such that* $0 = s_0 < s_1 < \cdots < s_m = t$,

$$v(t; x, y) \geq G_{s_1 - s_0} \cdots \bar{G}_{s_m - s_{m-1}}(x, y). \tag{4.17}$$

Proof. The proof is rather long. We devide it into three steps.

We put $u_{m-1}(x,y) = \bar{G}_{s_m - s_{m-1}}(x,y)$ and $u_i(x,y) = G_{s_{i+1}-s_i} u_{i+1}(x,y), i = 0, 1, \ldots, m-2$.

Step 1. *Fix* $\varepsilon > 0$. *For* $i = 0, 1, \ldots, m-2$ *there exist measurable functions* $\mathbf{b}_i \colon \mathcal{S} \to \mathbf{A}$ *such that* $u_i(x,y) \le G^{\mathbf{b}_i(x,y)}_{s_{i+1}-s_i} u_{i+1}(x,y) + \varepsilon$. This follows as below. Let $\{\mathbf{b}(k)\}_{k=1}^{\infty}$ be a dense set in $\mathcal{A}$. Now we have $u_i(x,y) = G_{s_{i+1}-s_i} u_{i+1}(x,y) = \sup_{\mathbf{b} \in \mathcal{A}} G^{\mathbf{b}}_{s_{i+1}-s_i} u_{i+1}(x,y)$. Since $\mathbf{b} \mapsto G^{\mathbf{b}}_{s_{i+1}-s_i} u_{i+1}(x,y)$ is continuous by the corollary to Proposition 4.1, we have

$$u_i(x,y) = \sup_{k \ge 1} G^{\mathbf{b}(k)}_{s_{i+1}-s_i} u_{i+1}(x,y).$$

For $(x,y) \in \mathbf{R}^2$ we put $k_i(x,y) \equiv \min\{k \ge 1; u_i(x,y) \le G^{\mathbf{b}(k)}_{s_{i+1}-s_i} u_{i+1}(x,y) + \varepsilon\}$. We put $\mathbf{b}_i(x,y) \equiv \mathbf{b}(k_i(x,y))$, then this satisfies the required condition.

Step 2. *For any given* $u = (u^1, u^2) : [0,\infty) \to \mathcal{S}$, *we put* $\mathbf{a}^* = (\pi^*, c^*, L^*) : [0,\infty) \to \mathbf{A}$ *by*

$$\begin{aligned}
&\mathbf{a}^*_s(u) = \mathbf{b}_0(u^1_0, u^2_0) \qquad \textit{if} \quad s \in [0, s_1),\\
&\mathbf{a}^*_s(u) = \mathbf{b}_i(u^1_{s_i}, u^2_{s_i}) \qquad \textit{if} \quad s \in [s_i, s_{i+1}), \quad i = 0, 1, \ldots, m-2,\\
&\mathbf{a}^*_s(u) = \mathbf{b}_0(0,0) \qquad \textit{if} \quad s \in (t, \infty).
\end{aligned}$$

Then we can construct a càdlàg process $w_\tau = (w^1_\tau, w^2_\tau)$ *which satisfies the following SDE*:

$$\begin{aligned}
w^1_\tau &= x - C^*_\tau + \int_0^\tau (r + (\hat{b} - r)\pi^*_s) w^1_s \, ds\\
&\quad + L^*_\tau + \int_0^\tau \int_{\{|z| \le 1\}} \pi^*_{s-} w^1_{s-} (e^z - 1) \tilde{N}(ds\,dz),\\
w^2_\tau &= y e^{-\beta\tau} + \beta \int_0^\tau e^{-\beta(\tau - s)} \, dC^*_s.
\end{aligned}$$

Indeed, we put $D_{p^{(i)}} = \{\tau; \tau + si \in D_p\}$, $p^{(i)}(\tau) = p(\tau + s_i)$, $\tau \in D_{p^{(i)}}$, $\mathcal{F}^{(i)}_\tau = \mathcal{F}_{\tau + s_i}$. Here $p(\cdot)$ denotes a Poisson point process corresponding to $N(ds\,dz)$. Let $z^{(i)}_\tau, i = 0, 1, 2, \ldots, m$, be the process defined by the following SDE, respectively:

$$\begin{aligned}
z^{(0)}_\tau &= x - C^*_\tau + \int_0^\tau (r + (\hat{b} - r)\pi^*_s) z^{(0)}_s \, ds + L^*_\tau\\
&\quad + \int_0^\tau \int_{\{|z| \le 1\}} \pi^*_{s-} w^1_{s-} (e^z - 1) \tilde{N}^{(0)}(ds\,dz),\\
z^{(i)}_\tau &= z^{(i-1)}_\tau - C^*_\tau + \int_0^\tau (r + (\hat{b} - r)\pi^*_s) z^{(i)}_s \, ds + L^*_\tau\\
&\quad + \int_0^\tau \int_{\{|z| \le 1\}} \pi^*_{s-} w^1_{s-} (e^z - 1) \tilde{N}^{(i)}(ds\,dz),
\end{aligned}$$

$i = 1, \ldots, m$. Here $\tilde{N}^{(0)}, \tilde{N}^{(i)}$ are compensated Poisson random measures corresponding to $p^{(0)}, p^{(i)}$, respectively.

We then put w^1_τ by

$$w^1_\tau = x \quad (\tau = 0), \qquad w^1_\tau = z^{(i)}_{\tau - s_i} \quad (s_i \le \tau < s_{i+1}),$$
$$w^1_\tau = z^{(m)}_{\tau - t} \quad (\tau \ge t).$$

The second component w^2 is given similarly. Then these satisfy the above condition.

Step 3.

$$v(t; x, y) \ge G_{s_1 - s_0} \cdots \bar{G}_{s_m - s_{m-1}}(x, y).$$

Indeed, we put $\mathbf{a}_s(\omega) \equiv \mathbf{a}^*_s(u)$ with $u = w(\omega)$. Then $\mathbf{a} = (\mathbf{a}_s(\omega)) \in \mathcal{A}$, $\mathbf{a}_s = \mathbf{b}_i(w_{s_i})$ for $s \in [s_i, s_{i+1})$, $i = 0, 1, \ldots, m-1$.

