Symbolic models for control systems

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Abstract In this paper we provide a bridge between the infinite state models used in control theory to describe the evolution of continuous physical processes and the finite state models used in computer science to describe software. We identify classes of control systems for which it is possible to construct equivalent (bisimilar) finite state models. These constructions are based on finite, but otherwise arbitrary, partitions of the set of inputs or outputs of a control system.

1 Introduction

1.1 Motivation

The formal specification, design and analysis of software systems is usually based on the use of finite state models such as transition systems, Buchi automata, etc. This fact creates a real challenge for embedded systems in which continuous components described by infinite state models, such as differential equations, interact with software modules. Although much is known about these hybrid systems, as can be seen from the proceedings of the latest workshops on the subject [15–19], systematic design methodologies are still lacking. In this paper we propose an important step towards a "correct by design" synthesis theory for embedded systems by *replacing infinite state models of physical systems with equivalent finite state models*. The possibility of expressing the dynamics of control systems through finite state models offers great promises for new design techniques addressing the requirements of both continuous and discrete components. Not only do these finite state models offer a common framework bridging control theory and computer science but they also enable the use of algorithmic synthesis techniques that are traditionally not considered in control theory.

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1.2 Contributions

The existence of finite state models is addressed through two different perspectives: an *input* and an *output* perspective. The input perspective is based on the specification of a desired input set as a finite partition of the existing infinite input set. Each element of this partition can be identified with a symbol and the objective is to construct an equivalent control system having these symbols as inputs. This symbolic view of inputs is natural since embedded software/hardware can only influence the dynamics of the physical world through a finite set of inputs. We show that these control systems with symbolic inputs always exist and that under suitable assumptions (linearity and controllability) they have a finite number of states. The output perspective is identical to the input perspective except that the starting point is a finite partition of the infinite output set of a control system. This is also a natural starting point since embedded software/hardware can only react to the physical environment through a finite description of its state. Starting from such symbolic description of the output of a control system the objective is to construct the simplest equivalent control system generating the same symbolic outputs. Under the relevant technical assumptions (linearity and controllability for example) we show existence and finiteness of these models. These finiteness results encompass existing results by the author and coworkers [40,41,43] and also provide new examples of systems which were previously not known to admit symbolic models.

1.3 Related work

The idea of using symbolic models for the control of continuous systems is not new and motivated much research in the area of hybrid systems [5,9,10,21,26,27,35,37,39,45]. Even though the use of symbolic models was advocated by these and many other researchers, the applicability of the proposed methods has always remained an open problem due to the lack of results ensuring existence of symbolic models for control systems. Initial attempts to the construction of symbolic models for control systems were based on integrals of motion [8,36]. Integrals or constants of motion are a quite natural way of defining state space partitions compatible with the continuous dynamics. The results in this paper, however, apply to classes of systems for which constants of motion do not necessarily exist and can therefore be seen as complementary to existing work, especially to [8]. Another line of inquiry, also related to the work presented in this paper, is the study of quantized control systems [6,30]. Symbolic models based on input quantization represent a different approach to symbolic control. Quantized input models are obtained by selecting a countable subset of the input space contrasting with finite bisimulations which are obtained by selecting a finite partition of the state or input and state space. Even though this observation seems to suggest a certain duality between these approaches, further research is still required to attest and formalize such duality. A different but related line of research consists of the study of bisimulations of purely continuous [28,42,44] or hybrid systems [20,33]. The work described in these references represents another approach to complexity reduction through the use of bisimulation based abstractions.

In the case of *dynamical* systems the situation is much better understood. Starting with Alur and Dill's work on timed automata [2], a fair amount of work was done to push the boundaries of the class of systems admitting finite bisimulations. This work culminated with the papers [1,13,14,32] introducing the decidable classes of multi-rate

and rectangular hybrid automata. On the purely continuous side, we mention the work of Lafferriere and co-workers [23] which used o-minimality to ensure existence of symbolic models. See also [7] for a simpler and more insightful proof of the same results. Unfortunately the techniques based on o-minimality do not extend to control systems since the space of inputs is usually infinite dimensional (for example the space of piece-wise constant functions) and thus fails to be captured by o-minimal structures.

