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Dirichlet Boundary Control of Semilinear Parabolic Equations Part 2: Problems with Pointwise State Constraints

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Abstract. This paper is the continuation of the paper "Dirichlet boundary control of semilinear parabolic equations. Part 1: Problems with no state constraints." It is concerned with an optimal control problem with distributed and Dirichlet boundary controls for semilinear parabolic equations, in the presence of pointwise state constraints. We first obtain approximate optimality conditions for problems in which state constraints are penalized on subdomains. Next by using a decomposition theorem for some additive measures (based on the Stone–Čech compactification), we pass to the limit and recover Pontryagin's principles for the original problem.

Key Words. Dirichlet boundary control, Unbounded distributed control, Pointwise state-constraint, Semilinear parabolic equation, Pontryagin's principle, Stone–Čech compactification.

AMS Classification. 49K20, 93C20, 35K20.

1. Introduction

This paper is the continuation of the paper "Dirichlet boundary control of semilinear parabolic equations. Part 1: Problems with no state constraints" [3]. In this part we study control problems for equations and cost functionals similar to Part 1, but with pointwise state constraints. Notation and assumptions are the ones of Part 1. Recall the state equation

$$\frac{\partial y}{\partial t}+Ay+\Phi(x,t,y,u)=0 \quad \text{in } Q, \qquad y=v \quad \text{on } \Sigma, \qquad y(0)=y_0 \quad \text{in } \Omega, \qquad (1)$$

where $Q = \Omega \times]0, T[$, $\Omega \subset \mathbb{R}^N$, $\Sigma = \Gamma \times]0, T[$, Γ is the boundary of Ω, $T > 0$, A is a second-order elliptic operator, the distributed control u belongs to $U_{\text{ad}} \subset L^q(Q)$, and the boundary control v belongs to $V_{\text{ad}} \subset L^\infty(\Sigma)$ (for simplicity, we here suppose that the initial conditions y_0 is fixed and belongs to $C(\bar{\Omega})$). We look for solutions of (1) satisfying constraints of the form

$$g(y) \in \mathcal{C}, \tag{2}$$

where g is a mapping from $C_{\text{b}}(\bar{Q}\backslash\bar{\Sigma})$ into $C_{\text{b}}(\bar{Q}\backslash\bar{\Sigma})$, and $\mathcal{C} \subset C_{\text{b}}(\bar{Q}\backslash\bar{\Sigma})$ is a closed convex subset with a nonempty interior in $C_{\text{b}}(\bar{Q}\backslash\bar{\Sigma})$. Consider the problem

$$(P) \quad \inf\{J(y, u, v) \mid (y, u, v) \in C_{\text{b}}(\bar{Q}\backslash\bar{\Sigma}) \times U_{\text{ad}} \times V_{\text{ad}} \text{ satisfying (1) and (2)}\},$$

with

$$J(y, u, v) = \int_Q F(x, t, y, u)\, dx\, dt + \int_\Sigma G(s, t, v)\, ds\, dt + \int_\Omega L(x, y(T))\, dx.$$

We have already obtained optimality conditions for Dirichlet boundary control problems of the form (P), when V_{ad} is convex and $G(s, t, \cdot)$ is differentiable, by using a Lagrange multiplier theorem [1]. Here, we are mainly interested in optimality conditions in the form of Pontryagin's principles.

As pointed out in [8] and [15], the main difficulty in proving optimality conditions for (P) is the following:

> Since the state constraint (2) is well posed in $C_{\text{b}}(\bar{Q}\backslash\bar{\Sigma})$ (or in $L^\infty(Q)$), the multiplier associated with this constraint belongs to $(C_{\text{b}}(\bar{Q}\backslash\bar{\Sigma}))'$ (or to $(L^\infty(Q))'$). Therefore, it is a finitely additive measure (and not a σ-additive one) and the corresponding adjoint equation cannot be studied in Sobolev spaces.

To bypass this difficulty, Fattorini and Murphy [8] consider a terminal constraint in $\Omega_\tau = \{x \in \Omega \mid d(x, \Gamma) \geq \tau\}$, with $\tau > 0$ (but the passage to the limit when τ tends to zero is not carried out). Mordukhovich and Zhang [15], [16] obtain an adjoint equation as the limit of adjoint equations for penalized problems, but the limit equation cannot be interpreted in the sense of distributions.

In [2] we have presented a new tool to overcome this kind of difficulty. By introducing the Stone–Čech compactification of the domain $\bar{Q}\backslash\bar{\Sigma}$, we have obtained a decomposition theorem for additive measures $\zeta \in (C_{\text{b}}(\bar{Q}\backslash\bar{\Sigma}))'$ (see Theorem 2.1). Roughly speaking, each $\zeta \in (C_{\text{b}}(\bar{Q}\backslash\bar{\Sigma}))'$ is decomposed in a regular part, which is a bounded Radon measure on $\bar{Q}\backslash\bar{\Sigma}$, and an additional part, which acts on the boundary $\bar{\Sigma}$. Due to this decomposition, we prove that only the regular part intervenes in the adjoint equation. The additional part intervenes only in Pontryagin's principle for the boundary control (Theorem 2.3).

As a consequence of these new optimality conditions, in the case of bilateral constraints of the form $a \leq y \leq b$ on Q (where a and b are continuous on $\bar{Q}$), we are able to obtain classical pointwise Pontryagin's principles (in other words, the nonregular part of the multiplier ζ associated with state constraints may be dropped out in the optimality conditions, see Theorem 6.1).

Optimality conditions in Theorem 2.3 are proved with the Ekeland variational principle. For this, we define a sequence of approximate problems in which state constraints are penalized only on $\overline{Q^{\tau_k}} = \{x \in \Omega \mid d(x,\Gamma) > \tau_k\} \times]\tau_k, T[$, with $\lim_k \tau_k = 0$ (see Section 5.2). Due to a suitable choice of a distance on the set of controls, which is different from the Ekeland distance, we obtain approximate optimality conditions (see Theorem 5.2) by using Taylor expansions stated in Theorem 5.2 of [3]. In these approximate optimality conditions, the multiplier associated with the penalization of state constraints is a Radon measure on $\overline{Q^{\tau_k}}$. Due to our decomposition theorem for additive measures in $(C_b(\bar{Q}\backslash\bar{\Sigma}))'$, to some stability condition (assumption **A7**, see also Proposition 2.4), and to the choice of the metric of the control set, we can pass to the limit in approximate optimality conditions when τ_k tends to zero.

Contrary to Theorem 2.1 of [1], we need neither differentiability assumptions on $\Phi(x,t,y,\cdot)$, $F(x,t,y,\cdot)$, and $G(s,t,y,\cdot)$, nor a convexity assumption on V_{ad}.

2. Assumptions and Main Result

Notation and assumptions **A1–A4** are the ones of Part 1 [3]. For the convenience of the reader, recall the notation $\Omega_\tau = \{x \in \Omega \mid d(x,\Gamma) > \tau\}$ (d is the Euclidean distance) and $Q^\tau = \Omega_\tau \times]\tau, T[$. Throughout what follows $\mathcal{L}^{N+1}$ denotes the $(N+1)$-dimensional Lebesgue measure and $\mathcal{L}^N$ denotes the N-dimensional Lebesgue measure. For simplicity, $\langle\cdot,\cdot\rangle_{*,\bar{Q}\backslash\bar{\Sigma}}$ stands for the duality pairing between $(C_b(\bar{Q}\backslash\bar{\Sigma}))'$ and $C_b(\bar{Q}\backslash\bar{\Sigma})$. If $\mu \in \mathcal{M}_b(\bar{Q}\backslash\bar{\Sigma})$ (the space of bounded Radon measures on $\bar{Q}\backslash\bar{\Sigma}$) and $y \in C_b(\bar{Q}\backslash\bar{\Sigma})$, we set $\langle\mu, y\rangle_{b,\bar{Q}\backslash\bar{\Sigma}} = \int_{\bar{Q}\backslash\bar{\Sigma}} y(x,t)\,d\mu(x,t)$.

2.1. *Additional Assumptions*

In addition to assumptions **A1–A4** of Part 1 [3], the following is assumed.

A5. L is a Carathéodory function from $\Omega \times \mathbb{R}$ into $\mathbb{R}$. For almost all $x \in \Omega$, $L(x,\cdot)$ is of class C^1. The following estimates hold:

$$|L(x,y)| \le L_1(x)\eta(|y|), \qquad |L'_y(x,y)| \le L_2(x)\eta(|y|),$$

where $L_1 \in L^1(\Omega)$, $L_2 \in L^p(\Omega)$, $p > 1$ is the same exponent as in **A3**, and η is as in **A2**. In addition, we assume that (P) admits solutions.

A6. In (2), g is a mapping of class C^1 from $C_b(\bar{Q}\backslash\bar{\Sigma})$ into $C_b(\bar{Q}\backslash\bar{\Sigma})$. Moreover, for all $M > 0$, there exist $\tau_0 > 0$, $0 < \gamma_0 \le 1$, and $0 < \gamma_1 \le 1$ such that

$$\|g(y_1) - g(y_2)\|_{C(\overline{Q^\tau})} \le C_1(M)(\|y_1 - y_2\|_{C(\overline{Q^\tau})} + \tau^{\gamma_0}) \qquad \text{for all} \quad 0 < \tau \le \tau_0,$$
$$\|g'(y_1) - g'(y_2)\|_{\mathcal{L}(C_b(\bar{Q}\backslash\bar{\Sigma}))} \le C_2(M)\|y_1 - y_2\|^{\gamma_1}_{C_b(\bar{Q}\backslash\bar{\Sigma})},$$

for all $y_1 \in C_b(\bar{Q}\backslash\bar{\Sigma})$, all $y_2 \in C_b(\bar{Q}\backslash\bar{\Sigma})$ satisfying $\|y_1\|_{C_b(\bar{Q}\backslash\bar{\Sigma})} \le M$, $\|y_2\|_{C_b(\bar{Q}\backslash\bar{\Sigma})} \le M$. ($\mathcal{L}(C_b(\bar{Q}\backslash\bar{\Sigma}))$ denotes the space of linear continuous mapppings from $C_b(\bar{Q}\backslash\bar{\Sigma})$ into $C_b(\bar{Q}\backslash\bar{\Sigma})$.)

We give some examples of state constraints satisfying **A6**.

Example 1. Let φ be a bounded continuous function on $(\bar{Q}\backslash\bar{\Sigma}) \times \mathbb{R}$. Suppose that φ'_y belongs to $C_b((\bar{Q}\backslash\bar{\Sigma}) \times \mathbb{R})$ and that, for some $\gamma_1 \in]0, 1]$, we have

$$|\varphi'_y(x, t, y_1) - \varphi'_y(x, t, y_2)| \le C|y_1 - y_2|^{\gamma_1}$$
$$\text{for all} \quad (x, t) \in \bar{Q}\backslash\bar{\Sigma} \quad \text{and all} \quad (y_1, y_2) \in \mathbb{R}^2.$$

The state constraint

$$\varphi(x, t, y(x, t)) \le 0 \qquad \text{on} \quad \bar{Q}\backslash\bar{\Sigma} \tag{3}$$

is of the form (2) by setting $\mathcal{C} = \{z \in C_b(\bar{Q}\backslash\bar{\Sigma}) \mid z \le 0 \text{ in } \bar{Q}\backslash\bar{\Sigma}\}$ and $g(y)(x, t) = \varphi(x, t, y)$. Moreover, **A6** is satisfied. Observe that if $a \in C_b(\bar{Q}\backslash\bar{\Sigma})$ and $b \in C_b(\bar{Q}\backslash\bar{\Sigma})$ satisfy $a + \varepsilon \le b$ on $\bar{Q}\backslash\bar{\Sigma}$ (for some $\varepsilon > 0$), then the constraints

$$a(x, t) \le y(x, t) \le b(x, t) \qquad \text{on} \quad \bar{Q}\backslash\bar{\Sigma}$$

may be written in the form (3).

Example 2. We can also construct other situations mixing pointwise and integral constraints. For example, the mapping $g\colon\ y \longrightarrow y \int_Q y(x, t)\, dx\, dt$ satisfies **A6**.

2.2. *Stability Conditions*

For every $\tau > 0$, set

$$(P_\tau) \quad \inf\left\{ J(y_{uv}, u, v) \mid (u, v) \in U_{\text{ad}} \times V_{\text{ad}} \text{ and } \inf_{z\in\mathcal{C}} \|z - g(y_{uv})\|_{C(\overline{Q^\tau})} = 0 \right\},$$

where y_{uv} is the solution of (1) corresponding to (u, v). We say that (P) is weakly stable on the right if assumption **A7** holds.

