Acta Informatica (2008) 45:279–320
DOI 10.1007/s00236-008-0071-6

ORIGINAL ARTICLE

# Relational structures model of concurrency

**Ryszard Janicki**

Received: 2 March 2007 / Accepted: 22 January 2008 / Published online: 11 March 2008
© Springer-Verlag 2008

**Abstract** The paper deals with the foundations of concurrency theory. We show how structurally complex concurrent behaviours can be modelled by relational structures $(X, <>, \sqsubset)$, where $X$ is a set (of event occurrences), and $<>$ (interpreted as *commutativity*) and $\sqsubset$ (interpreted as *weak causality*) are binary relations on $X$. The paper is a continuation of the approach initiated in Gaifman and Pratt (Proceedings of LICS'87, pp 72–85, 1987), Lamport (J ACM 33:313–326, 1986), Abraham et al. (Semantics for concurrency, workshops in computing. Springer, Heidelberg, pp 311–323, 1990) and Janicki and Koutny (Lect Notes Comput Sci 506:59–74, 1991), substantially developed in Janicki and Koutny (Theoretical Computer Science 112:5–52, 1993) and Janicki and Koutny (Acta Informatica 34:367–388, 1997), and recently generalized in Guo and Janicki (Lect Notes Comput Sci 2422:178–191, 2002) and Janicki (Lect Notes Comput Sci 3407:84–98, 2005). For the first time the full model for the most general case is given.

## 1 Introduction

It is often assumed that there are *two* major different (and often incompatible) attitudes towards abstracting non-sequential behaviour, one based on *interleaving abstraction* ([4,32], etc.), another based on *partially ordered causality* ([6,10,11,35], etc.). Interleaving models (for instance various types of *process algebras*[4]) are highly structured and compositional, but have difficulty in dealing with topics like fairness, confusion, etc. Partial order models can handle these problems better but are less compositional and less structured, although recent results [6,35] make that distance much smaller.

Nevertheless some aspects of concurrent behaviour are difficult or almost impossible to tackle using both process algebras and partially ordered causality based models. For example,

Partially supported by NSERC of Canada Grant.

R. Janicki (✉)
Department of Computing and Software, McMaster University,
Hamilton, ON L8S 4K1, Canada
e-mail: janicki@mcmaster.ca

the specification of priorities, error recovery, inhibitor nets, proper treatment of simultaneity, time testing, etc., are in some circumstances problematic [20,18,21,30,39].

There have been few attempts to go beyond interleaving and partially ordered causality, and so restricting concurrency to these two models is usually assumed without much discussion.

Lamport [29], Gaifman and Pratt [12] and Janicki and Koutny [16] have independently proposed, for specifying concurrent behaviour, a pair of relations, the first being just partially ordered causality, while the second is not (in general) a partial order, and can be interpreted as weak causality. The ideas of [29] have been further investigated by Abraham et al. [1], but have never been fully developed. Initially the similarities between Lamport's model [29], and Gaifman-Pratt's model [12] were considered accidental, as both the motivations and initial assumptions were different. The ideas of [16] were later fully developed in [18,20] and resulted in a general model of concurrency that includes both Lamport's [29] and Gaifman-Pratt's [12] models as well as the classical "interleaving" and "true concurrency" models, as distinctive special cases. The papers [18,20] not only provide the theoretical foundations of the model but prove its soundness as well.

In principle the model assumes that concurrent behaviour is fully described by a triple $(X, <, \sqsubset)$, where $X$ is the set of event occurrences, $<$ and $\sqsubset$ are relations on $X$, $<$ is "causality" (i.e. an abstraction of "earlier than"), and $\sqsubset$ is "weak causality" (an abstraction of "not later than" relation).

This model has been successfully applied to inhibitor systems [19,3,23,27,28], priority systems [21,26], asynchronous races [41,42], synthesis [24,33,36] and event automata [34], and has influenced many other approaches [5,40].

However, it was shown by Janicki and Koutny in [18] that relational structures of the type $(X, <, \sqsubset)$ *still cannot model the most general case of concurrent behaviour* and that the most general case requires relational structures of the type $(X, <\!\!>, \sqsubset)$, where $<\!\!>$, called "commutativity", is an abstraction of the pure "interleaving" relation ("either earlier than or later than, but never simultaneously"), but no axioms for $<\!\!>$ were given.

An axiomatic model for the structures of the type $(X, <\!\!>, \sqsubset)$ was recently proposed by Guo and Janicki [13] and Janicki [15] in two conference papers with rather sketchy proofs. This paper provides a full and substantially revised version of the results announced in [13,15].

The traditional models of concurrency, i.e. those based on the concepts of "interleaving" or "partially ordered causality" are mathematically much less complex and far more developed than the models with two relations, and *they suffice for the majority of standard applications*. Nonetheless some aspect of concurrent behaviour is difficult or almost impossible to tackle using either interleaving or partially ordered causality based models.

From a purely mathematical viewpoint the results of this paper can be seen as an extension of the Szpilrajn Theorem[1] [38] to orders that are not necessary total. Additionally, the results show how a set of equivalent partial orders can be uniquely represented by just two single relations.

The paper is structured into six parts, from Sects. 2 to 7. In Sect. 2, the main ideas will be presented informally, using five simple examples. Section 3 presents a revised version of the model of concurrency proposed in [18], including a few never published results; while Sect. 4 contains all of the necessary definitions and revised results of the theory of order structures from [20]. The new modified and improved version of the relationship between the order structures of [20] and the concurrent histories of [18] is discussed in Sect. 5. In Sect. 6, the longest section and the main portion of the paper, a theory of relational structures of the type

[1] Every partial order is the intersection of all its total order extensions.

$(X, <>, \sqsubset)$ is presented. The relationship between the results of Sect. 6 and the concurrent histories of [18] is discussed in detail in Sect. 7. Section 8 contains some final comments.

## 2 Motivation and intuition

To illustrate the main ideas, let us first consider the following four programs which are very simple, but nonetheless reflect the essence of the problem. (A fifth program will be presented later.) All the programs are written using a mixture of *cobegin, coend* and a (version of concurrent) *guarded commands*.

*Example 1* (programs P1, P2, P3 and P4)

```
P1: begin int x;
       a: x:=0;
       cobegin b: x:=x+1, c: x:=x+2 coend
       end P1.

  P2: begin int x,y;
       a: begin x:=0; y:=0 end;
       cobegin
       b: x=0 → y:=y+1, c: x:=x+1,
       coend
       end P2.

  P3: begin int x,y;
       a: begin x:=0; y:=0 end;
       cobegin
       b:y=0 → x:=x+1, c: x=0 → y:=y+1
       coend
       end P3.

  P4: begin int x,y;
       a: begin x:=0; y:=0 end;
       cobegin b: x:=x+1, c: y:=y+1 coend
       end P4.
```
□

Each program is a different composition of three events (actions) called $a$, $b$ and $c$ ($a_i$, $b_i$, $c_i$, $i = 1, \dots, 4$, to be exact, but a restriction to $a$, $b$, $c$, does not change the validity of the analysis below, while simplifying the notation). Alternative models of these programs are shown in Fig. 1.

What *concurrent behaviours* (*concurrent histories*) are generated by the above programs? Let us concentrate on behaviours that involve all three actions $a$, $b$, $c$ (sometimes such behaviours are called *proper*). Let $obs(P_i)$ denote the set of all program runs involving the actions $a$, $b$, $c$ that can be observed. Assume that simultaneous executions can be observed. In this simple case all runs (or observations) can be modelled by *step-sequences* (or equivalently *stratified orders*), with simultaneous execution of $a_1, \dots, a_n$ denoted by $\{a_1, \dots, a_n\}$. Let us denote $o_1 = abc$, $o_2 = acb$, $o_3 = a\{b, c\}$. Each $o_i$ can be seen as a partial order $o_i = (\{a, b, c\}, \stackrel{o_i}{\to})$, where: $o_1 = a \stackrel{o_1}{\to} b \stackrel{o_1}{\to} c$, $o_2 = a \stackrel{o_2}{\to} c \stackrel{o_2}{\to} b$, $o_3 = a \stackrel{o_3}{\to}$

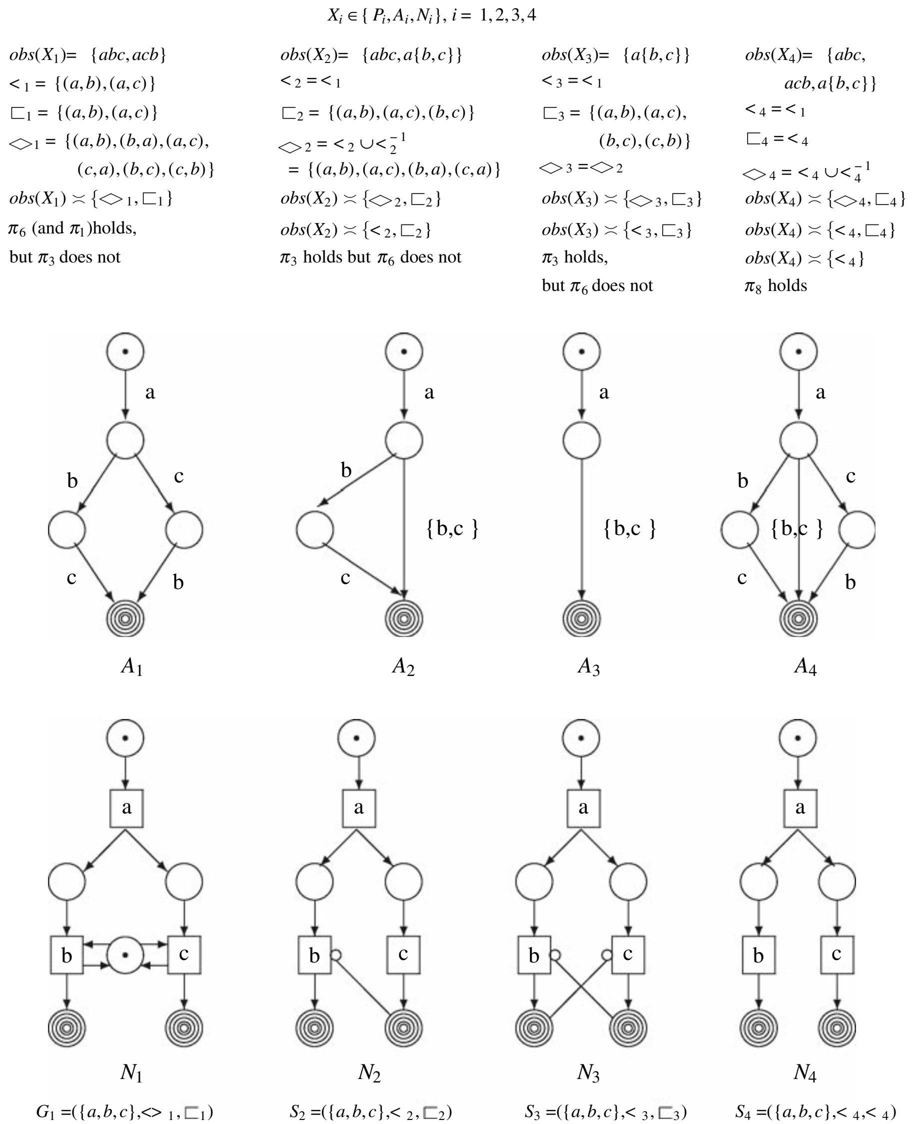


**Fig. 1** Models of behaviours, labelled transition systems and petri nets corresponding to programs $P_1$, $P_2$, $P_3$, $P_4$. Nets $N_2$ and $N_3$ are inhibitor nets (see [19]). $G_1$ is a generalised order structure while $S_i$, $i = 2, 3, 4$, are order structures. We assume *observations are stratified orders*

$b \wedge a \stackrel{o_3}{\rightarrow} c$. We can now write $obs(P_1) = \{o_1, o_2\}$, $obs(P_2) = \{o_1, o_3\}$, $obs(P_3) = \{o_3\}$, $obs(P_4) = \{o_1, o_2, o_3\}$. Note that for $i = 1, \ldots, 4$ all runs from $obs(P_i)$ yield exactly the same outcome. (This justifies calling $obs(P_i)$'s as *concurrent histories*.)

An abstract model of such an outcome is called a *concurrent behaviour*, but what entity constitutes such a model? Let us start with the set $obs(P_4)$. We may say that in this case for each run, $a$ always precedes both $b$ and $c$, and there is no *causal* relationship between $b$ and $c$. This *causality* relation, $<$, is the partial order defined as $< = \{(a, b), (a, c)\}$. In general $<$ is defined by: $x < y$ iff for each run $o$ we have $x \stackrel{o}{\rightarrow} y$. Hence for $P_4$, $<$ is an intersection of

$o_1, o_2$ and $o_3$, and $\{o_1, o_2, o_3\}$ is the set of all stratified extensions of the relation[2] $<$. Thus in this case the causality relation $<$ models the concurrent behaviour corresponding to the set of (equivalent) runs $obs(P_4)$.

We will say that $obs(P_4)$ and $<$ are *tantamount* and write $obs(P_4) \asymp \{<\}$ or $obs(P_4) \asymp (\{a, b, c\}, <)$. Having $obs(P_4)$ one may construct $<$ (as an intersection), and hence construct $obs(P_4)$ (as the set of all stratified extensions).

This is a classical case of the "true" concurrency approach, where concurrent behaviour is modelled by a causality relation.

Before considering the remaining cases, note that the causality relation $<$ is exactly the same in all four cases, i.e. $<_i = \{(a, b), (a, c)\}$, for $i = 1, \ldots, 4$, so we may omit the index $i$.

Let us consider now the set $obs(P_1)$. The causality relation $<$ does not model the concurrent behaviour correctly[3] since $o_3$ does not belong to $obs(P_1)$. Let $\diamond$ be a symmetric relation, called *commutativity*, defined as $x \diamond y$ iff for each run $o$ either $x \overset{o}{\to} y$ or $y \overset{o}{\to} x$. For the set $obs(P_1)$, the relation $\diamond_1$ looks like $\diamond_1 = \{(a, b), (b, a), (a, c), (c, a), (b, c), (c, b)\}$. The pair of relations $\{\diamond_1, <\}$ and the set $obs(P_1)$ are equivalent in the sense that each is definable from the other. (The set $obs(P_1)$ is the greatest set $PO$ of partial orders built from $a$, $b$ and $c$ satisfying $x \diamond_1 y \Rightarrow \forall o \in PO.\ x \overset{o}{\to} y \vee y \overset{o}{\to} x$ and $x < y \Rightarrow \forall o \in PO.\ x \overset{o}{\to} y$.) In other words, $obs(P_1)$ and $\{\diamond_1, <\}$ are *tantamount*, $obs(P_1) \asymp \{\diamond_1, <\}$, so we may say that in this case the relations $\{\diamond_1, <\}$ model the concurrent behaviour described by $obs(P_1)$.

Note also that $\diamond_4 = < \cup <^{-1}$ and the pair $\{\diamond_4, <\}$ also models the concurrent behaviour described by $obs(P_4)$.

To deal with $obs(P_2)$ and $obs(P_3)$ we need another relation, $\sqsubset$, called *weak causality*, defined as $x \sqsubset y$ iff for each run $o$ we have $\neg(y \overset{o}{\to} x)$ (*$x$ is never executed after $y$*). For our four cases we have $\sqsubset_2 = \{(a, b), (a, c), (b, c)\}$, $\sqsubset_1 = \sqsubset_4 = <$, and $\sqsubset_3 = \{(a, b), (a, c), (b, c), (c, b)\}$. Notice again that for $i = 2, 3$, the pair of relations $\{<, \sqsubset_i\}$ and the set $obs(P_i)$ are equivalent in the sense that each is definable from the other (The set $obs(P_i)$ can be defined as the greatest set $PO$ of partial orders built from $a$, $b$ and $c$ satisfying $x < y \Rightarrow \forall o \in PO.\ x \overset{o}{\to} y$ and $x \sqsubset_i y \Rightarrow \forall o \in PO.\ \neg(y \overset{o}{\to} x)$.)

Hence again in these cases ($i = 2, 3$) $obs(P_i)$ and $\{<, \sqsubset_i\}$ are *tantamount*, $obs(P_i) \asymp \{<, \sqsubset_i\}$, and so the pair $\{<, \sqsubset_i\}$, $i = 2, 3$, models the concurrent behaviour described by $obs(P_i)$. Note that $\sqsubset_i$ alone is not sufficient, since (for instance) $obs(P_2)$ and $obs(P_2) \cup \{\{a, b, c\}\}$ define the same $\sqsubset$. The relations $<, \diamond, \sqsubset$ are not independent, since it can be proven [18] that $< = \diamond \cap \sqsubset$.

Since $\sqsubset_1 = <, \sqsubset_4 = <$ and $\diamond_i = < \cup <^{-1}$ for $i = 2, 3, 4$, we also have $\{\diamond_1, <\} \asymp \{\diamond_1, \sqsubset_1\} \asymp Obs(P_1)$, $\{\diamond_i, \sqsubset_i\} \asymp \{<, \sqsubset_i\} \asymp Obs(P_i)$, for $i = 2, 3$, and $\{\diamond_4, \sqsubset_4\} \asymp \{<\} \asymp Obs(P_4)$.

Summing up we have (see the top of Fig. 1):

1. All sets of observations $obs(P_i)$, for $i = 1, 2, 3, 4$ are modelled by appropriate pairs of relations $\{\diamond_i, \sqsubset_i\}$, and $obs(P_i) \asymp \{\diamond_i, \sqsubset_i\}$.
2. $obs(P_i)$, for $i = 2, 3, 4$ can also be modelled by appropriate pairs of relations $\{<, \sqsubset_i\}$, and $obs(P_i) \asymp \{<, \sqsubset_i\}$.
3. $obs(P_4)$ can be modelled by the relation $<$ alone, and $obs(P_4) \asymp \{<\}$.

[2] The fact that $<$ equals $\overset{o_3}{\to}$ is coincidental, as there are not many partial orders that can be built from three elements. The absence of such an ordering is interpreted differently: it means no causal relationship for $<$, and simultaneous execution for $\overset{o_3}{\to}$.

[3] Unless we assume that simultaneity is not allowed, or not observed, in which case $obs(P_1) = obs(P_4) = \{o_1, o_2\}$, $obs(P_2) = \{o_1\}$, $obs(P_3) = \emptyset$.

The programs $P_1, P_2, P_3, P_4$ can be modelled by a variety of means. Figure 1 shows how they can be modelled by labelled transition systems (automata) $A_1, A_2, A_3, A_4$ and by Petri nets $N_1, N_2, N_3, N_4$ . Note that *all* behaviours that might be generated by two concurrent (i.e. non-sequential) events (actions) $b$ and $c$ are modelled by one of the four cases from Fig. 1.

The theory developed in [18] provides a hierarchy of models of concurrency, where each model corresponds to a so called “paradigm”, or general rule about the structures of concurrent histories. In principle, a paradigm describes how simultaneity is handled in concurrent histories. The paradigms are denoted by $\pi_1, \ldots, \pi_8$. It turns out that only paradigms $\pi_1, \pi_3$, $\pi_6$ and $\pi_8$ are (apparently) interesting from the point of view of concurrency theory. The most general paradigm, $\pi_1$, assumes no additional restrictions for concurrent histories. The most restrictive paradigm, $\pi_8$, simply says that if a set of partial orders $\Delta$ is a concurrent history (meaning in our case that $\Delta = obs(P_i)$ for $i = 1, 2, 3, 4$) then

$$\big(\exists o \in \Delta.\ x \overset{o}{\leftrightarrow} y\big) \iff \big(\exists o \in \Delta.\ x \overset{o}{\to} y\big) \wedge \big(\exists o \in \Delta.\ y \overset{o}{\to} x\big),$$

where $\overset{o}{\leftrightarrow}$ denotes simultaneity, i.e. $x \overset{o}{\leftrightarrow} y \iff \neg(x \overset{o}{\to} y) \wedge \neg(y \overset{o}{\to} x)$. The paradigm $\pi_3$ assumes that if a set $\Delta$ of partial orders is a concurrent history then

$$\big(\exists o \in \Delta.\ x \overset{o}{\to} y\big) \wedge \big(\exists o \in \Delta.\ y \overset{o}{\to} x\big) \Rightarrow \big(\exists o \in \Delta.\ x \overset{o}{\leftrightarrow} y\big).$$

The paradigm $\pi_6$ is symmetric to $\pi_3$, it assumes a concurrent history $\Delta$ satisfies

$$\big(\exists o \in \Delta.\ x \overset{o}{\leftrightarrow} y\big) \Rightarrow \big(\exists o \in \Delta.\ x \overset{o}{\to} y\big) \wedge \big(\exists o \in \Delta.\ y \overset{o}{\to} x\big).$$

The paradigms $\pi_1, \ldots, \pi_8$ create a kind of lattice [18], but when restricted to $\pi_1, \pi_3, \pi_6$, and $\pi_8$, the hierarchy is clear: $\pi_8 = \pi_3 \wedge \pi_6$, $\pi_3$ and $\pi_6$ are independent and both imply $\pi_1$. The relations $<$ and $\sqsubset$ can only model concurrent histories conforming to $\pi_3$, they cannot model concurrent histories that only conforms to $\pi_6$ or $\pi_1$. The relations $<\!\!>$ and $\sqsubset$ can handle all paradigms; however, if $\pi_6$ holds, then the relation $\sqsubset$ equals $<$, so the model is simpler.

In our case we have only three observations $o_1 = abc$, $o_2 = acb$, $o_3 = a\{b, c\}$, and of these only $o_3 = a\{b, c\}$ involves simultaneity. Recall that $o_1 = abc$ corresponds to $a \overset{o_1}{\to} b \overset{o_1}{\to} c$, $o_2 = acb$ corresponds to $a \overset{o_2}{\to} c \overset{o_2}{\to} b$, and $o_3 = a\{b, c\}$ corresponds to $a \overset{o_3}{\to} b \wedge a \overset{o_3}{\to} c \wedge b \overset{o_3}{\leftrightarrow} c$. Note that the only true atomic formula involving $\overset{o_i}{\leftrightarrow}$ (up to symmetry) is $b \overset{o_3}{\leftrightarrow} c$.

Obviously all $obs(P_i)$, $i = 1, 2, 3, 4$, conform to the paradigm $\pi_1$, but only $obs(P_2)$, $obs(P_3)$ and $obs(P_4)$ conform to paradigm $\pi_3$, $obs(P_1)$ conforms trivially to $\pi_6$, and only $obs(P_4)$ conforms to $\pi_8$. An example that conforms neither to $\pi_3$ nor to $\pi_6$, only to $\pi_1$, is analysed in Sect. 6 (Example 7, program $P_8$).