By Lemma 4.4, for $i = 0, 1, \ldots, m-1$,

$$\begin{aligned} E&\left[\int_{s_i}^{s_{i+1}} e^{-\alpha r} U(c_r)\, dr + e^{-\alpha s_{i+1}} u_{i+1}(x^{\mathbf{a},(x,y)}_{i+1})\right] \\ &= E[e^{-\alpha s_i} G^{\mathbf{b}_i(x^{\mathbf{a},(x,y)}_{s_i})}_{s_{i+1} - s_i} u_{i+1}(x^{\mathbf{a},(x,y)}_i)] \\ &\ge E[e^{-\alpha s_i}(u_i(x^{\mathbf{a},(x,y)}_{s_i}) - \varepsilon)] \qquad \text{(by Step 1)} \\ &\ge E[e^{-\alpha s_i} u_i(x^{\mathbf{a},(x,y)}_{s_i})] - \varepsilon. \end{aligned}$$

Hence for $i = 0, 1, \ldots, m-1$,

$$E\left[\int_{s_i}^{s_{i+1}} e^{-\alpha r} U(c_r)\, dr\right] \ge -E[e^{-\alpha s_{i+1}} u_{i+1}(x^{\mathbf{a},(x,y)}_{i+1})] + E[e^{-\alpha s_i} u_i(x^{\mathbf{a},(x,y)}_{s_i})] - \varepsilon.$$

This implies

$$E\left[\int_0^t e^{-\alpha r} U(c_r)\, dr\right] \ge u_0(x, y) - E[e^{-\alpha t} u_m(x^{\mathbf{a},(x,y)}_t)] - m\varepsilon.$$

Hence

$$v^{\mathbf{a}}(t; x, y) \ge G_{s_1 - s_0} \cdots \bar{G}_{s_m - s_{m-1}}(x, y) - m\varepsilon$$

by the definition of u_i, and

$$v(t; x, y) \ge G_{s_1 - s_0} \cdots \bar{G}_{s_m - s_{m-1}}(x, y) - m\varepsilon.$$

We have the assertion by letting $\varepsilon \to 0$. □

Corollary to Lemma 4.6. *For any series of partitions* $(\Delta_i([0, t]))$ *of* $[0, t]$ *such that* $\Delta_{i+1}([0, t]) \subset \Delta_i([0, t])$, $i = 1, 2, \ldots,$ *and* $\lim_{i\to\infty} |\Delta_i([0, t])| = 0$, *we have*

$$v(t; x, y) = \lim_{i\to\infty} G_{s^i_1 - s^i_0} \cdots \bar{G}_{s^i_{m(i)} - s^i_{m(i-1)}}(x, y). \tag{4.18}$$

Proof. By Lemma 4.5, $v(t; x, y) \le \liminf_{i\to\infty} G_{s_1^i - s_0^i} \cdots \bar{G}_{s_{m(i)}^i - s_{m(i-1)}^i}(x, y)$. By Lemma 4.6, $v(t; x, y) \ge \limsup_{i\to\infty} G_{s_1^i - s_0^i} \cdots \bar{G}_{s_{m(i)}^i - s_{m(i-1)}^i}(x, y)$.

These imply the assertion. □

Lemma 4.7. *For any* $(x, y) \in \mathcal{S}$ *and any* $0 \le s \le t \le u$,

$$G_{t-s} v(u - t; x, y) \le v(u - s; x, y). \tag{4.19}$$

Proof. We fix $\mathbf{b} \in \mathcal{A}$. Let $(\Delta_i([0, u-t]))$, $\Delta_i([0, u-t]) : 0 = s_0^i < s_1^i < \cdots < s_{m(i)}^i = u - t$ be a series of partitions of $[0, t]$ such that $\Delta_{i+1}([0, t]) \subset \Delta_i([0, t])$, $i = 1, 2, \ldots,$ and $\lim_{i\to\infty} |\Delta_i([0, t])| = 0$. By the corollary to Lemma 4.6 we have

$$v(t; x, y) = \lim_{i\to\infty} G_{s_1^i - s_0^i} \cdots \bar{G}_{s_{m(i)}^i - s_{m(i-1)}^i}(x, y).$$

By the definition of G and $\bar{G}$ in Section 1 and by the dominate convergence theorem

$$\begin{aligned}
&G_{t-s}^{\mathbf{b}} v(u - t; x, y) \\
&\quad = E\left[\int_0^{t-s} e^{-\alpha(t-s)} U(c_r^{(\mathbf{b})})\, dr + e^{-\alpha(t-s)} v(u - t; x_{t-s}^{\mathbf{b},(x,y)})\right] \\
&\quad = \lim_{i\to\infty} E\left[\int_0^{t-s} e^{-\alpha(t-s)} U(c_r^{(\mathbf{b})})\, dr + e^{-\alpha(t-s)} G_{s_1^i - s_0^i}\right. \\
&\qquad\qquad \left.\cdots \bar{G}_{s_{m(i)}^i - s_{m(i-1)}^i}(x_{u-t}^{\mathbf{b},(x,y)})\right] \\
&\quad = \lim_{i\to\infty} G_{t-s}^{\mathbf{b}} G_{s_1^i - s_0^i} \cdots \bar{G}_{s_{m(i)}^i - s_{m(i-1)}^i}(x, y) \\
&\quad \le \lim_{i\to\infty} G_{(t-s)-0} G_{s_1^i + (t-s) - (s_0^i + (t-s))} \cdots \bar{G}_{s_{m(i)}^i + (t-s) - (s_{m(i-1)}^i + (t-s))}(x, y) \\
&\quad \le v(u - s; x, y)
\end{aligned}$$

by Lemmas 4.4 and 4.6.

Since $\mathbf{b} \in \mathcal{A}$ is arbitrary, we have

$$G_{t-s} v(u - t; x, y) \le v(u - s; x, y). \qquad \square$$

We are close to the end of the proof. We next put two functionals $k_s^{\mathbf{a}} = k_s^{\mathbf{a},(x,y)}$, $\eta_s^{\mathbf{a}} = \eta_s^{\mathbf{a},(x,y)}$ by

$$k_s^{\mathbf{a}} \equiv \int_0^s e^{-r\alpha} U(c_r)\, dr + e^{-\alpha s} v(t - s, x_s^{\mathbf{a}}), \qquad 0 \le s \le t,$$

$$\begin{aligned}
\eta_s^{\mathbf{a}} \equiv &\int_0^s e^{-(\alpha+\mu)r} \cdot (U(c_r^{(\mathbf{a})}) + \mu v(t - r; x_r^{\mathbf{a}}))\, dr + e^{-(\alpha+\mu)s} v(t - s; x_s^{\mathbf{a}}), \\
&0 \le s \le t,
\end{aligned}$$

where $\mu \ge 0$ is a constant.