1.4 Organization

The results presented in this paper were developed in the setting of category theory [22] and in a framework very similar to the one used in [46] to compare different models for concurrency. However, for the sake of readability, we limited the use of category theory to the notion of morphism, essential for the results to be presented, and to the notions of completion. In particular, all the proofs are explicit and involve no category theoretic concepts. The readers averse to abstract constructions are invited to use the concrete results presented in Corollaries 4.9 and 5.6 as motivation for the abstract perspective taken in this paper.

The paper is organized as follows. The class of control systems considered in this paper and the corresponding morphisms are introduced in Sect. 2 while the notion of bisimulation is discussed in Sect. 3. Input induced bisimulations are the topic of Sect. 4 where their existence is proved. Finite bisimulations induced by inputs are also discussed and conditions for their existence established. Sect. 5 focuses on output induced bisimulations. Existence conditions are provided in this section for infinite and finite output induced bisimulations. The paper finishes with some discussion of the presented results in Sect. 6. For completeness, some results on equivalence relations are collected in an Appendix at the end of the paper.

2 Control systems and their morphisms

Control systems are usually defined by a differential equation:

$$\frac{d}{dt}x = F(x, u)$$

where $x \in \mathbb{R}^n$ denotes the state and $u \in \mathbb{R}^m$ denotes the input. Consider for example a block of mass m attached to a wall through a spring as displayed in Fig. 1. From basic physics we know that the sum of the forces acting on the block is given by ma where a is the acceleration of the block. The spring induces a force $-Kd$ proportional to the displacement d of the block and where K models the physical properties of the spring. If u is an external force also acting on the block we have:

$$ma = -Kd + u$$

Noting that acceleration is the time derivative of velocity v and that velocity is the time derivative of the displacement d we arrive at the following differential equation describing the evolution of the block:

$$\frac{d}{dt}v = -Kd + u$$
$$\frac{d}{dt}d = v$$

Fig. 1 Block attached to a
wall through a spring
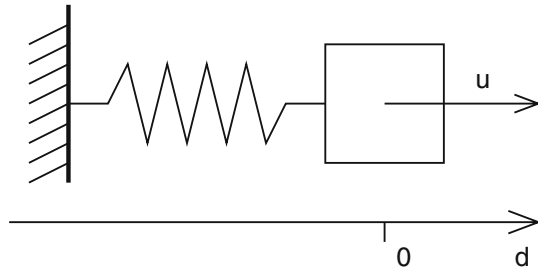
We can thus regard this systems as a control system with F and state x given by:

$$F(x, u) = \begin{bmatrix} -Kd + u \\ v \end{bmatrix}, \quad x = \begin{bmatrix} v \\ d \end{bmatrix} \in \mathbb{R}^2$$

and where $u \in \mathbb{R}$ is the input.

Under standard regularity conditions (see for example [38]) on the map $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and on a curve $\mathbf{u} : [0, \tau] \to \mathbb{R}^m$ with $\tau > 0$, there exists a curve $\mathbf{x} : [0, \tau] \to \mathbb{R}^n$ satisfying:

$$\frac{d}{dt}\mathbf{x}(t) = F(\mathbf{x}(t), \mathbf{u}(t)) \tag{2.1}$$

for almost all $t \in [0, \tau]$. The curve $\mathbf{u}$ is considered an input trajectory and it is chosen in order to ensure that the resulting state trajectory $\mathbf{x}$ satisfies certain desirable properties. We may be interested, for example, in choosing $\mathbf{u}$ in order to guarantee that the block in Fig. 1 moves from $d = 0$ to some other desired displacement. If we denote the set of all such input trajectories by U and if we denote by f the map sending the pair $(\mathbf{x}(0), \mathbf{u}) \in M \times U$ to $\mathbf{x}(\tau) \in M$ we can regard a control system simply as a map:

$$f : M \times U \to M$$

where $M = \mathbb{R}^n$ is the state set and U is the set of input trajectories. In this paper we will make two assumptions regarding the set U:

(1)   We will assume that elements of U are regular enough so that for each pair $(x, \mathbf{u}) \in M \times U$ there exists always a well defined curve $\mathbf{x}$ satisfying (2.1) with $\mathbf{x}(0) = x$. Examples of choices for U include piece-wise constant, piece-wise continuous and piece-wise smooth curves.
(2)   We will assume that curves in U are defined on intervals $[0, \tau]$ for some fixed, but otherwise arbitrary, $\tau > 0$. This is the case, for example, when control schemes are implemented in microprocessors through periodic tasks that are executed every $\tau > 0$ units of time.