In $P_1$ (and $A_1$, $N_1$ from Fig. 1), we have $b \overset{o_1}{\to} c$ and $c \overset{o_2}{\to} b$ but $o_3 \notin obs(P_1)$, so $\forall x, y \in \{a, b, c\}\ \forall o_i \in obs(P_1).\ \neg x \overset{o}{\leftrightarrow} y$, which means $obs(P_1)$ does not conform to $\pi_3$ (and consequently does not conform to $\pi_8$). Since $(\exists o \in obs(P_1).\ x \overset{o}{\leftrightarrow} y)$ is *false* for all $x, y \in \{a, b, c\}$, the set $obs(P_1)$ trivially satisfies $\pi_6$ (and $\pi_1$). Note that $<_1 = \sqsubset_1$.

For $obs(P_2)$ and $obs(P_3)$, the statement

$$\big(\exists o \in obs(P_i).\ x \overset{o}{\to} y\big) \wedge \big(\exists o \in obs(P_i).\ y \overset{o}{\to} x\big),$$

for $i = 2, 3$, is always false, so

$$\big(\exists o \in obs(P_i).\ x \overset{o}{\to} y\big) \wedge \big(\exists o \in obs(P_i).\ y \overset{o}{\to} x\big) \Rightarrow \big(\exists o \in obs(P_i).\ x \overset{o}{\leftrightarrow} y\big)$$

is always true for $i = 2, 3$. Thus, $obs(P_2)$ and $obs(P_3)$ conform to the paradigm $\pi_3$. Since $o_3 \in obs(P_2) \cap obs(P_3)$ therefore $b \stackrel{o_3}{\leftrightarrow} c$ holds, but

$$\big(\exists o \in obs(P_i).\ x \stackrel{o}{\to} y\big) \wedge \big(\exists o \in obs(P_i).\ y \stackrel{o}{\to} x\big)$$

for $i = 2, 3$, is always false, which means that

$$\big(\exists o \in obs(P_i).\ x \stackrel{o}{\leftrightarrow} y\big) \iff \big(\exists o \in obs(P_i).\ x \stackrel{o}{\to} y\big) \wedge \big(\exists o \in obs(P_i).\ y \stackrel{o}{\to} x\big)$$

is false for $i = 2, 3$. Thus, $obs(P_2)$ and $obs(P_3)$ do not conform to the paradigm $\pi_8$.

For the case of $obs(P_4)$, we have $obs(P_4) = \{o_1, o_2, o_3\}$ and $b \stackrel{o_1}{\to} c \wedge c \stackrel{o_2}{\to} b \wedge b \stackrel{o_3}{\leftrightarrow} c$, hence the formula

$$\big(\exists o \in obs(P_4).\ x \stackrel{o}{\to} y\big) \wedge \big(\exists o \in obs(P_4).\ y \stackrel{o}{\to} x\big) \iff \big(\exists o \in obs(P_4).\ x \stackrel{o}{\leftrightarrow} y\big)$$

holds, i.e. $obs(P_4)$ conforms to the paradigm $\pi_8$.

The programs $P_1, \ldots, P_4$ (and $P_5, \ldots, P_8$ in the remainder of the paper) are very simple, were invented for illustrative purposes only, and do not contain any kind of loops. Moreover the set of their (proper) executions creates exactly one concurrent history in each case.

Consider the following, slightly more realistic, concurrent system, *Priority*, described below.

*Example 2* (Priority)
The concurrent systems *Priority* comprises two sequential subsystems, such that:

- the first subsystem can cyclically engage in event $a$ followed by event $b$,
- the second subsystem can cyclically engage in event $b$ or in event $c$,
- the two systems synchronise by means of handshake communication,
- there is a priority constraint stating that if it is possible to execute event $b$ then $c$ must not be executed.

The example *Priority* follows from [37] and Fig. 2 shows its Petri net with priority specification [7,22], while corresponding COSY program [22] and corresponding CSP program [9]

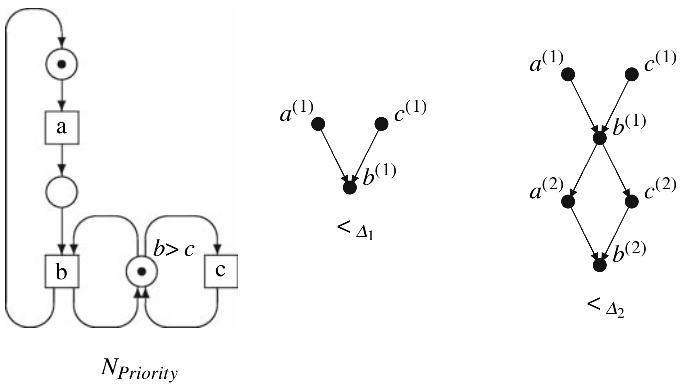


**Fig. 2** Petri net [7,22] specification of *Priority* from Example 2 and Hasse diagrams of $<_{\Delta_1}$ and $<_{\Delta_2}$, where $\Delta_1 = \{abc, \{a, c\}b\}$ and $\Delta_2 = \{abcabc, abc\{a, c\}b, \{a, c\}babc, \{a, c\}b\{a, c\}b\}$

are given below:

$$\begin{aligned} CSP_{\text{Priority}} &= \mu P.(a \to b \to P) \| \, \mu P.(b \to \Box_{>} c \to P) \\ COSY_{\text{Priority}} &= \texttt{priority b>c end} \\ &\quad\ \texttt{path a;b end} \\ &\quad\ \texttt{path b,c end}. \end{aligned}$$

This example has often been analysed in the literature ([7,20,22,37], usually under the interpretation that $a = ErrorMessage$, $b = StopAndRestart$ and $c = SomeAction$. In this case the set of all system runs can be defined as

$$obs(Priority) = Prefix((c^* \cup ab \cup \{a, c\}b)^*).$$

The set $obs(Priority)$ defines the operational semantics of the system *Priority* and is the set-theoretic union of all concurrent histories of *Priority*. Formal definition of concurrent histories will be given in the next section, but, for example, the sets $\Delta_1 = \{abc, \{a, c\}b\}$ and $\Delta_2 = \{abcabc, abc\{a, c\}b, \{a, c\}babc, \{a, c\}b\{a, c\}b\}$ are concurrent histories; $\Delta_1$ is the set of all observations such that each event (action) $a, b, c$ occurs once, and $\Delta_2$ is the set of all observations such that each event $a, b, c$ occurs twice. Since *Priority* contains loops, we need to distinguish between different occurrences of the same event in an observation. We will write $a^{(i)}$ to denote the $i$th occurrence of the event $a$. This notation is not needed when sequences or step sequences are used (a step sequence $\{a, c\}babc$ can be interpreted as an abbreviation of $\{a^{(1)}, c^{(1)}\}b^{(1)}a^{(2)}b^{(2)}c^{(2)}$), but is necessary for defining the relations $\Diamond$, $\sqsubset$ and $<$. For *Priority* we have "$c^{(i)}$ *not later than* $a^{(i)}$" for all $i$.

Hasse diagrams of partial orders $<_{\Delta_1}$ and $<_{\Delta_2}$ are presented in Fig. 2, while:

$$\sqsubset_{\Delta_1} = <_{\Delta_1} \cup \big\{\big(c^{(1)}, a^{(1)}\big)\big\}, \quad \sqsubset_{\Delta_2} = <_{\Delta_2} \cup \big\{\big(c^{(1)}, a^{(1)}\big), \big(c^{(2)}, a^{(2)}\big)\big\},$$
$$\Diamond_{\Delta_i} = <^{\text{sym}}_{\Delta_i}, \quad \text{for} \quad i = 1, 2.$$

It may easily be verified that for example the concurrent histories $\Delta_1$ and $\Delta_2$ are tantamount to the pairs of relations $\{<_{\Delta_1}, \sqsubset_{\Delta_1}\}$ and $\{<_{\Delta_2}, \sqsubset_{\Delta_2}\}$, respectively. We will call the triple $S^{\Delta_1}_{\text{Priority}} = (\{a^{(1)}, b^{(1)}, c^{(1)}\}, <_{\Delta_1}, \sqsubset_{\Delta_1})$ an *order structure* that models $\Delta_1$, and the triple $S^{\Delta_2}_{\text{Priority}} = (\{a^{(1)}, b^{(1)}, c^{(1)}, a^{(2)}, b^{(2)}, c^{(2)}\}, <_{\Delta_2}, \sqsubset_{\Delta_2})$ an *order structure* $\Delta_2$. The similar triples can be constructed for all concurrent histories of *Priority*. Note that *Priority* conforms to paradigm $\pi_3$, but not to $\pi_6$.

It can be proved [18] that paradigm $\pi_3$ implies $\Diamond = < \cup <^{-1}$, which means that $(X, \Diamond, \sqsubset)$ and $(X, <, \sqsubset)$ are *tantamount*. It can also be proven [18] that $\pi_6$ implies $<$ equals $\sqsubset$, which means that $(X, \Diamond, \sqsubset)$ and $(X, \Diamond, <)$ are tantamount. Since $\pi_8 = \pi_3 \wedge \pi_6$, the paradigm $\pi_8$ implies $\Diamond$ equals $< \cup <^{-1}$ and $<$ equals $\sqsubset$, i.e. in this case $(X, <)$, $(X, \Diamond, \sqsubset)$ and $(X, <, \sqsubset)$ are all tantamount.

The most restrictive case, $\pi_8$, corresponds to the classical "true concurrency" model where causal partial orders are sufficient to model all aspects of concurrent behaviour. In the "true concurrency" model, the formula that defines $\pi_8$ is equivalent to a "Diagonal Property" [8,10].

The paradigm $\pi_i$ is only one of the factors shaping concurrent histories (i.e. the sets $obs(P_i)$ in our example). Another important factor is the kind of partial orders that observable runs are allowed to be. It is argued in [18] that *observable runs of (discrete) software systems should be modelled by initially finite interval orders*; however, the results of [20] cover general partial orders as well. Observable runs are frequently assumed to be stratified orders or even total orders. This makes the modelling simpler, and such assumptions are often justified. It appears that the axioms for $(X, \Leftrightarrow, \sqsubset)$ and $(X, <, \sqsubset)$ depend heavily on what kind of partial orders the observable runs are allowed to be. Under the assumption that only totally ordered runs are allowed, the concept of a paradigm is irrelevant, since $<$ alone models concurrent behaviour and the relationship between sets of runs and the relation $<$ follows directly from Szpilrajn theorem [38]. The theory presented in this paper covers various types of partial orders interpreted as observable runs, including the case where being a partial order is the only assumption; however, the theory differs radically from case to case.

We can formulate our problem as follows:

*What axioms must the triple* $(X, \Leftrightarrow, \sqsubset)$, *(or* $(X, <, \sqsubset)$ *in case of* $\pi_3$ *or* $\pi_8$*) satisfy to be considered as models of concurrent behaviours?*

A detailed discussion of triples $(X, <, \sqsubset)$ that model concurrent behaviour under the assumption of paradigm $\pi_3$, is given in [20]. In this paper the case of $(X, \Leftrightarrow, \sqsubset)$, i.e. $\pi_1$ (the most general case), and the case of $(X, \Leftrightarrow, <)$, i.e. $\pi_6$, will both be analysed.

## 3 Observations, histories and paradigms

The mathematical model of concurrency that is used in this paper (and in [20]) was proposed by Janicki and Koutny in [18]. The model is based on three fundamental concepts: *observations of concurrent systems, concurrent histories, and paradigms of concurrency*. In order to make this paper self-contained; we briefly present a revised, adapted, and extended version of this model. All of the results presented in this section are needed either to formulate and prove the results of this paper or to improve their presentation. A few "new" results (ie., that have not been explicitly formulated or proved before) are also presented.

### 3.1 Partial orders

A *partial order* is a pair $po = (X, <)$ such that $X$ is a non-empty set and $<$ is an irreflexive ($\neg(x < x)$) and transitive ($x < y \wedge y < z \Rightarrow x < z$) relation on $X$. We say that $X$ is the domain of $po$. Sometime we also say that $<$ is a partial order on $X$. The following notation will be used throughout this paper. For two elements of $X$, $a$ and $b$, we write:

- two distinct incomparable elements $a$ and $b$ of $X$ will be denoted by $a \sim b$, i.e. $a \sim b \Leftrightarrow \neg(a < b) \wedge \neg(b < a) \wedge a \neq b$, $[\sim]$
- we will write $a <^{\frown} b$ if $a < b$ or $a \sim b$, and $[<^{\frown}]$
- we will write $a <^{\text{sym}} b$ if $a < b$ or $b < a$. $[<^{\text{sym}}]$

A partial order $po_1 = (X, <_1)$ is an *extension* of another partial order $po_2 = (X, <_2)$ if for all $x, y \in X$, $x <_2 y \Rightarrow x <_1 y$, i.e. if $<_2 \subseteq <_1$.

A partial order $(X, <)$ is said to be

- *total* if for all $a, b \in X$, either $a < b$ or $b < a$ or $a = b$;
- *stratified* if $\sim \cup\, id_X$, where $id_X$ is identity on $X$, is an equivalence relation;
- *interval* if for all $a, b, c, d \in X$, $a < c \wedge b < d \Longrightarrow a < d \vee b < c$;
- *initially finite* if for every $a \in X$, $\{b \mid b <^{\frown} a\}$ is finite [18].

We will denote the classes of total, stratified, interval and (arbitrary) partial orders by $TO$, $SO$, $IO$ and $PO$, respectively.

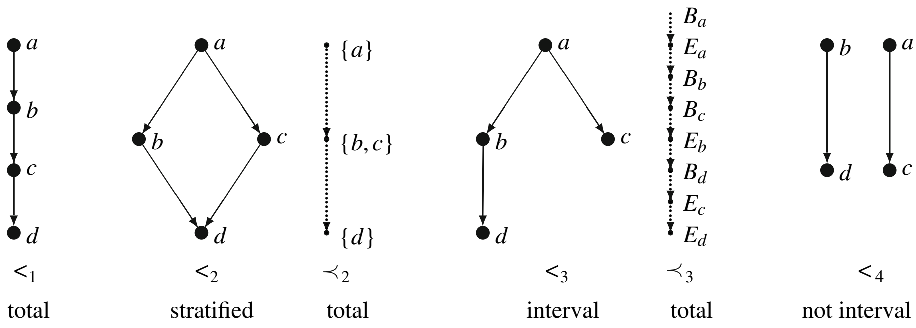


**Fig. 3** Various types of partial orders (represented as Hasse diagrams). The partial order $<_1$ is an extension of $<_2$, $<_2$ is an extension of $<_3$, and $<_3$ is and extension of $<_4$. Note that order $<_1$, being total, is uniquely represented by a sequence *abcd*, the stratified order $<_2$ is uniquely represented by a step sequence $a\{b, c\}d$. Finite interval orders do not have a universally accepted sequence representations

It is easy to see that a total order is stratified and a stratified order is interval. Stratified orders correspond to step sequences and they are often defined in an alternative way, namely, a poset $(X, <)$ is a stratified order iff there exists a total order $(Y, \prec)$ and a mapping $\phi : X \to Y$ such that

$$\forall x, y \in X.\ x < y \iff \phi(x) \prec \phi(y).$$

This definition is illustrated in Fig. 3, let $\phi : \{a, b, c, d\} \to \{\{a\}, \{b, c\}, \{d\}\}$ with $\phi(a) = \{a\}$, $\phi(b) = \phi(c) = \{b, c\}$, and $\phi(d) = \{d\}$. Note that for all $x, y \in \{a, b, c, d\}$ we have $x <_2 y \iff \phi(x) \prec_2 \phi(y)$.

For the interval orders, the name and intuition follow from Fishburn's Theorem:

**Theorem 1** [Fishburn [14] ([18] for initially finite case)]
*A partial order $po = (X, <)$ is* interval (interval and initially finite) *iff there exists a total (total and initially finite) order $(T, \prec)$ and two mappings $\varphi, \psi : X \to T$ such that for all $x, y \in X$,*

1. $\varphi(x) \prec \psi(y)$ *and*
2. $x < y \iff \psi(x) \prec \varphi(y)$. □

Usually $\varphi(x)$ is interpreted as the beginning and $\psi(x)$ as the end of an *interval* $x$. This theorem is also illustrated in Fig. 3 with $<_3$ and $\prec_3$. Let $\varphi$ and $\psi$ be defined as follows $\varphi, \psi : \{a, b, c, d\} \to \{B_a, E_a, B_b, E_b, B_c, E_c, B_d, E_d\}$, with $\varphi(a) = B_a$, $\varphi(b) = B_b$, $\varphi(c) = B_c$, $\varphi(d) = B_d$, and with $\psi(a) = E_a$, $\psi(b) = E_b$, $\psi(c) = E_c$, $\psi(d) = E_d$. Then for all $x, y \in \{a, b, c, d\}$, we have $\varphi(x) \prec_3 \psi(y)$ and $x <_3 y \iff \psi(x) \prec_3 \varphi(y)$.

Modelling concurrency usually assumes some form of discreteness, for instance the number of predecessors is finite, etc. This is captured by the concept of initial finiteness. It turns out that many results need separate proofs under the initial finiteness assumption. In general, if $\mathcal{C}$ is a class of partial orders then we will denote by $\mathcal{C}_{IF}$ the subclass of all initially finite partial orders in $\mathcal{C}$.

### 3.2 Observations, concurrent histories and tantamount entities

A *run* (*observation, instance of concurrent behaviour*) is an abstract model of the execution of a concurrent system. It was argued in [18] that

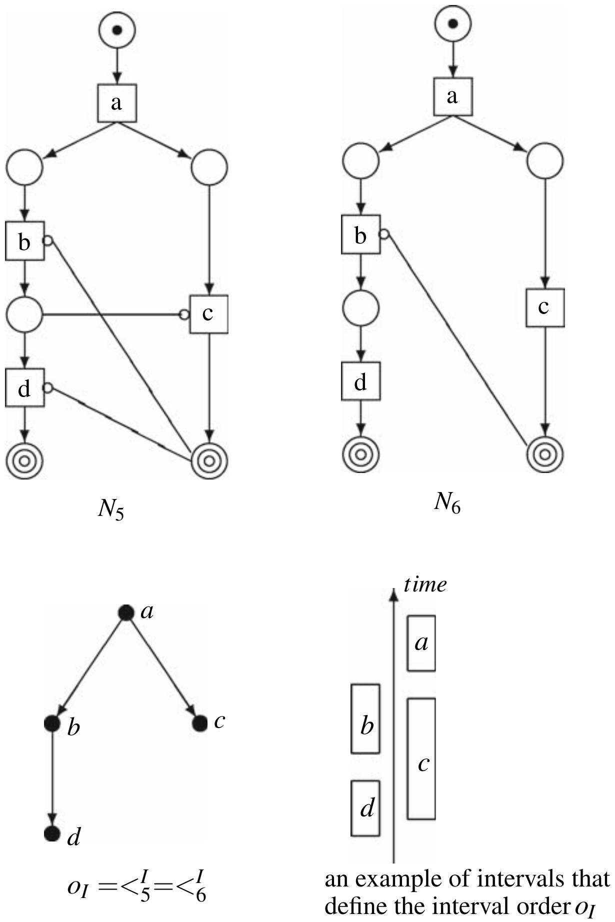


**Fig. 4** Petri net models of programs $P_5$ and $P_6$ from Example 3, an interval order $o_I$ which is a possible observation of both $P_5$ and $P_6$, and causality relations $<^I_5$, $<^I_6$. Even though $o_I = <^I_5 = <^I_6$, the interpretation is very different, in $o_I$ incomparability is interpreted as *simultaneity*, while in $<^I_5$ and $<^I_6$ as *no particular casual relationship*

> *an observation must be an initially finite order that is either total, or stratified, or interval.*

All observations from Fig. 1 are stratified (non-stratified partial orders require at least four elements). An example of an observation that is interval but not stratified is given in Fig. 4. The results of [20] are valid for all kinds of partial orders, not necessarily initially finite or interval; however, separate proofs are frequently required for different cases. Following [18,20] and the previous section, we will make a distinction in notation between general posets and those used as runs. We will use $o = (X, \rightarrow_o)$ rather than $po = (X, <)$ to denote an arbitrary run, and use $\leftrightarrow_o$ rather than $\sim$ to denote incomparability. Quite often we will assume that an observation $o$ is a special kind of partial order, for example stratified. We will use the symbol $O$ to denote any of partial orders (usually $O \in \{TO, SO, IO, PO\}$) and write $o \in O$.

*Concurrent history* is a *complete set of equivalent runs*.[4] To explain the concept, assume that all possible runs are total orders. A set $\Delta = \{abc, cba\}$ is *not* a concurrent history. Since the intersection of the runs $abc$ and $cba$ (which in this case are total orders), denoted by $<_\Delta$ or $<_{\{abc,cba\}}$, is the empty set, it implies that there is no causal relationship between $a$, $b$, and $c$. This means that for instance $bca$ is a possible run, but $bca \notin \Delta$, a contradiction. Let $\Delta^{cl}$ be the set of all total extensions of $<_{\{abc,cba\}}$, i.e. $\Delta^{cl} = \{abc, bac, acb, bca, cab, aba\}$. This set is complete, as it is the set of all total extensions of $<_{\Delta^{cl}} = \emptyset$, so it can be considered as a concurrent history.

If not all the runs are total, then a definition of concurrent histories requires using a more complex analysis of the runs. The set $<_\Delta$ from the example above can be seen as an *invariant* characterizing the set $\Delta$, as the set of *all elements of* $\Delta$ *satisfy the ordering relationship defined by* $<_\Delta$. It was argued in [18] that a concurrent history is a set of partial orders (of an appropriate type) with common domain that is fully characterized by all its *relational invariants*.

A *relational invariant* over a set of partial orders $\Delta$ is any relation $R \subseteq X \times X$ defined by a formula of the type

$$(x, y) \in R \iff \forall o \in \Delta.\ \phi_R(x, y, o),$$

where $\phi_R(x, y, o)$ is any propositional formula built from atoms $x \overset{o}{\to} y$, $y \overset{o}{\to} x$, $x \overset{o}{\leftrightarrow} y$ and $True$; for example $\phi_R(x, y, o) = x \overset{o}{\to} y \vee x \overset{o}{\leftrightarrow} y$.

Let $RInv(\Delta)$ denote the set of all relational invariants generated by $\Delta$.

Let $O$ be a class of partial orders[5] and let $\Delta \subseteq O$ be a set of partial orders with a common domain $X$.

- We will define $\Delta^{cl}_O$, the *closure of* $\Delta$ *with respect to set* $RInv(\Delta)$ as the set of all partial orders in $O$ with domain $X$ that satisfy:

$$o \in \Delta^{cl}_O \iff (\forall R \in RInv(\Delta).\ \forall x, y \in X.\ (x, y) \in R \Rightarrow \phi_R(x, y, o)).$$

We are now able to define formally the concept of concurrent history.