Proposition 4.8.

(1) $k_s^{\mathbf{a}}$ *is an* $\mathcal{F}_s$*-supermartingale.*
(2) $\eta_s^{\mathbf{a}}$ *is an* $\mathcal{F}_s$*-supermartingale.*
(3) $k_s^{\mathbf{a}} - \eta_s^{\mathbf{a}}$ *is an* $\mathcal{F}_s$*-supermartingale.*

Proof. (1) By Proposition 4.1, if $\mathbf{a}^m \to \mathbf{a}$ in $\mathcal{A}$, then for each $s \in [0, t]$ we have

$$k_s^{\mathbf{a}^m} \to k_s^{\mathbf{a}} \qquad \text{in } L^1.$$

Since $\bigcup_\Delta \mathcal{A}(\Delta)$ is dense in $\mathcal{A}$ by Proposition 4.2, it suffices to show (1) for $\mathbf{a} \in \mathcal{A}(\Delta)$.

Now let $\Delta[0, t] : 0 = t_0 < t_1 < \cdots < t_m = t$ be a partition of $[0, t]$, and let $\mathbf{a} \in \mathcal{A}(\Delta[0, t])$ be such that $\mathbf{a}_s \equiv \mathbf{a}_i$ $(s \in [t_i, t_{i+1}))$, $i = 0, 1, 2, \ldots, m-1$. Then for $s, s' \in [t_i, t_{i+1}]$, $s \leq s'$, we have $E[k_{s'}^{(\mathbf{a},(x,y))}|\mathcal{F}_s] \leq k_s^{(\mathbf{a},(x,y))}$.

Indeed,

$$\begin{aligned}
&E[k_{s'}^{\mathbf{a},(x,y)}|\mathcal{F}_s] \\
&\quad = E\left[\int_0^{s'} e^{-\alpha r} U(c_r)\, dr + e^{-\alpha s'} v(t-s'; x, y)|\mathcal{F}_s\right] \\
&\quad = \int_0^s e^{-\alpha r} U(c_r)\, dr + E\left[\int_s^{s'} e^{-\alpha r} U(c_r)\, dr + e^{-\alpha s'} v(t-s'; x, y)|\mathcal{F}_s\right].
\end{aligned}$$

Since $\mathbf{a}_r = \mathbf{a}_i$ $(r \in [s, s'))$, by Lemma 4.4 we have

$$\begin{aligned}
E[k_{s'}^{\mathbf{a},(x,y)}|\mathcal{F}_s] &= \int_0^s e^{-\alpha r} U(c_r)\, dr + e^{-\alpha s} G_{s'-s}^{\mathbf{a}_i} v(t-s'; x_s^{\mathbf{a},(x,y)}) \\
&\leq \int_0^s e^{-\alpha r} U(c_r)\, dr + e^{-\alpha s} G_{s'-s} v(t-s'; x_s^{\mathbf{a},(x,y)}) \\
&\leq \int_0^s e^{-\alpha r} U(c_r)\, dr + e^{-\alpha s} v(t-s; x_s^{\mathbf{a},(x,y)}) = k_s^{\mathbf{a},(x,y)}.
\end{aligned}$$

The last inequality follows from Lemma 4.7. This implies that $k_s^{\mathbf{a},(x,y)}$ is an $\mathcal{F}_s$-supermartingale.

To show (2) and (3), we prepare the following notation.

Let k_t be any supermartingale such that $E[\sup_{0\leq t\leq T} |k_t|] < +\infty$, and let μ be any non-negative finite random variable. We put

$$\eta_t \equiv k_t e^{-\mu t} + \int_0^t e^{-\mu s} k_s\, ds.$$

Following this notation we have:

Lemma 4.9. η_t *is a supermartingale and* $(k - \eta)_t$ *is a supermartingale.*

Proof.

Step 1. *We have*

$$\eta_t - \eta_0 = \lim_{r\to 0} \frac{1}{r}\int_0^t (k_{s+r} - k_s)e^{-\mu s}\, ds, \tag{4.20}$$

$$E\left[\sup_{0\le t\le T, r\in(0,1)} \left|\frac{1}{r}\int_0^t (k_{s+r} - k_s)e^{-\mu s}\, ds\right|\right] < +\infty. \tag{4.21}$$

Indeed, we have

$$\frac{1}{r}\int_0^t (\eta_{s+r} - \eta_s)\, ds = \frac{1}{r}\int_r^{t+r} \eta_s\, ds - \frac{1}{r}\int_0^t \eta_s\, ds = \frac{1}{r}\int_t^{t+r} \eta_s\, ds - \frac{1}{r}\int_0^r \eta_s\, ds$$

and

$$\begin{aligned}\left|\frac{1}{r}\int_0^t (\eta_{s+r} - \eta_s)\, ds\right| &\le \frac{1}{r}\int_t^{t+r} |\eta_s|\, ds + \frac{1}{r}\int_0^r |\eta_s|\, ds \le 2\sup_t |\eta_t| \\ &\le 4\sup_t |\eta_t| < +\infty.\end{aligned}$$

This implies

$$\lim_{r\to 0} \frac{1}{r}\int_0^t (\eta_{s+r} - \eta_s)\, ds = \eta_t - \eta_0,$$

$$E\left[\sup_{0\le t\le T, r\in(0,1)} \left|\lim_{r\to 0}\frac{1}{r}\int_0^t (\eta_{s+r} - \eta_s)\, ds\right|\right] < +\infty.$$

Hence it is enough to show

$$\lim_{r\to 0}\left(\frac{1}{r}\int_0^t (\eta_{s+r} - \eta_s)\, ds - \frac{1}{r}\int_0^t (k_{s+r} - k_s)e^{-\mu s}\, ds\right) = 0 \quad \text{a.s.}, \tag{4.22}$$

$$E\left[\sup_{0\le t\le T, r\in(0,1)} \left|\frac{1}{r}\int_0^t (\eta_{s+r} - \eta_s)\, ds - \frac{1}{r}\int_0^t (k_{s+r} - k_s)e^{-\mu s}\, ds\right|\right] < +\infty. \tag{4.23}$$