Given the above discussion we shall model control systems as a map:

$$f : M \times U \to M \tag{2.2}$$

describing the state $f(x, \mathbf{u}) \in M$ resulting from applying input curve $\mathbf{u} \in U$ at the state $x \in M$. To simplify notation we will denote an element of U simply by u instead of $\mathbf{u}$. Moreover, we will refer to this element u as an input rather than as an input curve.

In many situations the resulting state $f(x, u)$ cannot be observed directly but rather through an output map:

$$r : M \to P \tag{2.3}$$

transforming states $x \in M$ into outputs $r(x) \in P$. This is the case in our block example if we only have a sensor measuring the displacement of the block. Although we regard the state as an element $(v, d) \in \mathbb{R}^2$, we only have access to sensor measurements providing the displacement $r(v, d) = d \in \mathbb{R}$.

We will usually denote a control system by the pair (f, r) or by $M \times U \xrightarrow{f} M \xrightarrow{r} P$ when we want to emphasize the input, state and output sets. When $P = M$ and $r = 1_M$ (1_M is the identity function on M) we will simply denote a control system by $M \times U \xrightarrow{f} M$. Since our ultimate objective is to obtain finite models for (2.2) we shall not require any additional structure on the sets M, U and P or on the maps f and r.

Recall that in [46] a transition system is defined as a quadruple $(S, i, L, Tran)$ where S is a set of states, $i \in S$ is a distinguished state, L is a set of labels and $Tran \subseteq S \times L \times S$ is a transition relation. A control system $M \times U \xrightarrow{f} M$ also defines a transition relation given by the graph of f, that is, $\Gamma(f) = \{(x, u, y) \in M \times U \times M \mid f(x, u) = y\} \subset M \times U \times M$. We thus conclude that both models are very close in the way they capture the dynamics. However, control systems are not equipped with a distinguished initial state but they are equipped with an observation map $r : M \to P$. This distinguishes the models with respect to observations. For control systems, observations are obtained through the map r acting on the state while for transition systems, as defined in [46], observations are directly available as the labels $l \in L$ of transitions in $Tran$.

The notion of morphism between control systems will be instrumental for the results developed in this paper. Intuitively, a morphism between control systems is a triple of maps describing how inputs, states and outputs of one system are related to inputs, states and outputs of another system.

Definition 2.1 Let $M \times U \xrightarrow{f} M \xrightarrow{r} P$ and $N \times V \xrightarrow{g} N \xrightarrow{s} Q$ be control systems. A morphism $\alpha : (f, r) \to (g, s)$ from (f, r) to (g, s) is given by a triple of maps $\alpha = (\alpha_1, \alpha_2, \alpha_3) : M \times U \times P \to N \times V \times Q$ with $\alpha_1 : M \to N$, $\alpha_2 : M \times U \to V$ and $\alpha_3 : P \to Q$, making the following diagram commutative:

$$
\begin{array}{ccccc}
M \times U & \xrightarrow{\;f\;} & M & \xrightarrow{\;r\;} & P \\
\downarrow{\scriptstyle \alpha_1 \times \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_3} \\
N \times V & \xrightarrow{\;g\;} & N & \xrightarrow{\;s\;} & Q
\end{array}
\qquad (2.4)
$$

that is, satisfying the following equalities:

$$\alpha_1 \circ f(x, u) = g(\alpha_1(x), \alpha_2(x, u)) \qquad (2.5)$$

$$\alpha_3 \circ r = s \circ \alpha_1 \qquad (2.6)$$

When $P = M$, $Q = N$, $r = 1_M$ and $s = 1_N$ we shall denote a morphism α simply as a pair $\alpha = (\alpha_1, \alpha_2) : M \times U \to N \times V$.

The introduced notion of morphism is also very close to the notion of morphism between transition systems in [46]. The important difference is that we work with totally defined maps and that the map $\alpha_2 : M \times U \to V$ mapping inputs (or labels) to inputs (or labels) is state dependent.

In a morphism $\alpha = (\alpha_1, \alpha_2) : f \to g$, the map α_2 can be many times determined from the map α_1 as the next result shows.

Proposition 2.2 *Let* $M \times U \xrightarrow{f} M$ *and* $N \times V \xrightarrow{g} N$ *be control systems and let* $\alpha_1 : M \to N$ *be a map. If the following inclusion holds*:

$$\alpha_1 \circ f(x, U) \subseteq g(\alpha_1(x), V)$$

there exists a map $\alpha_2 : M \times U \to V$ *such that* (α_1, α_2) *is a morphism from* f *to* g. *Furthermore, if* $g_y : V \to N$ *is injective for every* $y \in \alpha_1(M)$, *then* α *is unique*.