A7. $\inf(P) = \lim_{\tau\searrow 0} \inf(P_\tau)$.

Assumption **A7** is satisfied in classical situations (see Proposition 2.4), and it is weaker than conditions ensuring the properness of the relaxation procedure by Young measures (see [4]). More precisely, if we associate with (P) a relaxed problem (RP) defined with Young measures, taking advantage of the linear structure of the relaxed problem with respect to control variables, we can obtain optimality conditions for (RP) in an easy way. We recover optimality conditions for (P) if $\inf(P) = \min(RP)$. In [4] we have proved that this properness condition is satisfied if, and only if,

$$\inf(P) = \sup_{\delta>0} \sup_{\tau>0} (\inf(P_{\delta,\tau})), \tag{4}$$

where

$$(P_{\tau,\delta}) \quad \inf\{J(y_{uv}, u, v) \mid (u, v) \in U_{\text{ad}} \times V_{\text{ad}} \text{ and } \inf_{z\in\mathcal{C}} \|z - g(y_{uv})\|_{C(\overline{Q^\tau})} \le \delta\}.$$

Observe that condition (4) is stronger than **A7**.

2.3. *The Stone–Čech Compactification*

As proved in [2], every $\zeta \in (C_b(\bar{Q}\backslash\bar{\Sigma}))'$ may be identified with a measure $\hat{\zeta} \in \mathcal{M}(\bar{Q} \times (\bar{Q}\backslash\bar{\Sigma})^\#)$, where $(\bar{Q}\backslash\bar{\Sigma})^\#$ denotes the Stone–Čech compactification of $\bar{Q}\backslash\bar{\Sigma}$. (For notational simplicity, we identify $\hat{\zeta}$ and ζ.) We denote by π the canonical projection from $\mathcal{M}(\bar{Q} \times (\bar{Q}\backslash\bar{\Sigma})^\#)$ onto $\mathcal{M}(\bar{Q})$ defined by

$$\pi\colon \zeta \in \mathcal{M}(\bar{Q} \times (\bar{Q}\backslash\bar{\Sigma})^\#) \longrightarrow \pi_\zeta \in \mathcal{M}(\bar{Q}),$$

$$\langle \pi_\zeta, \varphi\rangle_{\mathcal{M}(\bar{Q})\times C(\bar{Q})} = \langle \zeta, \varphi\rangle_{\mathcal{M}(\bar{Q}\times(\bar{Q}\backslash\bar{\Sigma})^\#)\times C(\bar{Q}\times(\bar{Q}\backslash\bar{\Sigma})^\#)} \qquad \text{for all} \quad \varphi \in C(\bar{Q}).$$

Throughout what follows, if $\zeta \in (C_b(\bar{Q}\backslash\bar{\Sigma}))'$, then $|\zeta|$ stands for the total variation of ζ.

Theorem 2.1 [2, Corollary 4.8]. *Let $\zeta \in (C_b(\bar{Q}\backslash\bar{\Sigma}))'$, there exists a bounded linear transformation $\Lambda_\zeta\colon C_b(\bar{Q}\backslash\bar{\Sigma}) \longrightarrow L^\infty_{\pi_{|\zeta|}}(\bar{Q})$ such that*

$$\langle \zeta, h\varphi\rangle_{*,\bar{Q}\backslash\bar{\Sigma}} = \langle \pi_\zeta, h\varphi\rangle_{b,\bar{Q}\backslash\bar{\Sigma}} + \int_{\bar{\Sigma}} \Lambda_\zeta(h)\varphi\, d\pi_{|\zeta|} \tag{5}$$

for all $(h, \varphi) \in C_b(\bar{Q}\backslash\bar{\Sigma}) \times C(\bar{Q})$. If $\tilde{h}$ is a nonnegative function in $C_b(\bar{Q}\backslash\bar{\Sigma})$, then $\int_{\bar{\Sigma}} \Lambda_\zeta(\tilde{h})\, d\pi_{|\zeta|} \geq 0$. Moreover, for every $\tilde{h}$ in $C(\bar{Q})$, we have $\int_{\bar{\Sigma}} \Lambda_\zeta(\tilde{h})\, d\pi_{|\zeta|} = \langle \pi_\zeta, \tilde{h}\rangle_{\mathcal{M}(\bar{\Sigma})\times C(\bar{\Sigma})}$.

Remark 2.2. Since $C(\bar{Q})$ is dense in $L^1_{\pi_{|\zeta|}}(\bar{Q})$, it is clear that for a given ζ, identity (5) uniquely defines Λ_ζ.

2.4. *Statement of the Main Result*

Define the Hamiltonian functions:

$$H_Q(x, t, y, u, p, \lambda) = \lambda F(x, t, y, u) + p\Phi(x, t, y, u)$$
$$\text{for all} \quad (x, t, y, u, p, \lambda) \in Q \times \mathbb{R}^4,$$
$$H_\Sigma(s, t, v, p, \lambda) = \lambda G(s, t, v) + pv \qquad \text{for all} \quad (s, t, v, p, \lambda) \in \Sigma \times \mathbb{R}^3.$$

Theorem 2.3. *If* **A1–A7** *are fulfilled and if $(\bar{y}, \bar{u}, \bar{v})$ is a solution of (P), then there exist $\bar{\lambda} \geq 0$, $\bar{p} \in L^1(0, T; W_0^{1,1}(\Omega))$, and $\bar{\zeta} \in (C_b(\bar{Q}\backslash\bar{\Sigma}))'$, such that the following conditions hold*:

- *Nontriviality condition*:

$$(\bar{\zeta}, \bar{\lambda}) \neq 0. \tag{6}$$

- *Complementarity condition*:

$$\langle \bar{\zeta}, z - g(\bar{y})\rangle_{*,\bar{Q}\backslash\bar{\Sigma}} = \langle \pi_{\bar{\zeta}}, z - g(\bar{y})\rangle_{b,\bar{Q}\backslash\bar{\Sigma}} + \int_{\bar{\Sigma}} \Lambda_{\bar{\zeta}}[z - g(\bar{y})]\, d\pi_{|\bar{\zeta}|} \leq 0 \tag{7}$$

for all $z \in \mathcal{C}$, where $\Lambda_{\bar{\zeta}}$ is the operator associated with $\bar{\zeta}$, defined in Theorem 2.1.

- *Adjoint equation*:

$$\begin{cases} -\dfrac{\partial \bar p}{\partial t} + A\bar p + \Phi'_y(x,t,\bar y,\bar u)\bar p + \bar\lambda F'_y(x,t,\bar y,\bar u) \\ \quad + [g'(\bar y)^*\pi_{\bar\zeta}]|_Q = 0 \qquad \text{in } Q, \\ \bar p(x,T) + \bar\lambda L'_y(x,\bar y(T)) + [g'(\bar y)^*\pi_{\bar\zeta}]|_{\Omega_T} = 0 \qquad \text{in } \Omega, \end{cases} \tag{8}$$

where $g'(\bar y)^*\pi_{\bar\zeta}$ *denotes the Radon measure on* $\bar Q\backslash\bar\Sigma$ *defined by* $\langle g'(\bar y)^*\pi_{\bar\zeta}, z\rangle_{b,\bar Q\backslash\bar\Sigma} = \langle \pi_{\bar\zeta}, g'(\bar y)z\rangle_{b,\bar Q\backslash\bar\Sigma}$ *for all* $z \in C_0(\bar Q\backslash\bar\Sigma)$, $[g'(\bar y)^*\pi_{\bar\zeta}]|_Q$ *denotes the restriction of* $g'(\bar y)^*\pi_{\bar\zeta}$ *to* Q, *and* $[g'(\bar y)^*\pi_{\bar\zeta}]_{|\Omega_T}$ *denotes the restriction of* $g'(\bar y)^*\pi_{\bar\zeta}$ *to* $\Omega\times\{T\}$.
- *Optimality condition for* $\bar u$:

$$H_Q(x,t,\bar y(x,t),\bar u(x,t),\bar p(x,t),\bar\lambda) \\ = \min_{u\in K_U(x,t)} H_Q(x,t,\bar y(x,t),u,\bar p(x,t),\bar\lambda) \tag{9}$$

for all $(x,t)\in\tilde Q$, *where* $\tilde Q$ *is a measurable subset of* Q *satisfying* $\mathcal{L}^{N+1}(\tilde Q) = \mathcal{L}^{N+1}(Q)$.
- *Optimality condition for* $\bar v$:

$$\int_\Sigma \left(H_\Sigma\left(s,t,v,\frac{\partial\bar p}{\partial n_A},\bar\lambda\right) - H_\Sigma\left(s,t,\bar v,\frac{\partial\bar p}{\partial n_A},\bar\lambda\right)\right) ds\,dt \\ + \int_{\bar\Sigma} \Lambda_{\bar\zeta}(g'(\bar y)(z_v - z_{\bar v}))\,d\pi_{|\bar\zeta|} \ge 0 \qquad \text{for all} \quad v\in V_{\rm ad}, \tag{10}$$

where $z_{\hat v}$ (*with* $\hat v = v$ *or* $\hat v = \bar v$) *is the solution of*

$$\frac{\partial z}{\partial t} + Az = 0 \quad \text{in } Q, \qquad z = \hat v \quad \text{on } \Sigma, \qquad z(\cdot,0) = 0 \quad \text{in } \Omega. \tag{11}$$

We can obtain optimality conditions for (P) without the weak stability condition **A7**, but under additional conditions on $V_{\rm ad}$ and G. In this case the optimality condition for boundary controls is stated in a Lagrangian form (see [1]).

In Proposition 2.4 below, we show that the weak stability condition **A7** is satisfied in classical situations.

Proposition 2.4. *Suppose that* **A1–A6** *are satisfied and that*

(i) $G(s,t,\cdot)$ *is convex*, $F(x,t,y,\cdot)$ *is convex*, $F(x,t,y,u) \ge C_3|u|^q$, $C_3 > 0$, $G(s,t,v)\ge 0$, *and* $L(x,y)\ge 0$,
(ii) $U_{\rm ad}$ *is convex*, $V_{\rm ad}$ *is convex, and*
(iii) $\Phi(x,t,y,u) = \varphi(y) + u$, *where* φ *is of class* C^1.

Then **A7** *is satisfied.*

Proof. First observe that $0 \le \inf(P_\tau) \le \inf(P)$. For every $\tau > 0$, let (y_τ, u_τ, v_τ) be a τ- solution of (P_τ). Due to assumption (i) in the proposition and to **A1**, $(u_\tau, v_\tau)_\tau$ is bounded in $L^q(Q)\times L^\infty(\Sigma)$. Let $(u_{\tau_k}, v_{\tau_k})_k$ be a subsequence of $(u_\tau, v_\tau)_\tau$, converging to $(\tilde{u}, \tilde{v})$ for the weak-$L^q(Q) \times$ weak*-$L^\infty(\Sigma)$ topology. Then using results in [3], and condition (iii), we can prove that $(y_{\tau_k})_k$ converges to $\tilde{y}$ (the solution of (1) corresponding to $(\tilde{u}, \tilde{v})$) uniformly on every subcylinder $\overline{Q^\varepsilon}$, for all $\varepsilon > 0$. Since $(\tilde{y}, \tilde{u}, \tilde{v})$ is admissible for (P), by using classical lower semicontinuity results we can prove that

$$\inf(P) \leq J(\tilde{y}, \tilde{u}, \tilde{v}) \leq \liminf_{k\to\infty} J(y_{\tau_k}, u_{\tau_k}, v_{\tau_k}),$$

and the proof is complete. □

3. Interior Estimates for the State Equation

Proposition 3.1. *Let φ be in $L^\ell(Q)$, with $1 < \ell < \infty$. The solution y of the equation*

$$\frac{\partial y}{\partial t} + Ay = \varphi \quad \text{in } Q, \qquad y = 0 \quad \text{on } \Sigma, \qquad y(0) = 0 \quad \text{in } \Omega,$$

belongs to $L^d(0, T; W^{1,d}(\Omega))$ for every $\ell < d < \infty$ satisfying $1/\ell - 1/d < 1/(N+2)$. Moreover,

$$\|y\|_{L^d(W^{1,d})} \leq C\|\varphi\|_{\ell,Q}.$$

Proof. This result may be proved as in Proposition 3.1 of [19] or by using maximal regularity results [11] and interpolation results [9], [20]. □

We now state estimates in the interior of the cylinder $\bar{Q}$ as a function of the distance to ∂Q. Such estimates are next used to obtain optimality conditions.