Since

$$\begin{aligned}&\frac{1}{r}\int_0^t (\eta_{s+r} - \eta_s)\, ds - \frac{1}{r}\int_0^t (k_{s+r} - k_s)e^{-\mu s}\, ds \\ &\quad = \frac{1}{r}\int_0^t ds\, \Big\{k_{s+r}e^{-\mu(s+r)} - k_s e^{-\mu s} \\ &\qquad\qquad + \int_s^{s+r} \mu k_\lambda e^{-\mu\lambda}\, d\lambda - (k_{s+r} - k_s)e^{-\mu s}\Big\}\end{aligned}$$

$$= \frac{1}{r}\int_0^t ds \left\{\int_s^{s+r} \mu k_\lambda e^{-\mu\lambda}\, d\lambda - \int_s^{s+r} \mu k_{s+r} e^{-\mu\lambda}\, d\lambda\right\}$$
$$= \frac{1}{r}\int_0^t ds \int_s^{s+r} \mu(k_\lambda - k_{s+r}) e^{-\mu\lambda}\, d\lambda,$$

we have

$$\left|\frac{1}{r}\int_0^t (\eta_{s+r} - \eta_s)\, ds - \frac{1}{r}\int_0^t (k_{s+r} - k_s) e^{-\mu s}\, ds\right|$$
$$\leq \frac{1}{r}\int_0^t ds \int_s^{s+r} |k_\lambda - k_{s+r}| \mu e^{-\mu\lambda}\, d\lambda \leq 2 \sup_t |k_t| \cdot \mu \cdot t.$$

Hence

$$E\left[\sup_{0\leq t\leq T, r\in(0,1)} \left|\frac{1}{r}\int_0^t (\eta_{s+r} - \eta_s)\, ds - \frac{1}{r}\int_0^t (k_{s+r} - k_s) e^{-\mu s}\, ds\right|\right] < +\infty.$$

On the other hand, by the right continuity of k_t,

$$\lim_{r\to 0} \frac{1}{r}\int_s^{s+r} (k_\lambda - k_{s+r})\mu e^{-\mu\lambda}\, d\lambda = 0 \quad \text{a.s.}$$

Combining this to the above, we have by the dominated convergence theorem

$$\lim_{r\to 0} \frac{1}{r}\int_0^t ds \int_s^{s+r} (k_\lambda - k_{s+r})\mu e^{-\mu\lambda}\, d\lambda = 0 \quad \text{a.s.}$$

That is,

$$\lim_{r\to 0}\left(\frac{1}{r}\int_0^t (\eta_{s+r} - \eta_s)\, ds - \frac{1}{r}\int_0^t (k_{s+r} - k_s) e^{-\mu s}\, ds\right) = 0.$$

Step 2. *We show that $t \mapsto \eta_t$ is an $\mathcal{F}_t$-supermartingale.* Indeed, by Step 1

$$\frac{1}{r}\int_0^t (k_{s+r} - k_s) e^{-\mu s}\, ds \to \eta_t - \eta_0 \qquad \text{in } L^1 \text{ as } \quad r \to 0.$$

Let $t_2 > t_1$. Since k_t is an $\mathcal{F}_t$-supermartingale, we have, for $s \geq t_1$,

$$E[(k_{s+r} - k_s) e^{-\mu s} | \mathcal{F}_{t_1}] = E[e^{-\mu s} E[k_{s+r} - k_s | \mathcal{F}_s] | \mathcal{F}_{t_1}] \leq 0.$$

Hence

$$E\left[\frac{1}{r}\int_{t_1}^{t_2} (k_{s+r} - k_s) e^{-\mu s}\, ds | \mathcal{F}_{t_1}\right] = \frac{1}{r}\int_{t_1}^{t_2} E[(k_{s+r} - k_s) e^{-\mu s} | \mathcal{F}_{t_1}]\, ds \leq 0.$$

We let $r \to 0$, then

$$E[\eta_{t_2} - \eta_{t_1}|\mathcal{F}_{t_1}] \leq 0,$$

that is,

$$E[\eta_{t_2}|\mathcal{F}_{t_1}] \leq \eta_{t_1}.$$

Step 3. *We show*

$$k_t - \eta_t = \lim_{r\to 0} \frac{1}{r} \int_0^t (k_{s+r} - k_s)(1 - e^{-\mu s})\, ds, \tag{4.24}$$

$$E\left[\sup_{0\leq t\leq T, r\in(0,1)} \left|\frac{1}{r}\int_0^t (k_{s+r} - k_s)(1 - e^{-\mu s})\, ds\right|\right] < +\infty. \tag{4.25}$$

Indeed, since $k_t - k_0 = \lim_{r\to 0} (1/r) \int_0^t (k_{s+r} - k_s)\, ds,\ k_0 = \eta_0$,

$$\begin{aligned} k_t - \eta_t &= (k_t - k_0) - (\eta_t - \eta_0) \\ &= \lim_{r\to 0} \frac{1}{r} \int_0^t (k_{s+r} - k_s)\, ds - \lim_{r\to 0} \frac{1}{r} \int_0^t (k_{s+r} - k_s) e^{-\mu s}\, ds, \end{aligned}$$

where

$$E\left[\frac{1}{r}\left|\int_0^t (k_{s+r} - k_s)\, ds\right|\right] \leq 2E\left[\sup_t |k_t|\right] < +\infty.$$

Hence the assertion follows.

Step 4. *Finally*, *we show* $t \mapsto k_t - \eta_t$ *is an* $\mathcal{F}_t$*-supermartingale*. Indeed, by Step 3,

$$\frac{1}{r} \int_0^t (k_{s+r} - k_s)(1 - e^{-\mu s})\, ds \to k_t - \eta_t \qquad \text{in } L^1. \tag{4.26}$$

For $t_2 \geq t_1$, since k_t is a supermartingale,

$$\begin{aligned} &E\left[\frac{1}{r}\int_{t_1}^{t_2} (k_{s+r} - k_s)(1 - e^{-\mu s})\, ds|\mathcal{F}_{t_1}\right] \\ &\quad = \frac{1}{r}\int_{t_1}^{t_2} E[(k_{s+r} - k_s)(1 - e^{-\mu s})|\mathcal{F}_{t_1}]\, ds \leq 0. \end{aligned}$$

Letting $r \to 0$, we have by (4.26)

$$E[(k_{t_2} - \eta_{t_2}) - (k_{t_1} - \eta_{t_1})|\mathcal{F}_{t_1}] \leq 0.$$

That is,

$$E[(k_{t_2} - \eta_{t_2})|\mathcal{F}_{t_1}] \leq k_{t_1} - \eta_{t_1}.$$

This proves the assertion. □

By Lemma 4.9, η_s and $k_s^{\mathbf{a}} - \eta_s$ are $\mathcal{F}_s$-supermatingales. To show (2) and (3) above, it suffices thus to show that $\eta_s = \eta_s^{\mathbf{a}}$.

