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Heat Equations with Fractional White Noise Potentials*

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Abstract. This paper is concerned with the following stochastic heat equations:

$$\frac{\partial u_t(x)}{\partial t} = \frac{1}{2}\Delta u_t(x) + w^H \cdot u_t(x), \qquad x \in \mathbb{R}^d, \quad t > 0,$$

where w^H is a time independent fractional white noise with Hurst parameter $H = (h_1, h_2, \ldots, h_d)$, or a time dependent fractional white noise with Hurst parameter $H = (h_0, h_1, \ldots, h_d)$. Denote $|H| = h_1 + h_2 + \cdots + h_d$. When the noise is time independent, it is shown that if $\frac{1}{2} < h_i < 1$ for $i = 1, 2, \ldots, d$ and if $|H| > d - 1$, then the solution is in $\mathcal{L}_2$ and the $\mathcal{L}_2$-Lyapunov exponent of the solution is estimated. When the noise is time dependent, it is shown that if $\frac{1}{2} < h_i < 1$ for $i = 0, 1, \ldots, d$ and if $|H| > d - 2/(2h_0 - 1)$, the solution is in $\mathcal{L}_2$ and the $\mathcal{L}_2$-Lyapunov exponent of the solution is also estimated. A family of distribution spaces $\mathcal{S}_\rho$, $\rho \in \mathbb{R}$, is introduced so that every chaos of an element in $\mathcal{S}_\rho$ is in $\mathcal{L}_2$. The Lyapunov exponents in $\mathcal{S}_\rho$ of the solution are also estimated.

Key Words. Heat equations, Fractional Brownian field, Multiple integral of Itô type, Stochastic integral of Itô type, Chaos expansion, Asymptotic behavior, Mittag–Leffler function.

AMS Classification. Primary 60H15, 60H05, Secondary 60G60, 35K05, 35R60, 60F25.

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1. Introduction

The following stochastic heat equations with white noise potentials have been studied by many authors:

$$\frac{\partial u_t(x)}{\partial t} = \frac{1}{2}\Delta u_t(x) + w \cdot u_t(x), \qquad 0 < t < \infty, \quad x \in \mathbb{R}^d,$$

where $\Delta = \sum_{i=1}^d (\partial^2/\partial x_i^2)$ and w is the white noise on some probability space $(\Omega, \mathcal{F}, P)$ (see [27], [26], [7], [28], [16], and the references therein). The expectation on $(\Omega, \mathcal{F}, P)$ is denoted by $\mathbb{E}$. Let

$$\mathcal{S}_\rho = \left\{ F\colon \Omega \to \mathbb{R}; \sum_{n=0}^{\infty} ((n+1)!)^\rho \mathbb{E} \ | F_n|^2 < \infty \right\},$$

where F_n is the nth chaos of F. These spaces have a nice property: for an element F in $\mathcal{S}_\rho$, $\rho \in \mathbb{R}$, each chaos F_n, $n = 0, 1, 2, \ldots$, is in $\mathcal{L}_2$. When $\rho = 0$, $\mathcal{S}_0 = \mathcal{L}_2$. In [16] the asymptotic behaviors in $\mathcal{L}_2$ and in $\mathcal{S}_\rho$ of the solution of an stochastic heat equation with white noise potentials have been studied. Specifically, if $w^H = w(x)$ is a time independent white noise and if $d \le 3$, it is shown that the solution is in $\mathcal{S}_\rho$ for certain ρ and the Lyapunov exponents of the solution in these spaces have been estimated. Similar results are also obtained if $w^H = w(t, x)$ is a time dependent white noise and if $d = 1$. Generally it is known that the solution is not in $\mathcal{S}_\rho$ and the stochastic heat equations are studied in more singular distribution spaces. There are many reasons why the stochastic heat equations with white noise potentials are important. One of these reasons is that the heat equation is one of the simplest second-order partial differential equations and the white noise is one of the simplest random fields.

On the other hand, there is growing interest in the study of fractional Brownian motions (FBM). A zero mean Gaussian process B_t^h, $0 \le t < \infty$, on some probability space $(\Omega, \mathcal{F}, P)$ is called an FBM with a given Hurst parameter $h \in (0, 1)$ if

$$\mathbb{E}(B_t^h B_s^h) = \frac{1}{2}\left(|t|^{2h} + |s|^{2h} - |t-s|^{2h}\right).$$

These processes have the following properties: on one hand they are simple Gaussian processes (though not as simple as standard Brownian motion); on the other hand they describe random phenomena with long range dependence. These properties enable them to be potential candidates in applications to various fields.

It is natural to extend the FBM to multiparameter cases (fractional Brownian fields = FBF). The fractional white noises can be defined accordingly. It is also natural to study the heat equation with *fractional* white noise potentials.

In [21] Lindstrøm defines an FBF, with a parameter $p \in (0, 2)$, as a Gaussian process $L^p(x)$, $x \in \mathbb{R}^d$. The basic feature of his FBF is

$$\mathbb{E}(L^p(x)L^p(y)) = \frac{1}{2}\left(|x|^{2p} + |y|^{2p} - |x-y|^{2p}\right). \tag{1.1}$$

References of some applications of these FBF are also given in [21]. However, the FBF $L^p(x)$, $x \in \mathbb{R}^d$, is not positively correlated. This is inconvenient for the purpose of this paper. For this reason, a new type of FBF is introduced. Since the FBM are fractional integrals of white noise, it is natural to define our FBF as partial fractional integrals of multiparameter white noise. Partial fractional integrals have been studied by many researchers (see [25], [29], and the references therein). Roughly speaking, we define the FBF with parameter $H = (h_1, \dots, h_d)$ as a Gaussian process $B^H(x)$, $x \in \mathbb{R}^d$, satisfying

$$\mathbb{E}(B^H(x)B^H(y)) = \frac{1}{2^d} \prod_{i=1}^{d} \left(|x_i|^{2h_i} + |y_i|^{2h_i} - |x_i - y_i|^{2h_i}\right). \tag{1.2}$$

In Section 2 we state some elementary properties of the FBF, $B^H(x)$, $x \in \mathbb{R}^d$.

In order to study the stochastic heat equation with fractional white noise potentials, we need to develop a stochastic calculus for FBF. We establish this using the ideas in [3], [9], and the references therein. Our presentation for this part will be brief. Only relevant results will be sketched.

In Section 3 we define the stochastic integral of deterministic functions with respect to FBF. We follow the ideas of [11] and [9].

In Section 4 we define multiple stochastic integrals of deterministic kernels with respect to FBF. The approach is standard (see [24], [23], [9], [19], [5], [6], and the references therein).

In Section 5 we define the stochastic integral of a general (random) integrand with respect to FBF in the spirit of the Malliavin calculus. It is an Itô (Skorohod) type integral. Some properties useful to this paper are also obtained.

After the introduction of stochastic calculus for FBF, we study the following stochastic heat equation with time independent fractional white noise potentials $w^H(x)$:

$$\frac{\partial u_t(x)}{\partial t} = \tfrac{1}{2}\Delta u_t(x) + w^H(x) \cdot u_t(x), \tag{1.3}$$

where $w^H(x)$ is the fractional white noise, i.e., formally $w^H(x) = (\partial^d/\partial x_1 \partial x_2 \cdots \partial x_d)$ $B^H(x)$. This equation will be solved in the mild form. Namely, we seek $u_t(x)$ such that

$$u_t(x) = u_0(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) u_s(y)\, dB^H(y)\, ds \tag{1.4}$$

(see Definition 6.1), where $p_t(x) = e^{-x^2/2t}/\sqrt{2\pi t}$. The above stochastic integral is in the sense of Itô–Skorohod and is defined in Section 5 of this paper.

Assume that $h_i > \frac{1}{2}$ for $i = 1, 2, \dots, d$. Let $|H| = h_1 + \cdots + h_d$. It is shown that when $|H| > d-1$, the solution is in $\mathcal{L}_2$, and the $\mathcal{L}_2$-Lyapunov exponent of the solution will be estimated. More precisely, we show that for all $0 < \kappa < 1 + |H| - d$,

$$\limsup_{t\to\infty} \frac{\log\bigl(\sup_{x\in\mathbb{R}^d} \mathbb{E}\ |\ u_t(x)|^2\bigr)}{t^{2/\kappa}} < \infty. \tag{1.5}$$

Note that when $H = (\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$, we obtain the usual white noise. The condition $|H| > d-1$ is optimal in the following sense: when $d = 2$, $H = (h, h)$, the condition implies that $h > \frac{1}{2}$. It is known (e.g., [27], [26], or [16]) that when $d = 2$ and when $h = \frac{1}{2}$ (the white noise case) the solution is not necessarily in $\mathcal{L}_2$. Our result also states that although when $d = 2$, $h = \frac{1}{2}$ (the white noise potential case), an $\mathcal{L}_2$ solution may not exist, once $h > \frac{1}{2}$, an $\mathcal{L}_2$ solution exists. This amuses the author.