Proposition 3.2. *Let a be a nonnegative function in $L^d(Q)$ ($d > N/2 + 1$), let φ be in $L^d(Q)$, and let (ψ, y_0) be in $L^\infty(\Sigma) \times L^\infty(\Omega)$. For all $\lambda \in [1, \infty[$ the solution y of*

$$\frac{\partial y}{\partial t} + Ay + ay = \varphi \quad \text{in } Q, \qquad y = \psi \quad \text{on } \Sigma, \qquad y(0) = y_0 \quad \text{in } \Omega$$

satisfies

$$\|y\|_{C(\overline{Q^\tau})} \leq C(\|\varphi\|_{d,Q} + \tau^{-2n_0}\|\psi\|_{2,\Sigma} + \tau^{-N/2\lambda}\|y_0\|_{\lambda,\Omega})$$

for all $\tau > 0$, where $C \equiv C(\Omega, A, q, \lambda)$ and n_0 denotes the first positive integer such that $\frac{1}{2} - n_0/(N+2) < 0$.

Proof. Let y_1 be the solution of

$$\frac{\partial y}{\partial t} + Ay + ay = \varphi \quad \text{in } Q, \qquad y = 0 \quad \text{on } \Sigma, \qquad y(0) = y_0 \quad \text{in } \Omega.$$

Since $d > N/2 + 1$, by using estimates on the semigroup in $L^\lambda(\Omega)$ generated by $-A$ (with Dirichlet boundary condition), and with the estimates in Chapter 3 of [10], we can prove that there exists a constant $C \equiv C(d, \lambda)$, not depending on a, such that

$$\|y_1\|_{C(\bar{Q}_{\tau T})} \le C(\|\varphi\|_{d,Q} + \tau^{-N/2\lambda}\|y_0\|_{\lambda,\Omega})$$
$$\text{for all} \quad \tau > 0 \quad \text{and all} \quad \lambda \in [1, \infty[. \tag{12}$$

Let y_2 be the solution of

$$\frac{\partial y}{\partial t} + Ay + ay = 0 \quad \text{in } Q, \qquad y = \psi \quad \text{on } \Sigma, \qquad y(0) = 0 \quad \text{in } \Omega.$$

We know that $\|y_2\|_{\infty,Q} \le \|\psi\|_{\infty,\Sigma}$. From Section 9.2 in Chapter 3 of [14] it follows that $\|y_2\|_{2,Q} \le C\|\psi\|_{2,\Sigma}$. Let n_0 be the first positive integer such that $\frac{1}{2} - n_0/(N+2) < 0$. Using interior estimates (as on pp. 172–173 of [13]), we have

$$\tau^2\|y_2\|_{L^{q_1}(\Omega_{\tau/n_0}\times(0,T))} \le C\tau^2\|y_2\|_{L^2(0,T;H^2(\Omega_{\tau/n_0}))} \le C\|y_2\|_{2,Q} \le C\|\psi\|_{2,\Sigma},$$

where $1/q_1 = \frac{1}{2} - 1/(N+2)$. Iterating this process n_0 times, we obtain

$$\|y_2\|_{C(\overline{Q^\tau})} \le C\tau^{-2n_0}\|\psi\|_{2,\Sigma}.$$

The proof is complete. □

4. Metric Spaces of Controls

Let (y, u, v) be in $L^\infty(Q) \times L^q(Q) \times L^\infty(\Sigma)$. Denote by z_{yuv} the solution to the equation

$$\frac{\partial z}{\partial t} + Az + \Phi'_y(\cdot, y, u)z = 0 \quad \text{in } Q, \qquad z = v \quad \text{on } \Sigma, \qquad z(0) = 0 \quad \text{in } \Omega. \tag{13}$$

Introduce the Ekeland distance on U_{ad}, V_{ad}, and $U_{\text{ad}} \times V_{\text{ad}}$:

$$d_U(u_1, u_2) = \mathcal{L}^{N+1}(\{(x,t) \in Q \mid u_1(x,t) \ne u_2(x,t)\}),$$
$$d_V(v_1, v_2) = \mathcal{L}^{N}(\{(s,t) \in \Sigma \mid v_1(s,t) \ne v_2(s,t)\}),$$
$$d_E((u_1, v_1), (u_2, v_2)) = d_U(u_1, u_2) + d_V(v_1, v_2).$$

Set

$$d_\tau((u_1, v_1), (u_2, v_2))$$
$$= d_E((u_1, v_1), (u_2, v_2)) + \|y_1 - y_2\|_{C(\overline{Q^\tau})} + \|z_1 - z_2\|_{C(\overline{Q^\tau})}, \tag{14}$$

where, for $i = 1, 2$, y_i is the solution of (1) corresponding to (u_i, v_i), and $z_i = z_{y_i u_i v_i}$ is the solution to (13) associated with (y_i, u_i, v_i). We can easily check that d_τ is a distance on $U_{\text{ad}} \times V_{\text{ad}}$. We explain in Remark 5.2 why we have chosen the distance d_τ in place of d_E, and why d_E cannot be used to prove Theorem 2.3.

Proposition 4.1. *Let* (u_1, v_1) *(resp.* (u_2, v_2)*) be in* $L^q(Q) \times V_{\text{ad}}$*, and let* y_1 *(resp.* y_2*) be the corresponding solution of* (1). *Then we have*

$$\|y_1 - y_2\|_{C(\overline{Q^\tau})} \leq C(\tau^{-2n_0} d_V(v_1, v_2)^{1/2} + \|\Phi(\cdot, y_2, u_2) - \Phi(\cdot, y_2, u_1)\|_{q,Q}),$$

for all $\tau > 0$*, where* n_0 *is the first positive integer such that* $\frac{1}{2} - n_0/(N+2) < 0$.

Proof. The function $y_1 - y_2$ is the solution of

$$\frac{\partial z}{\partial t} + Az + az = \Phi(\cdot, y_2, u_2) - \Phi(\cdot, y_2, u_1) \quad \text{in } Q, \qquad z = v_2 - v_1 \quad \text{on } \Sigma,$$
$$z(0) = 0 \quad \text{in } \Omega,$$

where $a = \int_0^1 \Phi'_y(\cdot, \theta y_1 + (1-\theta) y_2, u_1)\, d\theta \geq 0$. Due to Proposition 3.2, we have

$$\begin{aligned}\|y_1 - y_2\|_{C(\overline{Q^\tau})} &\leq C(\tau^{-2n_0} \|v_1 - v_2\|_{2,\Sigma} + \|\Phi(\cdot, y_2, u_2) - \Phi(\cdot, y_2, u_1)\|_{q,Q}) \\ &\leq C(\tau^{-2n_0} d_V(v_1, v_2)^{1/2} + \|\Phi(\cdot, y_2, u_2) - \Phi(\cdot, y_2, u_1)\|_{q,Q}).\end{aligned}$$

The proof is complete. □

Proposition 4.2. *Let* (y_1, u_1, v_1) *and* (y_2, u_2, v_2) *be in* $L^\infty(Q) \times L^q(Q) \times V_{\text{ad}}$. *Let* $z_1 = z_{y_1 u_1 v_1}$, $z_2 = z_{y_2 u_2 v_2}$ *be the corresponding solutions of* (13). *Then we have*

$$\|z_1 - z_2\|_{C(\overline{Q^\tau})} \leq C(\tau^{-2n_0} d_V(v_1, v_2)^{1/2} + \|\Phi'_y(\cdot, y_2, u_2) - \Phi'_y(\cdot, y_1, u_1)\|_{q,Q})$$

for all $\tau > 0$*, where* $C \equiv C(\Omega, A, q, M)$, $M \geq \|v_1\|_{\infty,\Sigma} + \|v_2\|_{\infty,\Sigma}$*, and* n_0 *is the first positive integer such that* $\frac{1}{2} - n_0/(N+2) < 0$.

Proof. The proof still relies on Proposition 3.2. □

When the set of distributed controls is bounded in $L^\infty(Q)$, all sequences converging in (U_{ad}, d_U) also converge in $L^q(Q)$, and (U_{ad}, d_U) is a complete metric space. For unbounded controls, these properties are no longer true. To overcome this difficulty, as in [18], we introduce a new metric space. For a given $\tilde{u} \in U_{\text{ad}}$ and $0 < M < \infty$, consider the set

$$U_{\text{ad}}(\tilde{u}, M) = \{u \in U_{\text{ad}} \mid |u(x,t) - \tilde{u}(x,t)| \leq M \text{ for almost every } (x,t) \in Q\}.$$

The mapping d_U is a distance on $U_{\text{ad}}(\tilde{u}, M)$ and if $(u_n)_n$ converges to u in $(U_{\text{ad}}(\tilde{u}, M), d_U)$, then $(u_n)_n$ converges to u in $L^q(Q)$. For a given $\tau > 0$, we endow $U_{\text{ad}}(\tilde{u}, M) \times V_{\text{ad}}$ with the metric d_τ defined in (14).

Lemma 4.3. *The metric space* $(U_{\text{ad}}(\tilde{u}, M) \times V_{\text{ad}}, d_\tau)$ *is complete. Moreover, the mapping which associates* $J(y_{uv}, u, v)$ *with* (u, v) *is bounded and continuous from* $(U_{\text{ad}}(\tilde{u}, M) \times V_{\text{ad}}, d_\tau)$ *into* $\mathbb{R}$.

Proof. Let $(u_n, v_n)_n \subset U_{\text{ad}}(\tilde{u}, M) \times V_{\text{ad}}$ be a Cauchy sequence in $(U_{\text{ad}}(\tilde{u}, M) \times V_{\text{ad}}, d_\tau)$. Following [18], we can prove that $(u_n, v_n)_n$ converges to some (u, v) in $U_{\text{ad}}(\tilde{u}, M) \times V_{\text{ad}}$

for d_E. Due to Propositions 4.1 and 4.2, it follows that $(y_{u_n v_n}, z_{u_n v_n})_n$ converges to (y_{uv}, z_{uv}) in $C(\overline{Q^\varepsilon})$, for every $\varepsilon > 0$. With assumptions on J and with these convergence results, we can prove the continuity result. □

5. Proof of Theorem 2.3

5.1. *The Distance Function*

Let $\tau > 0$ be fixed. Since the space $C(\overline{Q^\tau})$ is separable, there exists a norm $|\cdot|_{C(\overline{Q^\tau})}$ equivalent to the usual norm $\|\cdot\|_{C(\overline{Q^\tau})}$, such that $(C(\overline{Q^\tau}), |\cdot|_{C(\overline{Q^\tau})})$ is strictly convex and $(\mathcal{M}(\overline{Q^\tau}), |\cdot|_{\mathcal{M}(\overline{Q^\tau})})$ (where $|\cdot|_{\mathcal{M}(\overline{Q^\tau})}$ is the dual norm of $|\cdot|_{C(\overline{Q^\tau})}$) is also strictly convex. Moreover, there exist two positive constants $\tilde{C}$ and $\hat{C}$ such that

$$\tilde{C}|\cdot|_{C(\overline{Q^\tau})} \le \|\cdot\|_{C(\overline{Q^\tau})} \le \hat{C}|\cdot|_{C(\overline{Q^\tau})} \qquad \text{for all} \quad \tau > 0. \tag{15}$$

(See pp. 106–120 of [7] for the construction of the equivalent norm $|\cdot|_{C(\overline{Q^\tau})}$. In this construction, we can observe that the constants $\tilde{C}$ and $\hat{C}$ are independent of τ.) We denote by $\mathcal{C}_\tau$ the closure, for the usual topology of $C(\overline{Q^\tau})$, of the convex set $\{z|_{\overline{Q^\tau}} \mid z \in \mathcal{C}\}$ ($z|_{\overline{Q^\tau}}$ denotes the restriction of z to $\overline{Q^\tau}$). Consider the distance function to $\mathcal{C}_\tau$:

$$d_{\mathcal{C}_\tau}(\varphi) = \inf_{z\in\mathcal{C}_\tau} |\varphi - z|_{C(\overline{Q^\tau})} \qquad \text{for all} \quad \varphi \in C(\overline{Q^\tau}).$$

Observe that the mapping $\varphi \longrightarrow d_{\mathcal{C}_\tau}(\varphi)$ is convex and Lipschitz of rank 1.