We now write $k_s^{\mathbf{a}} = \int_0^s g_r\,dr + u_s$. Then

$$\begin{aligned}
\eta_s &= \left(\int_0^s g_r\,dr + u_s\right) e^{-\mu s} + \int_0^s \left(\int_0^r g_\theta\,d\theta + u_r\right) \mu e^{-\mu r}\,dr \\
&= \left(\int_0^s g_r\,dr + u_s\right) e^{-\mu s} + \int_0^s u_r \mu e^{-\mu r}\,dr + \int_0^s \left(\int_0^r g_\theta\,d\theta\right) (-e^{-\mu r})'\,dr \\
&= \left(\int_0^s g_r\,dr + u_s\right) e^{-\mu s} + \int_0^s u_r \mu e^{-\mu r}\,dr - \int_0^s g_\theta\,d\theta e^{-\mu s} + \int_0^s g_r e^{-\mu r}\,dr \\
&= \int_0^s (g_r + u_r\mu) e^{-\mu r}\,dr + u_s e^{-\mu s}.
\end{aligned}$$

This means that

$$\eta_s = \int_0^s (e^{-r(\alpha+\mu)}(U(c_r) + v(t-r; x_r^{\mathbf{a}})))\,dr + e^{-(\alpha+\mu)s} v(t-s; x_s^{\mathbf{a}}) = \eta_s^{\mathbf{a},(x,y)}.$$

The assertion of Proposition 4.8 is proved. □

Finally we prove:

Theorem 4.10. *Let τ be an $\mathcal{F}_t$-stopping time, $0 \le \tau \le t$, and let μ be any non-negative, progressively measurable random variable. Then*

$$\begin{aligned}
v(t; x, y) = \sup_{\mathbf{a}\in\mathcal{A}} E\Bigg[\int_0^{\tau\wedge t} & e^{-(\alpha+\mu)s}(U(c_s) + \mu v(t-s; x_s^{\mathbf{a}}))\,ds \\
&+ e^{-(\alpha+\mu)(\tau\wedge t)} v(t-\tau; x_\tau^{\mathbf{a}})\Bigg], \qquad (x, y) \in \mathcal{S}.
\end{aligned} \tag{4.27}$$

Proof. By Proposition 4.8,

$$k_s^{\mathbf{a}} = \int_0^s e^{-r\alpha} U(c_r)\,dr + e^{-\alpha s} v(t-s; x_s^{\mathbf{a}}) + e^{-\alpha s} v(t-s; X_s^{\mathbf{a}}, Y_s^{\mathbf{a}})$$

and

$$\eta_s^{\mathbf{a}} = \int_0^s e^{-(\alpha+\mu)r} \cdot (U(c_r) + \mu v(t-r; X_r^{\mathbf{a}}, Y_r^{\mathbf{a}}))\,dr + e^{-(\alpha+\mu)s} v(t-s; x_s^{\mathbf{a}}),$$

and $k_s^{\mathbf{a}} - \eta_s^{\mathbf{a}}$ are $\mathcal{F}_s$-supermatingales. Hence by the optional stopping time theorem,

$$v(t; x, y) = E[\eta_0^{\mathbf{a}}] \ge E[\eta_\tau^{\mathbf{a}}] \ge E[k_\tau^{\mathbf{a}}] \ge E[k_t^{\mathbf{a}}] = v^{\mathbf{a}}(t; x, y).$$

By taking the supremum,

$$v(t; x, y) \ge \sup_{\mathbf{a}\in\mathcal{A}} E[\eta_\tau^{\mathbf{a}}] \ge \sup_{\mathbf{a}\in\mathcal{A}} v^{\mathbf{a}}(t; x, y) = v(t; x, y).$$

Hence

$$v(t;x,y)=\sup_{\mathbf{a}\in\mathcal{A}} E[\eta_\tau^{\mathbf{a}}].\qquad\square$$

We choose $\mu \equiv 0$ and let $t \to \infty$ in the above, regard $\tau = \tau \wedge t$ for another finite $t > 0$, then Theorem 4.10 implies Lemma 1.5.

5. Proofs of the Lemmas

In this section we prove (1.7), (1.11), Proposition 1.3, and Lemma 2.1, and then provide a theorem (Theorem 5.1) which is used in Section 4 with its proof.

Sketch of Proof for (1.7). We follow Sections 3 and 4 of [6]. With respect to the assumptions on the utility function U, property (U.1) of [6] for the continuity and the concavity is the same. Property (U.2) of [6] on the sublinear growth is satisfied since our U is locally bounded and our variable c_t is bounded from above by M_1 (hence $U(c)$ can be replaced by the one which is sublinear for $c \geq M_1 + 1$).

We put, for $\delta > 0$,

$$H^\delta = \{d \in L^2(m_\delta \otimes P); d \geq 0, \exists(\pi, c, L) \in \mathcal{A} \text{ such that } d \leq c^{(\pi,c,L)}\ m_\delta \otimes P\text{-a.e.}\}. \tag{5.1}$$

Here we put $m_\delta(dt) = e^{-\delta t}dt$ and $c^{(\pi,c',L)} = c'$. We put

$$I(d_\cdot) = -E\left[\int_0^\infty e^{-\alpha t}U(d_t)\,dt\right]. \tag{5.2}$$

Just as in Lemmas 4.1–4.3 of [6], we have the following assertions:

(1) H^δ is a non-empty, bounded, and convex subspace of $L^2(m_\delta \otimes P)$.
(2) H^δ is closed in $L^2(m_\delta \otimes P)$.
(3) If $\alpha > \delta/2$, then the map I: $\mathrm{H}^\delta \to \mathbf{R}$ is proper, convex, and lower-semi-continuous with respect to the $L^2(m_\delta \otimes P)$-norm.