In general it is shown that if $|H| > d-2$, then there is ρ_0 such that the solution is in $\mathcal{S}_\rho$ for all $\rho < \rho_0$ and the Lyapunov exponents of $u_t(x)$ in these spaces are also obtained. To have an idea about the condition $|H| > d-2$, we consider $H = (h, h, \ldots, h)$. $|H| > d-2$ means that $h > (d-2)/d$. It is known that when $d < 4$, then the solution of the stochastic heat equation with time independent white noise potentials is in $\mathcal{S}_\rho$, and when $d \geq 4$, then the solution is not in any $\mathcal{S}_\rho$. When $d = 4$, $h > (d-2)/d$. This means that $h > \frac{1}{2}$. Thus although when $d = 4$, $h = \frac{1}{2}$ (the white noise potential case) an $\mathcal{S}_\rho$ solution may not exist, once $h > \frac{1}{2}$, an $\mathcal{S}_\rho$ solution exists.

The stochastic heat equation with time dependent fractional white noise potentials $w^H(t,x)$,

$$\frac{\partial u_t(x)}{\partial t} = \frac{1}{2}\Delta u_t(x) + w^H(t,x) \cdot u_t(x), \tag{1.6}$$

is studied in Section 7, where $H = (h_0, h_1, \ldots, h_d)$. Equation (1.6) is also solved in its mild form

$$u_t(x) = u_0(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) u_s(y)\, dB^H(s,y) \tag{1.7}$$

(see Definition 7.1). Again let $|H| = h_1 + \cdots + h_d$. It is shown that when

$$\rho_0 = \frac{2}{2h_0 - 1} + |H| - d > 0, \tag{1.8}$$

then the solution of (1.6) is in $\mathcal{L}_2$. Moreover, for any $0 < \kappa < \rho_0$,

$$\limsup_{t\to\infty} \frac{1}{t^{2h_0/\kappa}} \log \sup_{x\in\mathbb{R}^d} \mathbb{E}\ |u_t(x)|^2 < \infty. \tag{1.9}$$

In general it is shown that if $|H| > d + 2 - 4h_0$, then there is ρ_0 such that the solution is in $\mathcal{S}_\rho$ for all $\rho < \rho_0$ and the Lyapunov exponents of $u_t(x)$ in these spaces are also obtained.

If we formally think that the time independent FBF is a special case of the time dependent FBF with $h_0 = 1$, then condition (1.8) becomes $|H| > d-1$ and the condition $|H| > d + 2 - 4h_0$ becomes $|H| > d-2$. Thus (1.9) is an extension of (1.5).

Two relevant papers have been completed after the the first version of this paper. In [17], the stochastic partial differential of the form

$$\frac{\partial}{\partial t} u(t,x) = \frac{1}{2}\Delta u(t,x) + \mathcal{A}_t u(t,x) \cdot w^H(t,x), \qquad t \in \mathbb{R}_+, \quad x \in \mathbb{R}^d,$$

is studied, where $\mathcal{A}_t$ is a first-order partial differential operator. In [20] the Poisson type equation

$$\Delta U(x) = -w^H(x); \qquad x \in D, \quad U(x) = 0 \quad \text{for} \quad x \in \partial D,$$

is discussed. Here D is a bounded open set in $\mathbb{R}^d$ with a smooth boundary. We refer the reader to these papers for details.

2. Fractional Brownian Fields

Let $0 < h < 1$. It is well known that there is a Gaussian stochastic process $B_t^h, t \geq 0$, such that

$$\mathbb{E}\left(B_t^h\right) = 0, \qquad \mathbb{E}\left(B_t^h B_s^h\right) = \frac{1}{2}\left\{|t|^{2h} + |s|^{2h} - |t-s|^{2h}\right\} \tag{2.1}$$

for all $s, t \in \mathbb{R}_+$. This process is called the fractional Brownian motion (FBM) with Hurst parameter h. To simplify the presentation, it is always assumed that the FBM is 0 at $t = 0$.

If $h = \frac{1}{2}$, then the corresponding FBM is the usual standard Brownian motion. If $h > \frac{1}{2}$, then the process B_t^h exhibits a long-range dependence, that is, if $r(n) = \mathrm{cov}(B_1^h, B_{n+1}^h - B_n^h)$, then $\sum_{n=1}^\infty r(n) = \infty$. This process is self-similar in the sense that $B_{\alpha t}^h$ has the same probability law as $\alpha^h B_t^h$. A process satisfying this property is called self-similar with the Hurst parameter h.

Since, in many problems, the processes under study exhibit the self-similar and the long-range dependent properties and since the FBM are among the simplest processes of this kind, there have been many studies of these stochastic processes.

Let $B_t, t \in \mathbb{R}$, be a Brownian motion. Let $w_t = \dot{B}_t$ be the white noise. An FBM of Hurst parameter $h \in (0,1)$ can be given by the fractional integral of the white noise (up to a constant multiple)

$$\begin{aligned} B_t^h &= c_h \left[\int_{-\infty}^t (t-s)^{h-1/2}\, dB_s - \int_{-\infty}^0 (-s)^{h-1/2}\, dB_s\right] \\ &= c_h \left[\int_{-\infty}^t (t-s)^{h-1/2} w_s\, ds - \int_{-\infty}^0 (-s)^{h-1/2} w_s\, ds\right], \end{aligned} \tag{2.2}$$

where

$$c_h = \left(\int_0^\infty \left((1+x)^{h-1/2} - x^{h-1/2}\right)^2 dx + \frac{1}{2h}\right)^{-1/2}$$

and the subtraction of the term $\int_{-\infty}^0 (-s)^{h-1/2}\, dB_s$ is to ensure that $B_0^h = 0$. If we define

$$g_h(t,s) = c_h \begin{cases} (t-s)^{h-1/2} - (-s)^{h-1/2} & \text{when} \quad s < 0, \\ (t-s)^{h-1/2} & \text{when} \quad 0 < s < t, \\ 0 & \text{when} \quad s > t, \end{cases} \tag{2.3}$$

then $B_t^h = \int_{-\infty}^\infty g_h(t,s)\, dB_s$.

Recall that the Riemann–Liouville fractional integral of order α of a function $\varphi(t)$ over the whole axis is defined as (see, e.g., [29])

$$I_+^\alpha \varphi(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^t (t-s)^{\alpha-1} \varphi(s)\, ds,$$

where $\Gamma(\alpha) = \int_0^\infty s^{\alpha-1} e^{-s}\, ds$ is the gamma function. Thus the FBM of Hurst parameter h is the fractional integral of order $h + \frac{1}{2}$ of the white noise w_t (up to a constant multiple).

In this paper we are concerned with the FBF (or fractional Brownian motions of several parameters). It is natural to define the FBF as the partial fractional integrals of the white noise of multiparameters. Partial fractional integrals have been studied in the literature (see for instance, [29], [25], and the references therein).

We introduce some notations. Let $x = (x_1, x_2, \ldots, x_d)$, $y = (y_1, y_2, \ldots, y_d) \in \mathbb{R}^d$. Denote $dx = dx_1\, dx_2 \cdots dx_d$. If for $i = 1, 2, \ldots, d$, $x_i \le y_i$, then we denote $x \preceq y$ or $y \succeq x$. If $x \succeq 0$, then we denote $x^y := (x_1^{y_1}, x_2^{y_2}, \ldots, x_d^{y_d})$. Let $x \in \mathbb{R}^d$. Denote $\Gamma(x) = \Gamma(x_1)\Gamma(x_2)\cdots\Gamma(x_d)$, where $\Gamma(x_i)$ is the gamma function. If $x \preceq y$, then we denote

$$\int_x^y \varphi(u)\, du = \int_{x_1}^{y_1}\int_{x_2}^{y_2}\cdots\int_{x_d}^{y_d} \varphi(u_1, u_2, \ldots, u_d)\, du_1\, du_2 \cdots du_d.$$

Let $w(x)$ be the d parameter white noise (see [13] and [14] for more discussions).

Now we introduce an FBF with parameter $H = (h_1, \ldots, h_d)$, where $h_1, \ldots, h_d \in (0, 1)$.