Proof Assume that $\alpha_1 \circ f(x, U) \subseteq g(\alpha_1(x), V)$ holds. Then, for every $(x, u) \in M \times U$, $\alpha_1 \circ f(x, u) \in g(\alpha_1(x), V)$ so that we can define α_2 at (x, u) to be any $v \in V$ satisfying:

$$\alpha_1 \circ f(x, u) = g(\alpha_1(x), v) \tag{2.7}$$

This construction immediately provides a map $\alpha_2 : M \times U \to V$ making (α_1, α_2) a morphism from f to g. Assume now that g_y is injective for every $y \in \alpha_1(M)$. Then, the choice of $v \in V$ satisfying (2.7) is unique thus defining α_2 uniquely. □

We will also need to analyze the behavior of f in several steps and the following notation will be very useful in that regard. We denote by U^* the monoid freely generated by U, that is, the set of all finite length strings obtained by concatenating elements in U. Note that the string of zero length, denoted by ε, also belongs to U^*. We will also have the opportunity to use $U^{\leq k}$ to denote the set of all strings of length smaller or equal than $k \in \mathbb{N}$. If $u = u_1 u_2 \ldots u_n \in U^*$, we will use $u(i)$ to denote the ith element of the string u, that is, $u(i) = u_i$ with the convention that $u(0) = \varepsilon$ and we will use $u[i]$ to denote the first i elements of u, that is, $u[i] = u_1 u_2 \ldots u_i$. Several maps extend in a unique way to the sets U^* and $U^{\leq k}$. Map $f^* : M \times U^* \to M$ is defined by $f^*(x, \varepsilon) = x$ and by the recursion $f^*(x, u) = f^*(f(x, u(1)), u(2)u(3) \ldots u(n))$ for any string u of length n. Map $f^{\leq k} : M \times U^{\leq k} \to M$ is the restriction of f^* to $M \times U^{\leq k}$. We will also use $\alpha_2^* : M \times U^* \to V^*$ to denote the extension of $\alpha_2 : M \times U \to V$ defined by $\alpha_2^*(x, \varepsilon) = x$ and by the recursion $\alpha_2^*(x, u) = \alpha_2(x, u(1))\alpha_2^*(f(x, u(1)), u(2) \ldots u(n))$ for any string u of length n. The notation f_x and α_{2x} will be used to denote the functions defined by $f_x(u) = f(x, u)$ and $\alpha_{2x}(u) = \alpha_2(x, u)$, respectively. Other notational conventions used through the paper are as follows. For a map $\phi : A \to B$ we will denote by $\phi(A)$, $\phi^{-1}(b)$ and $\phi^{-1}(C)$ with $b \in B$ and $C \subseteq B$, the sets $\cup_{a \in A}\{\phi(a)\}$, $\{a \in A \mid \phi(a) = b\}$ and $\cup_{c \in C}\phi^{-1}(c)$, respectively, and $|A|$ will denote the cardinality of set A.

3 Morphisms and bisimulations

The notion of bisimulation [25,29] is usually defined with respect to transition systems as operational models. Although control systems also define transition relations we shall define bisimulation relations directly for control systems:

Definition 3.1 Let $M \times U \xrightarrow{f} M \xrightarrow{r} O$ and $N \times V \xrightarrow{g} N \xrightarrow{s} O$ be control systems with the same output set O. A relation $R \subseteq M \times N$ is said to be a bisimulation relation between (f, r) and (g, s) if for every $(x_1, y_1) \in R$ the following holds:

(1) $r(x_1) = s(y_1)$;
(2) for any $u \in U, f(x_1, u) = x_2$ implies the existence of $v \in V$ such that $g(y_1, v) = y_2$ and $(x_2, y_2) \in R$;

(3)   for any $v \in V, g(y_1, v) = y_2$ implies the existence of $u \in U$ such that $f(x_1, u) = x_2$
and $(x_2, y_2) \in R$.