Under these results we can conclude by using the result of Ekeland–Temam [14, Section V.3] that for $\alpha > \delta/2$ there exists a $d^* \in H^\delta$ such that

$$\inf_{d\in H^\delta} I(d) = I(d^*). \tag{5.3}$$

As in [6], we can show that

$$v(x,y) = E\left[\int_0^\infty e^{-\alpha t}U(c_t^{(\pi^*,c^*,L^*)})\,dt\right], \tag{5.4}$$

where $(\pi^*, c^*, L^*) \in \mathcal{A}$ is the one which appeared in (5.1) associated to d^*, and that

$$d^* = c^{(\pi^*,c^*,L^*)}. \tag{5.5}$$

In our setting $\delta > 0$ can be chosen arbitrarily small, hence the assertion follows. $\square$

Proof of (1.11). Since $dY_t = -\beta Y_t\,dt + \beta\,dC_t$, we have by the Itô formula

$$
\begin{aligned}
&E[e^{-\alpha t}\tilde{v}(X_t, Y_t)] \\
&\quad = \tilde{v}(x, y) + E\Bigg[\int_0^t e^{-\alpha s}\Bigg(-\alpha\tilde{v}\,ds + \tilde{v}_x\,dX_s + \tilde{v}_y\,dY_s \\
&\qquad\qquad + \int (\tilde{v}(x + \pi x(e^z - 1), y) - \tilde{v}(x, y) \\
&\qquad\qquad - \tilde{v}_x(x, y)\pi x(e^z - 1))\,d\nu(z)\,ds\Bigg)\Bigg|_{x=X_s, y=Y_s}\Bigg] \\
&\quad = \tilde{v}(x, y) + E\Bigg[\int_0^t e^{-\alpha s}\Bigg\{-\alpha\tilde{v}\,ds \\
&\qquad\qquad + \Bigg(\tilde{v}_x r x - \tilde{v}_x c_s + \beta\tilde{v}_y c_s - \beta y\tilde{v}_y + \tilde{v}_x\pi x(\hat{b} - r) \\
&\qquad\qquad + \int (\tilde{v}(x + \pi x(e^z - 1), y) \\
&\qquad\qquad - \tilde{v}(x, y) - \tilde{v}_x(x, y)\pi x(e^z - 1))\,d\nu(z)\Bigg)\,ds \\
&\qquad\qquad + \tilde{v}_x(x, y)dL_s\Bigg\}\Bigg|_{x=X_s, y=Y_s}\Bigg].
\end{aligned}
$$

If it holds that $N\tilde{v} \le 0$, $M\tilde{v} \le 0$, and that

$$
\begin{aligned}
&-\alpha\tilde{v} + \tilde{v}_x r x - \beta y\tilde{v}_y + \max_{0\le\pi\le 1}\Bigg\{\tilde{v}_x\pi x(\hat{b} - r) + \int (\tilde{v}(x + \pi x(e^z - 1), y) \\
&\qquad - \tilde{v}(x, y) - \tilde{v}_x(x, y)\pi x(e^z - 1))\,d\nu(z)\Bigg\} \\
&\quad + \sup_c (U(c) - c(\tilde{v}_x - \beta\tilde{v}_y)(x, y)) \le 0 \qquad \text{in } \mathcal{S},
\end{aligned}
$$

then it is easy to see that $v \le \tilde{v}$ holds by letting $t \to \infty$ since $\tilde{v} \in C_l(\bar{\mathcal{S}})$. □

Proof of Proposition 1.3. We observe

$$
\begin{aligned}
X_s^{x_1} - X_s^{x_2} &= x_1 - x_2 + \int_0^t \int \pi_{u-}(X_s^{x_1} - X_s^{x_2})(e^z - 1)\tilde{N}(ds\,dz) \\
&\quad + \int_0^t (r + (\hat{b} - r)\pi_s)(X_s^{x_1} - X_s^{x_2})\,ds \\
&= I_1 + I_2 + I_3 \qquad \text{(say)}.
\end{aligned} \tag{5.6}
$$

By the Burkholder–Davis–Gundy (BDG) inequality,

$$
\begin{aligned}
E\left[\sup_{t<T}|I_2(t)|^2\right] &\le K\int_0^T E\left[\sup_{u<t}|X_u^{x_1} - X_u^{x_2}|^2\right] dt|z|^2\nu(dz) \\
&\le K'\int_0^T E\left[\sup_{u<t}|X_u^{x_1} - X_u^{x_2}|^2\right] dt
\end{aligned}
$$

and

$$E\left[\sup_{t<T}|I_3(t)|^2\right] \le K\int_0^T E\left[\sup_{u<t}|X_u^{x_1} - X_u^{x_2}|^2\right] dt.$$

Hence

$$\begin{aligned} E\left[\sup_{t<T}|X_t^{x_1} - X_t^{x_2}|^2\right] &\le K_1|x_1 - x_2|^2 + K_2\int_0^T E\left[\sup_{u<t}|X_u^{x_1} - X_u^{x_2})|^2\right] dt \\ &\le K_1|x_1 - x_2|^2 e^{K_2.T} \end{aligned} \tag{5.7}$$

by Gronwall's lemma. Here, and below, the constants K, K', K_1, K_2 do not depend on T, (π_t, c_t, L_t), nor x, y.

Similarly, we have

$$E\left[\sup_{t<T}|Y_t^{y_1} - Y_t^{y_2}|^2\right] \le K_1|y_1 - y_2|^2 e^{K_2.T}. \tag{5.8}$$

On the other hand, we have by the Bellman principle (2), for $0 \le t < s \le T$,

$$\begin{aligned} &0 \le v(s; x, y) - v(t; x, y) \\ &= \sup_{(\pi,c,L)\in\mathcal{A}|_{[0,T]}} E\left[\int_0^{s-t} e^{-\alpha u}U(c_u)\,du + e^{-\alpha(s-t)}(v(t; X_{s-t}^x, Y_{s-t}^y) - v(t; x, y))\right. \\ &\qquad\left. + (e^{-\alpha(s-t)} - 1)v(s; x, y)\right]. \end{aligned}$$

Since $0 \le 1 - e^{-\alpha h} \le \alpha h$,

$$\begin{aligned} &|v(t; x, y) - v(s; x, y)| \\ &\quad\le K\left\{\int_0^{s-t} U(M_1)\,du + K' \sup_{(\pi,c,L)} E[|(X_{s-t}^x, Y_{s-t}^y) - (x, y)|]\right\} \end{aligned} \tag{5.9}$$

by (5.7), (5.8).