**Definition 1** [18] A set of runs $\Delta$ is a *concurrent history* in $O$ iff $\Delta = \Delta^{cl}_O$. □

Despite a relatively general definition, one may show (see [18] for details) that $RInv(\Delta)$ consists of at most eight different relations, and at most two of these are independent i.e. they cannot be calculated from each other by using the standard set theory operators union, intersection and complement.

Define the relations $<\!\!>_\Delta$, $\sqsubset_\Delta$, $<_\Delta$ and $\bowtie_\Delta$ on $X \times X$ as

- $x <\!\!>_\Delta y \iff \forall o \in \Delta.\ \big(x \overset{o}{\to} y \ \vee\ y \overset{o}{\to} x\big)$, [$<\!\!>_\Delta$]
- $x \sqsubset_\Delta y \iff \forall o \in \Delta.\ \big(x \overset{o}{\to} y \ \vee\ x \overset{o}{\leftrightarrow} y\big)$, [$\sqsubset_\Delta$]
- $x <_\Delta y \iff \forall o \in \Delta.\ x \overset{o}{\to} y$, [$<_\Delta$]
- $x \bowtie_\Delta y \iff \forall o \in \Delta.\ x \overset{o}{\leftrightarrow} y$. [$\bowtie_\Delta$]

The relation $<_\Delta$ is a *causality*, an abstraction of the “earlier than” relation, as $x <_\Delta y$ means that $x$ is performed earlier than $y$ in all observations from $\Delta$.

[4] The term “concurrent history” has been used by many authors, e.g., [10,25,31] and others, to denote formally different concepts (although intuitively close) in the idea of concurrency. The concept used in this paper was introduced in [16] and is close to that of [31].

[5] Typically, but not necessarily, one of $TO$, $SO$, $IO$, $PO$.

The relation $\sqsubset_\Delta$ is a *weak causality*, abstracted from the "not later than" relation, since $x \sqsubset_\Delta y$ means that $x$ is performed not later than $y$ in all observations from $\Delta$; in other words, $x$ may be performed earlier than $y$, or simultaneously with $y$, but not later than $y$.

The relation $<\!\!>_\Delta$ is called *commutativity* and can be seen as an abstraction of "interleaving" or the "not simultaneously" relation. In this case $x <\!\!>_\Delta y$ means that $x$ and $y$ are not performed simultaneously in any observation from $\Delta$.

The relation $\bowtie_\Delta$ is called *synchronisation*. It is an abstraction of simultaneity, where $x \bowtie_\Delta y$ means that $x$ and $y$ are simultaneous in all observations from $\Delta$. We define this relation only for completeness, as it will not be used in the rest of this paper. For more details the reader is referred to [18].

**Lemma 1** (from [18])

1. $RInv(\Delta) = \{\emptyset, <\!\!>_\Delta, \sqsubset_\Delta, \sqsubset_\Delta^{-1}, <_\Delta, <_\Delta^{-1}, \bowtie_\Delta, X \times X\}$.
2. $RInv(\Delta)$ *is the smallest set of relations containing* $\{<\!\!>_\Delta, \sqsubset_\Delta\}$ *and closed under the operations of union, intersection, complement and their inverse operations.*
3. $<_\Delta = <\!\!>_\Delta \cap \sqsubset_\Delta, \bowtie_\Delta = \sqsubset_\Delta \cap \sqsubset_\Delta^{-1}$.

The above results allow us to define the concept of a concurrent history in a slightly different but equivalent way.

We say that a partial order (run) $o = (X, \xrightarrow{o}) \in O$ is a *partial order extension* (or just an *extension*) of the relations $<\!\!>_\Delta$, $\sqsubset_\Delta$ and $<_\Delta$, respectively, in $O$, if and only if

$$\begin{array}{ll} (\textit{extension of } <\!\!>_\Delta) & \forall x, y \in X.\ x <\!\!>_\Delta y \Rightarrow \left(x \xrightarrow{o} y \ \vee\ y \xrightarrow{o} x\right) \\ (\textit{extension of } \sqsubset_\Delta) & \forall x, y \in X.\ x \sqsubset_\Delta y \Rightarrow \left(x \xrightarrow{o} y \ \vee\ x \overset{o}{\leftrightarrow} y\right), \\ (\textit{extension of } <_\Delta) & \forall x, y \in X.\ x <_\Delta y \Rightarrow x \xrightarrow{o} y. \end{array}$$

**Corollary 1**

1. *The set* $\Delta_O^{cl}$ *is the set of all partially ordered sets* $o = (X, \xrightarrow{o}) \in O$*, that are extensions in* $O$ *of both the relation* $<\!\!>_\Delta$ *and the relation* $\sqsubset_\Delta$.
2. $x <\!\!>_\Delta y \iff x <\!\!>_{\Delta_O^{cl}} y$ *and* $x \sqsubset_\Delta y \iff x \sqsubset_{\Delta_O^{cl}} y$

*Proof* 1. From Lemma 1(2).
2. Since $\Delta \subseteq \Delta_O^{cl}$ then $<\!\!>_{\Delta_O^{cl}} \subseteq <\!\!>_\Delta$ and $\sqsubset_{\Delta_O^{cl}} \subseteq \sqsubset_\Delta$. Suppose for some $x$ and $y$ we have $x <\!\!>_\Delta y$ and $\neg(x <\!\!>_{\Delta_O^{cl}} y)$. From the definition of $<\!\!>_{\Delta_O^{cl}}$, we can conclude $\neg(x <\!\!>_{\Delta_O^{cl}} y) \iff \exists \hat{o} \in \Delta_O^{cl}.\ x \overset{\hat{o}}{\leftrightarrow} y$. From Corollary 1(1) we have $\hat{o}$ is and extension of $<\!\!>_\Delta$, hence, from the definition of extension of $<\!\!>_\Delta$, $x <\!\!>_\Delta y$ implies $x \xrightarrow{\hat{o}} y \vee y \xrightarrow{\hat{o}} x$, a contradiction. Hence $<\!\!>_{\Delta_O^{cl}} = <\!\!>_\Delta$. For the equality $\sqsubset_{\Delta_O^{cl}} = \sqsubset_\Delta$ we proceed almost identically. □

*We may now say that a set of partially ordered sets* $\Delta$ *with a common domain is a concurrent history in* $O$ *if it equals the set of all extensions of* $<\!\!>_\Delta$ *and* $\sqsubset_\Delta$ *in* $O$. This definition was used in [13,15,20]; it makes our reasoning easier, but it does not explain the choice of $<\!\!>_\Delta$ and $\sqsubset_\Delta$. Corollary 1 provides this explanation.

The statement "in $O$" is important. The fact that $\Delta \subseteq O$ does *not* necessarily imply that $\Delta$ must be a history in $O$. For example $obs(P_1) \subseteq TO$ but it is interpreted as a history in $SO$. The sets $obs(P_i)$, $i = 2, 3, 4$, are concurrent histories in stratified and interval orders (all orders built from three elements are stratified), as $obs(P_i) = obs(P_i)_{SO}^{cl} = obs(P_i)_{IO}^{cl}$.

The set $\Delta_S = \{abcd, abdc, a\{b,d\}c, ab\{c,d\}\}$ is a concurrent history in the class of stratified orders since $\Delta_S = (\Delta_S)^{cl}_{SO}$, but it is *not* a concurrent history in the class of interval orders as $\Delta_S \neq (\Delta_S)^{cl}_{IO} = \{abcd, abdc, a\{b,d\}c, ab\{c,d\}, o_I\}$, where $o_I$ equals $<_3$ from Fig. 3, i.e. $<_3$ is an interval, but not stratified, order. The set $\Delta_I = \Delta_S \cup \{o_I\} = \{abcd, abdc, a\{b,d\}c, ab\{c,d\}, o_I\}$, obviously is not a concurrent history in the class of stratified orders, but it is in the class of interval orders as $\Delta_I = (\Delta_I)^{cl}_{IO}$.

The set $obs(P_1) = \{abc, acb\}$ is a concurrent history in both total orders and stratified orders, but the behaviours it models are different in each case. In the class of stratified orders $\{abc, acb\}$ models the behaviour of program $P_1$ (Petri net $N_1$, transition system $A_1$), while in the class of total orders it also models the behaviour of program $P_4$ (Petri net $N_4$, transition system $A_4$ since if we remove the transition labelled by $\{b,c\}$ from $A_4$, we get $A_1$). If only total orders are allowed, we cannot distinguish between the behaviours of $P_1$ and $P_4$.

For the concurrent system *Priority*, one may easily check that, for example, $\Delta_1 = \{abc, \{a,c\}b\}$ and $\Delta_2 = \{abcabc, abc\{a,c\}b, \{a,c\}babc, \{a,c\}b\{a,c\}b\}$ satisfy $\Delta_1 = (\Delta_1)^{cl}_{SO}$ and $\Delta_2 = (\Delta_2)^{cl}_{SO}$, so both $\Delta_1$ and $\Delta_2$ are concurrent histories in $SO$ generated by *Priority*.

For more detailed discussion of the theoretical properties of the above definition, the reader is referred to [18].

If a set of partial orders $\Delta$ is a concurrent history, then not only does it uniquely defines the triple $(X, <\!\!>_\Delta, \sqsubset_\Delta)$, but it can also be completely reconstructed from the triple $(X, <\!\!>_\Delta, \sqsubset_\Delta)$.

We say in such case that the set $\Delta$ and the triple $(X, <\!\!>_\Delta, \sqsubset_\Delta)$ are *tantamount*, and write $\Delta \asymp (X, <\!\!>_\Delta, \sqsubset_\Delta)$. (See examples in Sect. 2.)

In general, we will say that two entities $E_1$ and $E_2$ are *tantamount*, and write $E_1 \asymp E_2$, if $E_1$ can be transformed into $E_2$ and $E_2$ can be transformed into $E_1$. For instance, a minimum state automaton $A$, the regular language $L(A)$ accepted by $A$, and the labelled directed graph $G_A$ representing $A$, are all tantamount, $A \asymp L(A) \asymp G_A$. Any partial order $<$ and the set of all its total order extensions $T_<$ are also tantamount, $< \asymp T_<$. We will not use the word "equivalent", as it usually implies that entities are of the same type, as "equivalent automata", "equivalent expressions", etc. Tantamount entities can be of different types. From a formal perspective, the tantamount relation $\asymp$ is an equivalence relation.

### 3.3 Restriction to totally ordered observations

Most of the results presented in this paper do not assume that all observations are total orders, i.e. there exists an $o$ over $X$ and $x, y \in X$ such that $x \overset{o}{\leftrightarrow} y$. The situation when all observations over a given set of events $X$ are total orders (i.e. $O = TO$) is a valid special case, but many results presented in the remainder of the paper are then irrelevant. It turns out that if $O = TO$ every concurrent history $\Delta$ is tantamount to the partially ordered causality $(X, <_\Delta)$, so the model with only causality relations is sufficient.

**Lemma 2** (implicitly from [18])
*Let $\Delta$ be a concurrent history over $X$ in $TO$, i.e. $\Delta = \Delta^{cl}_{TO}$. Then we have*

1. $<\!\!>_\Delta = \{(x,y) \mid x, y \in X \wedge x \neq y\}$ *and* $\sqsubset_\Delta = <_\Delta$,
2. *$\Delta$ is the set of all total extensions of $<_\Delta$, i.e.* $\Delta = \{o \mid o \in TO \wedge <_\Delta \subseteq o\}$.
3. $\Delta \asymp (X, <_\Delta)$.

*Proof* 1. Clearly $<\!\!>_\Delta \subseteq \{(x,y) \mid x, y \in X \wedge x \neq y\}$. Let $x, y \in X$, $x \neq y$ and let $o \in \Delta$. Since $o$ is a total order then we have $x \overset{o}{\rightarrow} y$ or $y \overset{o}{\rightarrow} x$. This holds for all

$o \in \Delta$, hence $x \sqsubset\!\!\!\!\sqsupset_\Delta y$. From Lemma 1(3) we have $<_\Delta = \,\sqsubset\!\!\!\!\sqsupset_\Delta \cap \sqsubset_\Delta$. Clearly $\sqsubset_\Delta \subseteq \{(x, y) \mid x, y \in X \wedge x \neq y\}$, so $\sqsubset_\Delta = <_\Delta$.

2. From Corollary 1(2), $\Delta$ is the set of all extensions of $\sqsubset\!\!\!\!\sqsupset_\Delta$ and $\sqsubset_\Delta$. But every total order over is an extension of $\sqsubset\!\!\!\!\sqsupset_\Delta = \{(x, y) \mid x, y \in X \wedge x \neq y\}$, and from (1) above we have $\sqsubset_\Delta = <_\Delta$.
3. From (2) above and the definition of $\Delta^{cl}_O$. □

As was discussed in the previous subsection, the fact that $\Delta = \Delta^{cl}_O \subseteq TO$ does not necessarily imply $O = TO$. For example for $\Delta = obs(P_1)$ from Fig. 1, we would have $O = SO$, $obs(P_1) = \{abc, acb\} \subseteq TO$, and we have shown in Sect. 2 that $obs(P_1) \asymp (\{a, b, c\}, \sqsubset\!\!\!\!\sqsupset_1, \sqsubset_1)$ and $obs(P_1) \not\asymp (\{a, b, c\}, <_1)$. Under the $O = TO$ assumption, for Fig. 1 we have $obs(P_1) = obs(P_4) = \{abc, acb\}$, $obs(P_2) = \{abc\}$, $obs(P_3) = \emptyset$.

In [18] it is argued that, unless "team observers" are allowed and special circumstances are assumed, only *initially finite interval orders* can be observed. From Theorem 1 it follows that each *initially finite interval order* of events can be modelled by an *initially finite total order* of appropriate beginnings and ends of the events, so one may argue that we may always assume that all observations are total. This is true, however, one may similarly argue that complex numbers are redundant for they can be modelled by pairs or real numbers. But the formalism of complex numbers is much easier and more intuitive than calculus of pairs of reals. Modelling everything in terms of beginnings and ends would often result in models which are unnecessarily complex, difficult to understand, and unintuitive.

### 3.4 Paradigms

The problem is to find axioms for the relations $\sqsubset\!\!\!\!\sqsupset$ and $\sqsubset$ such that their partial order extensions can be interpreted as some $\Delta^{cl}$. To solve this problem the notion of a *paradigm* has been introduced.

As we mentioned earlier, a paradigm is a supposition or statement about the structure of a history involving a treatment of simultaneity. For instance, let $\Delta$ be a concurrent history. The classical causality based approach usually stipulates that if there is a run $o \in \Delta$ such that $a \overset{o}{\leftrightarrow} b$, then there must be a run such $a$ precedes $b$ and a run such that $b$ precedes $a$.

Formally, *paradigms*, $\omega \in Par$, are defined for event variables $x, y$, by a simple grammar

$$\omega := true | false | \exists o.\, x \overset{o}{\rightarrow} y | \exists o.\, x \overset{o}{\leftarrow} y | \exists o.\, x \overset{o}{\leftrightarrow} y | \neg\omega | \omega \vee \omega | \omega \wedge \omega | \omega \Rightarrow \omega,$$

A history $\Delta$ satisfies a paradigm $\omega \in Par$ if for all distinct $a, b \in dom(\Delta)$, $\omega(a, b)$ holds. It was shown [18] that in the study of concurrent histories, we only need to consider eight non-equivalent paradigms, denoted by $\pi_1, \ldots, \pi_8$. Of those eight, only $\pi_1, \pi_3, \pi_6$ and $\pi_8$ are important for our purposes. The most general paradigm, $\pi_1 = true$, admits all concurrent histories. The most restrictive paradigm, $\pi_8$, admits concurrent histories $\Delta$ such that

$$\left(\exists o \in \Delta.\, x \overset{o}{\leftrightarrow} y\right) \Leftrightarrow \left(\exists o \in \Delta.\, x \overset{o}{\leftarrow} y\right) \wedge \left(\exists o \in \Delta.\, x \overset{o}{\rightarrow} y\right).$$

The paradigm $\pi_3$, which is general enough to deal with most problems that cannot be dealt with under $\pi_8$, admits concurrent histories $\Delta$ such that

$$\left(\exists o \in \Delta.\, x \overset{o}{\leftarrow} y\right) \wedge \left(\exists o \in \Delta.\, x \overset{o}{\rightarrow} y\right) \Rightarrow \left(\exists o \in \Delta.\, x \overset{o}{\leftrightarrow} y\right).$$

The paradigm, $\pi_6$, symmetric to $\pi_3$, admits concurrent histories $\Delta$ such that

$$\left(\exists o \in \Delta.\, x \overset{o}{\leftrightarrow} y\right) \Rightarrow \left(\exists o \in \Delta.\, x \overset{o}{\leftarrow} y\right) \wedge \left(\exists o \in \Delta.\, x \overset{o}{\rightarrow} y\right).$$

Clearly, $\pi_8 = \pi_3 \wedge \pi_6$, $\pi_8 \Rightarrow \pi_3 \Rightarrow \pi_1$, and $\pi_8 \Rightarrow \pi_6 \Rightarrow \pi_1$. The paradigms will determine the way histories can be represented by their relational invariants.

**Lemma 3** [18]
*Let $\Delta$ be a concurrent history in $O$ over a set of events $X$.*

1. *If $\Delta$ conforms to $\pi_3$, then $<\!\!>_\Delta = <_\Delta \cup <_\Delta^{-1}$, and $\Delta$ equals the set of all $(o, \stackrel{o}{\rightarrow}) \in O$ that are extensions of both $<_\Delta$ and $\sqsubset_\Delta$, i.e. $\Delta \asymp (X, <_\Delta, \sqsubset_\Delta)$.*
2. *If $\Delta$ conforms to $\pi_8$, then $<\!\!>_\Delta = <_\Delta \cup <_\Delta^{-1}$, $\sqsubset_\Delta = <_\Delta$, and $\Delta$ equals the set of all $(o, \stackrel{o}{\rightarrow}) \in O$ that are extensions of $<_\Delta$ only, i.e. $\Delta \asymp (X, <_\Delta)$.*
3. *If $\Delta$ conforms to $\pi_6$, then $<_\Delta = \sqsubset_\Delta$.* □

Lemma 3 simply says that if the paradigm $\pi_3$ holds, causality, then $<_\Delta$, and weak causality (an abstraction of "not later than"), $\sqsubset_\Delta$, suffice to fully describe $\Delta$. If the paradigm $\pi_8$ holds, then a partial order $<_\Delta$ suffices to fully describe $\Delta$, so the use of partial orders only to model concurrent behaviour is justified. Finally, if $\pi_6$ holds, then weak causality equals causality. The axioms for relational structures $(X, <, \sqsubset)$, such that all their partial order extensions can be interpreted as concurrent histories $\Delta^{cl}$, were provided in [20]. We will briefly discuss them (revised and modified) in the next section.

Even if $O \neq TO$ it may happen that a concurrent history $\Delta$ consists of only totally ordered observations, i.e. $\Delta = \Delta_O^{cl} \subseteq TO$. Then $(\exists o \in \Delta.\ x \stackrel{o}{\leftrightarrow} y)$ equals $False$ for all $o \in \Delta$ and all $x, y \in X$, so this case requires a special consideration.

**Lemma 4** *Let $O$ be a class of partial orders, and let $\Delta$ be a concurrent history over $X$ satisfying $\Delta = \Delta_O^{cl} \subseteq TO$. Then we have*

1. *$\Delta$ conforms to $\pi_6$.*
2. *If $|\Delta| > 1$ then $\Delta$ does not conform to $\pi_3$ (and consequently not to $\pi_8$), and if $|\Delta| = 1$ then $\Delta$ conforms trivially to $\pi_8$.*
3. *$<\!\!>_\Delta = \{(x, y) \mid x, y \in X \wedge x \neq y\}$.*

*Proof* 1. Clearly $(\exists o \in \Delta.\ x \stackrel{o}{\leftrightarrow} y)$ is $False$ for all $o \in \Delta$ and all $x, y \in X$, which trivially implies $\pi_6$.
2. If $|\Delta| > 1$, then there are $o_1, o_2 \in \Delta$ such that $x \stackrel{o_1}{\rightarrow} y$ and $y \stackrel{o_2}{\rightarrow} x$ for some $x, y \in X$, i.e. $\pi_3$ does not hold. If $|\Delta| = 1$ then both $(\exists o \in \Delta.\ x \stackrel{o}{\leftrightarrow} y)$ and $(\exists o \in \Delta.\ x \stackrel{o}{\rightarrow} y \wedge \exists o \in \Delta.\ y \stackrel{o}{\rightarrow} x)$ are $False$, so $\pi_8$ does hold.
3. Similarly as the proof of Lemma 2(1). □

It is important to remember that the cases $O = TO$ and $\Delta_O^{cl} \subseteq TO \neq O$ are different. In the first case, by Lemma 2, $\Delta \asymp (X, <_\Delta)$, in the second case $\Delta \asymp (X, <_\Delta)$ only when $|\Delta| = 1$, otherwise $\Delta \not\asymp (X, <_\Delta)$.

## 4 Order structures

We now provide a formal theory of the relations $<$ and $\sqsubset$, interpreted as an abstraction of "earlier than" and "not later than". This is a revised version of main results of [20]. All of the notions and results presented in this section are necessary to formulate and prove our results.

Following [20], we will call triples $(X, <, \sqsubset)$ *order structures*.

**Definition 2** [20]

1. An *order structure*, or simply a *structure*, is a triple

$$S = (X, <, \sqsubset)$$

where $X$ is a non-empty set and $<, \sqsubset$ are two irreflexive binary relations on $X$ such that for all $a, b \in X$, we have

$$a < b \Rightarrow \neg(b \sqsubset a).$$

2. An order structure $S = (X, <, \sqsubset)$ is *saturated* if for all distinct $a, b \in X$ we have

$$\neg(a < b) \Rightarrow b \sqsubset a.$$

3. For any class $\Theta$ of ordered structures, $\Theta^{\text{sat}}$ denotes all of the saturated structures in $\Theta$.

□

The assertion $a < b \Rightarrow \neg(b \sqsubset a)$ above is motivated by the intended interpretation of $<$ and $\prec$ as, respectively, "earlier than" and "not later than" relations. At this point, however, we do no assume $<$ is a partial order. In all the special cases considered below $<$ implicitly becomes a partial order, but such a slightly more general definition simplifies some of the proofs. If $S = (X, <, \sqsubset)$ is saturated, then $<$ defines $\sqsubset$ and vice versa, i.e. $(X, <) \asymp (X, <, \sqsubset) \asymp (X, \sqsubset)$. If necessary we will use $X_S, <_S, \sqsubset_S$ to denote $X, <, \sqsubset$ such that $S = (X, <, \sqsubset)$.