Lemma 5.1. *For every $\tau > 0$ and every M, the mapping $(u, v) \longrightarrow d_{\mathcal{C}_\tau}(g(y_{uv}))$ is continuous from $(U_{\mathrm{ad}}(\bar{u}, M) \times V_{\mathrm{ad}}, d_\tau)$ into $\mathbb{R}$.*

Proof. Let $\tau > 0$, let $(u_n, v_n)_n$ be a sequence converging to (u, v) in $(U_{\mathrm{ad}}(\bar{u}, M) \times V_{\mathrm{ad}}, d_\tau)$, and let y_n and y_{uv} be the corresponding states. From Proposition 4.1 it follows that

$$\|y_n - y_{uv}\|_{C(\overline{Q^\tau})} \le C(\|\Phi(y_{uv}, u_n) - \Phi(y_{uv}, u)\|_{q,Q} + \tau^{-2n_0} d_V(v_n, v)^{1/2}).$$

Thus, for every $\tau > 0$,

$$\begin{aligned}
&|d_{\mathcal{C}_\tau}(g(y_n)) - d_{\mathcal{C}_\tau}(g(y_{uv}))| \\
&\quad \le |g(y_n) - g(y_{uv})|_{C(\overline{Q^\tau})} \le \frac{1}{\tilde{C}}\|g(y_n) - g(y_{uv})\|_{C(\overline{Q^\tau})} \le C\|y_n - y_{uv}\|_{C(\overline{Q^\tau})} \\
&\quad \le C(\|\Phi(y_{uv}, u_n) - \Phi(y_{uv}, u)\|_{q,Q} + \tau^{-2n_0} d_V(v_n, v)^{1/2}),
\end{aligned}$$

where C is a constant independent of τ. The continuity result follows from the convergence of $(u_n)_n$ to u in $L^q(Q)$ and the convergence of $(d_V(v_n, v))_n$ to zero. □

5.2. *Approximate Optimality Conditions*

Let $(\bar{y}, \bar{u}, \bar{v})$ be a solution of problem (P). For every $k > 1$, we set $\tau_k = 1/k$,

$$J_k(y,u,v) = \left\{ \left[\left(J(y,u,v) - \inf(P_{\tau_k}) + \frac{1}{k} \right)^+ \right]^2 + (d_{\mathcal{C}_{\tau_k}}(g(y)))^2 \right\}^{1/2},$$

and

$$(P_{\tau_k}) \quad \inf\{J_k(y_{uv},u,v) \mid (u,v) \in U_{\text{ad}} \times V_{\text{ad}}, d_{\mathcal{C}_{\tau_k}}(g(y_{uv})) = 0\},$$

where y_{uv} is the solution of (1) corresponding to (u, v). We have set $\tau_k = 1/k$, but any function τ_k such that $\lim_{k\to\infty} \tau_k = 0$ could be convenient. Let $J_k(\bar{y}, \bar{u}, \bar{v}) = \sigma_k^2 = J(\bar{y}, \bar{u}, \bar{v}) - \inf(P_{\tau_k}) + 1/k$. For every $k > 1$, the functional $(u, v) \to J_k(y_{uv}, u, v)$ is bounded and continuous on the metric space $(U_{\text{ad}}[\bar{u}, (\sigma_k)^{-1/2q}] \times V_{\text{ad}}, d_{\tau_k})$. Moreover,

$$J_k(y_{uv},u,v) > 0 \qquad \text{for every} \quad (u,v) \in U_{\text{ad}}[\bar{u}, (\sigma_k)^{-1/2q}] \times V_{\text{ad}},$$

$$J_k(\bar{y},\bar{u},\bar{v}) \le \inf_{U_{\text{ad}}[\bar{u},(\sigma_k)^{-1/2q}] \times V_{\text{ad}}} J_k(y_{uv},u,v) + \sigma_k^2.$$

Due to the Ekeland variational principle, there exists $(u_k, v_k) \in U_{\text{ad}}[\bar{u}, (\sigma_k)^{-1/2q}] \times V_{\text{ad}}$ such that

$$d_{\tau_k}((\bar{u},\bar{v}),(u_k,v_k)) \le \sigma_k, \tag{16}$$

$$J_k(y_k,u_k,v_k) \le J_k(y_{uv},u,v) + \sigma_k\, d_{\tau_k}((u,v),(u_k,v_k)), \tag{17}$$

for every $(u, v) \in U_{\text{ad}}[\bar{u}, (\sigma_k)^{-1/2q}] \times V_{\text{ad}}$, where y_k is the solution of (1) corresponding to (u_k, v_k).

Remark 5.2. Due to the choice of the distance d_{τ_k}, if (u_k, v_k) satisfies (16), then

$$\lim_{k\to\infty} \|\bar{y} - y_k\|_{C(\overline{Q^{\tau_k}})} = 0 \quad \text{and} \quad \lim_{k\to\infty} \|\bar{z}_{\bar{v}} - z_k\|_{C(\overline{Q^{\tau_k}})} = 0,$$

where $z_k = z_{y_k u_k v_k}$ is the solution of (13) associated with (y_k, u_k, v_k), and $\bar{z}_{\bar{v}} = z_{\bar{y}\bar{u}\bar{v}}$ is the solution of (13) associated with $(\bar{y}, \bar{u}, \bar{v})$. These convergence properties are needed in Section 5.3. They cannot be deduced from Propositions 4.1 and 4.2, and they cannot be obtained if we replace the distance d_{τ_k} by d_E.

Theorem 5.3. *Assume that* **A1–A6** *are fulfilled. Let* $k > 1$ *such that* $0 < \tau_k \le \tau_0$ (τ_0 *appears in* **A6**), *and let* (y_k, u_k, v_k) *in* $L^\infty(Q) \times U_{\text{ad}}[\bar{u}, (\sigma_k)^{-1/2q}] \times V_{\text{ad}}$ *satisfy* (16) *and* (17) (y_k *is the solution of* (1) *corresponding to* (u_k, v_k)). *Then there exist* $\lambda_k \ge 0$, $p_k \in L^1(0, T; W_0^{1,1}(\Omega))$, *and* $\mu_k \in \mathcal{M}(\overline{Q^{\tau_k}})$ *such that*

$$|\mu_k|^2_{\mathcal{M}(\overline{Q^{\tau_k}})} + (\lambda_k)^2 = 1, \qquad \langle \mu_k, z - g(y_k) \rangle_{\mathcal{M}(\overline{Q^{\tau_k}}) \times C(\overline{Q^{\tau_k}})} \le 0 \tag{18}$$

for all $z \in \mathcal{C}_{\tau_k}$,

$$-\frac{\partial p_k}{\partial t} + Ap_k + \Phi'_y(y_k,u_k)p_k + \lambda_k F'_y(y_k,u_k) + [g'(y_k)^*\mu_k]|_Q = 0 \qquad \text{in } Q,$$

$$p_k(\cdot,T) = -\lambda_k L'_y(\cdot, y_k(T)) - [g'(y_k)^*\mu_k]|_{\Omega_T} \qquad \text{in } \Omega, \tag{19}$$

$$\int_Q (H_Q(x,t,y_k,u_k^0,p_k,\lambda_k) - H_Q(x,t,y_k,u_k,p_k,\lambda_k))\,dx\,dt$$
$$\geq -\sigma_k(\mathcal{L}^{N+1}(Q) + \|\Phi(\cdot,y_k,u_k) - \Phi(\cdot,y_k,u_k^0)\|_{q,Q})$$
$$\textit{for all} \quad u^0 \in U_{\mathrm{ad}}, \tag{20}$$

$$\int_\Sigma \left(H_\Sigma\left(s,t,v,\frac{\partial p_k}{\partial n_A},\lambda_k\right) - H_\Sigma\left(s,t,v_k,\frac{\partial p_k}{\partial n_A},\lambda_k\right)\right)\,ds\,dt$$
$$\geq -\sigma_k(\mathcal{L}^N(\Sigma) + C\|v - v_k\|_{\infty,\Sigma}) \qquad \textit{for all} \quad v \in V_{\mathrm{ad}}, \tag{21}$$

where

$$u_k^0(x,t) = \begin{cases} u^0(x,t) & \textit{if} \quad |u^0(x,t) - \bar{u}(x,t)| \leq \left(\dfrac{1}{\sigma_k}\right)^{1/2q}, \\ \bar{u}(x,t) & \textit{otherwise}, \end{cases}$$

and $g'(y_k)^*\mu_k$ *is the bounded Radon measure defined on* $\overline{Q^{\tau_k}}$ *by*

$$z \longrightarrow \langle \mu_k, g'(y_k)z\rangle_{\mathcal{M}(\overline{Q^{\tau_k}})\times C(\overline{Q^{\tau_k}})} \qquad \textit{for all} \quad z \in C_0(\bar{Q}\backslash\bar{\Sigma}).$$

Proof. The proof is split into two steps.

Step 1: *Optimality Conditions for the Boundary Control* v_k. Let $v \in V_{\mathrm{ad}}$, and let $0 < \rho < 1$ be such that $\tau_\rho \leq \tau_k \leq \tau_0$, $\tau_\rho = \rho^{p'/\gamma_0 + q\bar{q}/(q-\bar{q})\gamma_0}$ with $q > \bar{q} > N/2 + 1$. Due to Theorem 5.2 and Remark 5.5 of [3], there exists a measurable subset $\Sigma^{k,\rho}$ such that $\mathcal{L}^N(\Sigma^{k,\rho}) = \rho\mathcal{L}^N(\Sigma)$, and

$$y_{k\rho} = y_k + \rho(z_{kv} - z_{kv_k}) + r_{k\rho}, \tag{22}$$

$$z_{k\rho} = z_{kv_k} + \rho(z_{kv} - z_{kv_k}) + \tilde{r}_{k\rho}, \tag{23}$$

with

$$\lim_{\rho\to 0} \frac{1}{\rho}\|r_{k\rho}\|_{C(\overline{Q^{\tau_k}})} \leq \lim_{\rho\to 0} \frac{1}{\rho}\|r_{k\rho}\|_{C(\overline{Q^{\tau_\rho}})} = 0,$$
$$\lim_{\rho\to 0} \frac{1}{\rho}\|\tilde{r}_{k\rho}\|_{C(\overline{Q^{\tau_k}})} \leq \lim_{\rho\to 0} \frac{1}{\rho}\|\tilde{r}_{k\rho}\|_{C(\overline{Q^{\tau_\rho}})} = 0,$$
$$J(y_{k\rho}, u_k, v_{k\rho}) - J(y_k, u_k, v_k) = \rho\Delta_k J + o(\rho), \tag{24}$$

where $v_{k\rho}$ is defined by

$$v_{k\rho}(s,t) = \begin{cases} v_k(s,t) & \text{on } \Sigma\backslash\Sigma^{k,\rho}, \\ v(s,t) & \text{on } \Sigma^{k,\rho}, \end{cases}$$

$y_{k\rho}$ is the solution of (1) corresponding to $(u_k, v_{k\rho})$, and z_{kv}, z_{kv_k}, and $z_{k\rho}$ stand for the solutions of (13) respectively corresponding to (y_k, u_k, v), (y_k, u_k, v_k), and $(y_k, u_k, v_{k\rho})$, and where

$$\Delta_k J = \int_Q F'_y(x,t,y_k,u_k)(z_{kv} - z_{kv_k})\,dx\,dt + \int_\Omega L'_y(x, y_k(T))(z_{kv} - z_{kv_k})(T)\,dx + \int_\Sigma (G(s,t,v) - G(s,t,v_k))\,ds\,dt.$$

On the other hand, since $\tau_\rho \le \tau_k$, with **A6** and (22), we have

$$\begin{aligned}
&\left\| \frac{g(y_{k\rho}) - g(y_k + \rho(z_{kv} - z_{kv_k}))}{\rho} \right\|_{C(\overline{Q^{\tau_k}})} \\
&\quad \le \left\| \frac{g(y_{k\rho}) - g(y_k + \rho(z_{kv} - z_{kv_k}))}{\rho} \right\|_{C(\overline{Q^{\tau_\rho}})} \\
&\quad \le C \frac{\|r_{k\rho}\|_{C(\overline{Q^{\tau_\rho}})} + (\tau_\rho)^{\gamma_0}}{\rho} \le C \left(\frac{\|r_{k\rho}\|_{C(\overline{Q^{\tau_\rho}})}}{\rho} + \rho^{p'-1+q\bar{q}/(q-\bar{q})} \right).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left\| \frac{g(y_{k\rho}) - g(y_k)}{\rho} - g'(y_k)(z_{kv} - z_{kv_k}) \right\|_{C(\overline{Q^{\tau_k}})} \\
&\quad \le \left\| \frac{g(y_k + \rho(z_{kv} - z_{kv_k})) - g(y_k)}{\rho} - g'(y_k)(z_{kv} - z_{kv_k}) \right\|_{C(\overline{Q^{\tau_k}})} \\
&\qquad + \left\| \frac{g(y_{k\rho}) - g(y_k + \rho(z_{kv} - z_{kv_k}))}{\rho} \right\|_{C(\overline{Q^{\tau_k}})} \\
&\quad \le \left\| \frac{g(y_k + \rho(z_{kv} - z_{kv_k})) - g(y_k)}{\rho} - g'(y_k)(z_{kv} - z_{kv_k}) \right\|_{C(\overline{Q^{\tau_k}})} \\
&\qquad + C \left(\frac{\|r_{k\rho}\|_{C(\overline{Q^{\tau_\rho}})}}{\rho} + \rho^{p'-1+q\bar{q}/(q-\bar{q})} \right).
\end{aligned}$$