Definition 2.1. A fractional Brownian field (FBF) with parameter $H = (h_1, \ldots, h_d)$, where $h_1, \ldots, h_d \in (0, 1)$, is defined as

$$B^H(x) = \int_{-\infty}^x g_H(x, y) w(y)\, dy, \qquad x \in \mathbb{R}^d, \tag{2.4}$$

where $-\infty = (-\infty, \ldots, -\infty)$ and

$$g_H(x, y) = \prod_{i=1}^d g_{h_i}(x_i, y_i). \tag{2.5}$$

(Recall that g_h is defined by (2.3).)

We compute the covariance of $B^H(x)$.

Proposition 2.1. *Let $B^H(x)$, $x \in \mathbb{R}^d$, be an FBF with Hurst parameter $H = (h_1, \ldots, h_d)$, where $h_1, \ldots, h_d \in (0, 1)$. Then*

$$\mathbb{E}\left(B^H(x)B^H(x')\right) = \frac{1}{2^d}\prod_{i=1}^d \left(|x_i|^{2h_i} + |x_i'|^{2h_i} - |x_i - x_i'|^{2h_i}\right). \tag{2.6}$$

Proof. It is easy to check the following identity (see, e.g., [30]):

$$\int_{-\infty}^{\min(t_1,t_2)} g_h(t_1, s) g_h(t_2, s)\, ds = \tfrac{1}{2}\left[t_1^{2h} + t_2^{2h} - |t_1 - t_2|^{2h}\right].$$

Let $x \wedge x' = (x_1 \wedge x_1', \ldots, x_d \wedge x_d')$. Then

$$\mathbb{E}\left(B^H(x)B^H(x')\right) = \int_{-\infty}^{x\wedge x'} g_H(x, y) g_H(x', y)\, dy$$

$$= \prod_{i=1}^{d} \int_{-\infty}^{x_i \wedge x_i'} g_{h_i}(x_i, y_i) g_H(x_i', y_i)\, dy_i$$
$$= \frac{1}{2^d} \prod_{i=1}^{d} \left(|x_i|^{2h_i} + |x_i'|^{2h_i} - |x_i - x_i'|^{2h_i} \right),$$

proving the proposition. □

Proposition 2.2. *$B^H(x)$ has a continuous version for all $H = (h_1, h_2, \ldots, h_d)$, where $0 < h_i < 1$, $i = 1, 2, \ldots, d$.*

Proof. Let $\Lambda_n = \left\{x \in \mathbb{R}^d; |x| \le n\right\}$. First we estimate $\mathbb{E}\left(B^H(x) - B^H(y)\right)^2$. Denote for $i = 0, 1, 2, \ldots, d$,

$$P(i; x, y) = (x_1, \ldots, x_i, y_{i+1}, \ldots, y_d).$$

Thus

$$\mathbb{E}\left(B^H(x) - B^H(y)\right)^2 = \mathbb{E}\left(\sum_{i=1}^{d} B^H(P(i; x, y)) - B^H(P(i-1; x, y))\right)^2$$
$$\le C \sum_{i=1}^{d} \mathbb{E}\left(B^H(P(i; x, y)) - B^H(P(i-1; x, y))\right)^2,$$

where, and in what follows, C denotes a constant whose value may vary from place to place. By (2.6),

$$\mathbb{E}\left(B^H(P(i; x, y)) - B^H(P(i-1; x, y))\right)^2$$
$$= \frac{1}{2^{d-1}} \prod_{j=1}^{i-1} |x_j|^{2h_j} \prod_{j=i+1}^{d} |y_j|^{2h_j} |y_i - x_i|^{2h_i} \le C|y_i - x_i|^{2h_i}.$$

Let $\bar{h} = \min(h_1, h_2, \ldots, h_d)$. Thus for all $x, y \in \Lambda_n$, there is a constant C such that

$$\mathbb{E}\left(B^H(x) - B^H(y)\right)^2 \le C \sum_{i=1}^{d} |y_i - x_i|^{2h_i} \le C|x - y|^{2\bar{h}}.$$

Since $B(x)$ is Gaussian, for all $2 \le p < \infty$,

$$\mathbb{E}\ |\ B^H(x) - B^H(y)|^p \le C_p \left\{\mathbb{E}\left(B^H(x) - B^H(y)\right)^2\right\}^{p/2}$$
$$\le C_p |x - y|^{p\bar{h}}.$$

Thus by Kolmogorov's continuity theorem (see, e.g., pp. 209–212 of [2]), $B^H(x)$ has a continuous version on Λ_n. The proposition follows from a routine argument. □

Let Ω be the set of all continuous functions from $\mathbb{R}^d$ to $\mathbb{R}$. Let $\mathcal{F}$ be the σ-algebra generated by all the coordinate functions $F(x)\colon\ \Omega \to \mathbb{R}$, $x \in \mathbb{R}^d$, where $F(x)(\omega) =$

$\omega(x)$, $\forall\omega \in \Omega$. Then the FBF, $B^H(x)$, $x \in \mathbb{R}^d$, induces a probability measure P^H on $(\Omega, \mathcal{F})$. The probability space $(\Omega, \mathcal{F}, P^H)$ is called the canonical fractional Brownian field space. The coordinate process on $(\Omega, \mathcal{F}, P^H)$ is called the canonical FBF with Hurst parameter $H = (h_1, \ldots, h_d)$. Sometimes we omit the explicit dependence on H.

The following proposition is easy to show.

Proposition 2.3. *Let $B^H(x)$, $x \in \mathbb{R}^d$, be an FBF with Hurst parameter $H = (h_1, \ldots, h_d)$, where $h_1, \ldots, h_d \in (0, 1)$. Then $B^H(x)$ is a self-similar process in the sense that for any diagonal matrix $A = \operatorname{diag}(a_1, a_2, \ldots, a_d)$, where $a_i > 0$ for all $1 \le i \le d$, the random field $B^H(Ax)$, $x \in \mathbb{R}^d$, has the same probability law as $a^H B^H(x)$, $x \in \mathbb{R}^d$, where $a^H = a_1^{h_1} \cdots a_d^{h_d}$. As a consequence, we obtain that $B^H(ax)$, $x \in \mathbb{R}^d$, has the same probability law as $a^{h_1+h_2+\cdots+h_d} B^H(x)$, $x \in \mathbb{R}^d$.*

Remark 1. In this paper we only deal with the FBF whose Hurst parameter $H = (h_1, \ldots, h_d)$ satisfies $h_i > \frac{1}{2}$ for $i = 1, 2, \ldots, d$.

3. Stochastic Integral of Deterministic Functions

Let $B^H(x)$ be an FBF with Hurst parameter $H = (h_1, h_2, \ldots, h_d)$. Assume that $h_i > \frac{1}{2}$ for $i = 1, 2, \ldots, d$. We first define the stochastic integral with respect to deterministic kernels. The methodology that we adopt is found in [3], [14], [13], and the references therein.

Introduce the following Hilbert space:

$$\mathcal{L}_H = \left\{ f\colon \mathbb{R}^d \to \mathbb{R};\ \|f\|_H^2 = \int_{\mathbb{R}^{2d}} \varphi_H(u, v) f(u) f(v)\, du\, dv < \infty \right\},$$

where and in the rest of this paper φ_H is defined as

$$\varphi_H(u, v) := \prod_{i=1}^d \varphi_{h_i}(u_i, v_i), \qquad u = (u_1, u_2, \ldots, u_d) \quad \text{and} \quad v = (v_1, v_2, \ldots, v_d)$$

with $\varphi_h(u, v) := h(2h-1)|u - v|^{2H-2}$. For any $f \in \mathcal{L}_H$, there are sequences $a_{n,k} \in \mathbb{R}$, $x_{n,k}, y_{n,k}$ in $\mathbb{R}^d$ with $x_{n,k} \preceq y_{n,k}$, $n = 1, 2, \ldots$, $1 \le k \le k_n < \infty$, such that $f_n \to f$ in $\mathcal{L}_H$, where

$$f_n = \sum_{k=1}^{k_n} a_{n,k} \chi_{(x_{n,k}, y_{n,k}]}. \tag{3.1}$$

Let $\mathcal{S}$ be the set of the simple functions from $\mathbb{R}^d$ to $\mathbb{R}$ of the above form.