The result below shows that the concept of an ordered structure can be viewed as a generalisation of the concept of a partial order.

**Corollary 2** *For every partial order* $(X, \prec)$ *we have:*

1. $(X, \prec, \prec^{\frown})$ *is a* saturated order structure.
2. $(X, \prec, \prec)$ *is an* order structure *and it is saturated if and only if* $\prec$ *is total.* □

The operations of union, intersection and subset for ordered structures with the same domain are defined component-wise. Let $S = (X, <, \sqsubset)$, $S_1 = (X, <_1, \sqsubset_1)$, $S_2 = (X, <_2, \sqsubset_2)$, $S_i = (X, <_i, \sqsubset_i)$, $i \in I$. Then

- $S = \bigcup_{i \in I} S_i \iff < = \bigcup_{i \in I} <_i \wedge \sqsubset = \bigcup_{i \in I} \sqsubset_i$,
- $S = \bigcap_{i \in I} S_i \iff < = \bigcap_{i \in I} <_i \wedge \sqsubset = \bigcap_{i \in I} \sqsubset_i$,
- $S_1 \subseteq S_2 \iff <_1 \subseteq <_2 \wedge \sqsubset_1 \subseteq \sqsubset_2$.

Note that $\bigcup_{i \in I} S_i$ may *not* be an order structure, but $\bigcap_{i \in I} S_i$ is always an order structure. Also note that stratified order structures are maximal w.r.t. to $\subseteq$.
We will also say that

- an order structure $S_2$ is an *extension* of an order structure $S_1$ if $S_1 \subseteq S_2$.

Now we introduce the important concept of extension completeness that will allow us to connect concurrent histories and ordered structures.

**Definition 3** [20]
A class $\Theta$ of ordered structures with possibly different domains is *extension complete* if for every $S = (X, <, \sqsubset) \in \Theta$, the set $ext_{\Theta}^{\text{sat}}(S) = \{T \in \Theta^{\text{sat}} \mid S \subseteq T\}$ is non-empty and

$$S = \bigcap_{T \in ext_{\Theta}^{\text{sat}}(S)} T.$$

□

In other words, $\Theta$ is extension complete if any ordered structure in $\Theta$ can be represented using its saturated extensions defined by $\Theta$. Extension completeness is an interesting and useful property for at least two reasons. Firstly, it provides a means of representing sets of partial orders by only two relations. Secondly, if gives a straightforward formula for calculating a structure from its extensions, much in the same way that a partial order can be derived from its total order extensions. Another, more sophisticated interpretation is the subject of the next section.

**Corollary 3** *If a class of structures $\Theta$ is extension complete, then for each $S \in \Theta$, we have* $S \asymp ext_{\Theta}^{\mathrm{sat}}(S)$. □

The following four concrete classes of structures are used to model concurrent histories.

**Definition 4** [20]

1. An order structure $S = (X, <, \sqsubset)$ is called a *total*, *stratified*, *interval* or *partial order structure*, if the following conditions T1–T3, S1–S4, I1–I6 or P1–P4, respectively, are satisfied:

   $T1\ \ a \not\sqsubset a$ $\qquad$ $T3\ \ a < b < c \Rightarrow a < c$ $\qquad$ [total]
   $T2\ \ a < b \Leftrightarrow a \sqsubset b$

   $S1\ \ a \not\sqsubset a$ $\qquad$ $S3\ \ a \sqsubset b \sqsubset c \Rightarrow a \sqsubset c \vee a = c$ $\qquad$ [stratified]
   $S2\ \ a < b \Rightarrow a \sqsubset b$ $\qquad$ $S4\ \ a \sqsubset b < c \vee a < b \sqsubset c \Rightarrow a < c.$

   $I1\ \ a \not\sqsubset a$ $\qquad$ $I4\ \ a < b \sqsubset c \vee a \sqsubset b < c \Rightarrow a \sqsubset c$
   $I2\ \ a < b \Rightarrow a \sqsubset b$ $\qquad$ $I5\ \ a < b \sqsubset c < d \Rightarrow a < d$ $\qquad$ [interval]
   $I3\ \ a < b < c \Rightarrow a < c$ $\qquad$ $I6\ \ a \sqsubset b < c \sqsubset d \Rightarrow a \sqsubset d \vee a = d$

   $P1\ \ a \not\sqsubset a$ $\qquad$ $P3\ \ a < b < c \Rightarrow a < c$ $\qquad$ [partial]
   $P2\ \ a < b \Rightarrow a \sqsubset b$ $\qquad$ $P4\ \ a \sqsubset b < c \vee a < b \sqsubset c \Rightarrow a \sqsubset c.$

2. A structure $(X, <, \sqsubset)$ is said to be *initially finite* if the set $\{b \mid b <^{\frown} a\}$ is finite for all $a \in X$. □

We will denote by **T**, **S**, **I** and **P**, respectively, the class of total, stratified, interval and partial order structures. As with partial orders, if $\Theta$ is a class of structures, we denote by $\Theta_{IF} \subseteq \Theta$ the subclass consisting of initially finite structures. One may verify easily that $\mathbf{T} \subset \mathbf{S} \subset \mathbf{I} \subset \mathbf{P}$. Total order structures are in fact classical partial orders in disguise (since $<$ equals to $\sqsubset$), and saturated total order structures are total orders in disguise (if a partial order $\prec$ satisfies $\prec = \prec^{\frown}$, it must be total). They were both introduced to show that the order structure theory covers partial order theory.

The comprehensive theory of the above classes of structures was provided in Janicki and Koutny [20]; however, all the four classes have been known for some time. Conditions I1–I5 follow from [29] where Lamport introduced a model for system execution using Lamport's concept of space-time relationship. The condition I6 was added (in Lamport's framework) in [2]. Initially finite interval structures were analysed and introduced in [18]. Stratified order structures were introduced by Gaifman and Pratt in [12] and by Janicki and Koutny in [16]. Finite stratified order structures are analysed in detail in [19]. Total and partial order structures were defined in [17]. The names were introduced in [20] and follow from the following result.

**Proposition 1** (from [20])
*Let $\Theta$ be a class of (total, stratified, interval, partial) order structures, respectively.*

1. $S_e = (X, \prec_e, \sqsubset_e) \in \Theta^{\text{sat}}$ *if and only if* $\sqsubset_e = \prec_e^{\frown}$ *and* $(X, \prec_e)$ *is a (total, stratified, interval, partial) order, respectively.*
2. *if* $S$ *is initially finite then its saturated extensions are also initially finite.* □

From Proposition 1(1) it follows that if $\Theta$ is one of $\{\mathbf{T}, \mathbf{S}, \mathbf{I}, \mathbf{P}\}$ then $\Theta^{\text{sat}}$ are also just partial orders (of appropriate type) in disguise. By a small abuse of notation we may treat the set $ext_{\Theta}^{\text{sat}}(S)$, for $S = (X, <, \sqsubset)$, as a set of partial orders. Formally we may define $poext_{\Theta}^{\text{sat}}(S) = \{\prec_e | \ (X, \prec_e, \prec_e^{\frown}) \in ext_{\Theta}^{\text{sat}}(S)\}$, and then use $ext_{\Theta}^{\text{sat}}(S)$ to denote $poext_{\Theta}^{\text{sat}}(S)$, when it does not lead to any discrepancy.

The main results of [20] can be formulated as follows.

**Theorem 2** [20, Theorem 2.9]
*The classes of order structures* $\mathbf{T}$, $\mathbf{S}$, $\mathbf{I}$, $\mathbf{P}$, $\mathbf{T}_{IF}$, $\mathbf{S}_{IF}$, $\mathbf{I}_{IF}$, $\mathbf{P}_{IF}$ *are extension complete.* □

**Corollary 4**
*Let* $\Theta_1$, $\Theta_2$ *be the two classes of ordered structures listed in Theorem 2 such that* $\Theta_1 \subseteq \Theta_2$ *and let* $S = (X, <, \sqsubset) \in \Theta_1$. *Then*

1. $ext_{\Theta_1}^{\text{sat}}(S) \subseteq ext_{\Theta_2}^{\text{sat}}(S)$, *and* $S = \bigcap_{T \in ext_{\Theta_1}^{\text{sat}}(S)} T = \bigcap_{T \in ext_{\Theta_2}^{\text{sat}}(S)} T$.
2. $S \asymp ext_{\Theta_1}^{\text{sat}}(S)$ *and* $S \asymp ext_{\Theta_2}^{\text{sat}}(S)$. □

Theorem 2 restricted to $\mathbf{T}$ is nothing more than the well known Szpilrajn Theorem[6] [38] in disguise. In general it can be seen as an extension of Szpilrajn's ideas to partial orders that are not necessarily total[7] (see [17,20]). In this section the triples $(X, <, \sqsubset)$ *do not have any special interpretation*.

## 5 Order structures and concurrent histories

Corollary 4 together with Lemma 3(1) suggest that ordered structures [i.e. the triples $(X, <, \sqsubset)$] might represent uniquely concurrent histories conforming to paradigm $\pi_3$, which is the subject of this section. The discussion of this problem is a little bit confusing in [20], as the fact that order structures may belong to different classes is not explicitly discussed. We hope this section will help elucidate this part of the theory proposed in [20].

For each class of structures $\Theta \in \{\mathbf{T}, \mathbf{S}, \mathbf{I}, \mathbf{P}, \mathbf{T}_{IF}, \mathbf{S}_{IF}, \mathbf{I}_{IF}, \mathbf{P}_{IF}\}$, let $po(\Theta)$ denote the corresponding class of partial orders. For instance $po(\mathbf{S}_{IF}) = SO_{IF}$, $po(\mathbf{I}) = IO$, etc. *In this subsection we will interpret the elements of* $ext_{\Theta}^{\text{sat}}(S)$ *as partial orders if* $S$ *is in one of the classes mentioned above.*

Using Theorem 2 and Lemmas 2, 3, 4 we get the following result.

**Proposition 2** [20,18]

1. *Let* $\Theta \in \{\mathbf{S}, \mathbf{I}, \mathbf{P}, \mathbf{S}_{IF}, \mathbf{I}_{IF}, \mathbf{P}_{IF}\}$ *be a class of order structures. For each order structure* $S \in \Theta$, *the set* $ext_{\Theta}^{\text{sat}}(S)$ *is a concurrent history in* $po(\Theta)$ *that conforms to* $\pi_3$.
2. *If* $\Theta \in \{\mathbf{T}, \mathbf{T}_{IF}\}$, *then* $ext_{\Theta}^{\text{sat}}(S)$ *is a concurrent history in* $po(\Theta)$ *that conforms to* $\pi_6$.

[6] Every partial is order equal to the intersection of all its total extensions [38].

[7] However, the sets of partial orders must satisfy and equivalent of paradigm $\pi_3$.

*Proof* 1. From Theorem 2 and Lemma 3(1).
2. From Theorem 2, Lemmas 4(1), 3(3), 2(3) and the definition of **T**. □

When using Proposition 2, it is important not to forget about Corollary 4(1) which indicates that in some cases the different concurrent histories might define identical order structures.

Let us now analyse programs $P_1$, $P_2$, $P_3$, $P_4$ and *Priority* from Examples 1 and 2, and Figs. 1 and 2.

The program $P_1$ from Fig. 1 does not conform to $\pi_3$ and $O = SO \neq TO$, so the theory presented above does not apply.

For the programs $P_2$ and $P_3$, the structures $S_2 = (\{a, b, c\}, <_2, \sqsubset_2)$ and $S_3 = (\{a, b, c\}, <_3, \sqsubset_3)$ belong to $\mathbf{S}_{IF}$ (but not to $\mathbf{T}_{IF}$), and $ext^{\mathrm{sat}}_{\mathbf{S}_{IF}}(S_i) = obs(P_i)$ for $i = 2, 3$.

For the program $P_4$ we have $S_4 = (\{a, b, c\}, <_4, \sqsubset_4) = (\{a, b, c\}, <_4, <_4)$, $S_4$ belongs to $\mathbf{T}_{IF} \subset \mathbf{S}_{IF} \subset \mathbf{I}_{IF} \subset \mathbf{P}_{IF}$, and $ext^{\mathrm{sat}}_{\mathbf{T}_{IF}}(S_4) = \{ab, ac\} \subset ext^{\mathrm{sat}}_{\mathbf{S}_{IF}}(S_4) = ext^{\mathrm{sat}}_{\mathbf{I}_{IF}}(S_4) = ext^{\mathrm{sat}}_{\mathbf{P}_{IF}}(S_4) = obs(P_4) = \{ab, ac, \{a, b\}\}$.

For *Priority* we have an infinite set of concurrent behaviours (concurrent histories) and an infinite set of appropriate stratified order structures that represent those behaviours. For instance the triples $S^{\Delta_1}_{\mathrm{Priority}} = (\{a^{(1)}, b^{(1)}, c^{(1)}\}, <_{\Delta_1}, \sqsubset_{\Delta_1})$, and $S^{\Delta_2}_{\mathrm{Priority}} = (\{a^{(1)}, b^{(1)}, c^{(1)}, a^{(2)}, b^{(2)}, c^{(2)}\}, <_{\Delta_2}, \sqsubset_{\Delta_2})$ belong to $\mathbf{S}_{IF}$, we have $ext^{\mathrm{sat}}_{\mathbf{S}_{IF}}(S^{\Delta_1}_{\mathrm{Priority}}) = ext^{\mathrm{sat}}_{\mathbf{I}_{IF}}(S^{\Delta_1}_{\mathrm{Priority}}) = ext^{\mathrm{sat}}_{\mathbf{P}_{IF}}(S^{\Delta_1}_{\mathrm{Priority}}) = \Delta_1$ and $ext^{\mathrm{sat}}_{\mathbf{S}_{IF}}(S^{\Delta_2}_{\mathrm{Priority}}) = ext^{\mathrm{sat}}_{\mathbf{I}_{IF}}(S^{\Delta_2}_{\mathrm{Priority}}) = ext^{\mathrm{sat}}_{\mathbf{P}_{IF}}(S^{\Delta_2}_{\mathrm{Priority}}) = \Delta_2$. Clearly $S^{\Delta_1}_{\mathrm{Priority}}$ and $S^{\Delta_2}_{\mathrm{Priority}}$ are extension complete.

In all of the cases discussed above, observations are either total or stratified orders. The next two examples involve interval order observations, while the third one is an example of a partial order structure that is not interval order structure.

*Example 3* (interval order observations I)
Consider the following program:

```
P5: begin int x,y,z;
      a: begin x:=0; y:=0; z:=0 end;
      cobegin
      begin
      b: x=0 → begin y:=1; z:=z+1 end;
      d: x=0 → begin z:=2*z; y:=0 end
      end,
      c: y=0 → x:=x+1
      coend
      end P5.
```

A Petri net with inhibitor arcs $N_5$ corresponding to $P_5$ is in Fig. 4. If *interval order observations* are allowed (i.e. $O = IO$), and we are interested only in observations involving the whole set $\{a, b, c, d\}$ of events, then we have:

$$obs_{IO}(P_5) = \{abdc, o_I\},$$

where $o_I$ is an interval order observation defined by a Hasse diagram in Fig. 4. The set $obs_{IO}(P_5) = \{abdc, o_I\}$ conforms to $\pi_3$ but not $\pi_8$. The causality relation $<^I_5$ defined by $obs_{IO}(P_5)$ is also presented in Fig. 4, while weak causality $\sqsubset^I_5$ and commutativity $\diamond^I_5$ are the following:

$$\sqsubset^I_5 = <^I_5 \cup \{(b, c), (d, c)\}, \quad \diamond^I_5 = <^I_5 \cup \left(<^I_5\right)^{-1}.$$

The order structure $S_5^I = (\{a,b,c,d\}, <_5^I, \sqsubset_5^I)$ belongs to $\mathbf{I}_{IF}$ but not to $\mathbf{S}_{IF}$ and $ext^{\text{sat}}_{\mathbf{I}_{IF}}(S_5^I) = obs_{IO}(P_5)$. Hence $S_5^I \asymp obs_{IO}(P_5)$.

However, if only *stratified observations* are allowed (i.e. $O = SO$), we have:

$$obs_{SO}(P_5) = \{abcd\}, \quad <_5^S = \sqsubset_5^I = abdc,$$
$$\lessdot\!\!\!>_5^S = <_5^S \cup \left(<_5^S\right)^{-1} = \{abdc, cdba\}.$$

The order structure $S_5^S = (\{a,b,c,d\}, <_5^S, \sqsubset_5^S) = (\{a,b,c,d\}, <_5^S, <_5^S)$ belongs to $\mathbf{T}_{IF} \subset \mathbf{S}_{IF} \subset \mathbf{I}_{IF} \subset \mathbf{P}_{IF}$, and $ext^{\text{sat}}_{\mathbf{T}_{IF}}(S_5^S) = \{abdc\} = obs_{SO}(P_5) = ext^{\text{sat}}_{\mathbf{S}_{IF}}(S_5^S) = ext^{\text{sat}}_{\mathbf{I}_{IF}}(S_5^S) = ext^{\text{sat}}_{\mathbf{P}_{IF}}(S_5^S)$. In this case we have $S_5^S \asymp (\{a,b,c,d\}, <_5^S) \asymp obs_{SO}(P_5)$, as the case of stratified orders is reducible to the case of total orders. The set $obs_{SO}(P_5)$ conforms to $\pi_8$ and $obs_{IO}(P_5) \not\asymp obs_{SO}(P_5)$. □

*Example 4* (interval order observations II)
Consider the following program:

```
P6: begin int x,y,z;
      a: begin x:=0; y:=0; z:=0 end;
      cobegin
      begin
      b: x=0 → y:=y+1;
      d: z:=z+1
      end,
      c: x:=x+1
      coend
      end P6.
```

A Petri net with inhibitor arcs $N_6$ corresponding to $P_6$ is also in Fig. 4. If *interval order observations* are allowed (i.e. $O = IO$), and we are interested only in observations involving the whole set $\{a,b,c,d\}$ of events, then we have:

$$obs_{IO}(P_6) = \{abdc, abdc, a\{b,c\}d, ab\{c,d\}, o_I\},$$

where $o_I$ is an interval order observation defined by the Hasse diagram in Fig. 4, similarly to program $P_5$. The set $obs_{IO}(P_6)$ conforms to $\pi_3$ but not $\pi_8$. if only *stratified observations* are allowed (i.e. $O = SO$), we have:

$$obs_{SO}(P_6) = \{abdc, abdc, a\{b,c\}d, ab\{c,d\}\}.$$

The set $obs_{SO}(P_6)$ also conforms to $\pi_3$ but not $\pi_8$. Clearly $obs_{IO}(P_6)$ is not equal to $obs_{SO}(P_6)$, however, both sets define *identical* causality, weak causality and commutativity relations, namely $<_6^I = <_6^S = <_5^I$ (see Fig. 4 for an appropriate Hasse diagram), and

$$\sqsubset_6^I = \sqsubset_6^S = <_6^I \cup \{(b,c)\}, \quad \lessdot\!\!\!>_6^I = \lessdot\!\!\!>_6^S = <_6^I \cup \left(<_6^I\right)^{-1}.$$

The order structures $S_6^I = S_6^S = (\{a,b,c,d\}, <_6^S, \sqsubset_6^S)$ belong to $\mathbf{S}_{IF} \subset \mathbf{I}_{IF} \subset \mathbf{P}_{IF}$, $ext^{\text{sat}}_{\mathbf{S}_{IF}}(S_6) = obs_{SO}(P_6) \subset ext^{\text{sat}}_{\mathbf{I}_{IF}}(S_6) = ext^{\text{sat}}_{\mathbf{P}_{IF}}(S_6) = obs_{IO}(P_6)$. In this case we have $S_6^S = S_6^I$, both $S_6^S$ and $S_6^I$ are extension complete, but $ext^{\text{sat}}_{\mathbf{S}_{IF}}(S_6) \neq ext^{\text{sat}}_{\mathbf{I}_{IF}}(S_6)$. Hence $obs_{IO}(P_6) \asymp obs_{SO}(P_6)$, even though $obs_{IO}(P_6) \neq obs_{SO}(P_6)$. □

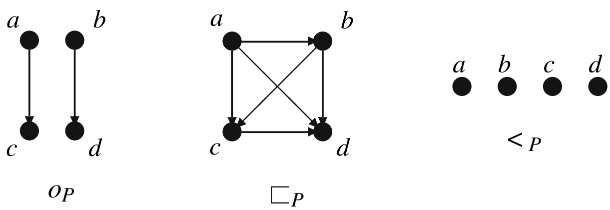


**Fig. 5** Partial orders $o_P$, $\sqsubset_P$ and $<_P$ for Example 5 ($<_P$ is the empty relation on $\{a, b, c, d\}$)

*Example 5* (partial order structure but NO interval order structure)
Let $o_P$, $<_P$ and $\sqsubset_P$ be relations from Fig. 5. Consider the set of partial orders $\Delta = \{\{a, b, c\}d, a\{b, c, d\}, o_P\}$ and the triple $S_P = (\{a, b, c, d\}, <_P, \sqsubset_P)$. Note that $\Delta = \Delta^{cl}_{PO}$, i.e. $\Delta$ satisfies the definition of a *history* (Definition 1), and $<_\Delta = <_P$, $\sqsubset_\Delta = \sqsubset_P$. Furthermore $\Delta$ conforms to $\pi_3$ and $o_P$ is not an interval order. The triple $S_P$ is a partial order structure but not interval order structure, i.e. $S_P \in \mathbf{P}_{IF} \setminus \mathbf{I}$, and $ext^{\text{sat}}_{\mathbf{P}_{IF}} = \Delta$.

The set $\Delta$ is technically a concurrent history; however, its elements cannot be interpreted as observations. It was argued in [18] that *an observation must always be an initially finite interval order*, and $o_P \in \Delta$ is *not* an interval order. Hence we cannot provide any concurrent system that generates $\Delta$ and is modelled by $S_P$. □

It is important to point out that the results of this section and these of [12,20,29] either assume or imply the paradigm $\pi_3$, or $O = TO$, which suffices for most applications, but it is not the most general case. Under the assumption of $\pi_3$ we *have to* model the programs $P_1$ and $P_7$ (Fig. 6) by two sequential behaviours instead of a more natural modelling involving concurrent behaviour.