Therefore, setting $\hat{r}_{k\rho} = g(y_{k\rho}) - g(y_k) - \rho\, g'_y(y_k)(z_{kv} - z_{kv_k})$, we have proved that

$$\lim_{\rho \to 0} \frac{1}{\rho} \|\hat{r}_{k\rho}\|_{C(\overline{Q^{\tau_k}})} = 0. \tag{25}$$

Set $(u, v) = (u_k, v_{k\rho})$ in (17). From (22)–(25), it follows that

$$\begin{aligned}
&-\lambda_k \Delta_k J - \langle \mu_k, g'(y_k)(z_{kv} - z_{kv_k}) \rangle_{\mathcal{M}(\overline{Q^{\tau_k}}) \times C(\overline{Q^{\tau_k}})} \\
&\quad \le \limsup_{\rho \to 0} \frac{J_k(y_k, u_k, v_k) - J_k(y_{k\rho}, u_k, v_{k\rho})}{\rho} \\
&\quad \le \sigma_k(\mathcal{L}^N(\Sigma) + \|z_{kv} - z_k\|_{C(\overline{Q^{\tau_k}})}) \le \sigma_k(\mathcal{L}^N(\Sigma) + C\|v - v_k\|_{\infty,\Sigma}),
\end{aligned} \tag{26}$$

where

$$\lambda_k = \frac{(J(y_k, u_k, v_k) - \inf(P_{\tau_k}) + 1/k)^+}{J_k(y_k, u_k, v_k)},$$

$$\mu_k = \begin{cases} \dfrac{d_{\mathcal{C}_{\tau_k}}(g(y_k))\nabla d_{\mathcal{C}_{\tau_k}}(g(y_k))}{J_k(y_k, u_k, v_k)} & \text{if} \quad d_{\mathcal{C}_{\tau_k}}(g(y_k)) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the weak solution $p_k \in L^1(0, T; W_0^{1,1}(\Omega))$ of (19). With the Green formula of Theorem 4.2 of [3], we obtain

$$- \int_Q \lambda_k F'_y(x, t, y_k, u_k)(z_{kv} - z_{kv_k})\, dx\, dt - \int_\Omega \lambda_k L'_y(x, y_k(T))(z_{kv} - z_{kv_k})(T)\, dx$$
$$- \langle \mu_k, g'(y_k)(z_{kv} - z_{kv_k})\rangle_{\mathcal{M}(\overline{Q^{\tau_k}})\times C(\overline{Q^{\tau_k}})} = - \int_\Sigma \frac{\partial p_k}{\partial n_A}\,(v - v_k)\, ds\, dt.$$

Taking the definition of $\Delta_k J$ and (26) into account, we have

$$\int_\Sigma \left(H_\Sigma \left(s, t, v_k, \frac{\partial p_k}{\partial n_A}, \lambda_k \right) - H_\Sigma \left(s, t, v, \frac{\partial p_k}{\partial n_A}, \lambda_k \right) \right) ds\, dt$$
$$\leq \sigma_k(\mathcal{L}^N(\Sigma) + C\|v - v_k\|_{\infty,\Sigma}).$$

Finally, from the definition of μ_k and λ_k, it follows that

$$|\mu_k|^2_{\mathcal{M}(\overline{Q^{\tau_k}})} + (\lambda_k)^2 = 1 \quad \text{and} \quad \langle \mu_k, z - g(y_k)\rangle_{\mathcal{M}(\overline{Q^{\tau_k}})\times C(\overline{Q^{\tau_k}})} \leq 0$$

for all $z \in \mathcal{C}_{\tau_k}$.

Step 2. *Optimality Conditions for the Distributed Control* u_k. The approximate Pontryagin's principle (20) may be obtained in the same way as in [18]. □

5.3. *Proof of the Optimality Conditions*

Step 1: *Convergence Results*. Observe that the weak stability condition on the right, stated in **A7** (Section 2.2), implies the convergence of the sequence $(\sigma_k)_k$ to zero, when k tends to infinity. Indeed,

$$0 \leq \lim_{k\to\infty} \sigma_k^2 = \lim_{k\to\infty} \left(\inf(P) - \inf(P_{\tau_k}) + \frac{1}{k} \right) = 0.$$

Moreover, since the sequence $(\lambda_k)_k$ is bounded in $\mathbb{R}^+$, there exists a subsequence converging to some $\bar\lambda \geq 0$. From Theorem 4.2 of [3], it results that

$$\|p_k\|_{L^{d'}(W^{1,\delta'})}$$
$$\leq C(\|F'_y(\cdot, y_k, u_k)\|_{1,Q} + \|L'_y(\cdot, y_k(T))\|_{1,\Omega} + \|g'(y_k)\|_{\mathcal{L}(C_b(\bar Q\setminus\bar\Sigma))}|\mu_k|_{\mathcal{M}(\overline{Q^{\tau_k}})}),$$

for every $\delta > 1$, $d > 1$, satisfying $N/2d + 1/\delta < \frac{1}{2}$ (where $\mathcal{L}(C_b(\bar{Q}\backslash\bar{\Sigma}))$ is the space of linear continuous mappings from $C_b(\bar{Q}\backslash\bar{\Sigma})$ into $C_b(\bar{Q}\backslash\bar{\Sigma})$). Since the sequences $(\mu_k)_k$, $(y_k)_k$, and $(u_k)_k$ are bounded in $\mathcal{M}(\overline{Q^{\tau_k}})$, $C_b(\bar{Q}\backslash\bar{\Sigma})$, and $L^q(Q)$, the sequence $(p_k)_k$ is bounded in $L^{\delta'}(0, T; W^{1,d'}(\Omega))$ for every $\delta > 1$, $d > 1$, satisfying $N/2d+1/\delta < \frac{1}{2}$. Then there exist a subsequence, still indexed by k, and $\bar{p}$ such that $(p_k)_k$ weakly converges to $\bar{p}$ in $L^{\delta'}(0, T; W^{1,d'}(\Omega))$ for every $\delta > 1$, $d > 1$, satisfying $N/2d + 1/\delta < \frac{1}{2}$. From the embedding theorems, it follows that $(p_k)_k$ weakly converges to $\bar{p}$ in $L^{q'}(Q)$. Observe that $(u_k)_k$ and $(u_k^0)_k$ converge respectively to $\bar{u}$ and u^0 in $L^q(Q)$. (Indeed $\int_Q |\bar{u}(x,t)-u_k(x,t)|^q\,dx\,dt \le (1/(\sigma_k)^{1/2})\,d_U(\bar{u}, u_k) \le (\sigma_k)^{1/2}$.) Moreover, $(d_V(v_k, \bar{v}))_k$ converges to zero, $(y_k)_k$ converges to $\bar{y}$ in $C(\overline{Q^\varepsilon})$, for every $\varepsilon > 0$. From assumptions on Φ, F, and L, with Lebesgue's theorem of dominated convergence, we obtain

$$\lim_{k\to\infty} \|\Phi'_y(\cdot, y_k, u_k) - \Phi'_y(\cdot, \bar{y}, \bar{u})\|_{q,Q} = 0,$$
$$\lim_{k\to\infty} \|F'_y(\cdot, y_k, u_k) - F'_y(\cdot, \bar{y}, \bar{u})\|_{1,Q} = 0,$$
$$\lim_{k\to\infty} \|L'_y(\cdot, y_k(T)) - L'_y(\cdot, \bar{y}(T))\|_{1,\Omega} = 0.$$

The measure μ_k induces a measure $\zeta_k \in (C_b(\bar{Q}\backslash\bar{\Sigma}))'$ via the formula

$$\langle \zeta_k, h\rangle_{*,\bar{Q}\backslash\bar{\Sigma}} = \langle \mu_k, h\rangle_{\mathcal{M}(\overline{Q^{\tau_k}})\times C(\overline{Q^{\tau_k}})} \qquad \text{for every} \quad h \in C_b(\bar{Q}\backslash\bar{\Sigma}).$$

It follows that $\|\mu_k\|_{\mathcal{M}(\overline{Q^{\tau_k}})} = \|\zeta_k\|_{(C_b(\bar{Q}\backslash\bar{\Sigma}))'}$. On the other hand, for every h in $C^1(\bar{Q}) \cap C_0(\bar{Q}\backslash\bar{\Sigma})$, we have

$$\begin{aligned}
&\int_Q \left(p_k \frac{\partial h}{\partial t} + \sum_{i,j} a_{ij} D_i h D_j p_k + \Phi'_y(x, t, y_k, u_k) p_k h \right) dx\,dt \\
&\quad = -\langle \mu_k, g'(y_k)h\rangle_{\mathcal{M}(\overline{Q^{\tau_k}})\times C(\overline{Q^{\tau_k}})} - \int_Q \lambda_k F'_y(x, t, y_k, u_k) h\,dx\,dt \\
&\qquad - \int_\Omega \lambda_k L'_y(y_k(T)) h(T)\,dx \\
&\quad = -\langle \zeta_k, g'(y_k)h\rangle_{*,\bar{Q}\backslash\bar{\Sigma}} - \int_Q \lambda_k F'_y(x, t, y_k, u_k) h\,dx\,dt \\
&\qquad - \int_\Omega \lambda_k L'_y(x, y_k(T)) h(T)\,dx.
\end{aligned} \tag{27}$$

Since the sequence $(\zeta_k)_k$ is bounded in $(C_b(\bar{Q}\backslash\bar{\Sigma}))'$, there exists a generalized sequence, still indexed by k, such that $(\zeta_k)_k$ weak-star converges to a limit $\bar{\zeta}$ in $(C_b(\bar{Q}\backslash\bar{\Sigma}))'$. From Theorem 2.1, there exists a bounded linear transformation $\Lambda_{\bar{\zeta}}\colon C_b(\bar{Q}\backslash\bar{\Sigma}) \longrightarrow L_{\pi_{|\bar{\zeta}|}}(\bar{Q})$, such that

$$\langle \bar{\zeta}, \varphi h\rangle_{*,\bar{Q}\backslash\bar{\Sigma}} = \langle \pi_{\bar{\zeta}}, \varphi h\rangle_{b,\bar{Q}\backslash\bar{\Sigma}} + \int_{\bar{\Sigma}} \varphi \Lambda_{\bar{\zeta}}(h)\,d\pi_{|\bar{\zeta}|} \tag{28}$$

for every $\varphi \in C(\bar{Q})$ and every $h \in C_{\mathrm{b}}(\bar{Q}\setminus\bar{\Sigma})$. Besides, for every $h \in C_{\mathrm{b}}(\bar{Q}\setminus\bar{\Sigma})$, we have

$$\begin{aligned}
&|\langle\bar{\zeta}, g'(\bar{y})h\rangle_{*,\bar{Q}\setminus\bar{\Sigma}} - \langle\zeta_k, g'(y_k)h\rangle_{*,\bar{Q}\setminus\bar{\Sigma}}| \\
&\quad \leq |\langle\bar{\zeta} - \zeta_k, g'(\bar{y})h\rangle_{*,\bar{Q}\setminus\bar{\Sigma}}| + |\langle\zeta_k, (g'(\bar{y}) - g'(y_k))h\rangle_{*,\bar{Q}\setminus\bar{\Sigma}}| \\
&\quad \leq |\langle\bar{\zeta} - \zeta_k, g'(\bar{y})h\rangle_{*,\bar{Q}\setminus\bar{\Sigma}}| + |\langle\mu_k, (g'(\bar{y}) - g'(y_k))h\rangle_{\mathcal{M}(\overline{Q^{\tau_k}})\times C(\overline{Q^{\tau_k}})}| \\
&\quad \leq |\langle\bar{\zeta} - \zeta_k, g'(\bar{y})h\rangle_{*,\bar{Q}\setminus\bar{\Sigma}}| + C\|g'(\bar{y}) - g'(y_k)\|_{C(\overline{Q^{\tau_k}})} \\
&\quad \leq |\langle\bar{\zeta} - \zeta_k, g'(\bar{y})h\rangle_{*,\bar{Q}\setminus\bar{\Sigma}}| + C\|\bar{y} - y_k\|^{\gamma_1}_{C(\overline{Q^{\tau_k}})} \\
&\quad \leq |\langle\bar{\zeta} - \zeta_k, g'(\bar{y})h\rangle_{*,\bar{Q}\setminus\bar{\Sigma}}| + C(d_{\tau_k}((u_k, v_k), (\bar{u}, \bar{v})))^{\gamma_1} \\
&\quad \leq |\langle\bar{\zeta} - \zeta_k, g'(\bar{y})h\rangle_{*,\bar{Q}\setminus\bar{\Sigma}}| + C(\sigma_k)^{\gamma_1},
\end{aligned}$$

where C is a positive constant independent of k. It follows that

$$\lim_{k\to\infty}\langle\zeta_k, g'(y_k)h\rangle_{*,\bar{Q}\setminus\bar{\Sigma}} = \langle\bar{\zeta}, g'(\bar{y})h\rangle_{*,\bar{Q}\setminus\bar{\Sigma}} \qquad \text{for every} \quad h \in C_{\mathrm{b}}(\bar{Q}\setminus\bar{\Sigma}).$$

With (28), we obtain

$$\lim_{k\to\infty}\langle\zeta_k, g'(y_k)h\rangle_{*,\bar{Q}\setminus\bar{\Sigma}} = \langle\bar{\zeta}, g'(\bar{y})h\rangle_{*,\bar{Q}\setminus\bar{\Sigma}} = \langle\pi_{\bar{\zeta}}, g'(\bar{y})h\rangle_{\mathrm{b},\bar{Q}\setminus\bar{\Sigma}}$$

for every $h \in C_0(\bar{Q}\setminus\bar{\Sigma})$. Therefore, by passing to the limit when k tends to infinity in (27), we prove that $\bar{p}$ is the solution of (8).