Lemma 3.1. *$\mathcal{S}$ is a linear space. $\mathcal{S}$ is also dense in $\mathcal{L}_H$.*

Proof. It suffices to show that $\chi_{(x_{n,k}, y_{n,k}]} - \chi_{(x'_{n,k}, y'_{n,k}]}$ is also a simple function. We only prove this lemma for $d = 2$. There are three possibilities: one of $(x_{n,k}, y_{n,k}]$ and $(x'_{n,k}, y'_{n,k}]$ is contained in the other; they are overlapped but no one is contained in the other; they are disjointed. It is easy to see that in all these three cases, the lemma is true. □

Let $f_n \in \mathcal{S}$ be of the form (3.1). We define

$$\int_{\mathbb{R}^d} f_n(x)\, dB^H(x) = \sum_{k=1}^{k_n} a_{n,k} B^H((x_{n,k}, y_{n,k}]).$$

It is easy to check that

$$\mathbb{E}\left\{\int_{\mathbb{R}^d} f_n(x)\, dB^H(x)\right\}^2 = \int_{\mathbb{R}^{2d}} \varphi_H(u,v)\, f_n(u)\, f_n(v)\, du\, dv.$$

(See [11] and [9] for the one-parameter case. The multiple parameter case is similar.) From this identity, it follows that

$$\int_{\mathbb{R}^d} f_n(x)\, dB^H(x), \qquad n = 1, 2, \ldots,$$

is a Cauchy sequence in $\mathcal{L}_2$. The limit is independent of the choice of the sequence f_n which converges to f. This limit, denoted by $\int_{\mathbb{R}^d} f(x)\, dB^H(x)$, is called the stochastic integral of $f \in \mathcal{L}_H$ with respect to the FBF $B^H(x)$, $x \in \mathbb{R}^d$.

This integral has the usual properties as a stochastic integral.

Proposition 3.2. *Let $f, g \in \mathcal{L}_H$. Then*

(i)

$$\int_{\mathbb{R}^d} [f(x) \pm g(x)]\, dB^H(x) = \int_{\mathbb{R}^d} f(x)\, dB^H(x) \pm \int_{\mathbb{R}^d} g(x)\, dB^H(x).$$

(ii) $\mathbb{E}\left(\int_{\mathbb{R}^d} f(x)\, dB^H(x)\right) = 0.$

(iii) *The following isometric identity holds*:

$$\mathbb{E}\left(\int_{\mathbb{R}^d} f(x)\, dB^H(x)\right)^2 = \int_{\mathbb{R}^{2d}} \varphi_H(u,v)\, f(u)\, f(v) du\, dv. \tag{3.2}$$

4. Multiple Stochastic Integral

This section and the next section follow the idea in [24] and [18]. For the (one-parameter) FBM, similar results have already been established in [5], [6], [9], and [19].

The Hermite polynomial of degree $n \geq 0$ is defined by

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}, \qquad x \in \mathbb{R}. \tag{4.1}$$

The generating function of these polynomials is

$$e^{tx - t^2/2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x), \qquad t \in \mathbb{R}.$$

For example,

$$H_0(x) = 1, \qquad H_1(x) = x, \qquad H_2(x) = x^2 - 1, \qquad H_3(x) = x^3 - 3x.$$

In general,

$$H_n(x) = \sum_{k \le n/2} \frac{(-1)^k n!}{2^k k!\,(n-2k)!} x^{n-2k}, \qquad n = 0, 1, 2, \dots.$$

It is also easy to see that

$$x^n = \sum_{k \le n/2} \frac{n!}{2^k k!\,(n-2k)!} H_{n-2k}(x), \qquad n = 0, 1, 2, \dots.$$

Denote by $\otimes$ the symmetric tensor product. Let $\{e_1, e_2, \dots\}$ be an orthonormal basis of $\mathcal{L}_H$. Then $\left\{e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}, 1 \le i_1, i_2, \dots, i_n < \infty\right\}$ is an orthonormal basis of $\mathcal{L}_H(\mathbb{R}^n) := \mathcal{L}_H^{\otimes n}$. It is easy to see that

$$\mathcal{L}_H(\mathbb{R}^n) = \left\{ f\colon \mathbb{R}^{nd} \to \mathbb{R};\ \int_{\mathbb{R}^{2nd}} \prod_{i=1}^n \varphi_H(u_i, v_i) \cdot f(u) f(v)\, du_1\, dv_1 \cdots du_n\, dv_n < \infty \right\},$$

where we denote

$$u = (u_1, \dots, u_n), \qquad v = (v_1, \dots, v_n),$$

and $f(u_1, \dots, u_n)$ is symmetric with respect to $u_1, \dots, u_n$. Let $e \in \mathcal{L}_H$ be a unit vector in $\mathcal{L}_H$. Then $e^{\otimes n}$ is a function of $x_1, \dots, x_n$. It is in $\mathcal{L}_H(\mathbb{R}^{nd})$. We define the multiple integral of Itô type of this function by

$$\int_{\mathbb{R}^{nd}} e^{\otimes n}(x_1, \dots, x_n)\, dB^H(x_1)\, dB^H(x_2) \cdots dB^H(x_n) = H_n\left(\int e(x)\, dB^H(x) \right).$$

More generally we define the multiple integral of Itô type of the function $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}$,

$$I_n(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}) = \int_{\mathbb{R}^{nd}} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}(x_1, \dots, x_n) \cdot dB^H(x_1)\, dB^H(x_2) \cdots dB^H(x_n),$$

by the polarization procedure (see, e.g., [12] and [23]).

To illustrate, we take one example. Since

$$e_1 \otimes e_2 = \tfrac{1}{4}\left[(e_1 + e_2)^{\otimes 2} - (e_1 - e_2)^{\otimes 2}\right],$$

we have

$$\int_{\mathbb{R}^{2d}} e_1 \otimes e_2(x_1, x_2)\, dB^H(x_1)\, dB^H(x_2) = \tfrac{1}{4}\left\{ k_1^2 H_2\left(\frac{e_1 + e_2}{k_1}\right) - k_2^2 H_2\left(\frac{e_1 - e_2}{k_2}\right) \right\},$$

where H_2 is defined by (4.1) and

$$k_1 = \|e_1 + e_2\|_H \quad \text{and} \quad k_2 = \|e_1 - e_2\|_H .$$

Denote

$$\mathcal{C}_n = \left\{ f \in \mathcal{L}_H(\mathbb{R}^n);\ f = \sum_{\text{finite sum}} a_{i_1,i_2,\ldots,i_n} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n} \right\}.$$

Let $f \in \mathcal{C}_n$ have the above form. Define

$$I_n(f) = \sum_{\text{finite sum}} a_{i_1,i_2,\ldots,i_n} I_n \left(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n} \right).$$

For $f, g \in \mathcal{L}(\mathbb{R}^{nd})$ we denote

$$\langle f, g\rangle_H = \int_{\mathbb{R}^{2nd}} \prod_{i=1}^n \varphi_H(u_i, v_i) f(u) g(v)\, du_1\, dv_1 \cdots du_n\, dv_n$$

and $\|f\|_H = (\langle f, f\rangle_H)^{1/2}$. It is easy to check that when $f \in \mathcal{C}_n$, the following isometric inequality holds:

$$\mathbb{E}\,(I_n(f))^2 = n!\, \|f\|_H^2 .$$

Now let f be an element of $\mathcal{L}_H$. There is a sequence $f_k \in \mathcal{C}_n$ such that $f_k \to f$ in $\mathcal{C}_n$. It is easy to see by the isometric equality that $I_n(f_k)$, $k = 1, 2, \ldots,$ is a Cauchy sequence in $\mathcal{L}_2$. The limit can be shown independent of the choice of the sequence f_k. The limit is called the multiple integral of Itô type and is denoted by

$$I_n(f) = \int_{\mathbb{R}^n d} f(x_1, x_2, \ldots, x_n)\, dB^H(x_1)\, dB^H(x_2) \cdots dB^H(x_n) = \lim_{k\to\infty} I_n(f_k).$$

It is easy to see that

$$\mathbb{E}\,(I_n(f) I_n(g)) = n!\, \langle f, g\rangle_H . \tag{4.2}$$

The following theorem can be proved in a similar way as in [9].

Theorem 4.1. *Let $H = (h_1, \ldots, h_d)$ with $h_i > \frac{1}{2}$. Let $B^H(x)$, $x \in \mathbb{R}^d$, be the canonical FBF in the canonical probability space $(\Omega, \mathcal{F}, P^H)$. Then for any square integrable random variable F on $(\Omega, \mathcal{F}, P^H)$, i.e., $F \in L^2(\Omega, \mathcal{F}, P^H)$, it admits the following chaos expansion*:

$$F = \sum_{n=0}^{\infty} I_n(f_n), \tag{4.3}$$

where f_n is in $\mathcal{L}(\mathbb{R}^{nd})$, $n = 0, 1, 2, \ldots$. The above series is convergent in $L^2(\Omega, \mathcal{F}, P^H)$. Moreover,

$$\mathbb{E} F^2 = \sum_{n=0}^{\infty} \|f_n\|_H^2 . \tag{4.4}$$

Remark 2. f_n is called the nth chaos coefficient of F. $F_n = I_n(f_n)$ is called the nth chaos of F.