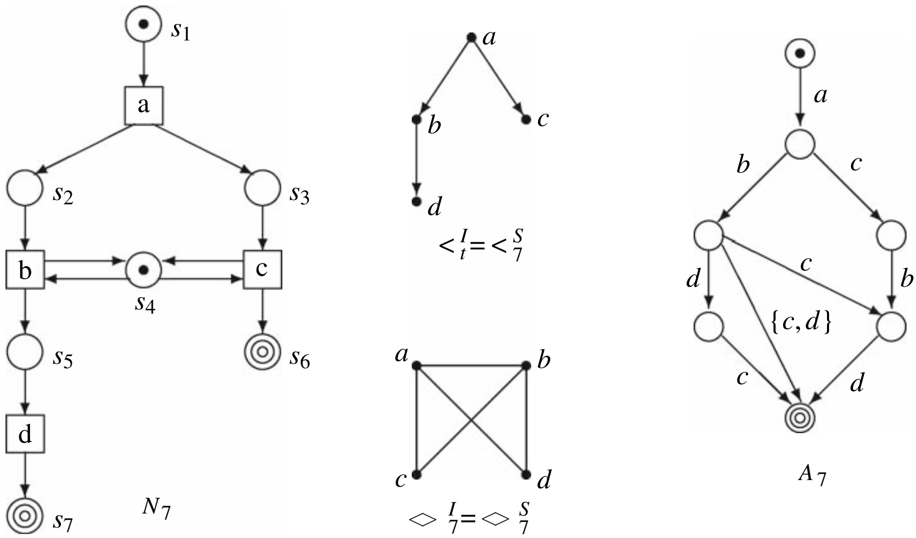


**Fig. 6** Petri net $N_7$ and transition system $A_7$ corresponding to program $P_7$ from Example 6, and appropriate casuality and commutativity relations. The transition system $A_7$ models $P_7$ only if observations are restricted to stratified orders ($O = PO$). Causality is represented as a Hasse diagram, while commutativity as the whole relation

## 6 Generalised order structures

The previous two sections were mainly devoted to the theory of triples $(X, <, \sqsubset)$ where '$<$' could be interpreted as "causality" and '$\sqsubset$' as "weak causality". Such triples can adequately model concurrent histories, but only if some conditions, constituting the paradigm $\pi_3$ are satisfied. Lemma 1(2) suggests that all concurrent histories (as defined by Definition 1) could be modelled by triples of the type $(X, <>, \sqsubset)$, where $<>$ is interpreted as "commutativity". In this section we develop a theory of such triples.

**Definition 5** [13] A *generalised order structure* is a triple

$$G = (X, <>, \sqsubset)$$

such that $X$ is a non-empty set, $<>$ and $\sqsubset$ are two irreflexive relations on $X$, $<>$ is symmetric, and the triple $S_G = (X, <_G, \sqsubset)$, where $<_G = <> \ \cap \ \sqsubset$, is an order structure (i.e. $<_G$ is irreflexive and for all $x, y \in X, x <_G y \Rightarrow \neg(y \sqsubset a)$).
For each generalised order structure $G = (X, <>, \sqsubset)$, the order structure $S_G = (X, <_G, \sqsubset)$, where $<_G = <> \ \cap \ \sqsubset$, will be called the *order structure induced by G*. □

- For each relation $R$ (not necessarily a partial order), let $R^{\text{sym}}$ denote the symmetric closure of $R$, i.e. $R^{\text{sym}} = R \ \cup R^{-1}$.

The corollary below shows that the name "generalised" is justified, and slightly modified order structures create a subclass of generalised order structures.

**Corollary 5**

1. *For each order structure $S = (X, <, \sqsubset)$, the triple $G_S = (X, <^{\text{sym}}, \sqsubset)$ is a generalised order structure.*
2. *For every partial order $(X, <)$, the triples $(X, <^{\text{sym}}, <^{\frown})$ and $(X, <^{\text{sym}}, <)$ are generalised order structures.* □

**Definition 6** A generalised order structure $G = (X, <>, \sqsubset)$ is called *saturated* if $<> = <_G^{\text{sym}}$, where $<_G = <> \cap \sqsubset$, and $S_G = (X, <_G, \sqsubset)$ is saturated (i.e. $x <_G y \Leftrightarrow \neg(y \sqsubset x)$). □

If $G = (X, <>, \sqsubset)$ is saturated, then both $<>$ and $\sqsubset$ are uniquely defined by the relation (not necessarily a partial order here) $<_G = <> \ \cap \ \sqsubset$, i.e. $(X, <>, \sqsubset) \asymp (X, <_G)$.

- For any class $\Theta$ of generalised ordered structures, $\Theta^{\text{sat}}$ is the class of all saturated generalised structures in $\Theta$.

The result below shows that the generalised order structures can be seen as a further generalisation of the concept of a partial order.

**Corollary 6** *For every partial order $(X, \prec)$ we have:*

1. $(X, \prec^{\text{sym}}, \prec^{\frown})$ *is a* saturated generalised order structure.
2. $(X, \prec^{\text{sym}}, \prec)$ *is a* generalised order structure *and it is saturated if and only if $\prec$ is total.* □

The operations of union, intersection and subset for generalised ordered structures with the same domain are defined component-wise. Let $G = (X, <>, \sqsubset)$, $G_1 = (X, <>_1, \sqsubset_1)$, $G_2 = (X, <>_2, \sqsubset_2)$, $G_i = (X, <>_i, \sqsubset_i)$, $i \in I$. Then

- $G=\bigcup_{i\in I} G_i \iff \diamond = \bigcup_{i\in I} \diamond_i \wedge \sqsubset = \bigcup_{i\in I} \sqsubset_i$,
- $G=\bigcap_{i\in I} G_i \iff \diamond = \bigcap_{i\in I} \diamond_i \wedge \sqsubset = \bigcap_{i\in I} \sqsubset_i$,
- $G_1 \subseteq G_2 \iff \diamond_1 \subseteq \diamond_2 \wedge \sqsubset_1 \subseteq \sqsubset_2$.

Note that $\bigcup_{i\in I} G_i$ may *not* be a generalised order structure, but $\bigcap_{i\in I} G_i$ always is a generalised order structure. We will also say that

- a generalised order structure $G_2$ is an *extension* of a generalised order structure $G_1$ if $G_1 \subseteq G_2$.

We will now define extension completeness in the same manner as it was done for ordered structures.

**Definition 7** A class $\Theta$ of generalised order structures with possible different domains is *extension complete* if for every generalised order structure $G \in \Theta$, the set $ext_{\Theta}^{\text{sat}}(G) = \{T \in \Theta^{\text{sat}} \mid G \subseteq T\}$ is non-empty and

$$G = \bigcap_{T\in ext_{\Theta}^{\text{sat}}(G)} T$$

□

In other words, $\Theta$ is extension complete if any generalised order structure in $\Theta$ can be represented using its saturated extensions defined by $\Theta$. All comments made after Definition 3 (extension completeness for ordered structures) are also valid for generalised order structures.

**Corollary 7** *If a class of generalised structures $\Theta$ is extension complete, then for each $S \in \Theta$, we have $S \asymp ext_{\Theta}^{\text{sat}}(S)$.* □

Now we restrict our attention to four concrete classes of generalised order structures.

**Definition 8** A generalised order structure $G = (X, \diamond, \sqsubset)$ is called a *total*, *stratified*, *interval*, *partial* or *initially finite* if the order structure $S_G = (X, <_G, \sqsubset)$, where $<_G = \diamond \cap \sqsubset$, is total (axioms T1–T3), stratified (axioms S1–S4), interval (axioms I1–I6), partial (axioms P1–P4), or initially finite, respectively. □

We shall use **GT**, **GS**, **GI** and **GP** to denote, respectively, the classes of total, stratified, interval and partial order generalised structures. One may verify easily that **GT** $\subset$ **GS** $\subset$ **GI** $\subset$ **GP**. If $\Theta$ is a class of generalised structures, we denote by $\Theta_{IF} \subseteq \Theta$ the subclass consisting of initially finite generalised structures. From Definition 8 we obtain immediately the following result.

**Corollary 8** *If $<_G = \diamond \cap \sqsubset = \sqsubset$ then $G = (X, \diamond, \sqsubset)$ is a generalised total order structure.* □

**Proposition 3** *Let $\Theta$ be a class of generalised* (*total, stratified, interval, partial*) *order structures, respectively.*

1. *$G_e = (X, \diamond_e, \sqsubset_e) \in \Theta^{\text{sat}}$ if and only if there is a* (*total, stratified, interval, partial*) *order $(X, \prec_e)$ such that $\diamond_e = \prec_e^{\text{sym}}$ and $\sqsubset_e = \prec_e^{\frown}$.*
2. *if $G$ is initially finite then its saturated extensions are also initially finite.*

*Proof* From Definition 5 and Proposition 1. □

From Proposition 3(1) it follows that if $\Theta$ is one of **GT**, **GS**, **GI**, **GP** then the elements of $\Theta^{\text{sat}}$ are also just partial orders (of appropriate type) in disguise. By a small abuse of notation we may treat the set $ext^{\text{sat}}_{\Theta}(G)$, for $G = (X, \frown, \sqsubset)$, as a set of partial orders. Formally we may define $poext^{\text{sat}}_{\Theta}(G) = \{\prec_e | \, (X, \prec_e^{\text{sym}}, \prec_e^{\frown}) \in ext^{\text{sat}}_{\Theta}(G)$, and then use $ext^{\text{sat}}_{\Theta}(G)$ to denote $poext^{\text{sat}}_{\Theta}(G)$, when it does not lead to any confusion.

The generalised order structure $G_1$ from Fig. 1 corresponding to the program $P_1$ from the Introduction is total, i.e. it belongs to $\mathbf{GT}_{IF}$, and $ext^{\text{sat}}_{\mathbf{GT}_{IF}}(G_1) = ext^{\text{sat}}_{\mathbf{GS}_{IF}}(G_1) = ext^{\text{sat}}_{\mathbf{GI}_{IF}}(G_1) = ext^{\text{sat}}_{\mathbf{GP}_{IF}}(G_1) = obs(P_1)$. One can easily verify by inspection that $G_1$ of Fig. 1 is extension complete.

We will now discuss three examples of more complex cases than $G_1$. All the properties discussed below can easily be verified by inspection.

*Example 6* (generalised total order structures)
Consider the following program:

```
P7: begin int x,y;
      a: begin x:=0; y:=0 end;
      cobegin
      begin
      b: x:=x+1;
      d: y:=y+1
      end,
      c: x:=x+2
      coend
      end P7.
```

A Petri net with inhibitor arcs $N_7$ and a transition system $A_7$, both corresponding to $P_7$ are in Fig. 6. The program $P_7$ can produce only stratified observations, even if observing interval orders is allowed, i.e. the cases $O = IO$ and $O = SO$ are identical. If we are only interested in observations involving the whole set $\{a, b, c, d\}$ of events, then we have:

$$obs_{IO}(P_7) = obs_{SO}(P_7) = \{abcd, abdc, acbd, ab\{cd\}\}.$$

The concurrent history $obs_{SO}(P_7)$ conforms to $\pi_6$ but not to $\pi_3$, causality and commutativity relations, $<^I_7 = <^S_7$ and $\frown^I_7 = \frown^S_7$, are depicted in Fig. 6, while weak causality $\sqsubset^I_7 = \sqsubset^S_7$ equals $<^I_7$ (since $\pi_6$ is satisfied).

The triple $G_7 = (\{a, b, c, d\}, \frown^S_7, \sqsubset^S_7)$ is a generalised *total* order structure, i.e. it belongs to $\mathbf{GT}_{IF}$. However, $ext^{\text{sat}}_{\mathbf{GT}_{IF}}(G_7) = \{abcd, abdc, acbd\} \neq obs_{SO}(P_7) = obs_{IO}(P_7) = \{abcd, abdc, acbd, ab\{c, d\}\}$, and $G_7$ (treated as generalised total order structure) is *not* extension complete. On the other hand $G_7$ is also a generalised stratified (and interval, partial) order structure, and $ext^{\text{sat}}_{\mathbf{GS}_{IF}}(G_7) = ext^{\text{sat}}_{\mathbf{GI}_{IF}}(G_7) = obs_{SO}(P_7) = obs_{IO}(P_7)$. It can easily be verified that $G_7$ *is* extension complete in $\mathbf{GS}_{IF}$ (and in $\mathbf{GI}_{IF}$ or $\mathbf{GP}_{IF}$ ). Hence we can write $G_7 \asymp obs_{SO}(P_7)$.

Let us now assume that only totally ordered observations are allowed (i.e. $O = TO$). In this case:

$$obs_{TO}(P_7) = \{abcd, abdc, acbd\}$$

and this is exactly the set of all total extension of the causality $<^T_7$ that equals $<^I_7$ from Fig. 6. The triple $G^T_7 = (\{a, b, c, d\}, \frown^T_7, \sqsubset^T_7)$, where

$$\frown^T_7 = \{(x, y) \mid x, y \in \{a, b, c, d\} \wedge x \neq y\}, \text{ and } \sqsubset^T_7 = <^T_7,$$

is a generalised total order structure, but there is no need to construct and analyse it, as we have here $G_7^T \asymp (\{a,b,c,d\}, <_7^T) \asymp obs_{TO}(P_7)$ [see Lemma 2(3)]. Note a fundamental difference between $G_7^T$ and $G_1$ from Fig. 1 (despite the fact that $<>_1$ and $<>_7^T$ are structurally identical, $X \times X$ minus identity). In the case of $G_1$ we assumed $O = SO \neq TO$, so $G_1 \not\asymp (\{a,b,c\}, <_1)$, for $G_7^T$ we assumed $O = TO$, so clearly $G_7^T \asymp (\{a,b,c,d\}, <_7^T)$.

□

*Example 7* (generalised stratified and interval order structures)
Consider the following program:

```
P8: begin int x,y;
    a: begin x:=0; y:=0 end;
    cobegin
    begin
    b: x:=x+1;
    d: y=0 → y:=y+1
    end,
    c: x:=x+2
    d: y:=y+2
    coend
    end P8.
```

A Petri net with inhibitor arcs $N_8$ and a transition system $A_8$, both corresponding to $P_8$ are in Fig. 7. The transition system $A_8$ corresponds to $P_8$ only if all observations must be stratified orders (i.e. when $O = SO$).

If *interval order observations* are allowed (i.e. $O = IO$), we have

$$obs_{IO}(P_8) = \{abcde, abdce, acbde, ab\{c,d\}e, abc\{d,e\}, o_1^I, o_2^I\},$$

where $o_1^I$ and $o_2^I$ are interval orders defined by Hasse diagrams in top right of Fig. 7. The concurrent history $obs_{IO}(P_8)$ conforms to neither $\pi_3$ nor to $\pi_6$.

Causality $<_8^I$ and commutativity $<>_8^I$ relations are also presented in Fig. 7, while weak causality $\sqsubset_8^I$ satisfies:

$$\sqsubset_8^I = <_8^I \cup \{(d,e)\}.$$

The triple $G_8^I = (\{a,b,c,d,e\}, <>_8^I, \sqsubset_8^I)$ is a *generalised interval order structure* that belongs to the class $\mathbf{GI}_{IF} \backslash \mathbf{GS}$ and $ext^{\mathrm{sat}}_{\mathbf{GI}_{IF}}(G_8^I) = obs_{IO}(P_8)$. One may easily verify by inspection that $G_8^I$ is extension complete. Clearly $G_8^I \asymp obs_{IO}(P_8)$.

If only *stratified observations* are allowed (i.e. $O = SO$) we have:

$$obs_{SO}(P_8) = \{abcde, abdce, acbde, ab\{c,d\}e, abc\{d,e\}\},$$

and $obs_{SO}(P_8)$ also conforms to neither $\pi_3$ nor to $\pi_6$.

Causality $<_8^S$ and commutativity $<>_8^S$ relations are described in Fig. 7, and weak causality $\sqsubset_8^S$ is given by:

$$\sqsubset_8^S = <_8^S \cup \{(d,e)\}.$$

Note that in this case we have $<_8^I \neq <_8^S$, $\sqsubset_8^I \neq \sqsubset_8^S$ and $<>_8^I \neq <>_8^S$. The triple $G_8^S = (\{a,b,c,d,e\}, <>_8^S, \sqsubset_8^S)$ is a *generalised stratified order structure* that belongs to $\mathbf{GS}_{IF} \backslash \mathbf{GT}$

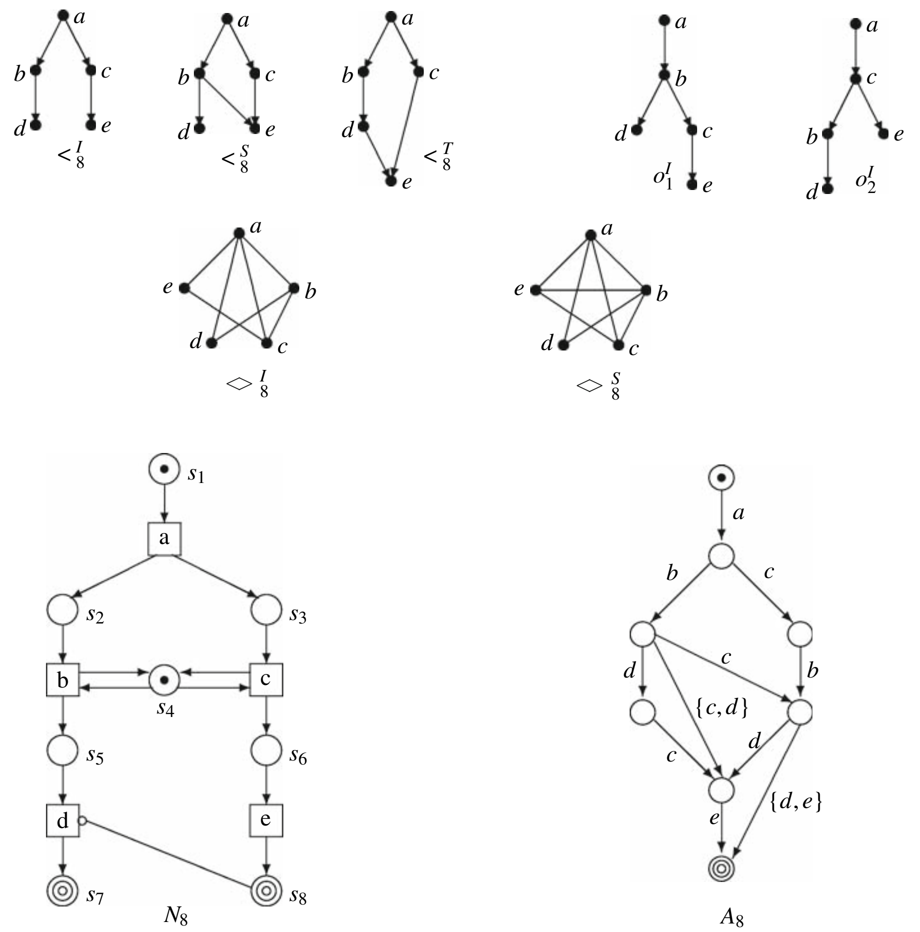


**Fig. 7** Petri net $N_8$, transition system $A_8$ and appropriate relations corresponding to the program $P_8$ from Example 7. Partial orders are represented as Hasse diagrams, the graphs for $<\!\!>^I_8$ and $<\!\!>^S_8$ represent whole relations

and $G^S_8$ is extension complete. We also have $ext^{\text{sat}}_{\mathbf{GS}_{IF}}(G^S_8) = obs_{SO}(P_8)$ and $G^S_8 \asymp obs_{SO}(P_8)$.

If only *totally ordered observations* are allowed (i.e. $O = TO$) we have:

$$obs_{TO}(P_8) = \{abcde, abdce, acbde\},$$

and $obs_{TO}(P_8)$ is just the set of all total extensions of the partial order $<^T_8$ from Fig. 7. By Lemma 2(3) we have $(\{a, b, c, d, e\}, <^T_8) \asymp obs_{TO}(P_8)$, so a construction of any generalised order structure is not needed. Note that $<^T_8 \neq <^I_8$ and $<^T_8 \neq <^S_8$. □

*Example 8* (generalised partial order structures)
Let $o_1, o_2, o_3, o_4, <_P$ and $\overline{<\!\!>}_P$ be relations defined in Fig. 8. Define $X = \{a, b, c, d, e, f, g, h\}$, $<\!\!>_P = X \times X \setminus \overline{<\!\!>}_P$, $\sqsubset_P = <_P \cup \{(g, h)\}$. Consider the set of partial orders $\Delta = \{o_1, o_2, o_3, o_4\}$ and the triple $G_P = (\{a, b, c, d\}, <\!\!>_P, \sqsubset_P)$. Note that $\Delta = \Delta^{cl}_{PO}$, i.e.

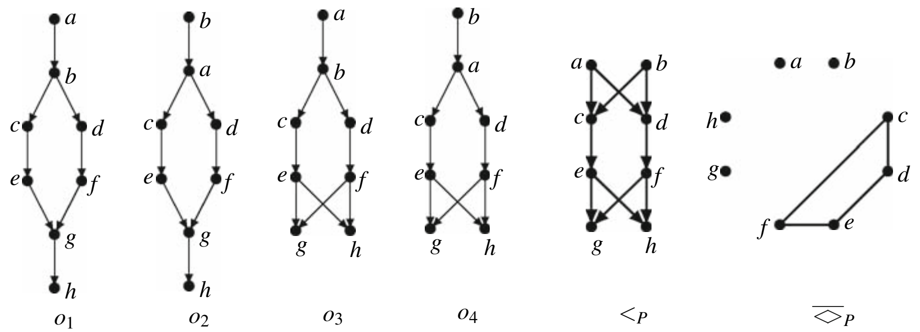

**Fig. 8** The relations for Example 8 partial orders are represented as Hasse diagrams, the graph for $\overline{\Diamond}$ represents the whole relation

$\Delta$ satisfies the definition of a *history* (Definition 1), and $<_\Delta = <_P, \sqsubset_\Delta = \sqsubset_P, \Diamond_\Delta = \Diamond_P$. Furthermore $\Delta$ conforms only to $\pi_1$ and none of $o_1, o_2, o_3, o_4$, is not an interval order. The triple $G_P$ is a generalised partial order structure but not generalised interval order structure, i.e. $G_P \in \mathbf{GP}_{IF} \backslash \mathbf{GI}$, and $ext^{sat}_{\mathbf{GP}_{IF}} = \Delta$. Furthermore $G_P$ is extension complete.

Similarly as in the case of $S_P$ from Example 5 (and for similar reasons), we cannot provide any concurrent system that generates $\Delta = \{o_1, o_2, o_3, o_4\}$. □

Before further analysis of extension completeness of appropriate generalised order structures, we need another definition of "extension".

- A partial order $(X, \prec)$ is an *extension* of a generalised order structure $(X, \Diamond, \sqsubset)$ if and only if $\Diamond \subseteq \prec^{sym}$ and $\sqsubset \subseteq \prec^{\frown}$.

The following lemma gives some simple necessary and sufficient conditions for extension completeness. We will use it often in the remainder of this paper.