Step 2: *Integral Pontryagin's Principle for* $\bar{u}$. With the convergence results previously stated and using classical arguments, by passing to the limit in (20) we obtain

$$\begin{aligned}
&\int_Q H_Q(x, t, \bar{y}(x,t), \bar{u}(x,t), \bar{p}(x,t), \bar{\lambda})\, dx\, dt \\
&\qquad \leq \int_Q H_Q(x, t, \bar{y}(x,t), u(x,t), \bar{p}(x,t), \bar{\lambda})\, dx\, dt
\end{aligned}$$

for every $u \in U_{\mathrm{ad}}$. The pointwise Pontryagin's principle may be obtained as in Section 5.2 of [18].

Step 3: *Integral Pontryagin's Principle for* $\bar{v}$. Inequality (10) is obtained by passing to the limit in (21), with the following convergence result:

$$\begin{aligned}
&\lim_{k\to\infty}\int_\Sigma \frac{\partial p_k}{\partial n_A}(v - v_k)\, ds\, dt \\
&\qquad = \int_\Sigma \frac{\partial \bar{p}}{\partial n_A}(v - \bar{v})\, ds\, dt + \int_{\bar{\Sigma}} \Lambda_{\bar{\zeta}}(g'(\bar{y})(z_v - z_{\bar{v}}))\, d\pi_{|\bar{\zeta}|}
\end{aligned} \tag{29}$$

for every $v \in V_{\mathrm{ad}}$ (recall that z_v and $z_{\bar{v}}$ are the solutions to (11) corresponding to v and $\bar{v}$).

We prove (29). With the Green formula in Theorem 4.2 of [3], we have

$$\int_Q \lambda_k F'_y(x,t,y_k,u_k)(z_{kv}-z_{kv_k})\,dx\,dt+\int_\Omega \lambda_k L'_y(x,y_k(T))(z_{kv}-z_{kv_k})\,dx$$
$$=-\langle \mu_k, g'(y_k)(z_{kv}-z_{kv_k})\rangle_{\mathcal{M}(\overline{Q^{\tau_k}})\times C(\overline{Q^{\tau_k}})}+\int_\Sigma \frac{\partial p_k}{\partial n_A}(v-v_k)\,ds\,dt$$
$$=-\langle \zeta_k, g'(y_k)(z_{kv}-z_{kv_k})\rangle_{*,\bar{Q}\setminus\bar{\Sigma}}+\int_\Sigma \frac{\partial p_k}{\partial n_A}(v-v_k)\,ds\,dt,$$

where z_{kv} (respectively z_{kv_k}) is the solution to (13) corresponding to (y_k, u_k, v) (respectively (y_k, u_k, v_k)). In the same way, we have

$$\int_Q \bar{\lambda} F'_y(x,t,\bar{y},\bar{u})(\bar{z}_v-\bar{z}_{\bar{v}})\,dx\,dt+\int_\Omega \bar{\lambda} L'_y(x,\bar{y}(T))(\bar{z}_v-\bar{z}_{\bar{v}})\,dx$$
$$=-\langle \pi_{\bar{\zeta}}, g'(\bar{y})(\bar{z}_v-\bar{z}_{\bar{v}})\rangle_{\mathrm{b},\bar{Q}\setminus\bar{\Sigma}}+\int_\Sigma \frac{\partial \bar{p}}{\partial n_A}(v-\bar{v})\,ds\,dt$$
$$=-\langle \bar{\zeta}, g'(\bar{y})(\bar{z}_v-\bar{z}_{\bar{v}})\rangle_{*,\bar{Q}\setminus\bar{\Sigma}}+\int_{\bar{\Sigma}} \Lambda_{\bar{\zeta}}(g'(\bar{y})(\bar{z}_v-\bar{z}_{\bar{v}}))\,d\pi_{|\bar{\zeta}|}$$
$$+\int_\Sigma \frac{\partial \bar{p}}{\partial n_A}(v-\bar{v})\,ds\,dt,$$

where $\bar{z}_v$ (respectively $\bar{z}_{\bar{v}}$) is the solution of (13) corresponding to $(\bar{y}, \bar{u}, v)$ (respectively $(\bar{y}, \bar{u}, \bar{v})$). By setting

$$I_k=\left|\int_\Sigma \left(\frac{\partial p_k}{\partial n_A}(v-v_k)-\frac{\partial \bar{p}}{\partial n_A}(v-\bar{v})\right)ds\,dt-\int_\Sigma \Lambda_{\bar{\zeta}}(g'(\bar{y})(\bar{z}_v-\bar{z}_{\bar{v}}))\,d\pi_{|\bar{\zeta}|}\right|,$$

it follows that

$$I_k \le \|\bar{\lambda}F'_y(\cdot,\bar{y},\bar{u})(\bar{z}_v-\bar{z}_{\bar{v}})-\lambda_k F'_y(x,t,y_k,u_k))(z_{kv}-z_{kv_k})\|_{1,Q}$$
$$+\|\bar{\lambda}L'_y(\cdot,\bar{y}(T))(\bar{z}_v-\bar{z}_{\bar{v}})(T)-\lambda_k L'_y(\cdot,y_k(T))(z_{kv}-z_{kv_k})(T)\|_{1,\Omega}$$
$$+|\langle \bar{\zeta}, g'(\bar{y})(\bar{z}_v-\bar{z}_{\bar{v}})\rangle_{*,\bar{Q}\setminus\bar{\Sigma}}-\langle \zeta_k, g'(y_k)(z_{kv}-z_{kv_k})\rangle_{*,\bar{Q}\setminus\bar{\Sigma}}|$$
$$\le (|\bar{\lambda}-\lambda_k|\|F'_y(\cdot,\bar{y},\bar{u})\|_{1,Q}+|\lambda_k|\|F'_y(\cdot,\bar{y},\bar{u})-F'_y(\cdot,y_k,u_k)\|_{1,Q})\|\bar{z}_v-\bar{z}_{\bar{v}}\|_{\infty,Q}$$
$$+\|F'_y(\cdot,y_k,u_k)((\bar{z}_v-\bar{z}_{\bar{v}})-(z_{kv}-z_{kv_k}))\|_{1,Q}$$
$$+(|\bar{\lambda}-\lambda_k|\;\|L'_y(\cdot,\bar{y}(T))\|_{1,\Omega}+|\lambda_k|\;\|L'_y(\cdot,\bar{y}(T))$$
$$-L'_y(\cdot,y_k(T))\|_{1,\Omega})\|\bar{z}_v-\bar{z}_{\bar{v}}\|_{\infty,Q}$$
$$+\|L'_y(\cdot,y_k(T))((\bar{z}_v-\bar{z}_{\bar{v}})-(z_{kv}-z_{kv_k}))(T)\|_{1,\Omega}$$
$$+|\langle \bar{\zeta}, g'(\bar{y})(\bar{z}_v-\bar{z}_{\bar{v}})\rangle_{*,\bar{Q}\setminus\bar{\Sigma}}-\langle \zeta_k, g'(y_k)(z_{kv}-z_{kv_k})\rangle_{*,\bar{Q}\setminus\bar{\Sigma}}|.$$

Using the previous convergence results, the convergence of $(y_k, z_{kv}, z_{kv_k})_k$ to $(\bar{y}, \bar{z}_v, \bar{z}_{\bar{v}})$ in $C(\overline{Q^\varepsilon})$, for every $\varepsilon > 0$, we obtain

$$\lim_{k\to\infty} \|F'_y(\cdot, \bar{y}, \bar{u}) - F'_y(\cdot, y_k, u_k)\|_{1,Q}$$
$$= \lim_{k\to\infty} \|L'_y(\cdot, \bar{y}(T)) - L'_y(\cdot, y_k(T))\|_{1,\Omega} = 0, \tag{30}$$

$$\lim_{k\to\infty} \|F'_y(\cdot, y_k, u_k)((\bar{z}_v - \bar{z}_{\bar{v}}) - (z_{kv} - z_{kv_k}))\|_{1,Q} = 0, \tag{31}$$

$$\lim_{k\to\infty} \|L'_y(\cdot, y_k(T))((\bar{z}_v - \bar{z}_{\bar{v}}) - (z_{kv} - z_{kv_k}))(T)\|_{1,\Omega} = 0. \tag{32}$$

On the other hand, notice that

$$|\langle\bar{\zeta}, g'(\bar{y})(\bar{z}_v - \bar{z}_{\bar{v}})\rangle_{*,\bar{Q}\setminus\bar{\Sigma}} - \langle\zeta_k, g'(y_k)(z_{kv} - z_{kv_k})\rangle_{*,\bar{Q}\setminus\bar{\Sigma}}|$$
$$\le |\langle\zeta_k, g'(y_k)(z_{kv} - z_{kv_k}) - g'(\bar{y})(\bar{z}_v - \bar{z}_{\bar{v}})\rangle_{*,\bar{Q}\setminus\bar{\Sigma}}|$$
$$+ |\langle\zeta_k - \bar{\zeta}, g'(\bar{y})(\bar{z}_v - \bar{z}_{\bar{v}})\rangle_{*,\bar{Q}\setminus\bar{\Sigma}}|,$$

and due to **A6**, (16), and Proposition 4.2

$$|\langle\zeta_k, g'(y_k)(z_{kv} - z_{kv_k}) - g'(\bar{y})(\bar{z}_v - \bar{z}_{\bar{v}})\rangle_{*,\bar{Q}\setminus\bar{\Sigma}}|$$
$$= |\langle\mu_k, g'(y_k)(z_{kv} - z_{kv_k}) - g'(\bar{y})(\bar{z}_v - \bar{z}_{\bar{v}})\rangle_{\mathcal{M}(\overline{Q^{\tau_k}})\times C(\overline{Q^{\tau_k}})}|$$
$$\le |\langle\mu_k, (g'(y_k) - g'(\bar{y}))(\bar{z}_v - \bar{z}_{\bar{v}})\rangle_{\mathcal{M}(\overline{Q^{\tau_k}})\times C(\overline{Q^{\tau_k}})}|$$
$$+ |\langle\mu_k, g'(y_k)((\bar{z}_v - \bar{z}_{\bar{v}}) - (z_{kv} - z_{kv_k}))\rangle_{\mathcal{M}(\overline{Q^{\tau_k}})\times C(\overline{Q^{\tau_k}})}|$$
$$\le C(\|g'(y_k) - g'(\bar{y})\|_{\mathcal{L}(C(\overline{Q^{\tau_k}}))} + \|z_{kv_k} - \bar{z}_{\bar{v}}\|_{C(\overline{Q^{\tau_k}})} + \|z_{kv} - \bar{z}_v\|_{C(\overline{Q^{\tau_k}})})$$
$$\le C(\|y_k - \bar{y}\|^{\gamma_1}_{C(\overline{Q^{\tau_k}})} + \|z_{kv_k} - \bar{z}_{\bar{v}}\|_{C(\overline{Q^{\tau_k}})} + \|z_{kv} - \bar{z}_v\|_{C(\overline{Q^{\tau_k}})})$$
$$\le C((\sigma_k)^{\gamma_1} + \sigma_k + \|\Phi'_y(\cdot, y_k, u_k) - \Phi'_y(\cdot, \bar{y}, \bar{u})\|_{q,Q}).$$

From (30)–(32) and these estimates, we deduce that

$$\lim_{k\to\infty} I_k = 0. \tag{33}$$

On the other hand, since $z_v - \bar{z}_v = z_{\bar{v}} - \bar{z}_{\bar{v}} = 0$ on $\bar{\Sigma} \cup \Omega_0$ (z_v and $z_{\bar{v}}$ are the solutions of (11) corresponding to v and $\bar{v}$, and $\bar{z}_v$ and $\bar{z}_{\bar{v}}$ are the solutions of (13) corresponding to $(\bar{y}, \bar{u}, v)$ and $(\bar{y}, \bar{u}, \bar{v})$), then

$$\int_{\bar{\Sigma}} \Lambda_{\bar{\zeta}}(g'(\bar{y})(z_v - z_{\bar{v}}))\, d\pi_{|\bar{\zeta}|} = \int_{\bar{\Sigma}} \Lambda_{\bar{\zeta}}(g'(\bar{y})(\bar{z}_v - \bar{z}_{\bar{v}}))\, d\pi_{|\bar{\zeta}|}. \tag{34}$$

Therefore, (29) follows from (33) and (34).