5. Stochastic Integral of Itô Type

Let F and G be elements in $\mathcal{L}_2$ given by

$$F = \sum_{n=0}^{k} I_n(f_n) \quad \text{and} \quad G = \sum_{n=0}^{k} I_n(g_n), \tag{5.1}$$

where $f_n, g_n \in \mathcal{L}_H(\mathbb{R}^{nd}), n = 1, 2, \ldots, k,$ are symmetric functions. Define the following Hilbert scalar product:

$$\langle F, G \rangle_{\mathcal{D}} = \sum_{n=0}^{k} (n+1) \mathbb{E}\left[I_n(f_n) I_n(g_n) \right].$$

Set $\|F\|_{\mathcal{D}} = \langle F, G \rangle_{\mathcal{D}}^{1/2}$. We define $\mathcal{D}$ as the closure of all random variables of form (5.1) under the norm $\|\cdot\|_{\mathcal{D}}$:

$$\mathcal{D} = \{F \in \mathcal{L}_2; \|F\|_{\mathcal{D}} < \infty\}.$$

$\mathcal{D}$ is a special kind of Meyer–Watanabe distribution space. Let $f(x)$ be a random field such that for all $x \in \mathbb{R}^d$, $f(x) \in \mathcal{D}$. Thus $f(x)$ can be written as

$$f(x) = \sum_{n=0}^{\infty} I_n(f_n(x)),$$

where $f_n(x)$ is of the form $f_n(x; x_1, x_2, \ldots, x_n)$, $x, x_1, x_2, \ldots, x_n \in \mathbb{R}^d$, and

$$\sum_{n=0}^{\infty} (n+1) \mathbb{E}\left(I_n(f_n(x)) \right)^2 < \infty.$$

Let $\tilde{f}_n$ be the symmetrization of $f_n(x; x_1, x_2, \ldots, x_n)$ with respect to the $n+1$ (d-dimensional) vectors $x; x_1, x_2, \ldots, x_n$:

$$\tilde{f}_n(x_1, x_2, \ldots, x_{n+1}) = \frac{1}{n} \sum_{i=1}^{n} f(x_i; x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}).$$

We define

$$\int_{\mathbb{R}^d} f(x)\, dB^H(x) = \sum_{n=1}^{\infty} I_n(\tilde{f}_{n-1}). \tag{5.2}$$

Lemma 5.1. *Let $g_n \in \mathcal{L}_H(\mathbb{R}^{nd})$. Let f_n be the symmetrization of g_n. Then $f_n \in \mathcal{L}_H(\mathbb{R}^{nd})$ and*

$$\|f_n\|_H \le \|g_n\|_H.$$

The following theorem is easy to prove.

Theorem 5.2. *Let $f(x)$, $x \in \mathbb{R}^d$, be a random field such that $f(x) \in \mathcal{D}$ for all $x \in \mathbb{R}^d$ and the following assumption holds*:

$$\int_{\mathbb{R}^{2d}} \varphi_H(x, y)\langle f(x),\ f(y)\rangle_{\mathcal{D}}\, dx\, dy < \infty, \tag{5.3}$$

then $\int_0^t \int_{\mathbb{R}^d} f(x)\, dB^H(x)$ is well-defined as an element in $\mathcal{L}_2$ and

$$\mathbb{E}\left|\int_{\mathbb{R}^d} f(x)\, dB^H(x)\right|^2 \le \int_{\mathbb{R}^{2d}} \varphi_H(x, y)\langle f(x),\ f(y)\rangle_{\mathcal{D}}\, dx\, dy. \tag{5.4}$$

The following lemma is a Fubini type lemma.

Lemma 5.3. *Let $f(s, x)$, $s \in \mathbb{R}^m$, $x \in \mathbb{R}^d$, be a random field on $(\Omega, \mathcal{F}, P)$. Assume that*

$$\int_{\mathbb{R}^m} \int_{\mathbb{R}^{2d}} \varphi(x, y)|\langle f(s, x),\ f(s, y)\rangle_{\mathcal{D}}|\, ds\, dx\, dy < \infty. \tag{5.5}$$

Then

$$\int_{\mathbb{R}^d} \left\{\int_{\mathbb{R}^m} f(s, x)\, ds\right\} dB^H(x) = \int_{\mathbb{R}^m} \left\{\int_{\mathbb{R}^d} f(s, x)\, dB^H(x)\right\} ds. \tag{5.6}$$

We introduce the following “flat” Hilbert space as in [16]:

$$\mathcal{S}_\rho := \left\{F = \sum_{n=0}^{\infty} F_n;\ \sum_{n=0}^{\infty}((n+1)!)^\rho \mathbb{E}|F_n|^2 < \infty\right\},$$

where $\rho \in \mathbb{R}$ and F_n is the nth chaos of F. Since we do not introduce weights as in [26], we call these spaces the “flat” $\mathcal{L}_2$ type of distribution (or test) spaces. An element in $\mathcal{S}_\rho$ has the following property: each chaos is in $\mathcal{L}_2$.

Lemma 5.4. *Let $f(x)$, $x \in \mathbb{R}^d$, be a stochastic process on $(\Omega, \mathcal{F}, P)$ such that*

$$\int_{\mathbb{R}^{2d}} \varphi_H(x, y)\langle f(x),\ f(y)\rangle_{\mathcal{S}_\rho}\, dx\, dy < \infty.$$

Then $\int_{\mathbb{R}^d} f(x)\, dB^H(x)$ exists as an element of $\mathcal{S}_\rho$.

6. Heat Equations with Stationary Fractional Noise Potentials

Let $\Delta = \sum_{i=1}^d (\partial^2/\partial x_i^2)$ be the Laplacian. Let $H = (h_1, h_2, \ldots, h_d)$ with $\frac{1}{2} < h_i < 1$, $1 \le i \le d$. Let $B^H(x)$ be the canonical FBF with parameter H on the canonical probability space $(\Omega, \mathcal{F}, P)$. (For simplicity, we omit the explicit dependence on H.) The expectation with respect to this probability space is denoted by $\mathbb{E}$. The fractional white noise is denoted by $w^H(x)$, i.e., formally $w^H(x) = (\partial^d/\partial x_1 \partial x_2 \cdots \partial x_d) B^H(x)$.

Consider the following stochastic heat equation with time independent fractional white noise as the potentials:

$$\frac{\partial u_t(x)}{\partial t} = \tfrac{1}{2}\Delta u_t(x) + w^H(x)\cdot u_t(x), \tag{6.1}$$

where $u_0(x)$ is given and (for simplicity) is deterministic.

Let $P_t(x) = (1/(2\pi t)^{d/2})e^{-|x|^2/2t}$, $t\in\mathbb{R}_+$, $x\in\mathbb{R}^d$, be the Gaussian kernel associated with $\Delta/2$. Denote

$$P_t f(x) = \int_{\mathbb{R}^d} P_t(x-y) f(y)\,dy.$$

Definition 6.1. A random field $u\colon \mathbb{R}_+\times\mathbb{R}^d\times\Omega\to\mathbb{R}$ is called a solution of (6.1) if

(i) $u\colon [0,\infty)\times\mathbb{R}^d\times\Omega\to\mathbb{R}$ is jointly measurable;
(ii) $\int_0^t\int_{\mathbb{R}^d} P_{t-s}u_s(x-z)\,dB^H(z)\,ds$ is well-defined for all $t\in\mathbb{R}_+$ and $x\in\mathbb{R}^d$ as an element of $\mathcal{S}_\gamma$ for some $\gamma\in\mathbb{R}$.
(iii) the following equation holds in $\mathcal{S}_\gamma$:

$$u_t(x) = P_t u_0(x) + \int_0^t\int_{\mathbb{R}^d} P_{t-s}(x-z)u_s(z)\,dB^H(z)\,ds. \tag{6.2}$$

We denote $T_n = \{0<s_1<s_2<\cdots<s_n<t\}$ and $ds = ds_1\,ds_2\cdots ds_n$. If we iterate the above equation, we obtain formally that

$$u_t(x) = P_t u_0(x) + \sum_{n=1}^\infty I_n(f_n(t,x)), \tag{6.3}$$

where

$$\begin{aligned} f_n(t,x;x_1,x_2,\ldots,x_n) &= \operatorname{Sym}\Big\{\int_{T_n}\int_{\mathbb{R}^d} P_{t-s_n}(x-x_n) \\ &\qquad \cdots P_{s_2-s_1}(x_2-x_1)P_{s_1}(x_1-y)u_0(y)\,dy\,ds\Big\}, \end{aligned} \tag{6.4}$$

where Sym denotes the symmetrization with respect to $x_1,x_2,\ldots,x_n$.