**Lemma 5** *Let $G = (X, \Diamond, \sqsubset)$ be a generalised structure and $\Omega$ any non-empty set of partial orders that extend $G$. Then $\Diamond = \bigcap_{\prec \in \Omega} \prec^{sym}$, and $\sqsubset = \bigcap_{\prec \in \Omega} \prec^{\frown}$ if and only if for all distinct $x, y \in X$ we have:*

1. $\neg(x \Diamond y) \Rightarrow \exists \prec \in \Omega.\ \neg(x \prec y) \wedge \neg(y \prec x)$
2. $\neg(x \sqsubset y) \Rightarrow \exists \prec \in \Omega.\ y \prec x$.

*Proof* Note that (1) is equivalent to $(\bigcap_{\prec \in \Omega} \prec^{sym}) \subseteq \Diamond$, and (2) is equivalent to $(\bigcap_{\prec \in \Omega} \prec^{\frown}) \subseteq \sqsubset$, so the implication ($\Rightarrow$) holds.
($\Leftarrow$) Suppose (1) and (2) hold. From the definition of extension, for all $x, y \in X$, $x \Diamond y \Rightarrow \forall \prec \in \Omega.\ x \prec^{sym} y$ and $x \sqsubset y \Rightarrow \forall \prec \in \Omega.\ x \prec^{sim} y$, i.e. $\Diamond \subseteq \bigcap_{\prec \in \Omega} \prec^{sym}$ and $\sqsubset \subseteq \bigcap_{\prec \in \Omega} \prec^{\frown}$. □

The next lemma states when the union of order structures is an order structure.

**Lemma 6** *Let $(Q, \prec)$ be an upper semi-lattice,*[8] *and $\{S_r \mid r \in Q\}$, where $S_r = (X, <_r, \sqsubset_r)$, be a class of (total, stratified, interval, partial) order structures such that $r_1 \preceq r_2 \Rightarrow (<_{r_1} \subseteq <_{r_2} \wedge \sqsubset_{r_1} \subseteq \sqsubset_{r_2})$. Then $S = (X, <, \sqsubset)$, where $< = \bigcup_{r \in Q} <_r$ and $\sqsubset = \bigcup_{r \in Q} \sqsubset_r$, is also a (total, stratified, interval, partial) order structure.*

[8] A partial order $(X, <)$ is an upper semi-lattice if for any $x, y \in X$ there is $z \in X$ such that $x \leq z \wedge y \leq z$. In particular any total order is an upper semi-lattice.

*Proof* (Total) In this case $<_r = \sqsubset_r$ for each $r \in Q$. Since $x \not<_r x$ for any $r \in Q$, $x \not< x$. If $x < y < z$, then by definition there exist $r_1$ and $r_2$ such that $x <_{r_1} y$ and $y <_{r_2} z$. By the hypothesis, $Q$ is an upper semi-lattice, so there is a $r \in Q$ such that $r_1 \preceq r$ and $r_2 \preceq r$, hence $<_{r_1} \subseteq <_r$ and $<_{r_2} \subseteq <_r$. It follows that $x <_r y$ and $y <_r z$, hence $x <_r z$. Therefore $x < z$.
(Stratified) S1: Since $x \not\sqsubset_r x$ for any $t \in Q$, $x \not\sqsubset_S x$.
S2: If $x < y$, then by definition there is $r \in Q$ such that $x <_r y$, hence $x \sqsubset_r y$. So $x \sqsubset y$.
S3: If $x \sqsubset y \sqsubset z$, then by definition there exist $r_1$ and $r_2$ such that $x \sqsubset_{r_1} y$ and $y \sqsubset_{r_2} z$. By the hypothesis, $Q$ is an upper semi-lattice, so there is a $r \in Q$ such that $r_1 \preceq r$ and $r_2 \preceq r$, hence $\sqsubset_{r_1} \subseteq \sqsubset_r$ and $\sqsubset_{r_2} \subseteq \sqsubset_r$. It follows that $x \sqsubset_r y$ and $y \sqsubset_r z$, hence $x \sqsubset_r z \vee x = z$. Therefore $x \sqsubset y \vee x = z$.
S4: Similarly to the proof of S3.
(Interval) I1 as S1, I2 as S2, I3 as (Total), I4, I5, I6 similarly to S3.
(Partial) P1 as S1, P2 as S2, P3 as (Total), P4 similarly to S3. ◻

The above two lemmas will be used in proofs of our main results. Lemma 5 is a natural extension of Lemma 2.1 from [20]. Unfortunately, as opposed to [20], we are unable to give one set of proofs that fits all cases; we have to go case by case. We will start with the case of generalised stratified order structures, and end with generalised total order structures.

For later use, for any order structure $S = (X, <, \sqsubset)$ we define the following sets [20]:

$$\begin{aligned} (W_S)_a &= \{a\} \cup \{c | c \sqsubset a\} & (W_S)^a &= \{a\} \cup \{c | a \sqsubset c\} \\ (Y_S)_a &= \{a\} \cup \{c | c < a\} & (Y_S)^a &= \{a\} \cup \{c | a < c\}. \end{aligned}$$

The above sets will be used in several proofs in the remainder of the paper.

## 6.1 Generalised stratified order structures

In order to prove that generalised stratified order structures are extension complete, it suffices to show that conditions (1) and (2) of Lemma 5 are satisfied.

The proof below is intuitively quite straightforward, we just construct appropriate partial orders, but the constructions are quite complex and lengthy, and employ transfinite induction. The technique we will use was developed in [20] to prove Theorem 2. Preliminary version of this subsection has been published as [13].

We will start with proving condition (2).

**Lemma 7** *Let $G = (X, \lessdot\!\!\!\gtrdot, \sqsubset) \in \mathbf{GS}_{IF}$ ($G \in \mathbf{GS}$) and let $a, b \in X$ be any two distinct elements such that $\neg(b \sqsubset a)$. Then there exists $\prec \in SO_{IF}$ ($\prec \in SO$) such that $\prec$ extends $G$ (i.e. $\lessdot\!\!\!\gtrdot \subseteq \prec^{\text{sym}}$ and $\sqsubset \subseteq \prec^{\frown}$) and $a \prec b$.*

*Proof* By Proposition 1(1), it suffices to show there exists a $T = (X, <_T, \sqsubset_T) \in \mathbf{S}^{\text{sat}}_{IF}$ such that $(X, <_T^{\text{sym}}, \sqsubset_T)$ extends $G$ and $x \prec y \iff x <_T y$. By definition, $S_G = (X, <_G, \sqsubset)$, where $<_G = \lessdot\!\!\!\gtrdot \cap \sqsubset$, is in $\mathbf{S}_{IF}$. Consider the order structure $S_0 = (X, <_0, \sqsubset_0)$ where

$$<_0 = <_G \cup (W_S)_a \times (W_S)^b,$$

and

$$\sqsubset_0 = \sqsubset \cup (W_S)_a \times (W_S)^b.$$

Clearly $a <_0 b$ and $<_0 \cap \lessdot\!\!\!\gtrdot = \sqsubset_0 \cap \lessdot\!\!\!\gtrdot$. We show that $S_0 \in \mathbf{S}_{IF}$.

S1 and S2 hold trivially. To show that S3 holds, suppose $x \sqsubset_0 y \sqsubset_0 z$. Since we have $\neg(b \sqsubset a)$ and $(W_S)_a \cap (W_S)^b = \emptyset$, then

$$(x, y) \in (W_S)_a \times (W_S)^b \wedge (y, z) \in (W_S)_a \times (W_S)^b$$

is impossible. Thus we have either

(i) $x \sqsubset y \sqsubset z$, or
(ii) $x \sqsubset y \wedge (y, z) \in (W_S)_a \times (W_S)^b$, or
(iii) $(x, y) \in (W_S)_a \times (W_S)^b \wedge y \sqsubset z$.

If (i) holds, then $x \sqsubset z \vee x = z$, so $x \sqsubset_0 z \vee x = z$.
If (ii) holds, then $x \sqsubset y \sqsubset a$, which implies $x \sqsubset a \vee x = a$, hence $x \in (W_S)_a$. Then we have $(x, z) \in (W_S)_a \times (W_S)^b$, therefore $x \sqsubset_0 z$.
Similarly, if (iii) holds, then $z \in (W_S)^b$ and $(x, z) \in (W_S)_a \times (W_S)^b$, so $x \sqsubset_0 z$.
Thus in any case, $x \sqsubset_0 y \sqsubset_0 z \Longrightarrow x \sqsubset_0 z \vee x = z$, that is S3 holds.

The proof for S4 is very similar. Suppose $x \sqsubset_0 y <_0 z$. Since $\neg(b \sqsubset a)$ and $(W_S)_a \cap (W_S)^b = \emptyset$ then

$$(x, y) \in (W_S)_a \times (W_S)^b \wedge (y, z) \in (W_S)_a \times (W_S)^b$$

does not hold. Thus, either

(i) $x \sqsubset y < z$, or
(ii) $x \sqsubset y \wedge (y, z) \in (W_S)_a \times (W_S)^b$, or
(iii) $(x, y) \in (W_S)_a \times (W_S)^b \wedge y < z$.

If (i) holds, then we have $x < z$, so $x <_0 z$.
If (ii) holds, then $x \sqsubset y < a$, which implies $x < a$, hence $x \in (W_S)_a$. This means $(x, z) \in (W_S)_a \times (W_S)^b$, therefore $x <_0 z$.
Similarly, if (iii) holds, then $z \in (W_S)^b$ and $(x, z) \in (W_S)_a \times (W_S)^b$, so $x <_0 z$.
Therefore in any case, $x \sqsubset_0 y <_0 z \Rightarrow x <_0 z$.
The proof of the case $x <_0 y \sqsubset_0 z$ is almost the same, hence S4 holds.

Finally, it is easy to see that for any $h \in X$,

$$(W_{S_0})^h \subseteq (W_S)^h \cup (W_S)^a.$$

Therefore $S_0$ is also initially finite. The rest of the proof is a refinement of the proof of Lemma 2.2 in [20].

Since $S$ is initially finite, $X$ is countable, hence we may assume that there is a well order, $\lhd$ say, of $X$ such that each element in $X$ is preceded only by finitely many elements. We extend $\lhd$ lexicographically to $X \times X$ and we define $\unlhd = \lhd \cup id_X$.

We now show, by $\lhd$-induction, that for each pair $(c, d) \in X \times X$, we can associate $S_{cd} = (X, <_{cd}, \sqsubset_{cd}) \in \mathbf{S}_{IF}$ such that for all pairs $(c, d), (e, f) \in X \times X$, there exists $S_{ef} = (X, <_{ef}, \sqsubset_{ef}) \in \mathbf{S}_{IF}$ and the following are satisfied

$$<_0 \subseteq <_{cd} \wedge \sqsubset_0 \subseteq \sqsubset_{cd}, \tag{1}$$

$$(e, f) \lhd (c, d) \Longrightarrow <_{ef} \subseteq <_{cd} \wedge \sqsubset_{ef} \subseteq \sqsubset_{cd}, \tag{2}$$

$$c \neq d \Longrightarrow c <_{cd} d \vee d \sqsubset_{cd} c, \tag{3}$$

$$<_{cd} \cap \Diamond = \sqsubset_{cd} \cap \Diamond \ . \tag{4}$$

Moreover, for all $g \in X$,

$$(W_{S_{cd}})^g \quad \subseteq \quad (W_{S_0})^g \cup \left( \bigcup_{h \unlhd c} (W_{S_0})^h \right). \tag{5}$$

Let $(m, m)$ be the minimal pair w.r.t. the well ordering $\lhd$ in $X \times X$. Set $S_{mm} = S_0$. Of course $S_{mm} \in \mathbf{S}_{IF}$, and it trivially satisfies (1), (2), (3), (4) and (5).

Now let $(e, f) \in X \times X$ be a pair such that $S_{cd} \in \mathbf{S}_{IF}$ are defined for all $(c, d) \lhd (e, f)$ and satisfy (1), (2), (3), (4), and (5). There are only finite such $(c, d)$'s, so there is a maximal pair, $(\hat{c}, \hat{d})$ say. If $(e = f) \vee (e <_{\hat{c}\hat{d}} f) \vee (f \sqsubset_{\hat{c}\hat{d}} e)$, then define $S_{ef} = S_{\hat{c}\hat{d}}$; else set $S_{ef} = (X, <_{ef}, \sqsubset_{ef})$ where

$$<_{ef} = <_{\hat{c}\hat{d}} \cup \big(W_{\hat{c}\hat{d}}\big)_e \times (W_{\hat{c}\hat{d}})^f,$$

and

$$\sqsubset_{ef} = \sqsubset_{\hat{c}\hat{d}} \cup \big(W_{\hat{c}\hat{d}}\big)_e \times (W_{\hat{c}\hat{d}})^f.$$

Similarly as for $S_0$ we can show that $S_{ef} \in \mathbf{S}_{IF}$. It is also clear that $S_{ef}$ satisfies (1), (2), (3) and (4) above. For any $g \in X$,

$$\big(W_{S_{ef}}\big)^g \subseteq \big(W_{S_{\hat{c}\hat{d}}}\big)^g \cup \big(W_{S_{\hat{c}\hat{d}}}\big)^e.$$

By (5) for all $(c, d) \lhd (e, f)$, we obtain

$$\begin{aligned}\big(W_{S_{ef}}\big)^g &\subseteq \left((W_{S_0})^g \cup \left(\bigcup_{h \unlhd \hat{c}} (W_{S_0})^h\right)\right) \cup \left(\big(W_{S_0}\big)^e \cup \left(\bigcup_{h \unlhd \hat{c}} \big(W_{S_0}\big)^h\right)\right) \\ &\subseteq \big(W_{S_0}\big)^g \cup \left(\bigcup_{h \unlhd e} (W_{S_0})^h\right).\end{aligned}$$

Thus $S_{ef} = (X, <_{ef}, \sqsubset_{ef})$ also satisfies (5). Next we define

$$T = (X, <_T, \sqsubset_T), \quad \text{where } <_T = \bigcup_{(c,d) \in X \times X} <_{cd}, \quad \text{and} \quad \sqsubset_T = \bigcup_{(c,d) \in X \times X} \sqsubset_{cd}.$$

By Lemma 6, $T = (X, <_T, \sqsubset_T) \in \mathbf{S}$. Clearly, $a <_T b$. By (3) $T$ is saturated, i.e. $x <_T y \iff \neg(y \sqsubset_T x)$, or, equivalently $x \sqsubset_T y \iff x <_T^{\text{sim}} y$. To show that $T$ is also initially finite, observe that for any $g \in X$,

$$(W_T)^g \subseteq \bigcup_{(c,d) \in X \times X} \big(W_{S_{cd}}\big)^g = \bigcup_{d \in X} \left(\left(\bigcup_{c \lhd g} \big(W_{S_{cd}}\big)^g\right) \cup \big(W_{S_{gd}}\big)^g \cup \left(\bigcup_{g \lhd c} \big(W_{S_{cd}}\big)^g\right)\right).$$

By (2), we can prove that $\bigcup_{c \lhd g} (W_{S_{cd}})^g \subseteq (W_{S_{gd}})^g$. Suppose $x \in \bigcup_{g \lhd c} (W_{S_{cd}})^g$. If $x = g$, then $x \in (W_{S_{gd}})^g$. Else there is a pair $(c, d)$ with $g \lhd c$ such that $x \sqsubset_{S_{cd}} g$ which implies $\neg(g <_{cd} x)$. By (2), $\neg(g <_{gd} x)$, which in turn, by (3), implies that $x \sqsubset_{gd} g$. It follows that $x \in (W_{S_{gd}})^g$. Thus,

$$\bigcup_{g \lhd c} (W_{S_{cd}})^g \subseteq (W_{S_{gd}})^g.$$

Therefore

$$(W_T)^g \subseteq \bigcup_{d \in X} (W_{S_{gd}})^g \subseteq \bigcup_{h \unlhd g} (W_{S_0})^h. \tag{6}$$

Since there are only finitely many elements proceeding $g$ and $S_0$ is initially finite, we see that the last union in (6) is a finite set. This shows that $T$ is initially finite.

Finally, we show that $<_T$ is an extension of $G$. By construction of $T$, we have $\sqsubset \subseteq \sqsubset_T = <_T^{\sim}$. We only need to show that $<\!\!> \subseteq <_T^{\text{sym}} = (<_T \cup (<_T)^{-1})$. Suppose $c <\!\!> d$, then $c \neq d$. By (3), $c <_T d \vee d \sqsubset_T c$. If $c <_T d$, we are done. Otherwise $d \sqsubset_T c$. By (4), $<_T \cap <\!\!> = \sqsubset_T \cap <\!\!>$, hence $d <_T c$. So in any case, $c <_T d \vee d <_T c$. Hence we set $\prec = <_T$ and this completes the proof of Lemma 7 for *initially finite stratified* generalised order structures. For just *stratified* generalised order structure we proceed identically, just omitting all references to initial finiteness. □

Now we will show that the condition (1) of Lemma 5 is also satisfied.

**Lemma 8** *Let* $G = (X, \Leftrightarrow, \sqsubset) \in \mathbf{GS}_{IF}$ $(G \in \mathbf{GS})$ *and* $a, b \in X$ *be any distinct pair such that* $\neg(b \Leftrightarrow a)$. *Then there exists* $\prec \in SO_{IF}$ $(SO)$ *such that* $\prec$ *extends* $G$ *(i.e.* $\Leftrightarrow \subseteq \prec^{\text{sym}}$ *and* $\sqsubset \subseteq \prec^{\frown}$*) and* $a \prec^{\frown} b \wedge b \prec^{\frown} a$.

*Proof* The idea of the proof is the same as in the proof of Lemma 7. Since $\Leftrightarrow$ is symmetric, $\neg(b \Leftrightarrow a)$ implies $\neg(a \Leftrightarrow b)$. By definition, $S_G = (X, <_G, \sqsubset)$, where $<_G = \Leftrightarrow \cap \sqsubset$, is in $\mathbf{S}_{IF}$. We first show that there is an $S_0 = (X, <_0, \sqsubset_0) \in \mathbf{S}_{IF}$ such that $S_0$ extends $S$, $a \sqsubset_0 b$, $b \sqsubset_0 a$ and $<_0 \cap \Leftrightarrow = \sqsubset_0 \cap \Leftrightarrow$.

Define $S_0' \in \mathbf{S}_{IF}$ as follows: If $a \sqsubset b$, then $S_0' = S$; else $S_0' = (X, <_0', \sqsubset_0')$ where

$$<_0' = <_G \ \cup (Y_S)_a \times (Y_S)^b - \{(a, b)\},$$

and

$$\sqsubset_0' = \sqsubset \ \cup (Y_S)_a \times (Y_S)^b.$$

Clearly $S \subseteq S_0'$ and $a \sqsubset_0' b$. By virtually repeating the part of proof of Lemma 7 that shows "$S_0 \in \mathbf{S}_{IF}$" we show $S_0' \in \mathbf{S}_{IF}$. Moreover, it is clear that $<_0' \ \cap \ \Leftrightarrow = \sqsubset_0' \ \cap \ \Leftrightarrow$.

Define $S_0$ as follows: If $b \sqsubset_0' a$, then $S_0 = S_0'$; else $S_0 = (X, <_0, \sqsubset_0)$ where

$$<_0 = <_0' \ \cup \left(Y_{S_0'}\right)_b \times \left(Y_{S_0'}\right)^a - \{(b, a)\},$$

and

$$\sqsubset_0 = \sqsubset_0' \ \cup (Y_{S_0'})_b \times (Y_{S_0'})^a.$$

Again it is clear that $S_G \subseteq S_0' \subseteq S_0$ and $a \sqsubset_0 b$ and $b \sqsubset_0 a$. And a routine check, as in the beginning of the proof of Lemma 7, shows that $S_0' \in \mathbf{S}_{IF}$. Moreover, it is also clear that $<_0 \ \cap \ \Leftrightarrow = \sqsubset_0 \ \cap \ \Leftrightarrow$.

Now by the same construction as the one in the second part of the proof of Lemma 7 above (which is a refinement of the proof of Lemma 2.2 in [20]), we see that there exists $T \in \mathbf{S}_{IF}^{\text{sat}}$ such that $T = (X, <_T, \sqsubset_T)$ extends $G$ and $a \sqsubset_T b$ and $b \sqsubset_T a$. Since $T$ is saturated this implies $a <_T^{\frown} b$ and $b <_T^{\frown} a$. Hence we set $\prec = <_T$.

This completes the proof of Lemma 8 for *initially finite stratified* generalised order structures. For just *stratified* generalised order structures we proceed identically, just omitting all references to initial finiteness. □

Lemmas 7 and 8 imply that the necessary and sufficient conditions in Lemma 5 for extension completeness are satisfied by the class $\mathbf{GS}_{IF}$ and $\mathbf{GS}$. Therefore we can formulate the following theorem—the main result of this subsection.

**Theorem 3** *The classes of generalised ordered structures* $\mathbf{GS}$ *and* $\mathbf{GS}_{IF}$ *are extension complete.*

*Proof* From Lemmas 5,7 and 8. □

### 6.2 Generalised interval order structures

The triple $(\{a, b, c, d\}, <>^I_8, \sqsubset^I_8)$ from Fig. 7 is an example of a generalised interval order structure which is not stratified.

It seems the technique used in previous section is not enough to prove an analogue of Theorem 3 for generalised interval order structures. To do that we need Theorem 1 (Fishburn's Theorem [14,18]) and the following results from [1,14,38].

**Lemma 9** [38] *Let $po = (X, <)$ be a partial order and $a, b \in X$, $a \sim b$. Define $Y = \{a\} \cup \{y \mid y < a\}$, $Z = \{b\} \cup \{z \mid b < z\}$. Then $(X, <^{ab}) = (X, < \cup\, Y \times Z)$ is a partial order and $a <^{ab} b$.* □

**Theorem 4** [1] *Let $S = (X, <, \sqsubset)$ be an interval order structure. Then there is a partial order $(T, \prec)$ and two mappings $\varphi, \psi : X \to T$ such that $T = \varphi(X) \cup \psi(X)$, and for all distinct $a, b \in X$,*

1. $\varphi(a) \prec \psi(a)$,
2. $\{\varphi(a), \psi(a)\} \cap \{\varphi(b), \psi(b)\} = \emptyset$,
3. $a < b \iff \psi(a) \prec \varphi(b)$,
4. $a \sqsubset b \iff \varphi(a) \prec \psi(b)$. □

The above theorem can be seen as a generalisation of Fishburn's Theorem (Theorem 1) to order structures. It turns out that an analogue of Theorem 4 also holds for initially finite interval order structures.

**Theorem 5** *Let $S = (X, <, \sqsubset)$ be an* initially finite *interval order structure. Then there is an* initially finite *partial order $(T, \prec)$ and two mappings $\varphi, \psi : X \to T$ such that $T = \varphi(X) \cup \psi(X)$, and for all distinct $a, b \in X$,*

1. $\varphi(a) \prec \psi(a)$,
2. $\{\varphi(a), \psi(a)\} \cap \{\varphi(b), \psi(b)\} = \emptyset$,
3. $a < b \iff \psi(a) \prec \varphi(b)$,
4. $a \sqsubset b \iff \varphi(a) \prec \psi(b)$.