Step 4: *Complementarity Condition*. From the definition of ζ_k and from (18), we deduce

$$\langle\zeta_k, z - g(y_k)\rangle_{*,\bar{Q}\setminus\bar{\Sigma}} \le 0 \qquad \text{for every} \quad z \in \mathcal{C}. \tag{35}$$

As above, we can write

$$
\begin{aligned}
&|\langle\bar{\zeta}, g(\bar{y})\rangle_{*,\bar{Q}\setminus\bar{\Sigma}} - \langle\zeta_k, g(y_k)\rangle_{*,\bar{Q}\setminus\bar{\Sigma}}| \\
&\quad \le |\langle\bar{\zeta} - \zeta_k, g(\bar{y})\rangle_{*,\bar{Q}\setminus\bar{\Sigma}}| + |\langle\zeta_k, g(\bar{y}) - g(y_k)\rangle_{*,\bar{Q}\setminus\bar{\Sigma}}| \\
&\quad \le \langle\bar{\zeta} - \zeta_k, g(\bar{y})\rangle_{*,\bar{Q}\setminus\bar{\Sigma}}| + |\langle\mu_k, g(\bar{y}) - g(y_k)\rangle_{\mathcal{M}(\overline{Q^{\tau_k}})\times C(\overline{Q^{\tau_k}})}| \\
&\quad \le |\langle\bar{\zeta} - \zeta_k, g(\bar{y})\rangle_{*,\bar{Q}\setminus\bar{\Sigma}}| + C\|g(\bar{y}) - g(y_k)\|_{C(\overline{Q^{\tau_k}})} \\
&\quad \le |\langle\bar{\zeta} - \zeta_k, g(\bar{y})\rangle_{*,\bar{Q}\setminus\bar{\Sigma}}| + C\|\bar{y} - y_k\|_{C(\overline{Q^{\tau_k}})} \\
&\quad \le |\langle\bar{\zeta} - \zeta_k, g(\bar{y})\rangle_{*,\bar{Q}\setminus\bar{\Sigma}}| + C\sigma_k,
\end{aligned}
$$

where C is a positive constant independent of k. Using this estimate and passing to the limit in (35), when k tends to infinity, we obtain

$$
\langle\bar{\zeta}, z - g(\bar{y})\rangle_{*,\bar{Q}\setminus\bar{\Sigma}} = \langle\pi_{\bar{\zeta}}, z - g(\bar{y})\rangle_{\mathrm{b},\bar{Q}\setminus\bar{\Sigma}} + \int_{\bar{\Sigma}} \Lambda_{\bar{\zeta}}(z - g(\bar{y}))\, d\pi_{|\bar{\zeta}|}
$$

for all $z \in \mathcal{C}$.

Step 5: *Nontriviality Condition*. Using (15) and passing to the limit in

$$
1 = \lambda_k^2 + |\mu_k|^2_{\mathcal{M}(\overline{Q^{\tau_k}})} \le \lambda_k^2 + (\hat{C}\|\zeta_k\|_{(C_{\mathrm{b}}(\bar{Q}\setminus\bar{\Sigma}))'})^2,
$$

we have $1 \le \bar{\lambda}^2 + \hat{C}^2(\lim_k\|\zeta_k\|_{(C_{\mathrm{b}}(\bar{Q}\setminus\bar{\Sigma}))'})^2$ ($\hat{C}$ is the constant in (15)). If $\bar{\lambda} > 0$, the proof is complete. If $\bar{\lambda} = 0$, we must prove that $\|\bar{\zeta}\|_{(C_{\mathrm{b}}(\bar{Q}\setminus\bar{\Sigma}))'} > 0$. Since $\mathrm{int}_{C_{\mathrm{b}}(\bar{Q}\setminus\bar{\Sigma})}\mathcal{C} \ne \emptyset$, there exists a ball $B(z, 2\rho) \subset \mathcal{C}$ in $C_{\mathrm{b}}(\bar{Q}\setminus\bar{\Sigma})$, with center z and radius $2\rho > 0$. We can choose $\tilde{z}_k \in B(0, 2\rho)$ such that $\langle\zeta_k, \tilde{z}_k\rangle_{*,\bar{Q}\setminus\bar{\Sigma}} = \rho\|\zeta_k\|_{(C_{\mathrm{b}}(\bar{Q}\setminus\bar{\Sigma}))'}$. Since $z + \tilde{z}_k \in \mathcal{C}$, we have

$$
\langle\zeta_k, z + \tilde{z}_k - g(y_k)\rangle_{*,\bar{Q}\setminus\bar{\Sigma}} \le 0.
$$

By passing to the limit,

$$
\hat{C}\rho + \langle\bar{\zeta}, z - g(\bar{y})\rangle_{*,\bar{Q}\setminus\bar{\Sigma}} \le 0,
$$

and it follows that $\bar{\zeta} \ne 0$. Observe that $\bar{\zeta} \ne 0$ is equivalent to $(\pi_{\bar{\zeta}}|_{\bar{Q}\setminus\bar{\Sigma}}, \pi_{|\bar{\zeta}|}|_{\bar{\Sigma}}) \ne 0$ (it is a direct consequence of Theorem 2.1). ☐

6. Application to the Case of Bilateral Constraints

Consider state-constraints of the form

$$
a(x,t) \le y(x,t) \le b(x,t) \qquad \text{for all} \quad (x,t) \in \bar{Q}\setminus\bar{\Sigma}, \tag{36}
$$

where a and b are two functions in $C(\bar{Q})$ satisfying $a(x,t) < b(x,t)$ on $\bar{Q}$. The state constraints (36) may be written in the form (2) by setting

$$y \in \mathcal{C} = \{z \in C_{\mathrm{b}}(\bar{Q}\setminus\bar{\Sigma}) \mid a \le z \le b\}. \tag{37}$$

Theorem 6.1. *Suppose that the assumptions of Theorem* 2.3 *are satisfied. Suppose in addition that* $a(x,0)+\tilde{\varepsilon} \le y_0(x) \le b(x,0)-\tilde{\varepsilon}$ *in* Ω *and that there exists* $\tilde{v} \in V_{\mathrm{ad}}$ *satisfying* $a(s,t)+\tilde{\varepsilon} \le \tilde{v}(s,t) \le b(s,t)-\tilde{\varepsilon}$ *on* Σ (*for some* $\tilde{\varepsilon} > 0$). *Then there exist* $\bar{\lambda} \ge 0$, $\bar{p} \in L^1(0,T;W_0^{1,1}(\Omega))$, $\bar{\mu}_a \in \mathcal{M}_{\mathrm{b}}(\bar{Q}\setminus\bar{\Sigma})$, *and* $\bar{\mu}_b \in \mathcal{M}_{\mathrm{b}}(\bar{Q}\setminus\bar{\Sigma})$ *such that*

$$\bar{\mu}_a \ge 0, \qquad \bar{\mu}_b \ge 0, \qquad (\bar{\mu}_a, \bar{\mu}_b, \bar{\lambda}) \ne 0, \tag{38}$$

$$\langle \bar{\mu}_b, \bar{y}\rangle_{\mathrm{b},\bar{Q}\setminus\bar{\Sigma}} = \langle \bar{\mu}_b, b\rangle_{\mathrm{b},\bar{Q}\setminus\bar{\Sigma}}, \qquad \langle \bar{\mu}_a, \bar{y}\rangle_{\mathrm{b},\bar{Q}\setminus\bar{\Sigma}} = \langle \bar{\mu}_a, a\rangle_{\mathrm{b},\bar{Q}\setminus\bar{\Sigma}}, \tag{39}$$

$$\bar{p} \textit{ satisfies } (8) \textit{ with } g'(\bar{y})^*\pi_{\bar{\zeta}} \equiv \bar{\mu}_b - \bar{\mu}_a, \tag{40}$$

$$H_Q(x,t,\bar{y}(x,t),\bar{u}(x,t),\bar{p}(x,t),\bar{\lambda}) = \min_{u \in K_U(x,t)} H_Q(x,t,\bar{y}(x,t),u,\bar{p}(x,t),\bar{\lambda})$$

for all $(x,t) \in \tilde{Q}$,

$$H_\Sigma\left(s,t,\bar{y}(s,t),\bar{v}(s,t),\frac{\partial \bar{p}}{\partial n_A},\bar{\lambda}\right) = \min_{v \in \tilde{K}_V(s,t)} H_\Sigma\left(s,t,\bar{y}(s,t),v,\frac{\partial \bar{p}}{\partial n_A},\bar{\lambda}\right)$$

for all $(s,t) \in \tilde{\Sigma}$, *where* $\tilde{Q}$ *is a measurable subset of* Q *satisfying* $\mathcal{L}^{N+1}(Q) = \mathcal{L}^{N+1}(\tilde{Q})$, $\tilde{\Sigma}$ *is a measurable subset of* Σ *satisfying* $\mathcal{L}^N(\Sigma) = \mathcal{L}^N(\tilde{\Sigma})$, *and* $\tilde{K}_V(s,t) = K_V(s,t) \cap \{v \in V \mid a(s,t) \le v \le b(s,t)\}$.

Proof. With Theorem 2.3 and with arguments similar to those in the proof of Theorem 7.3 in [2], we can prove that there exist $\bar{\lambda} \ge 0$, $\bar{p} \in L^1(0,T;W_0^{1,1}(\Omega))$, $\bar{\zeta} \in (C_{\mathrm{b}}(\bar{Q}\setminus\bar{\Sigma}))'$, two bounded linear transformations $\Lambda_+\colon C_{\mathrm{b}}(\bar{Q}\setminus\bar{\Sigma}) \longrightarrow L^\infty_{\pi_{\bar{\zeta}^+}}(\bar{Q})$ and $\Lambda_-\colon C_{\mathrm{b}}(\bar{Q}\setminus\bar{\Sigma}) \longrightarrow L^\infty_{\pi_{\bar{\zeta}^-}}(\bar{Q})$ such that

$$\begin{aligned}
&(\pi_{\bar{\zeta}^-}, \pi_{\bar{\zeta}^+}, \bar{\lambda}) \ne 0,\\
&\langle \pi_{\bar{\zeta}^+}, \bar{y}\rangle_{\mathrm{b},\bar{Q}\setminus\bar{\Sigma}} = \langle \pi_{\bar{\zeta}^+}, b\rangle_{\mathrm{b},\bar{Q}\setminus\bar{\Sigma}}, \qquad \langle \pi_{\bar{\zeta}^-}, \bar{y}\rangle_{\mathrm{b},\bar{Q}\setminus\bar{\Sigma}} = \langle \pi_{\bar{\zeta}^-}, a\rangle_{\mathrm{b},\bar{Q}\setminus\bar{\Sigma}},
\end{aligned} \tag{41}$$

$$\begin{aligned}
&\int_{\bar{\Sigma}} \Lambda_+(\bar{y})\, d\pi_{\bar{\zeta}^+} = \langle \pi_{\bar{\zeta}^+}, b\rangle_{\mathcal{M}(\bar{\Sigma})\times C(\bar{\Sigma})}, \qquad \int_{\bar{\Sigma}} \Lambda_-(\bar{y})\, d\pi_{\bar{\zeta}^-} = \langle \pi_{\bar{\zeta}^-}, a\rangle_{\mathcal{M}(\bar{\Sigma})\times C(\bar{\Sigma})},\\
&\bar{p} \text{ satisfies (8) with } g'(\bar{y})^*\pi_{\bar{\zeta}} \equiv \pi_{\bar{\zeta}^+} - \pi_{\bar{\zeta}^-},\\
&H_Q(x,t,\bar{y}(x,t),\bar{u}(x,t),\bar{p}(x,t),\bar{\lambda})\\
&\quad = \min_{u \in K_U(x,t)} H_Q(x,t,\bar{y}(x,t),u,\bar{p}(x,t),\bar{\lambda})
\end{aligned} \tag{42}$$

for all $(x, t) \in \tilde{Q}$, where $\tilde{Q}$ is a measurable subset of Q satisfying $\mathcal{L}^{N+1}(Q) = \mathcal{L}^{N+1}(\tilde{Q})$, and

$$\int_{\Sigma} \left(H_{\Sigma}\left(\cdot, \bar{y}, \bar{v}, \frac{\partial \bar{p}}{\partial n_A}, \bar{\lambda}\right) - H_{\Sigma}\left(\cdot, \bar{y}, v, \frac{\partial \bar{p}}{\partial n_A}, \bar{\lambda}\right)\right) ds\, dt \\ \leq \int_{\bar{\Sigma}} \Lambda_{-}(z_{\bar{v}} - z_v)\, d\pi_{\bar{\zeta}^-} - \int_{\bar{\Sigma}} \Lambda_{+}(z_{\bar{v}} - z_v)\, d\pi_{\bar{\zeta}^+} \tag{43}$$

for all $v \in V_{\text{ad}}$, z_v is the solution of (11) corresponding to v. For $v \in V_{\text{ad}}$, let φ_v be the solution of

$$\frac{\partial y}{\partial t} + Ay = 0 \quad \text{in } Q, \qquad y = v \quad \text{on } \Sigma, \qquad y(0) = y_0 \quad \text{in } \Omega.$$