The right-hand side of (5.9) tends to 0 as $|s - t| \to 0$ uniformly to (x, y) by the right continuity of $(X_\cdot^x, Y_\cdot^y)$. This and the local boundedness of U leads to assertion (1).

Assertion (2) for the continuity with respect to (x, y) uniformly to t follows from (5.7)–(5.8) and the local boundedness of $U(\cdot)$.

Assertion (3) for $G_t u(x, y)$ follows similarly. □

Proof of Lemma 2.1. We denote the position of (X_t^*, Y_t^*) after the initial jump from (x, y) caused by $(\Delta L; \Delta L > 0)$, by

$$(\hat{X}, \hat{Y}) = (\hat{X}(\Delta L), \hat{Y}(\Delta L)) = (x + \Delta L, y).$$

By the Bellman principle, we may assume without loss of generality that $(\hat{X}, \hat{Y}) \in \overline{B_r \cap \mathcal{S}}$ for some $r > 0$.

We have by the Bellman principle

$$v(x,y) = E[v(\hat{X},\hat{Y})] = \int_{A_L} v(\hat{X},\hat{Y})\,dP + \int_{\Omega\setminus A_L} v(x,y)\,dP.$$

Hence

$$\int_{A_L} (v(\hat{X},\hat{Y}) - v(x,y))\,dP = 0. \tag{5.10}$$

Since $v \le \varphi$ and $(v-\varphi)(x,y) = 0$,

$$\int_{A_L} (\varphi(\hat{X},\hat{Y}) - \varphi(x,y))\,dP \ge 0. \tag{5.11}$$

We denote ΔL_t at $t=0$ by ε_L. By the assumption,

$$\varphi(\hat{X}(\varepsilon_L),\hat{Y}(\varepsilon_L)) \le \varphi(\hat{X}(\varepsilon),\hat{Y}(\varepsilon)) \tag{5.12}$$

for $0 < \varepsilon \le \varepsilon_L$.

Suppose first $\varepsilon_L > 0$. Then

$$\varphi_x(x,y)\cdot P(A_L) \ge 0 \qquad \text{for} \quad \varepsilon \le \varepsilon_L. \tag{5.13}$$

Indeed, by (5.11) and (5.12)

$$\int_{A_L} (\varphi(\hat{X}(\varepsilon),\hat{Y}(\varepsilon)) - \varphi(x,y))\,dP \ge 0$$

for $\varepsilon \le \varepsilon_L$. Hence by Fatou's lemma

$$\int_{A_L} \limsup_{\varepsilon\to 0} \left(\frac{1}{\varepsilon}(\varphi(x+\varepsilon,y) - \varphi(x,y))\right)\,dP \ge 0.$$

Hence (5.13) follows.

Hence, in view of the assumption of Lemma 2.1 and (5.13), we have

$$P(A_L) = 0$$

as long as $\varepsilon_L > 0$. □

Theorem 5.1. *Let $\xi\colon \Omega \to \mathbf{R}$ be an $\mathcal{F}_0$-measurable function such that $E[|\xi|^2] < +\infty$, and let $\zeta\colon \Omega \to \mathbf{A}$ be another $\mathcal{F}_0$-measurable function. We put, for $\mathbf{b} = (\pi, c, L)$,*

$$x_t^{\mathbf{b},x} = x + \int_0^t b(\mathbf{b}, x_s^{\mathbf{b},x})\,ds + \int_0^{t+}\int h^{\mathbf{b}}(x_s^{\mathbf{b},x}, z)\tilde{N}(ds\,dz), \tag{5.14}$$

$$y_t = \xi + \int_0^t b(\xi, y_s)\,ds + \int_0^{t+}\int h^{\zeta}(y_s, z)\tilde{N}(ds\,dz).$$

Let the mapping $F\colon \mathbf{A} \times D \to [0, \infty)$ *be a* $(B_A \times B_D)$*-measurable function. Then we have*

$$E[F(\zeta, y)|\mathcal{F}_0] = E[F(\mathbf{b}, x_{\cdot}^{\mathbf{b},x})]|_{\mathbf{b}=\zeta, x=\xi}. \tag{5.15}$$

Proof. First we prepare the following lemma.

Lemma 5.2. *Let* $\mathbf{b}$ *be* $\mathcal{F}_0$*-measurable. Then* $x_t^{\mathbf{b},x}$ *is independent of* $\mathcal{F}_0$.

This lemma can be shown by an approximating argument. See the Appendix of [34] for the details.

Let $\{\beta(i); i = 1, 2, 3, \ldots\}$ be any countable dense subset of $\mathbf{A}$. For each $\beta \in \mathbf{A}$, we put

$$i_n(\beta) \equiv \min\{j; d(\beta, \beta(j)) \le 2^{-n}\}$$

and

$$\bar{k}_n(\beta) \equiv \beta(i_n(\beta)).$$

For each $x \in \mathbf{R}$, let $k_n(x)$ be defined as $k_n(x) = j/2^n$ if $j/2^n < x \le (j+1)/2^n$ for $j = 0, \pm 1, \pm 2, \ldots$.

It is sufficient to prove the assertion in the case, for $t_1, \ldots, t_m \in [0, \infty)$,

$$E[F(\zeta, y_{t_1}, \ldots, y_{t_m})|\mathcal{F}_0] = E[F(\mathbf{b}, x_{t_1}^{\mathbf{b},x}, \ldots, x_{t_m}^{\mathbf{b},x})]|_{\mathbf{b}=\zeta, x=\xi}, \tag{5.16}$$

where $F(\beta, x_1, x_2, \ldots, x_m)$ is bounded continuous in $(\beta, x_1, x_2, \ldots, x_m) \in \mathbf{A} \times \mathbf{R}^{2m}$. We divide the proof into four steps.