*Proof* Let $(T, \prec)$ be a poset from Theorem 4 and let $y \in T$. Then there is $b \in X$ such that $y = \varphi(b)$ or $y = \psi(b)$. Since $\varphi(b) \prec \psi(b)$, it suffices to consider the case $y = \psi(b)$. Since $S$ is initially finite, then $\{a \mid a <^\frown b\} = \{a \mid \neg(b < a)\}$ is finite. We also have

$$\neg(b < a) \iff \neg(\psi(a) \prec \varphi(b)) \iff \varphi(a) \prec \psi(b) \vee \varphi(a) \sim \psi(b),$$

which means if $\{a \mid \neg(b < a\}$ is finite, the set

$$\{\varphi(a) \mid \varphi(a) \prec \psi(b)\} \cup \{\varphi(a) \mid \varphi(a) \sim \psi(b)\}$$

is also finite.

We will show that the set $\{x \mid x \prec^\frown y\}$ is finite.
Since $T = \varphi(X) \cup \psi(X)$ and $y = \psi(b)$, we have

$$\{x \mid x \prec^\frown y\} = \{\varphi(a) \mid \varphi(a) \prec \psi(b)\} \cup \{\psi(a) \mid \psi(a) \prec \psi(b)\} \cup \{\varphi(a) \mid \varphi(a) \sim \psi(b)\} \cup \{\psi(a) \mid \psi(a) \sim \psi(b)\}.$$

We have already shown that the set $\{\varphi(a) \mid \varphi(a) \prec \psi(b)\} \cup \{\varphi(a) \mid \varphi(a) \sim \psi(b)\}$ is finite.

Consider the set $\{\psi(a) \mid \psi(a) \prec \psi(b)\}$.
Since $\varphi(a) \prec \psi(a)$ then we have $\psi(a) \prec \psi(b) \Rightarrow \varphi(a) \prec \psi(b)$, so the cardinality of the set $\{\psi(a) \mid \psi(a) \prec \psi(b)\}$ is smaller than or equal to the cardinality of the set $\{\varphi(a) \mid \varphi(a) \prec \psi(b)\}$, i.e. $\{\psi(a) \mid \psi(a) \prec \psi(b)\}$ is finite. Consider now the set $\{\psi(a) \mid \psi(a) \sim \psi(b)\}$. Since $\varphi(a) \prec \psi(a)$, then

$$\psi(a) \sim \psi(b) \Rightarrow \neg(\psi(b) \prec \varphi(a)) \Leftrightarrow \varphi(a) \prec \psi(b) \vee \varphi(a) \sim \psi(b).$$

Hence the cardinality of $\{\psi(a) \mid \psi(a) \sim \psi(b)\}$ is smaller than the cardinality of $\{\varphi(a) \mid \varphi(a) \prec \psi(b)\} \cup \{\varphi(a) \mid \varphi(a) \sim \psi(b)$, which means $\{\psi(a) \mid \psi(a) \sim \psi(b)\}$ is also finite, which ends the proof. □

The results of Theorems 4 and 5 can be extended to generalized structures.

- A relation $R \subseteq X \times X$ is *acyclic* if and only if for all $x \in X$, $\neg(xR^*x)$.

**Theorem 6** *Let $G = (X, \Leftrightarrow, \sqsubset)$ be a generalised interval order structure. Then there is a partial order $(T, \prec)$, two mappings $\varphi, \psi : X \to T$, and an acyclic relation $\lhd \subseteq T \times T$ such that $T = \varphi(X) \cup \psi(X)$, and for all distinct $a, b \in X$ and $x, y \in T$,*

1. $\varphi(a) \prec \psi(a)$,
2. $\{\varphi(a), \psi(a)\} \cap \{\varphi(b), \psi(b)\} = \emptyset$,
3. $x \prec y \ \Rightarrow\ x \lhd y$,
4. $a \Leftrightarrow b \iff \psi(a) \lhd \varphi(b) \vee \psi(b) \lhd \varphi(a)$
5. $a <_G b \iff \psi(a) \prec \varphi(b)$, *where* $<_G = \Leftrightarrow \cap \sqsubset$
6. $a \sqsubset b \iff \varphi(a) \prec \psi(b)$.

*Furthermore, if $G$ is initially finite then $(T, \prec)$ is also initially finite.*

*Proof* Let $<_G = \Leftrightarrow \cap \sqsubset$, and let $<_t$ be any total extension of $<_G$. Define a relation $R$ on $X$ as $R = \Leftrightarrow \cap <_t$. Note that $\Leftrightarrow = R \cup R^{-1}$, $<_G \subseteq R$ and $R \setminus <_G = R \setminus \sqsubset$. Let $T = (X, \prec)$ be a partial order given by Theorem 4. Then (1), (2), (5) and (6) above are satisfied. We define the relation $\lhd_R$ as follows

$$x \lhd_R y \iff (\exists a, b \in X.\ x = \psi(a) \wedge y = \varphi(b) \wedge aRb \wedge \neg(a <_G b)).$$

First note that $\psi(a) \neq \varphi(b)$ since $aRb$ implies $a \neq b$, so $\lhd_R$ is irreflexive. We also have $x \lhd_R y \Rightarrow \neg(a <_G b) \Leftrightarrow \neg(\psi(a) \prec \varphi(b)) \Leftrightarrow \neg(x \prec y)$, hence $\lhd_R \cap \prec = \emptyset$. Assume that $x \lhd_R y \wedge y \prec x$. This implies $y = \varphi(b) \prec x = \psi(a)$, and, by (6) above, $b \sqsubset a$. By $R \subseteq \Leftrightarrow$, we also have $a \Leftrightarrow b$, so altogether we have $b \sqsubset a \wedge \neg(a <_G b) \wedge a \Leftrightarrow b$, which imply $b <_G a$. But $b <_G a$ implies $bRa$, so we have $aRb$ and $bRa$, a contradiction as $R \subseteq <_t$. Hence $x \lhd_R y \Rightarrow \neg(y \prec x)$, i.e. $\lhd_R \cap \prec^{\text{sym}} = \emptyset$. Similarly we show $\lhd_R^{-1} \cap \prec^{\text{sym}} = \emptyset$, hence $\lhd_R^{\text{sym}} \cap \prec^{\text{sym}} = \emptyset$. Now we show that for all $x, y, z \in T$, $\neg(x \lhd_R y \lhd_R z)$. Note that $x \lhd_R y \lhd_R z$ implies $\exists a_x, b_y, a_y, b_z \in X.\ x = \psi(a_x) \wedge y = \varphi(b_y) = \psi(a_y) \wedge z = \psi(b_z)$. By Theorem 4 we have $\varphi(a) \prec \psi(a)$ for all $a \in X$, and $a \neq b \Rightarrow \{\varphi(a), \psi(a)\} \cap \{\varphi(b), \psi(b)\} = \emptyset$ for all $a, b \in X$, so the statement $\exists a_y, b_y \in X.\ y = \varphi(a_y) = \psi(b_y)$ is false. Hence the relation $\lhd_R$ is acyclic. Define the relation $\lhd$ on $T$ as follows:

$$x \lhd y \iff x \prec y \vee x \lhd_R y.$$

Since both $\prec$ and $\lhd$ are acyclic ($\prec$ is a partial order) and $\lhd_R^{\text{sym}} \cap \prec^{\text{sym}} = \emptyset$, then $\lhd$ is acyclic as well, and furthermore (3) above is satisfied.
Clearly $aRb \iff ((aRb \wedge \neg(a <_G b)) \vee a <_G b)$, which means

$$aRb \iff \psi(a) \prec \varphi(b) \vee \psi(a) \lhd_R \varphi(b) \iff \psi(a) \lhd \varphi(b).$$

Since $<\!\!> \; = \; R \cup R^{-1}$, we have proved point (4) of this Theorem. This means all (1)–(6) are satisfied. Initial finiteness of $(T, \prec)$ follows from Theorem 5. □

We can now show extension completeness of generalised interval order structures.

**Theorem 7** *The classes of generalised interval order structures* **GI** *and of generalised initially finite interval order structures* $\mathbf{GI}_{IF}$*, are extension complete.*

*Proof* Let $G = (X, <\!\!>, \sqsubset)$ be a generalised interval order structure and let $\Omega$ be a set of interval orders that extend $G$. First we show that $\Omega \neq \emptyset$. Let $(T, \prec)$ be a partial order from Theorem 6, and let $(T, \prec_t)$ be any total order satisfying $x \lhd y \Rightarrow x \prec_t y$, where $\lhd$ is a relation given by Theorem 6. Since $\lhd$ is acyclic, such $\prec_t$ always exists. Clearly $x \prec y \Rightarrow x \lhd y \Rightarrow x \prec_t y$. Define $po = (X, <_p)$ as $a <_p b \iff \psi(a) \prec_t \varphi(b)$. By Theorem 1, $<_p$ is an interval order. Now, let $a, b \in X$. We have

$$\begin{aligned} & a <\!\!> b \Rightarrow \psi(a) \lhd \varphi(b) \vee \psi(b) \lhd \varphi(a) \\ & \quad \Rightarrow \psi(a) \prec_t \varphi(b) \wedge \psi(b) \prec_t \varphi(a) \Rightarrow a <_p b \vee b <_p a, \\ & a \sqsubset b \Rightarrow \varphi(a) \prec \psi(b) \Rightarrow \varphi(a) \prec_t \psi(b) \\ & \quad \Leftrightarrow \neg(\psi(b) \prec_t \varphi(a)) \Leftrightarrow \neg(b <_p a) \Rightarrow a <_p^{\frown} b. \end{aligned}$$

Hence $\Omega$ is non-empty. We now show that the conditions (1) and (2) of Lemma 5 are satisfied.

Define $<_G = <\!\!> \cap \sqsubset$ and let $\neg(a <\!\!> b)$. Since $<_G^{\mathrm{sym}} \subseteq <\!\!>$, we have $\neg(a <_G^{\mathrm{sym}} b)$, i.e. $a \sim_G b$. We will consider two cases $(a \sqsubset b \wedge b \sqsubset a)$ and $\neg(a \sqsubset b \wedge b \sqsubset a)$.

*Case* $(a \sqsubset b \wedge b \sqsubset a)$. In this case then any element $<_T$ of $\Omega$ satisfies $a <_T^{\frown} b$ and $b <_T^{\frown} a$, i.e. $a \sim_T b$, so (1) of Lemma 5 holds.

*Case* $\neg(a \sqsubset b \wedge b \sqsubset a)$. In this case by Theorem 6, we have

$$a \sim_G b \iff \psi(a) \sim \varphi(b) \wedge \varphi(a) \sim \psi(b).$$

Define the partial order $\hat{\prec} = (\prec^{\varphi(b)\psi(a)})^{\varphi(a)\psi(b)}$ by applying the construction from Lemma 9 twice. By Lemma 9 we now have $\varphi(a)\hat{\prec}\psi(b)$ and $\varphi(b)\hat{\prec}\psi(a)$. Let $(T, \hat{\prec}_t)$ be any total extension of $(T, \hat{\prec})$. Define the poset $po = (X, <_p)$ by

$$c <_p d \iff \psi(c)\hat{\prec}_t\varphi(d).$$

By Theorem 1, $po$ is an interval order, and proceeding similarly as at the beginning of this proof, one may show that $po \in \Omega$. Moreover we have

$$\begin{aligned} & \varphi(a)\hat{\prec}_t\psi(b) \Rightarrow \neg(\psi(b)\hat{\prec}_t\varphi(a)) \Rightarrow \neg(b <_p a) \Leftrightarrow a <_p^{\frown} b, \\ & \varphi(b)\hat{\prec}_t\psi(a) \Rightarrow \neg(\psi(b)\hat{\prec}_t\varphi(a)) \Rightarrow \neg(a <_p b) \Leftrightarrow b <_p^{\frown} a, \end{aligned}$$

and so Lemma 5(1) is also satisfied in this case.

Suppose $\neg(a \sqsubset b)$. Hence $\neg(a <_G b)$, i.e. $b <_G a \vee a \sim_G b$. If $b <_G a$ then any element $<_T$ from $\Omega$ satisfies $b <_T a$, so Lemma 5(2) holds. Assume that $a \sim_G b$. From Theorem 6,

$$\begin{aligned} & \neg(a \sqsubset b) \Leftrightarrow \neg(\varphi(a) \prec \psi(b)) \\ & \quad \Leftrightarrow (\psi(b) \prec \varphi(a) \vee \varphi(a) \sim \psi(b)) \\ & \quad \Leftrightarrow b <_G a \vee \varphi(a) \sim \psi(b)). \end{aligned}$$

This means in this case we have:

$$a \sim_G b \Rightarrow \varphi(a) \sim \psi(b)).$$

Define the partial order $\dot{\prec} = \prec^{\psi(b)\varphi(a)}$ on $T$ using the construction from Lemma 9. By Lemma 9 we have $\psi(b)\dot{\prec}\varphi(a)$. Let $(T, \dot{\prec}_t)$ be any total extension of $(T, \dot{\prec})$. Define the poset $qo = (X, <_q)$ by

$$c <_q d \iff \psi(c)\dot{\prec}_t\varphi(d).$$

By Theorem 1, $qo$ is an interval order, and proceeding similarly as at the beginning of this proof, one may show that $qo \in \Omega$. In this case we have:

$$\psi(b)\dot{\prec}_t\varphi(a) \Rightarrow b <_q a,$$

and so Lemma 5(2) is satisfied.

Thus, by Lemma 5, the class **GI** is extension complete. Now suppose that $G$ is also initially finite. From Theorem 6 the partial order $(T, \prec)$ is initially finite, and from the definition of initial finiteness every total extension is also initially finite. Consider the poset $po$ from the first part of the proof, i.e. $a <_p b \iff \psi(a) \prec_t \varphi(b)$. Note that $a <_p^{\frown} b \iff \neg(b <_p a) \iff \varphi(b) \prec_t \psi(a)$. Hence $\{a \mid a <_p^{\frown} b\}$ is initially finite because $\{x \mid x \prec_t \psi(a)\}$ is finite. Similarly for the remaining constructions. Therefore the entire proof can be formulated using initially finite posets only. Thus $\mathbf{GI}_{IF}$ is also extension complete. □

The proof of Theorem 7 is less straightforward and elegant than the proof of Theorem 3, but it is also mathematically less sophisticated and much easier to follow. Unfortunately this technique cannot be applied to generalised stratified order structures since an equivalence of Theorem 4 does not exist for stratified order structures (see [18]).

### 6.3 Generalised partial order structures

An equivalence of Theorem 4 does not exist for general partial orders either, but the technique used for generalised stratified order structures can be applied in this case. We will show that conditions (1) and (2) of Lemma 5 are satisfied, using very similar reasoning as in the case of stratified order structures. We start with proving condition (2).

**Lemma 10** *Let $G = (X, <>, \sqsubset) \in \mathbf{GP}_{IF}$ ($G \in \mathbf{GP}$) and let $a, b \in X$ be any two distinct elements such that $\neg(b \sqsubset a)$. Then there exists $\prec \in PO_{IF}$ ($\prec \in PO$) such that $\prec$ extends $G$ (i.e. $<> \subseteq \prec^{\text{sym}}$ and $\sqsubset \subseteq \prec^{\frown}$) and $a \prec b$.*

*Proof* The proof is very similar to the proof of Lemma 7. We will only present the parts that are different. By Proposition 1(1), it suffices to show there exists a $T = (X, <_T, \sqsubset_T) \in \mathbf{P}_{IF}^{\text{sat}}$ such that $(X, <_T^{\text{sym}}, \sqsubset_T)$ extends $G$ and $x \prec y \iff x <_T y$. By definition, $S_G = (X, <_G, \sqsubset)$, where $<_G = <> \cap \sqsubset$, is in $\mathbf{P}_{IF}$. Consider the order structure $S_0 = (X, <_0, \sqsubset_0)$ where

$$<_0 = <_G \cup (W_S)_a \times (W_S)^b,$$

and

$$\sqsubset_0 = \sqsubset \cup (W_S)_a \times (W_S)^b.$$

Clearly $a <_0 b$ and $<_0 \cap <> = \sqsubset_0 \cap <>$. We show that in this case $S_0$ satisfies P1–P4 and is initially finite.

P1 and P2 hold trivially. To show that P3 holds, suppose that $x <_0 y <_0 z$. Since we have $\neg(b \sqsubset a)$ and $(W_S)_a \cap (W_S)^b = \emptyset$,

$$(x, y) \in (W_S)_a \times (W_S)^b \wedge (y, z) \in (W_S)_a \times (W_S)^b$$

is impossible. Thus we have either

(i) $x < y < z$, or
(ii) $x < y \wedge (y, z) \in (W_S)_a \times (W_S)^b$, or
(iii) $(x, y) \in (W_S)_a \times (W_S)^b \wedge y < z$.

If (i) holds, then $x < z$, thus $x <_0 z$.
If (ii) holds, we have $x < y < a$, which implies $x < a$, i.e. $x \in (W_S)_a$. Hence $(x, z) \in (W_S)_a \times (W_S)^b$, therefore $x <_0 z$.
Similarly, if (iii) holds, then $z \in (W_S)^b$ and $(x, z) \in (W_S)_a \times (W_S)^b$, so $x <_0 z$.
So in any case, $x <_0 y <_0 z \Longrightarrow x <_0 z \vee x = z$, that is P3 holds.

The proof of P4 is very similar. Suppose $x \sqsubset_0 y <_0 z$. Since $\neg(b \sqsubset a)$ and $(W_S)_a \cap (W_S)^b = \emptyset$, then

$$(x, y) \in (W_S)_a \times (W_S)^b \wedge (y, z) \in (W_S)_a \times (W_S)^b$$

does not hold. Thus we have either

(i) $x \sqsubset y < z$, or
(ii) $x \sqsubset y \wedge (y, z) \in (W_S)_a \times (W_S)^b$, or
(iii) $(x, y) \in (W_S)_a \times (W_S)^b \wedge y < z$.

If (i) holds, then $x \sqsubset z$, thus we have $x <\sqsubset_0 z$.
If (ii) holds, we have $x \sqsubset y < a$, which implies $x \sqsubset a$, hence $x \in (W_S)_a$. Then $(x, z) \in (W_S)_a \times (W_S)^b$, therefore $x \sqsubset_0 z$.
Similarly, if (iii) holds, then $z \in (W_S)^b$ and $(x, z) \in (W_S)_a \times (W_S)^b$, so $x \sqsubset_0 z$.
So in any case we have $x \sqsubset_0 y <_0 z \Rightarrow x \sqsubset_0 z$.
The proof of the case $x <_0 y \sqsubset_0 z$ is almost the same, hence P4 holds.

Finally, it is easy to see that for any $h \in X$,

$$(W_{S_0})^h \subseteq (W_S)^h \cup (W_S)^a.$$

Therefore $S_0$ is also initially finite. The rest of the proof is an exact copy of the appropriate part of the proof of Lemma 7. □

Now we will show that condition (1) of Lemma 5 is also satisfied.

**Lemma 11** *Let $G = (X, <\!\!>, \sqsubset) \in \mathbf{GP}_{IF}$ ($G \in \mathbf{GP}$) and let $a, b \in X$ be any distinct pair such that $\neg(b <\!\!> a)$. There exists $\prec \in PO_{IF}$ ($PO$) such that $\prec$ extends $G$ (i.e. $<\!\!> \subseteq \prec^{\text{sym}}$ and $\sqsubset \subseteq \prec^{\frown}$) and $a \prec^{\frown} b \wedge b \prec^{\frown} a$.*

*Proof* This is very similar to the proof of Lemma 8. Since $<\!\!>$ is symmetric, we have $\neg(b <\!\!> a)$ implies $\neg(a <\!\!> b)$. By definition, $S_G = (X, <_G, \sqsubset)$, where $<_G = <\!\!> \cap \sqsubset$, is in $\mathbf{P}_{IF}$. We first show that there is an $S_0 = (X, <_0, \sqsubset_0) \in \mathbf{P}_{IF}$ such that $S_0$ extends $S$, $a \sqsubset_0 b$, $b \sqsubset_0 a$ and $<_0 \cap <\!\!> = \sqsubset_0 \cap <\!\!>$.

Define $S'_0 \in \mathbf{S}_{IF}$ as follows: If $a \sqsubset b$, then $S'_0 = S$; else $S'_0 = (X, <'_0, \sqsubset'_0)$ where

$$<'_0 = <_G \cup (Y_S)_a \times (Y_S)^b - \{(a, b)\},$$

and

$$\sqsubset'_0 = \sqsubset \cup (Y_S)_a \times (Y_S)^b.$$

Clearly $S \subseteq S'_0$ and $a \sqsubset'_0 b$. By virtually repeating the part of proof of Lemma 10 that shows "$S_0 \in \mathbf{P}_{IF}$", we show that $S'_0 \in \mathbf{P}_{IF}$. Moreover, it is also clear that $<'_0 \cap <\!\!> = \sqsubset'_0 \cap <\!\!>$.

Define $S_0$ as follows: If $b \sqsubset'_0 a$, then $S_0 = S'_0$; else $S_0 = (X, <_0, \sqsubset_0)$ where

$$<_0 = <'_0 \cup \left(Y_{S'_0}\right)_b \times \left(Y_{S'_0}\right)^a - \{(b, a)\},$$

and

$$\sqsubset_0 = \sqsubset_0' \cup \left(Y_{S_0'}\right)_b \times \left(Y_{S_0'}\right)^a.$$

Again it is clear that $S_G \subseteq S_0' \subseteq S_0$ and $a \sqsubset_0 b$ and $b \sqsubset_0 a$. And a routine check, as the one at the beginning of the proof of Lemma 10, shows that $S_0' \in \mathbf{P}_{IF}$. Moreover, it is clear that $<_0 \cap \mathrel{<\!\!>} = \sqsubset_0 \cap \mathrel{<\!\!>}$. The rest of the proof is an exact copy of the appropriate part of the proof of Lemma 8. □

Lemmas 10 and 11 imply that the necessary and sufficient conditions in Lemma 5 for extension completeness are satisfied by the classes $\mathbf{GP}_{IF}$ and $\mathbf{GP}$. Therefore the we can formulate the following Theorem.

**Theorem 8** *The classes of generalised ordered structures* $\mathbf{GP}$ *and* $\mathbf{GP}_{IF}$ *are extension complete.*

*Proof* From Lemmas 5, 10 and 11. □

### 6.4 Generalised total order structures

While all classes of order structures considered in Sect. 3 are extension complete, for generalised order structures this is true only for **GS**, **GI** and **GP**. Total order structures from **GT** *are not always* extension complete. The total order structure $G_1$ of Fig. 1 is extension complete, while $G_7$ from Example 6 is not extension complete when treated as an element of **GT**.