We compute the $\mathcal{L}_2$ norm of each chaos. Denote

$$\Theta_n(t,x) = \mathbb{E}\,(I_n(f_n(t,x)))^2\,.$$

Assume that $|u_0(x)|\le C<\infty$, where, and in what follows, C denotes a positive constant whose value may differ in different appearances. By the isometric equality (4.2), we obtain

$$\begin{aligned} \Theta_n(t,x) = n!\int_{T_n^2}\int_{\mathbb{R}^{2(d+1)}} &\prod_{i=1}^n \varphi_H(\xi_i-\eta_i)P_{t-s_n}(x-\xi_n) \\ &\cdots P_{s_2-s_1}(\xi_2-\xi_1)P_{s_1}(\xi_1-y)u_0(y)P_{t-r_n}(x-\eta_n) \end{aligned}$$

$$
\begin{aligned}
&\cdots P_{r_2-r_1}(\eta_2-\eta_1)P_{r_1}(\eta_1-\tilde{y})u_0(\tilde{y})\,dy\,d\tilde{y}\,d\xi\,d\eta\,ds\,dr \\
&\le Cn!\int_{T_n^2}\int_{\mathbb{R}^{2d}}\prod_{i=1}^n \varphi_H(\xi_i-\eta_i)P_{t-s_n}(x-\xi_n)\cdots P_{s_2-s_1}(\xi_2-\xi_1) \\
&\quad \cdot P_{t-r_n}(x-\eta_n)\cdots P_{r_2-r_1}(\eta_2-\eta_1)\,d\xi\,d\eta\,ds\,dr.
\end{aligned}
$$

Let $p_t(x)$ be the one-dimensional Gaussian kernel $(2\pi t)^{-1/2}e^{-x^2/2t}$. Thus

$$
\Theta_n(t,x)\le Cn!\int_{T_n^2}\prod_{i=1}^d \Theta_{i,n}(t,x_i)\,ds\,dr,
$$

where

$$
\begin{aligned}
\Theta_{i,n}(t,x_i) = &\int_{\mathbb{R}^2}\prod_{i=1}^n \varphi_{h_i}(\rho_i-\tau_i)p_{t-s_n}(x_i-\rho_n)\cdots p_{s_2-s_1}(\rho_2-\rho_1) \\
&\cdot p_{t-r_n}(x_i-\tau_n)\cdots p_{r_2-r_1}(\tau_2-\tau_1)\,d\rho\,d\tau.
\end{aligned}
$$

Lemma 6.1. *Let* $0<\alpha<1$. *Then there is a positive constant* C_α *such that*

$$
\int_{\mathbb{R}} p_t(x-y)|y|^{-\alpha}\,dy \le C_\alpha t^{-\alpha/2}.
$$

Proof. It is easy to see that

$$
\begin{aligned}
\int_{\mathbb{R}} p_t(x-y)|y|^{-\alpha}\,dy &= C\int_{\mathbb{R}} e^{-|x-y|^2/2t}t^{-1/2}|y|^{-\alpha}\,dy \\
&= C\int_{\mathbb{R}} e^{-|\xi-x/\sqrt{2t}|^2|^2}t^{-\alpha/2}|\xi|^{-\alpha}\,d\xi \\
&\le Ct^{-\alpha/2}\left(\int_{\{|\xi|\le 1\}}|\xi|^{-\alpha}\,d\xi+\int_{\{|\xi|>1\}}e^{-|\xi-x/\sqrt{2t}|^2|^2}\right)d\xi \\
&\le Ct^{-\alpha/2}.
\end{aligned}
$$

This shows the lemma. □

Now we estimate $\Theta_{i,n}(t,x_i)$.

Lemma 6.2. *There is a constant* C_{h_i} *such that*

$$
\Theta_{i,n}(t,x_i)\le C_{h_i}^n\prod_{k=1}^n (s_{k+1}-s_k+r_{k+1}-r_k)^{h_i-1}\,, \tag{6.5}
$$

where $s_{n+1}=r_{n+1}=t$.

Proof. Let B_t, $\tilde{B}_t$ be two independent real-valued standard Brownian motions. Then

$$
\int_{\mathbb{R}} p_t(x-y)f(y)\,dy = \mathbb{E}\,f(x+B_t).
$$

Thus

$$\begin{aligned}
&\int_{\mathbb{R}^2} \varphi_{h_i}(\rho_1-\tau_1)\, p_{s_2-s_1}(\rho_2-\rho_1)\, p_{r_2-r_1}(\tau_2-\tau_1)\, d\rho_1\, d\tau_1 \\
&\quad = \int_{\mathbb{R}} \mathbb{E}\, \varphi_{h_i}(\rho_2+B_{s_2-s_1}-\tau_1)\, p_{r_2-r_1}(\tau_2-\tau_1)\, d\tau_1 \\
&\quad = \mathbb{E}\varphi_{h_i}\left(\rho_2+B_{s_2-s_1}-(\tau_2+\tilde{B}_{r_2-r_1})\right) \\
&\quad = \mathbb{E}\varphi_{h_i}\left(\rho_2-\tau_2+B_{s_2-s_1}-\tilde{B}_{r_2-r_1}\right) \\
&\quad = \mathbb{E}\varphi_{h_i}\left(\rho_2-\tau_2+B_{s_2-s_1+r_2-r_1}\right) \\
&\quad = \int_{\mathbb{R}} p_t(\rho_2-\tau_2-y)|y|^{2h_i-2}\, dy \\
&\quad \le C_{h_i} t^{h_i-1},
\end{aligned}$$

where $t=s_2-s_1+r_2-r_1$. Thus we show that (6.5) is true when $n=1$. The lemma follows from iteration . □

Now we estimate Θ_n. Denote by $\mathcal{C}_x$ a generic function of $x>0$ such that there is constant $\lambda>1$ such that $1/\lambda^x \le \mathcal{C}_x \le \lambda^x$. $\mathcal{C}_x$ may be different in different appearances. λ may depend on the initial condition u_0, H, d, and other parameters. However, λ is independent of n and t.

Lemma 6.3. *Let $|H|=h_1+h_2+\cdots+h_d$; let $d-|H|<2$ and let $\gamma=2d-2|H|$. For any $\gamma<4$, we have*

$$\Theta_n \le \mathcal{C}_n t^{2n} n^{(1-2(1-\gamma/4))n}. \tag{6.6}$$

Proof. Let $C_H=C_{h_1}\cdots C_{h_d}$. Applying Lemma 6.2, we obtain

$$\begin{aligned}
\Theta_n &\le n!\, C_H^n \int_{T_n^2} \prod_{k=1}^n (s_{k+1}-s_k+r_{k+1}-r_k)^{-\gamma/2}\, ds\, dr \\
&\le n!\, C_H^n \int_{T_n^2} \prod_{k=1}^n (s_{k+1}-s_k)^{-\gamma/4}\, (r_{k+1}-r_k)^{-\gamma/4}\, ds\, dr \\
&\le n!\, C_H^n \left\{ \int_{T_n} \prod_{k=1}^n (s_{k+1}-s_k)^{-\gamma/4}\, ds \right\}^2 .
\end{aligned}$$

Now we estimate the integral inside the above { }:

$$\mathcal{I}_n := \int_{T_n} \prod_{k=1}^n (s_{k+1}-s_k)^{-\gamma/4}\, ds.$$

It is easy to see that the above integral exists when $\gamma < 4$. We obtain an explicit bound. Making the substitution $u_1 = s_1$, $u_2 = s_2 - s_1, \ldots, u_n = s_n - s_{n-1}$, we obtain

$$\begin{aligned}\mathcal{I}_n &= \int_{\mathcal{U}_n} \prod_{k=2}^{n-1} u_k^{-\gamma/4} (t - u_1 - \cdots - u_n)^{-\gamma/4} \, du \\ &\le t^n \int_{\mathcal{V}_n} v_2^{-\gamma/4} \cdots v_{n-1}^{-\gamma/4} (1 - v_1 - \cdots - v_n)^{-\gamma/4} dv \\ &= \frac{t^n \Gamma(1-\gamma/4)^n}{\Gamma(n(1-\gamma/4)+2)},\end{aligned}$$

where

$$\mathcal{U}_n = \{(u_1, u_2, \ldots, u_n); u_1 > 0, \ldots, u_n > 0, u_1 + u_2 + \cdots + u_n < t\}$$

and

$$\mathcal{V}_n = \{(v_1, v_2, \ldots, v_n); v_1 > 0, \ldots, v_n > 0, v_1 + v_2 + \cdots + v_n < 1\}.$$

Thus

$$\Theta_n \le \frac{C_n t^{2n} n!}{\Gamma(n(1-\gamma/4)+2)^2}.$$

By the Stirling formula [1], we obtain

$$n! = n^n \mathcal{C}_n$$

and

$$\Gamma(n(1-\gamma/4)+1)^2 = \mathcal{C}_n n^{2(1-\gamma/4)n}.$$

Thus the lemma is proved. □

The following lemma deals with the asymptotics of $u_t(x)$ in $\mathcal{S}_\rho$.