Step 1. *We have*

$$\begin{aligned} &E[F(\bar{k}_n(\zeta), x_{t_1}^{\bar{k}_n(\zeta), k_n(\xi)}, \ldots, x_{t_m}^{\bar{k}_n(\zeta), k_n(\xi)})|\mathcal{F}_0] \\ &\quad = E[F(\beta, x_{t_1}^{\beta,x}, \ldots, x_{t_m}^{\beta,x})]|_{\beta=\bar{k}_n(\zeta), x=k_n(\xi)}. \end{aligned}$$

Let $\Gamma_n \equiv \{k_n(x); x \in \mathbf{R}\}$. Then

$$\begin{aligned} \text{L.H.S.} &= E\left[\sum_{\beta\in\{\beta(j)\}} \sum_{x\in\Gamma_n} 1_{\{\bar{k}_n(\zeta)=\beta\}} 1_{\{k_n(\xi)=x\}} F(\beta, x_{t_1}^{\beta,x}, \ldots, x_{t_m}^{\beta,x})|\mathcal{F}_0\right] \\ &= \sum_{\beta\in\{\beta(j)\}} \sum_{x\in\Gamma_n} 1_{\{\bar{k}_n(\zeta)=\beta\}} 1_{\{k_n(\xi)=x\}} F(\beta, x_{t_1}^{\beta,x}, \ldots, x_{t_m}^{\beta,x})|\mathcal{F}_0. \end{aligned} \tag{5.17}$$

Since ζ, ξ are $\mathcal{F}_0$-measurable, we have, by Lemma 5.2,

$$\text{L.H.S.} = \sum_{\beta\in\{\beta(j)\}} \sum_{x\in\Gamma_n} 1_{\{\bar{k}_n(\zeta)=\beta\}} 1_{\{k_n(\xi)=x\}} F(\beta, x_{t_1}^{\beta,x}, \ldots, x_{t_m}^{\beta,x}) = \text{R.H.S.}$$

Step 2. *There exists a subsequence* (n') *such that*

$$P\left(\lim_{n'\to\infty}\sup_{0\le t\le T}|x_t^{\bar{k}_{n'}(\zeta),k_{n'}(\xi)}-y_t|>0\right)=0. \tag{5.18}$$

Indeed, we have

$$\begin{aligned}
&1_{\{\bar{k}_n(\zeta)=\beta,k_n(\xi)=x\}}\cdot x_t^{\beta,x}\\
&\quad=1_{\{\bar{k}_n(\zeta)=\beta,k_n(\xi)=x\}}\cdot x+\int_0^t 1_{\{\bar{k}_n(\zeta)=\beta,k_n(\xi)=x\}}b(\beta,x_s^{\beta,x})\,ds\\
&\qquad+\int_0^t\int 1_{\{\bar{k}_n(\zeta)=\beta,k_n(\xi)=x\}}h^{\bar{k}_n(\zeta)}(x_s^{\bar{k}_n(\zeta),k_n(\xi)},z)\tilde{N}(ds\,dz).
\end{aligned}$$

Hence

$$\begin{aligned}
x_t^{\bar{k}_n(\zeta),k_n(\xi)}&=k_n(\xi)+\int_0^t b(\bar{k}_n(\zeta),x_s^{\bar{k}_n(\zeta),k_n(\xi)})\,ds\\
&\quad+\int_0^t\int h^{\bar{k}_n(\zeta)}(x_s^{\bar{k}_n(\zeta),k_n(\xi)},z)\tilde{N}(ds\,dz).
\end{aligned}$$

By this and the definition of y_t,

$$\begin{aligned}
&E\left[\sup_{0\le t\le T}|x_t^{\bar{k}_n(\zeta),k_n(\xi)}-y_t|^2\right]\\
&\quad\le 2E[|k_n(\zeta)-\xi|^2]\\
&\qquad+C_T\int_0^T\{E[|k_n(\xi)-\xi|^2]+E[|b(k_n(\xi),y_s)-b(\zeta,y_s)|^2]\\
&\qquad\qquad+E[|h^{\bar{k}_n(\zeta)}(y_s,z)-h^{\zeta}(y_s,z)|^2\nu(dz)\}\,ds.
\end{aligned}$$

Hence by Gronwall's inequality,

$$E\left[\sup_{0\le t\le T}|x_t^{\bar{k}_n(\zeta),k_n(\xi)}-y_t|^2\right]\to 0$$

as $n\to\infty$. Hence we have the assertion.

Step 3. *We put* $\Phi(\beta,x)\equiv E[F(\beta,x_{t_1}^{\beta,x},\ldots,x_{t_m}^{\beta,x})]$. *Then* $(\beta,x)\mapsto\Phi(\beta,x)$ *is continuous*. Assume $(\beta_k,x_k)\to(\beta,x)$. We have as in the calculation at Proposition 4.1 that

$$E\left[\sup_{0\le t\le T}|x_t^{\beta_k,x_k}-x_t^{\beta,x}|^2\right]\to 0$$

as $k\to\infty$. Since $F(\cdot,\cdot)$ is bounded continuous in $(\beta,x_1,\ldots,x_m)$, we have the assertion.

Step 4.

$$E[F(\zeta,y_{t_1},\ldots,y_{t_m})|\mathcal{F}_0]=E[F(\beta,x_{t_1}^{\beta,x},\ldots,x_{t_m}^{\beta,x})]|_{\beta=\zeta,x=\xi}. \tag{5.19}$$

We have by Step 2

$$E[F(\zeta, y_{t_1}, \dots, y_{t_m})|\mathcal{F}_0] = E\left[\lim_{n'\to\infty} F(\bar{k}_{n'}(\zeta), x_{t_1}^{\bar{k}_{n'}(\zeta),k_{n'}(\xi)}, \dots, x_{t_m}^{\bar{k}_{n'}(\zeta),k_{n'}(\xi)})|\mathcal{F}_0\right]$$
$$= \lim_{n'\to\infty} E[F(\bar{k}_{n'}(\zeta), x_{t_1}^{\bar{k}_{n'}(\zeta),k_{n'}(\xi)}, \dots, x_{t_m}^{\bar{k}_{n'}(\zeta),k_{n'}(\xi)})|\mathcal{F}_0].$$

By Step 1,

$$\text{R.H.S.} = \lim_{n'\to\infty} E[F(\beta, x_{t_1}^{\beta,x}, \dots, x_{t_m}^{\beta,x})]|_{\beta=\bar{k}_{n'}(\zeta), x=k_{n'}(\xi)}$$

which is equal to

$$E[F(\beta, x_{t_1}^{\beta,x}, \dots, x_{t_m}^{\beta,x})]|_{\beta=\zeta, x=\xi}$$

by Step 3. □

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