**Proposition 4** *A generalised total order structure* $G = (X, \mathrel{<\!\!>}, \sqsubset)$ *is extension complete if and only if* $\mathrel{<\!\!>} = \{(x, y) \mid x, y \in X \wedge x \neq y\}$.

*Proof* If $G = (X, \mathrel{<\!\!>}, \sqsubset)$ is a generalised total order structure then $\sqsubset = \prec_G$ is a partial order.
($\Rightarrow$) A saturated generalised total order structure is of the form $(X, \prec^{\text{sym}}, \prec)$ where $\prec$ is a total order. But for every total order $\prec$ on $X$, we have

$$\prec^{\text{sym}} = \{(x, y) \mid x, y \in X \wedge x \neq y\}.$$

Hence $\mathrel{<\!\!>} = \bigcap_{T \in ext^{\text{sat}}_{\mathbf{GT}}(G)} \prec_T^{\text{sym}} = \{(x, y) \mid x, y \in X \wedge x \neq y\}$.
($\Leftarrow$) If $\mathrel{<\!\!>} = \{(x, y) \mid x, y \in X \wedge x \neq y\}$ and $\sqsubset = \prec_G$ is a partial order, then the triple $(X, \mathrel{<\!\!>}_T, \prec_T)$ belongs to $ext^{\text{sat}}_{\mathbf{GT}}(G)$ if and only if $\mathrel{<\!\!>}_T = \{(x, y) \mid x, y \in X \wedge x \neq y\}$ and $\prec_T$ is a total extension of $\prec_G$. Thus $G$ is extension complete. □

Since every generalised total order structure also belongs to **GS**, if $\mathrel{<\!\!>} \neq \{(x, y) \mid x, y \in X \wedge x \neq y\}$ then $ext^{\text{sat}}_{\mathbf{GT}}(G) \subset ext^{\text{sat}}_{\mathbf{GS}}(G)$, and from Theorem 3,

$$G = \bigcap_{T \in ext^{\text{sat}}_{\mathbf{GS}}(G)} T,$$

i.e. $G$ is extension complete in **GS**. For instance for $G_7$ from Fig. 6, we have $ext^{\text{sat}}_{\mathbf{GT}}(G_7) = \{abcd, abdc, acbd\} \subset ext^{\text{sat}}_{\mathbf{GS}}(G_7) = \{abcd, abdc, acbd, ab\{c, d\}\}$, and $G_7$ is extension complete when treated as a generalised stratified order structure.

Finally note that even extension complete generalised order structures are *not* partial orders in disguise, as opposed to total order structures from the previous section. The generalised total order structure $G_1$ from Fig. 1 is extension complete, but it is not tantamount to any partial order.

In this entire section the triples $(X, \mathrel{<\!\!>}, \sqsubset)$ *do not have any special interpretation*, and the results can be seen as another extension of Szpilrajn's ideas [38] to partial orders that are not necessarily total, and this time no assumption about the sets of partial orders is made.

## 7 Generalised order structures and concurrent histories

Structurally the relationship between Generalised Order Structures and Concurrent Histories is similar to that of between Order Structures and Concurrent Histories which was discussed in Sect. 5. The only substantial difference is the case of Generalised Total Order Structures.

First note that if a generalised order structure $G = (X, <>, \sqsubset)$ satisfies $<> = \prec_G^{\text{sym}}$ where $\prec_G = <> \cap \sqsubset$, then each property of $G$ is fully described by its induced order structure $S_G = (X, \prec_G, \sqsubset)$. In particular we have the following result.

**Corollary 9** *Let $G = (X, <>, \sqsubset) \in \Theta$ be a generalised order structure and let $<_G = < \cap \sqsubset$, $S_G = (X, <_G, \sqsubset)$. If $<> = <_G^{\text{sym}}$, then*

$$(X, \prec_e^{\text{sym}}, \prec_e^{\frown}) \in ext_\Theta^{\text{sat}}(G) \iff (X, \prec_e, \prec_e^{\frown}) \in ext_\Theta^{\text{sat}}(S_G).$$

□

The above result together with Lemma 3(1) and Proposition 2 indicate that when the paradigm $\pi_3$ is satisfied, we do not need generalized order structures at all, the modelling power of order structures from Sect. 3 is sufficient. If the paradigm $\pi_6$ holds we only need generalized total order structures.

**Proposition 5** *Let $\Delta$ be a concurrent history with a domain $X$ conforming to $\pi_6$ in $O$, where $O$ is any class of partial orders. Then the triple $G = (X, <>_\Delta, \sqsubset_\Delta)$ is a generalised total order structure*

*Proof* From the definitions of $<>_\Delta$, $\sqsubset_\Delta$ and Lemma 3(3). □

Because the class of total order structures is not extension complete, the counterpart of Corollary 4 is a bit more elaborate. It can be formulated as follows.

**Corollary 10**

1. *Let $\Theta_1, \Theta_2 \in \{\mathbf{GS}, \mathbf{GI}, \mathbf{GP}, \mathbf{GS}_{IF}, \mathbf{GI}_{IF}, \mathbf{GP}_{IF}\}$, $\Theta_1 \subseteq \Theta_2$, and let the generalised order structure $G = (X, <>, \sqsubset) \in \Theta_1$. Then*
   (a) $ext_{\Theta_1}^{\text{sat}}(G) \subseteq ext_{\Theta_2}^{\text{sat}}(G)$, *and* $S = \bigcap_{T \in ext_{\Theta_1}^{\text{sat}}(G)} T = \bigcap_{T \in ext_{\Theta_2}^{\text{sat}}(G)} T$.
   (b) $S \asymp ext_{\Theta_1}^{\text{sat}}(G)$ *and* $S \asymp ext_{\Theta_2}^{\text{sat}}(G)$.
2. *Let $\Theta_1 \in \{\mathbf{GT}, \mathbf{GT}_{IF}\}$, $\Theta_2 \in \{\mathbf{GS}, \mathbf{GI}, \mathbf{GP}, \mathbf{GS}_{IF}, \mathbf{GI}_{IF}, \mathbf{GP}_{IF}\}$, and $\Theta_1 \subseteq \Theta_2$. If $G = (X, <>, \sqsubset) \in \Theta_1$ is extension complete, then*
   (a) $ext_{\Theta_1}^{\text{sat}}(G) \subseteq ext_{\Theta_2}^{\text{sat}}(G)$, *and* $S = \bigcap_{T \in ext_{\Theta_1}^{\text{sat}}(G)} T = \bigcap_{T \in ext_{\Theta_2}^{\text{sat}}(G)} T$.
   (b) $S \asymp ext_{\Theta_1}^{\text{sat}}(G)$ *and* $S \asymp ext_{\Theta_2}^{\text{sat}}(G)$.
3. *Let $\Theta_1 \in \{\mathbf{GT}, \mathbf{GT}_{IF}\}$, $\Theta_2 \in \{\mathbf{GS}, \mathbf{GI}, \mathbf{GP}, \mathbf{GS}_{IF}, \mathbf{GI}_{IF}, \mathbf{GP}_{IF}\}$, and $\Theta_1 \subseteq \Theta_2$. If $G = (X, <>, \sqsubset) \in \Theta_1$ is* not *extension complete, then*
   (a) $ext_{\Theta_1}^{\text{sat}}(G) \subset ext_{\Theta_2}^{\text{sat}}(G)$, *and* $\bigcap_{T \in ext_{\Theta_1}^{\text{sat}}(G)} T \neq S = \bigcap_{T \in ext_{\Theta_2}^{\text{sat}}(G)} T$.
   (b) $S \not\asymp ext_{\Theta_1}^{\text{sat}}(G)$ *but* $S \asymp ext_{\Theta_2}^{\text{sat}}(G)$. □

For each class of generalised order structures $\Theta \in \{\mathbf{GT}, \mathbf{GS}, \mathbf{GI}, \mathbf{GP}, \mathbf{GT}_{IF}, \mathbf{GS}_{IF}, \mathbf{GI}_{IF}, \mathbf{GP}_{IF}\}$, let $po(\Theta)$ denotes the class of partial orders with the corresponding name. For instance $po(\mathbf{GS}_{IF}) = SO_{IF}$, $po(\mathbf{GI}) = IO$, etc. *For the rest of this subsection, we will interpret the elements of $ext_\Theta^{\text{sat}}(G)$ as partial orders if $G$ is in one of the classes mentioned above* (see comments after Propositions 1 and 3).

We have the following analogue of Proposition 2 for generalised order structures.

**Proposition 6**

1. *Let $\Theta \in \{\mathbf{GS}, \mathbf{GI}, \mathbf{GP}, \mathbf{GS}_{IF}, \mathbf{GI}_{IF}, \mathbf{GP}_{IF}\}$ be a class of generalised order structures. For each $G \in \Theta$, the set $ext^{\mathrm{sat}}_{\Theta}(G)$ is a concurrent history in $po(\Theta)$ that conforms to $\pi_1$.*
2. *Let $\Theta \in \{\mathbf{GT}, \mathbf{GT}_{IF}\}$. If $G \in \Theta$ is extension complete, then the set $ext^{\mathrm{sat}}_{\Theta}(G)$ is a concurrent history in $po(\Theta)$ that conforms to $\pi_6$.*
3. *Let $\Theta_1 \in \{\mathbf{GT}, \mathbf{GT}_{IF}\}$, $\Theta_2 \in \{\mathbf{GS}, \mathbf{GI}, \mathbf{GP}, \mathbf{GS}_{IF}, \mathbf{GI}_{IF}, \mathbf{GP}_{IF}\}$, and $\Theta_1 \subseteq \Theta_2$. If $G \in \Theta_1$ is* not *extension complete, then the set $ext^{\mathrm{sat}}_{\Theta_2}(G)$ is a concurrent history in $po(\Theta_2)$ that conforms to $\pi_6$.*

*Proof* 1. From the definition of extension completeness and Theorems 3, 7 and 8.
2. From the definition of extension completeness and Proposition 4.
3. From part (1) and Corollary 10(3). □

When using Proposition 6 it is important to keep Corollary 10 in mind. Let us now analyse appropriate examples from Examples 1–8. The generalised order structure $G_1 = (\{a, b, c\}, <>_1, \sqsubset_1)$ from Fig. 1 that corresponds to the program $P_1$ is extension complete and belongs to $\mathbf{GT}_{IF}$ and $ext^{\mathrm{sat}}_{\mathbf{GT}_{IF}}(G_1) = obs(P_1) = \{abc, acb\}$.

For the program $P_7$ of Example 6 we have $G_7 = (\{a, b, c, d\}, <>^S_7, \sqsubset^S_7)$, and $G_7 \in \mathbf{GT}_{IF} \subset \mathbf{GS}_{IF}$ is *not* extension complete ($<>^S_7$ does not satisfy Proposition 4) if viewed as an element of $\mathbf{GT}_{IF}$. However, it is extension complete if viewed as an element of $\mathbf{GS}_{IF}$, and $ext^{\mathrm{sat}}_{\mathbf{GS}_{IF}}(G_7) = obs_{SO}(P_7) = \{abcd, abdc, acbd, ab\{c, d\}\} = obs_{IO}(P_7)$. It can easily be checked that the concurrent history $ext^{\mathrm{sat}}_{\mathbf{GS}_{IF}}(G_7)$ conforms to the paradigm $\pi_6$.

The generalised order structure $G^I_8 = (\{a, b, c, d, e\}, <>^I_8, \sqsubset^I_8)$ that corresponds to the program $P_8$ from Example 7 belongs to $\mathbf{GI}_{IF}$ but not to $\mathbf{GS}_{IF}$, and we have $ext^{\mathrm{sat}}_{\mathbf{GI}_{IF}}(G_8) = obs_{IO}(P_8)$.

However, the generalised order structure $G^S_8 = (\{a, b, c, d, e\}, <>^S_8, \sqsubset^S_8)$ that also corresponds to the program $P_8$ from Example 7 under the assumption that only stratified orders are allowed as observations, belongs to $\mathbf{GS}_{IF} \subset \mathbf{GI}_{IF}$, and $ext^{\mathrm{sat}}_{\mathbf{GS}_{IF}}(G_8) = obs_{SO}(P_8)$.

An example of a generalised partial order structure that is not a generalised interval order structure is analysed in Example 8 (the triple $G_P$). We have $G_P \in \mathbf{GP}_{IF} \setminus \mathbf{GI}$. The set $\Delta$ from Fig. 5 is technically a concurrent history (it satisfies Definition 1); however, its elements cannot be interpreted as observations. As we mentioned when analysing Example 5, it was argued in [18] that *an observation must always be an initially finite interval order*, and *neither* of $o_i \in \Delta$, $i = 1, \ldots, 4$ is an interval order. Hence we cannot provide any concurrent system that generates $\Delta$ from Fig. 8.

## 8 Final comments

In this paper, we refined the notion of generalised structures introduced in [13], and proved that the classes of *stratified, interval and partial* generalised order structures *are extension complete*, while *total* generalised order structures are extension complete only if the relation $<>$ has a special form. The total generalised order structures here differ from those in [13,15], and are more in the spirit used in [20].

The concept of a *concurrent history* from [18] was revisited, redefined, and analysed. The special but common case, when only totally ordered observations are allowed, was analysed in detail. We have shown (in a more convincing way than in [18,20]) that if only totally ordered observations are allowed, neither the theory presented in this paper, nor that of [20], is needed.

The theory of *order structures* of [20] was also revisited and reformulated in the new framework. We emphasize the fact that order structures may belong to various different classes, and this needs to be taken into consideration when concurrent histories are modelled. This issue is not discussed in [20]. The theory presented in this paper is therefore an extension of that of [20]. If a generalized order structure $G = (X, \frown, \sqsubset)$ satisfies $\frown = (\frown \cap \sqsubset)^{\text{sym}}$, then all properties of $G$ are fully described by the appropriate properties of the order structure $S_G = (X, \frown \cap \sqsubset, \sqsubset)$, and the complete theory of order structures is given in [20].

The generalised order structures model all kinds of concurrent histories (see [18]); however, if the paradigm $\pi_3$ does hold, appropriate generalised order structures are just order structures in disguise, so the simpler model of [20] can be used. If the paradigm $\pi_6$ holds (and $\pi_3$ does not), we have a relatively simple but also tricky case of generalised total order structures.

We have illustrated all the concepts by a variety of simple examples.

From a purely mathematical point of view the results of this paper can be seen as generalisations of Szpilrajn's Theorem [38], from total orders to stratified, interval and general partial orders, *without any restrictions on the structure of sets of partial orders*, and as the full generalisation of the results of [20]. The generalised partial order structures (and partial order structures of [20]) do not have an obvious interpretation in concurrency theory, but they represent the most unrestricted case of the theory.

An immediate application of the results obtained here seems to be in the concurrent system synthesis problem area. We believe that the approach introduced in [33] could now, after employing the results of this paper, handle cases like program $P_1$ from the introduction.

We also believe that all results that use the model of [19] (finite version of [18]) and [18] ([23,27,36], etc.) can be relatively easily extended to the model presented in this paper. In some cases this should extend the area of applications.

The main results of this paper, Theorems 3, 7, and 8, although highly motivated by concurrency theory, are entirely independent of any particular interpretation.

**Ackowledgments** The author gratefully acknowledges the anonymous referees, whose comments significantly contributed to the final version of this paper. The formal definition of generalised order structures is due to Guangyuan Guo. Wolfram Kahl is thanked for a hint that helped to prove Theorem 6. Jeff Zucker and Marek R. Janicki are thanked for correcting many errors.

## References

1. Abraham, U., Ben-David, S., Magodor, M.: On global-time and inter-process communication. In: Semantics for Concurrency, Workshops in Computing, pp. 311–323. Springer, Heidelberg (1990)
2. Anger, F.D.: On Lamport's interprocess communication model. ACM TOPLAS **11**(3), 404–417 (1989)
3. Baldan, P., Busi, N., Corradini, A., Pinna, M.: Domain and event structure semantics for petri nets with read and inhibitor arcs. Theor. Comput. Sci. **323**, 129–189 (2004)
4. Begstra J.A., et al. (eds.): The Handbook of Process Algebras. Elsevier, Amsterdam (2000)
5. Best, E., de Boer, F., Palamedissi, C.: Partial order and SOS semantics for linear constraint programs. In: Lecture Notes in Computer Science, vol. 1282, pp. 256–273. Springer, Heidelberg (1997)
6. Best, E., Devillers, R., Koutny, M.: Petri Net Algebra. Springer, Heidelberg (2001)
7. Best, E., Koutny, M.: Petri net semantics of priority systems. Theor. Comput. Sci. **94**, 141–158 (1992)
8. Best, E., Koutny, M.: Operational and denotational semantics for the box algebras. Theor. Comput. Sci. **211**, 1–83 (1999)
9. Brookes, S.D., Hoare, C.A.R., Roscoe, A.W.: A theory of communicating systems. J. ACM **31**, 560–599 (1984)
10. Degano, P., Montanari, U.: Concurrent histories; a basis for observing distributed systems. J. Comput. Syst. Sci. **34**, 422–467 (1987)
11. Diekert, V., Rozenberg, G. (eds.) The Book of Traces. World Scientific, Singapore (1995)

12. Gaifman, H., Pratt, V.: Partial order models of concurrency and the computation of functions. In: Proceedings of LICS'87, pp. 72–85
13. Guo, G., Janicki, R.: Modelling Concurrent Behaviours by Commutativity and Weak Causality Relations. In: Proceedings of AMAST'02. Lecture Notes in Computer Science, vol. 2422, pp. 178–191 (2002)
14. Fishburn, P.C.: Intransitive indifference with unequal indifference intervals. J. Math. Psych. **7**, 144–149 (1970)
15. Janicki, R.: A generalisation of a relational structures model of concurrency. In: Proceedings of ICTAC'04. Lecture Notes in Computer Science, vol. 3407, pp. 84–98 (2005)
16. Janicki, R., Koutny, M.: Invariants and paradigms of concurrency theory. In: Proceedings of PARLE'91. Lecture Notes in Computer Science, vol. 506, pp. 59–74 (1991)
17. Janicki, R., Koutny, M.: Order structures and generalisation of Szpilrajn's theorem. In: Proceedings of FSTTCS'93. Lecture Notes in Computer Science, vol. 761, pp. 348–357 (1993)
18. Janicki, R., Koutny, M.: Structure of concurrency. Theor. Comput. Sci. **112**, 5–52 (1993)
19. Janicki, R., Koutny, M.: Semantics of inhibitor nets. Inf. Comput. **123**(1), 1–16 (1995)
20. Janicki, R., Koutny, M.: Fundamentals of modelling concurrency using discrete relational structures. Acta Informatica **34**, 367–388 (1997)
21. Janicki, R., Koutny, M.: On causality semantics of nets with priorities. Fundam. Inf. **38**, 222–255 (1999)
22. Janicki, R., Lauer, P.E.: Specification and Analysis of Concurrent Systems: The COSY Approach. Springer, Heidelberg (1992)
23. Juhás, G., Lorenz, R., Mauser, S.: Synchronous + Concurrent = Earlier Than + Not Later Than. In: Proceedings of ACSD'06 (Application of Concurrency To System Design), pp. 261–272. IEEE Press, New York (2006)
24. Juhás, G., Lorenz, R., Neumair, C.: Synthesis of controlled behaviour with modules of signal nets. In: Proceedings of ATPN'04. Lecture Notes in Computer Science, vol. 3099, pp. 233–257. Springer, Heidelberg (2004)
25. Katz, S., Peled, D.: Defining conditional independence using collapses. In: Semantics for Concurrency, Workshops in Computing, pp. 262–290. Springer, Heidelberg (1990)
26. Klaudel, H., Pommereau, F.: A class of composable and preemptible high-level petri nets witn and application to a multi-tasking system. Fundam. Inf. **50**, 33–55 (2002)
27. Kleijn, H.C.M., Koutny, M.: Process semantics of P/T-nets with inhibitor arcs. In: Lecture Notes in Computer Science, vol. 1825, pp. 261–281. Springer, Heidelberg (2000)
28. Kleijn, H.C.M., Koutny, M.: Process semantics of general inhibitor nets. Inf. Comput. **190**, 18–69 (2004)
29. Lamport, L.: The mutual exclusion problem: Part I—A theory of interprocess communication; Part II—Statements and solutions. J. ACM **33**(2), 313–326 (1986)
30. Lamport, L.: What it means for a concurrent programm to satisfy a specification: why no one has specified priority. In: Proceedings of the 12th ACM Symposium on Programming Languages, pp. 78–83 (1985)
31. Mazurkiewicz, A.: Trace theory. In: Lecture Notes in Computer Science, vol. 225, pp. 297-324. Springer, Heidelberg (1986)
32. Milner, R.: Operational and algebraic semantics of concurrent processes. In: van Leuween, J. (ed.) Handbook of Theoretical Computer Science, vol. 2, pp. 1201-1242. Elsevier, Amsterdam (1993)
33. Pietkiewicz-Koutny, M.: The synthesis problem for elementary net systems. Fundam. Inf. **40**(2,3), 310–327 (1999)
34. Pinna, G.M.: Event structures with disabling/enabling relation and event automata. Funadam. Inf. **73**(3), 409–430 (2006)
35. Reisig, W.: Elements of Distributed Algorithms. Springer, Heidelberg (1998)
36. Roux, O.H., Lime, D.: Time petri nets with inhibitor arcs. Formal semantics and state space complexity. In: Proceedings of ATPN'04. Lecture Notes in Computer Science, vol. 3099, pp. 370–390. Springer, Heidelberg (2004)
37. Shields, M.W.: On the non-sequential behaviours of systems possessing a general free-choice-property, CSR-92-81. Department of Computer Science, University of Edinburgh (1981)
38. Szpilrajn, E.: Sur l'extension de l'ordre partial. Fundam. Math. **16**, 386–389 (1930)
39. Vogler, W.: Timed testing of concurrent systems. Inf. Comput. **121**, 149–171 (1995)
40. Vogler, W.: Partial order semantics and inhibitor arcs. In: Lecture Notes in Computer Science, vol. 1295, pp. 508–517. Springer, Heidelberg (1997)
41. Wollowski, R., Beister, J.: Precise petri net modelling of critical races in asynchronous arbiters and synchronizers. In: Proceedings of 1st Workshop on Hardware Design and Petri Nets, Lisbon, pp. 46-65 (1998)
42. Wollowski, R., Beister, J.: Comprehensive causal specification of asynchronous controller and arbiter behaviour. In: Yakovlev, A., Gomes, L., Lavagno, L. (eds.) Hardware Design and Petri Nets. Kluwer, Dordrecht (2000)