There is an important difference between this notion and the one introduced in [25,29]. We require related states to match on outputs but we do not require transitions to match on labels. This modeling choice is motivated by control applications where the state carries important physical information about the system and the desired behavior is described in terms of the state evolution. The input, however, is a signal that we can arbitrarily manipulate in order to enforce the desired output behavior. Nevertheless, it is not difficult to see that bisimilar control systems still exhibit the same output behavior.

Instead of working with Definition 3.1 it will be more convenient, at the technical level, to work with the following class of morphisms.

Definition 3.2 A morphism $\alpha : (f, r) \to (g, s)$ from control system $M \times U \xrightarrow{f} M$ $\xrightarrow{r} P$ to control system $N \times V \xrightarrow{g} N \xrightarrow{s} Q$ is said to be fiber surjective if $\alpha_2 : M \times U \to V$ is fiber surjective, that is, $\alpha_{2x} : U \to V$ is surjective for every $x \in M$. Morphism α is said to be surjective if it is fiber surjective and both $\alpha_1 : M \to N$ and $\alpha_3 : P \to Q$ are surjective maps.

The choice of fiber-surjective morphisms is justified by the following result.

Proposition 3.3 *Let $\alpha : (f, r) \to (g, s)$ be a fiber surjective morphism from control system $M \times U \xrightarrow{f} M \xrightarrow{r} P$ to control system $N \times V \xrightarrow{g} N \xrightarrow{s} Q$. The graph of the map α_1, defined by $\Gamma(\alpha_1) = \{(x, y) \in M \times N \mid \alpha_1(x) = y\}$, is a bisimulation relation between control system $M \times U \xrightarrow{f} M \xrightarrow{\alpha_3 \circ r} Q$ and control system $N \times V \xrightarrow{g} N \xrightarrow{s} Q$.*

Proof Let $(x, y) \in \Gamma(\alpha_1)$. Then $\alpha_1(x) = y$ and it follows from (2.6) that $\alpha_3 \circ r(x_1) = s \circ \alpha_1(x) = s(y)$ thus showing that requirement (1) in Definition 3.1 is satisfied.

Assume now that $f(x_1, u) = x_2$ and $\alpha_1(x_1) = y_1$. Then by (2.5) we have $\alpha_1(x_2) = \alpha_1 \circ f(x_1, u) = g(\alpha_1(x_1), \alpha_2(x_2, u)) = g(y_1, v)$ thus showing that requirement (2) in Definition 3.1 also holds with $v = \alpha_2(x_1, u)$.

Finally we prove requirement (3). Let $\alpha_1(x_1) = y_1$ and $g(y_1, v) = y_2$. Since α is fiber surjective, there exists $u \in U$ satisfying $\alpha_2(x_1, u) = v$. If we denote by x_2 the state $f(x_1, u)$, it follows from (2.5) that $\alpha_1(x_2) = \alpha_1 \circ f(x_1, u) = g(\alpha_1(x_1), \alpha_2(x_1, u)) = g(y_1, v) = y_2$ which concludes the proof.   □

Note that not every bisimulation can be obtained as the graph of a fiber-surjective morphism. However, this kind of bisimulation will suffice for the results presented in this paper.

4 Input induced bisimulations

In this section we discuss the construction of bisimilar control systems induced by inputs. Consider the block example described in the previous section and assume we want to develop a controller, to be implemented in embedded hardware/software, whose objective is to force the block to a desired displacement different from zero.

This controller will change the displacement of the block through an actuator exerting a force u on the block. However, the actuator can only enforce[1] a finite number of force values. Let us assume that these values are -1, 0 and 1 and that the actuator transforms requests into forces according to the following map:

$$\alpha_2(x, u) = \begin{cases} -1 & \text{if } u(0) \leq -1 \\ 0 & \text{if } -1 < u(0) < 1 \\ 1 & \text{if } u(0) \geq 1 \end{cases} \qquad (4.1)$$

It is therefore natural to construct a bisimilar control system with input set $V = \{-1, 0, 1\}$ describing the dynamics of the original control system under these new inputs.

4.1 Existence of input induced bisimulations

Starting from a control system $M \times U \xrightarrow{f} M \xrightarrow{r} P$ and a map $\alpha_2 : M \times U \to V$ there may exist several possible ways to obtain a control system $N \times V \xrightarrow{g} N \xrightarrow{s} Q$ and a morphism $\alpha : (f, r) \to (g, s)$. Among these we will be interested in the minimally restrictive one which can be singled out through the following unique factorization property.