Observing that $\bar{y} - \varphi_{\bar{v}}$ belongs to $C_0(\bar{Q}\backslash\bar{\Sigma})$, and that, for every $v \in V_{\text{ad}}$, $z_v - z_{\bar{v}} \equiv \varphi_v - \varphi_{\bar{v}}$ on $\bar{Q}\backslash\bar{\Sigma}$, we obtain

$$\int_{\bar{\Sigma}} \Lambda_{+}(\varphi_{\bar{v}} - \bar{y})\, d\pi_{\bar{\zeta}^+} = \int_{\bar{\Sigma}} \Lambda_{-}(\varphi_{\bar{v}} - \bar{y})\, d\pi_{\bar{\zeta}^-} = 0, \tag{44}$$

$$\int_{\bar{\Sigma}} \Lambda_{+}(z_{\bar{v}} - z_v)\, d\pi_{\bar{\zeta}^+} = \int_{\bar{\Sigma}} \Lambda_{+}(\varphi_{\bar{v}} - \varphi_v)\, d\pi_{\bar{\zeta}^+} \qquad \text{for all} \quad v \in V_{\text{ad}}, \tag{45}$$

$$\int_{\bar{\Sigma}} \Lambda_{-}(z_{\bar{v}} - z_v)\, d\pi_{\bar{\zeta}^-} = \int_{\bar{\Sigma}} \Lambda_{-}(\varphi_{\bar{v}} - \varphi_v)\, d\pi_{\bar{\zeta}^-} \qquad \text{for all} \quad v \in V_{\text{ad}}. \tag{46}$$

From (41), (43)–(46), it follows that

$$\int_{\Sigma} \left(H_{\Sigma}\left(\cdot, \bar{y}, \bar{v}, \frac{\partial \bar{p}}{\partial n_A}, \bar{\lambda}\right) - H_{\Sigma}\left(\cdot, \bar{y}, v, \frac{\partial \bar{p}}{\partial n_A}, \bar{\lambda}\right)\right) ds\, dt \\ \leq \langle \pi_{\bar{\zeta}^-}, a\rangle_{\mathcal{M}(\bar{\Sigma})\times C(\bar{\Sigma})} - \langle \pi_{\bar{\zeta}^+}, b\rangle_{\mathcal{M}(\bar{\Sigma})\times C(\bar{\Sigma})} \\ + \int_{\bar{\Sigma}} \Lambda_{+}(\varphi_v)\, d\pi_{\bar{\zeta}^+} - \int_{\bar{\Sigma}} \Lambda_{-}(\varphi_v)\, d\pi_{\bar{\zeta}^-} \tag{47}$$

for all $v \in V_{\text{ad}}$.

1. We claim that $(\pi_{\bar{\zeta}^+|\bar{Q}\backslash\bar{\Sigma}}, \pi_{\bar{\zeta}^-|\bar{Q}\backslash\bar{\Sigma}}, \bar{\lambda}) \neq 0$. Arguing by contradiction, we suppose that $(\pi_{\bar{\zeta}^+|\bar{Q}\backslash\bar{\Sigma}}, \pi_{\bar{\zeta}^-|\bar{Q}\backslash\bar{\Sigma}}, \bar{\lambda}) = 0$. It follows that $\bar{p} \equiv 0$. With (41), we have $(\pi_{\bar{\zeta}^-|\bar{\Sigma}}, \pi_{\bar{\zeta}^+|\bar{\Sigma}}) \neq 0$, and with (47), we have

$$0 \leq \int_{\bar{\Sigma}} \Lambda_{+}(\varphi_v)\, d\pi_{\bar{\zeta}^+} - \int_{\bar{\Sigma}} \Lambda_{-}(\varphi_v)\, d\pi_{\bar{\zeta}^-} + \langle \pi_{\bar{\zeta}^-}, a\rangle_{\mathcal{M}(\bar{\Sigma})\times C(\bar{\Sigma})} - \langle \pi_{\bar{\zeta}^+}, b\rangle_{\mathcal{M}(\bar{\Sigma})\times C(\bar{\Sigma})}$$

for all $v \in V_{\text{ad}}$. In particular,

$$0 \leq \int_{\bar{\Sigma}} \Lambda_{+}(\varphi_{\bar{v}})\, d\pi_{\bar{\zeta}^+} - \int_{\bar{\Sigma}} \Lambda_{-}(\varphi_{\bar{v}})\, d\pi_{\bar{\zeta}^-} \\ + \langle \pi_{\bar{\zeta}^-}, a\rangle_{\mathcal{M}(\bar{\Sigma})\times C(\bar{\Sigma})} - \langle \pi_{\bar{\zeta}^+}, b\rangle_{\mathcal{M}(\bar{\Sigma})\times C(\bar{\Sigma})}. \tag{48}$$

With a comparison principle, we prove that

$$\psi_{a+\tilde{\varepsilon}}(x,t) \leq \varphi_{\tilde{v}}(x,t) \leq \psi_{b-\tilde{\varepsilon}}(x,t) \qquad \text{for all} \quad (x,t) \in \bar{Q}\backslash\bar{\Sigma},$$

where ψ_h (for $h = a + \tilde{\varepsilon}$ or $h = b - \tilde{\varepsilon}$) is the solution of

$$\frac{\partial \psi}{\partial t} + A\psi = 0 \quad \text{in } Q, \qquad \psi = h_{|\Sigma} \quad \text{on } \Sigma, \qquad \psi(0) = h(0) \quad \text{in } \Omega.$$

Notice that $\psi_{b-\tilde{\varepsilon}}$ and $\psi_{a+\tilde{\varepsilon}}$ belong to $C(\bar{Q})$. From Theorem 2.1, it follows that

$$\begin{aligned}
&\langle \pi_{\bar{\zeta}^+}, a\rangle_{\mathcal{M}(\bar{\Sigma})\times C(\bar{\Sigma})} + \tilde{\varepsilon}\pi_{\bar{\zeta}^+}(\bar{\Sigma}) \\
&\qquad = \int_{\bar{\Sigma}} \Lambda_+(\psi_{a+\tilde{\varepsilon}})\, d\pi_{\bar{\zeta}^+} \leq \int_{\bar{\Sigma}} \Lambda_+(\varphi_{\tilde{v}})\, d\pi_{\bar{\zeta}^+} \leq \int_{\bar{\Sigma}} \Lambda_+(\psi_{b-\tilde{\varepsilon}})\, d\pi_{\bar{\zeta}^+} \\
&\qquad = \langle \pi_{\bar{\zeta}^+}, b\rangle_{\mathcal{M}(\bar{\Sigma})\times C(\bar{\Sigma})} - \tilde{\varepsilon}\pi_{\bar{\zeta}^+}(\bar{\Sigma}),
\end{aligned}$$

and

$$\begin{aligned}
&\langle \pi_{\bar{\zeta}^-}, a\rangle_{\mathcal{M}(\bar{\Sigma})\times C(\bar{\Sigma})} + \tilde{\varepsilon}\pi_{\bar{\zeta}^-}(\bar{\Sigma}) \\
&\qquad = \int_{\bar{\Sigma}} \Lambda_-(\psi_{a+\tilde{\varepsilon}})\, d\pi_{\bar{\zeta}^-} \leq \int_{\bar{\Sigma}} \Lambda_-(\varphi_{\tilde{v}})\, d\pi_{\bar{\zeta}^-} \leq \int_{\bar{\Sigma}} \Lambda_-(\psi_{b-\tilde{\varepsilon}})\, d\pi_{\bar{\zeta}^-} \\
&\qquad = \langle \pi_{\bar{\zeta}^-}, b\rangle_{\mathcal{M}(\bar{\Sigma})\times C(\bar{\Sigma})} - \tilde{\varepsilon}\pi_{\bar{\zeta}^-}(\bar{\Sigma}).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_{\bar{\Sigma}} \Lambda_+(\varphi_{\tilde{v}})\, d\pi_{\bar{\zeta}^+} - \int_{\bar{\Sigma}} \Lambda_-(\varphi_{\tilde{v}})\, d\pi_{\bar{\zeta}^-} + \langle \pi_{\bar{\zeta}^-}, a\rangle_{\mathcal{M}(\bar{\Sigma})\times C(\bar{\Sigma})} - \langle \pi_{\bar{\zeta}^+}, b\rangle_{\mathcal{M}(\bar{\Sigma})\times C(\bar{\Sigma})} \\
&\qquad \leq -\tilde{\varepsilon}[\pi_{\bar{\zeta}^-}(\bar{\Sigma}) + \pi_{\bar{\zeta}^+}(\bar{\Sigma})],
\end{aligned}$$

which is in contradiction with (48). By setting $\bar{\mu}_a \equiv \pi_{\bar{\zeta}^-}{}_{|\bar{Q}\backslash\bar{\Sigma}}$ and $\bar{\mu}_b \equiv \pi_{\bar{\zeta}^+}{}_{|\bar{Q}\backslash\bar{\Sigma}}$, we obtain (38), (39), and (40).

2. By a comparison principle, we prove that, for every $v \in V_{\text{ad}}$ obeying $a \leq v \leq b$, we have

$$\int_{\bar{\Sigma}} \Lambda_+(\varphi_v)\, d\pi_{\bar{\zeta}^+} \leq \int_{\bar{\Sigma}} \Lambda_+(\psi_b)\, d\pi_{\bar{\zeta}^+} = \langle \pi_{\bar{\zeta}^+}, b\rangle_{\mathcal{M}(\bar{\Sigma})\times C(\bar{\Sigma})},$$

$$\int_{\bar{\Sigma}} \Lambda_-(\psi_a) d\pi_{\bar{\zeta}^-} = \langle \pi_{\bar{\zeta}^-}, a\rangle_{\mathcal{M}(\bar{\Sigma})\times C(\bar{\Sigma})} \leq \int_{\bar{\Sigma}} \Lambda_-(\varphi_v)\, d\pi_{\bar{\zeta}^-}.$$

Taking (47) into account, we obtain

$$\begin{aligned}
&\int_{\Sigma} \left(H_\Sigma\left(\cdot, \bar{y}, \bar{v}, \frac{\partial \bar{p}}{\partial n_A}, \bar{\lambda}\right) - H_\Sigma\left(\cdot, \bar{y}, v, \frac{\partial \bar{p}}{\partial n_A}, \bar{\lambda}\right)\right) ds\, dt \\
&\leq \langle \pi_{\bar{\zeta}^-}, a\rangle_{\mathcal{M}(\bar{\Sigma})\times C(\bar{\Sigma})} - \langle \pi_{\bar{\zeta}^+}, b\rangle_{\mathcal{M}(\bar{\Sigma})\times C(\bar{\Sigma})} + \int_{\bar{\Sigma}} \Lambda_+(\varphi_v)\, d\pi_{\bar{\zeta}^+} - \int_{\bar{\Sigma}} \Lambda_-(\varphi_v)\, d\pi_{\bar{\zeta}^-} \\
&\leq 0 \qquad \text{for all} \quad v \in V_{\text{ad}} \quad \text{with} \quad a \leq v \leq b.
\end{aligned}$$

The pointwise Pontryagin's principle may be obtained as in Section 5.2 of [18]. The proof is complete. □

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