Lemma 6.4. *Let $d - |H| < 2$ and let $u_t(x)$ be defined by* (6.2). *Denote $\rho_0 = 1 + |H| - d$. If $\rho < \rho_0$, then $u_t(x) \in \mathcal{S}_\rho$ for all $t > 0$ and $x \in \mathbb{R}^d$. Moreover, for all $0 < \kappa < \rho_0 - \rho$,*

$$\limsup_{t \to \infty} \frac{\log\left(\sup_{x \in \mathbb{R}^d} \|u_t(x)\|_\rho^2\right)}{t^{2/\kappa}} < \infty. \tag{6.7}$$

Proof. We continue to use the notation introduced previously. By Lemma 6.3 and the definition of $\mathcal{S}_\rho$,

$$\|u_t(x)\|_\rho^2 \le \sum_{n=0}^{\infty} n!^\rho \Theta_n$$

$$\leq \sum_{n=0}^{\infty} \frac{C^n t^{2n}}{\Gamma\left((2(1-\gamma/4)-1-\rho)n+1\right)}$$
$$\leq E_{2(1-\gamma/4)-1-\rho}(Ct^2),$$

where $E_r(x)$ is the Mittag–Leffler function of x with parameter r. From the asymptotic property of the Mittag–Leffler function [10], it follows that

$$\|u_t(x)\|_\rho^2 \leq \exp(Ct^{2/(2(1-\gamma/4)-1-\rho)}).$$

Now it is easy to check that $r > 2d - 2|H|$ implies that $2(1-\gamma/4) - 1 < \rho_0$. Namely, $\kappa = 2(1-\gamma/4) - 1 - \rho < \rho_0 - \rho$. This proves the lemma. □

Lemma 6.5. *$u_t(x)$ defined by* (6.3) *is the solution of* (6.1) *in the sense of Definition* 6.1.

Proof. Let $u_t(x)$ be given by (6.3). It suffices to verify (6.2). By the Fubini lemma and the definition of the integral and the definition of $f_n(t, x)$, we have

$$\int_0^t \int_{\mathbb{R}^d} P_{t-s}(x-z) I_n(f_n(s,z))\, dB^H(z)\, ds$$
$$= \int_{\mathbb{R}^d} I_n\left(\int_0^t P_{t-s}(x-z)\, f_n(s,z)\, ds\right) dB^H(z)$$
$$= I_{n+1}(f_{n+1}(t,x)).$$

Thus the lemma follows. □

From the above lemmas, we get

Theorem 6.6. *Let B^H be the canonical FBF with Hurst parameter $H = (h_1, h_2, \ldots, h_d)$. Denote $|H| = h_1 + h_2 + \cdots + h_d$ and $\rho_0 = 1 + |H| - d$. Let $\frac{1}{2} < h_i < 1$ for all $1 \leq i \leq d$. If $d - |H| < 2$, then the solution of* (6.1) *is in $\mathcal{S}_\rho$ for all $t \in \mathbb{R}_+$, $x \in \mathbb{R}^d$, and $\rho < \rho_0$. Moreover, if $0 < \kappa < \rho_0 - \rho$, then*

$$\limsup_{t\to\infty} \frac{\log\left(\sup_{x\in\mathbb{R}^d} \|u_t(x)\|_\rho^2\right)}{t^{2/\kappa}} < \infty. \tag{6.8}$$

Since, when $\rho = 0$, $\mathcal{S}_0$ is the $\mathcal{L}_2$ space, we have

Corollary 6.7. *Let B^H be the canonical FBF with parameter $H = (h_1, h_2, \ldots, h_d)$. Denote $|H| = h_1 + h_2 + \cdots + h_d$. Let $\frac{1}{2} < h_i < 1$ for all $1 \leq i \leq d$. If $|H| > d - 1$, then the solution of* (6.1) *is in $\mathcal{L}_2$ for all $t \in \mathbb{R}_+$, $x \in \mathbb{R}^d$. Moreover, for any $0 < \kappa < 1 + |H| - d$,*

$$\limsup_{t\to\infty} \frac{\log\left(\sup_{x\in\mathbb{R}^d} \mathbb{E}|u_t(x)|^2\right)}{t^{2/\kappa}} < \infty. \tag{6.9}$$

We take a look at the case that all h_i are the same: $h_1 = h_2 = \cdots = h_d = h$. In this case $|H| = hd$. Thus the condition $|H| > d - 1$ becomes $h > 1 - 1/d$. Thus

Corollary 6.8. *Let B^H be a canonical FBF in $\mathbb{R}^d$ ($d \geq 2$) with parameter $H = (h, h, \dots, h)$ with $\frac{1}{2} < h < 1$. If $h > 1 - 1/d$, then the solution of* (6.1) *is in $\mathcal{L}_2$ for all $t \in \mathbb{R}_+$, $x \in \mathbb{R}^d$. Moreover, for any $0 < \kappa < 1 + d(h-1)$,*

$$\limsup_{t\to\infty} \frac{\log\bigl(\sup_{x\in\mathbb{R}^d} \mathbb{E}|u_t(x)|^2\bigr)}{t^{2/\kappa}} < \infty. \tag{6.10}$$

Remark 3. When $d = 2$ we see that the condition in the above corollary becomes $h > \frac{1}{2}$. When $d = 2$ and when the noise is white ($h = \frac{1}{2}$), the solution is not in $\mathcal{L}_2$ when t is large [16]. This result demonstrates that in the ($d = 2$)-dimensional case, although when $h = \frac{1}{2}$ the solution of (6.1) is not in $\mathcal{L}_2$ for all t, once $h > \frac{1}{2}$, then the solution is in $\mathcal{L}_2$ for all t. This explanation also shows us that the conditions in the above corollary are the best we can get (in some sense).

7. Heat Equations with Nonstationary Fractional Noise Potentials

In this section we consider the stochastic heat equation with time dependent fractional white noise potentials. We continue to use the notation previously introduced.

Let $B^H(t, x)$, $(t, x) \in \mathbb{R}^{d+1}$, be the canonical FBF with Hurst parameter H on the canonical probability space $(\Omega, \mathcal{F}, P)$, where $H = (h_0, h_1, \dots, h_d)$ with $\frac{1}{2} < h_i < 1$, $0 \leq i \leq d$. The fractional white noise is denoted by $w^H(t, x)$, i.e., formally $w^H(t, x) = (\partial^{d+1}/\partial t \partial x_1 \partial x_2 \cdots \partial x_d) B^H(t, x)$.

Consider the following stochastic heat equation with time dependent fractional white noise as the potentials:

$$\frac{\partial u_t(x)}{\partial t} = \tfrac{1}{2}\Delta u_t(x) + w^H(t, x) \cdot u_t(x), \tag{7.1}$$

where the initial function $u_0(x)$ is given and is deterministic.

Let $P_t(x) = (1/(2\pi t)^{d/2}) e^{-|x|^2/2t}$ be the Gaussian kernel associated with $\Delta/2$. Denote

$$P_t f(x) = \int_{\mathbb{R}^d} P_t(x-y) f(y)\, dy.$$

Definition 7.1. A random field $u\colon \mathbb{R}_+ \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ is called a solution of (7.1) if

(i) $u\colon \mathbb{R}_+ \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ is jointly measurable;
(ii) $\int_0^t \int_{\mathbb{R}^d} P_{t-s} u_s(x-z)\, dB^H(s, z)$ is well-defined for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^d$ as an element of $\mathcal{S}_\gamma$ for certain $\gamma \in \mathbb{R}$;
(iii) the following equation holds in $\mathcal{S}_\gamma$:

$$u_t(x) = P_t u_0(x) + \int_0^t \int_{\mathbb{R}^d} P_{t-s}(x-z) u_s(z)\, dB^H(s, z). \tag{7.2}$$

The formal chaos expansion of the solution is

$$u_t(x) = P_t u_0(x) + \sum_{n=1}^{\infty} I_n(f_n(t, x)), \tag{7.3}$$

where

$$\begin{aligned} &f_n(t,x;s_1,x_1,s_2,x_2,\dots,s_n,x_n)\\ &\quad=\mathrm{Sym}\left\{\int_{\mathbb{R}^d}P_{t-s_n}(x-x_n)\cdots P_{s_2-s_1}(x_2-x_1)P_{s_1}(x_1-y)u_0(y)\,dy\right\}, \end{aligned} \tag{7.4}$$

where Sym is the symmetrization with respect to n (($d+1$)-dimensional) variables $(s_1,x_1),\dots,(s_n,x_n)$ and

$$\begin{aligned} I_n(f_n(t,x))=\int_{([0,t]\times\mathbb{R}^d)^n}&f_n(t,x;s_1,x_1,s_2,x_2,\dots,s_n,x_n)\\ &\cdot dB^H(s_1,x_1)\,dB^H(s_2,x_2)\cdots dB^H(s_n,x_n). \end{aligned}$$

We compute the $\mathcal{L}_2$ norm of each chaos, i.e., $\Theta_n(t,x)=\mathbb{E}\,(I_n(f_n(t,x)))^2$. Assume that $|u_0(x)|\le C<\infty$. By the isometric inequality, we obtain

$$\begin{aligned} \Theta_n(t,x)&=\int_{T_n^2}\int_{\mathbb{R}^{2(d+1)}}\prod_{i=1}^n\varphi_{h_0}(s_i-r_i)\varphi_H(\xi_i-\eta_i)P_{t-s_n}(x-\xi_n)\\ &\qquad\cdots P_{s_2-s_1}(\xi_2-\xi_1)P_{s_1}(\xi_1-y)u_0(y)P_{t-r_n}(x-\eta_n)\\ &\qquad\cdots P_{r_2-r_1}(\eta_2-\eta_1)P_{r_1}(\eta_1-y)u_0(\tilde y)\,dy\,d\tilde y\,d\xi\,d\eta\,ds\,dr\\ &\le C\int_{T_n^2}\int_{\mathbb{R}^{2d}}\prod_{i=1}^n\varphi_{h_0}(s_i-r_i)\varphi_H(\xi_i-\eta_i)P_{t-s_n}(x-\xi_n)\\ &\qquad\cdots P_{s_2-s_1}(\xi_2-\xi_1)P_{t-r_n}(x-\eta_n)\cdots P_{r_2-r_1}(\eta_2-\eta_1)\,d\xi\,d\eta\,ds\,dr. \end{aligned}$$

Let $p_t(x)$ be the one-dimensional Gaussian kernel $(2\pi t)^{-1/2}e^{-x^2/2t}$. Thus

$$\Theta_n(t,x)=\int_{T_n^2}\prod_{i=1}^n\varphi_{h_0}(s_i-r_i)\prod_{i=1}^d\Theta_{i,n}(t,x_i,s,r)\,ds\,dr,$$

where

$$\begin{aligned} \Theta_{i,n}(t,x_i,s,r)=\int_{\mathbb{R}^2}&\prod_{i=1}^n\varphi_{h_i}(\rho_i-\tau_i)P_{t-s_n}(x_i-\rho_n)\cdots P_{s_2-s_1}(\rho_2-\rho_1)\\ &\cdot P_{t-r_n}(x_i-\tau_n)\cdots P_{r_2-r_1}(\tau_2-\tau_1)\,d\rho\,d\tau. \end{aligned}$$

Similar to (6.5), $\Theta_{i,n}(t,x_i,s,r)$ is estimated as follows:

$$\Theta_{i,n}(t,x_i,s,r)\le C_{h_i}^n\prod_{k=1}^n(s_{k+1}-s_k+r_{k+1}-r_k)^{h_i-1},$$

where $s_{n+1}=t_{n+1}=t$. As before, let $\gamma=2d-2|H|$. Then

$$\begin{aligned} \Theta_n&\le C_H^n\int_{T_n^2}\prod_{k=1}^n|s_k-r_k|^{2h_0-2}\,(s_{k+1}-s_k+r_{k+1}-r_k)^{-\gamma/2}\,ds\,dr\\ &\le C_H^n\left\{\int_{T_n^2}\prod_{k=1}^n(s_{k+1}-s_k+r_{k+1}-r_k)^{-p\gamma/2}\,ds\,dr\right\}^{1/p}\\ &\quad\cdot\left\{\int_{T_n^2}\prod_{k=1}^n|s_k-r_k|^{(2h_0-2)q}\,ds\,dr\right\}^{1/q}. \end{aligned}$$

Making the substitution $s_k = tu_k$ and $r_k = tv_k$, we obtain that there is a constant C_{p,h_0} such that for any value of q such that $1 \le q < 1/(2-2h_0)$,

$$\begin{aligned}&\left\{\int_{T_n^2}\prod_{k=1}^n |s_k - r_k|^{(2h_0-2)q}\, ds\, dr\right\}^{1/q}\\ &\quad = \left\{t^{(2+(2h_0-2)q)n}\int_{\mathcal{V}_n^2}\prod_{k=1}^n |u_k - v_k|^{(2h_0-2)q}\, du\, dv\right\}^{1/q}\\ &\quad \le C_{p,h_0}^n t^{((2+2(h_0-2)q)/q)n}.\end{aligned}$$

Similar to the proof of Lemma 6.3, we obtain that if $p\gamma < 4$, then

$$\left(\int_{T_n^2}\prod_{k=1}^n (s_{k+1}-s_k+r_{k+1}-r_k)^{-p\gamma/2}\, ds\, dr\right)^{1/p} \le \frac{C^n t^{2n/p}}{\Gamma((2/p)(1-p\gamma/4)n+1)}.$$

Thus we obtain

$$\begin{aligned}\Theta_n &\le C_{p,H,\gamma}^n t^{((2+2(h_0-1)q)/q)n}\left\{\int_{T_n^2}\prod_{k=1}^n (s_{k+1}-s_k+r_{k+1}-r_k)^{-p\gamma/2} ds\, dr\right\}^{1/p}\\ &\le \frac{C_{p,H,\gamma}^n t^{2(h_0-1)n}}{\Gamma((2/p-\gamma/2)n+1)}.\end{aligned}$$

When $1 \le q < 1/(2-2h_0)$,

$$p = \frac{q}{q-1} > \frac{1}{2h_0-1}.$$

The above condition is equivalent to the following condition:

$$d < 4h_0 + |H| - 2. \tag{7.5}$$

Thus

$$\frac{2}{p} - \frac{\gamma}{2} < \frac{2}{2h_0-1} + |H| - d.$$

Hence, similar to the argument for Theorem 6.6, we obtain

Theorem 7.1. *Let B^H be the canonical FBF with parameter $H = (h_0, h_1, h_2, \ldots, h_d)$. Denote $|H| = h_1 + h_2 + \cdots + h_d$. Let $\frac{1}{2} < h_i < 1$ for all $0 \le i \le d$. Let $\rho_0 = 2/(2h_0-1) + |H| - d$. If $d < 4h_0 + |H| - 2$, then the solution of* (7.1) *is in $\mathcal{S}_\rho$ for all $\rho < \rho_0$. Moreover, for any $0 < \kappa < \rho_0 - \rho$,*

$$\limsup_{t\to\infty} \frac{1}{t^{2h_0/\kappa}} \log \sup_{x\in\mathbb{R}^d} \|u_t(x)\|_\rho < \infty. \tag{7.6}$$

Since, when $\rho = 0$, $\mathcal{S}_0$ is the $\mathcal{L}_2$ space, we have

Corollary 7.2. *Let B^H be the canonical FBF with parameter $H = (h_1, h_2, \ldots, h_d)$. Denote $|H| = h_1 + h_2 + \cdots + h_d$. Let $\frac{1}{2} < h_i < 1$ for all $1 \le i \le d$. If*

$$\rho_0 = \frac{2}{2h_0 - 1} + |H| - d > 0, \tag{7.7}$$

then the solution of (7.1) *is in $\mathcal{L}_2$. Moreover, for any $0 < \kappa < \rho_0$,*

$$\limsup_{t\to\infty} \frac{1}{t^{2h_0/\kappa}} \log \sup_{x\in\mathbb{R}^d} \mathbb{E}|u_t(x)|^2 < \infty. \tag{7.8}$$

We take a look at the case that all h_i are the same: $h_0 = h_1 = h_2 = \cdots = h_d = h$. In this case $|H| = hd$. Thus condition (7.7) becomes $h > 1 - 2/(2h_0 - 1)d$. Thus

Corollary 7.3. *Let B^H be the canonical FBF in $\mathbb{R}^d$ $(d \ge 2)$ with parameter $H = (h, h, \ldots, h)$ with $\frac{1}{2} < h < 1$. If $h > 1 - 2/(2h_0 - 1)d$, then the solution of* (7.1) *is in $\mathcal{L}_2$.*

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