Definition 4.1 Let $M \times U \xrightarrow{f} M \xrightarrow{r} P$ be a control system and $\alpha_2 : M \times U \to V$ a map. The α_2-completion of (f, r) is a triple $((g, s), \alpha_1, \alpha_3)$ where $N \times V \xrightarrow{g} N \xrightarrow{s} Q$ is a control system, $\alpha_1 : M \to N$ and $\alpha_3 : P \to Q$ are maps making $\alpha = (\alpha_1, \alpha_2, \alpha_3) : M \times U \times P \to N \times V \times Q$ a morphism from (f, r) to (g, s) with the following property: for any control system $N' \times V \xrightarrow{g'} N' \xrightarrow{s'} Q'$ and morphism $\alpha' = (\alpha'_1, \alpha_2, \alpha'_3) : (f, r) \to (g', s')$ there exists a unique morphism $\beta = (\beta_1, 1_V, \beta_3) : (g, s) \to (g', s')$ satisfying $\alpha' = \beta \circ \alpha$.

Note that bisimilarity between f and its α_2-completion now follows trivially if one choses α_2 to be fiber surjective. However, we shall study α_2-completions for maps α_2 that are not necessarily fiber-surjective since this extra generality adds no additional complications.

To understand the unique factorization property required by Definition 4.1 let us consider a control system $M \times U \xrightarrow{f} M$ with $M = \{x_1, x_2, x_3\}$, $U = \{u_1, u_2, u_3\}$ and f defined on Table 1 and also represented in Fig. 2. If the map $\alpha_2 : M \times U \to V$ is defined by $\alpha_2(x, u_1) = v_1 = \alpha_2(x, u_2)$ and $\alpha_2(x, u_3) = v_2$ for every $x \in M$ and if $V = \{v_1, v_2\}$, then control systems $N \times V \xrightarrow{g} N$ and $O \times V \xrightarrow{h} O$ defined on Table 2 and represented in Fig. 3 can be equipped with morphisms $\alpha = (\alpha_1, \alpha_2) : f \to g$ and $\gamma = (\gamma_1, \alpha_2) : f \to h$ also defined on Table 2.

Among all the possible control systems equipped with a morphism from f, of which $N \times V \xrightarrow{g} N$ and $O \times V \xrightarrow{h} O$ are two examples, the unique factorization property in Definition 4.1 isolates the "least constrained" solution. To illustrate this point, let us assume that $O \times V \xrightarrow{h} O$ satisfies the unique factorization

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[1] Alternatively we can assume that although the actuator can generate a force with an infinite number of intensities, it can only receive finitely many different requests from the software implementation of a controller.
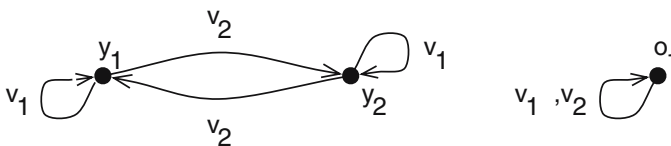
Table 1 Definition of control system f

x	u	$f(x,u)$
x_1	u_1	x_1
x_1	u_2	x_3
x_1	u_3	x_2
x_2	u_1	x_2
x_2	u_2	x_2
x_2	u_3	x_2
x_3	u_1	x_1
x_3	u_2	x_1
x_3	u_3	x_2

Fig. 2 Graphical representation of control system $f : M \times U \to M$
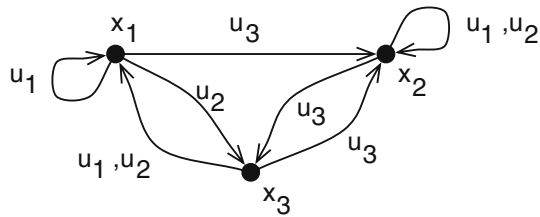
Table 2 Definition of control systems g and h, and of maps α_1 and γ_1

x	y	o	v	$g(y,v)$	$h(o,v)$	α_1	γ_1
	y_1		v_1	y_1			
	y_1		v_2	y_2			
	y_2		v_1	y_2			
	y_2		v_2	y_1			
		o_1	v_1		o_1		
		o_1	v_2		o_1		
x_1						y_1	
x_2						y_2	
x_3						y_1	
x_1							o_1
x_2							o_1
x_3							o_1

Fig. 3 Graphical representation of control systems $g : N \times V \to N$ and $h : O \times V \to O$

property. Then, there must exist a unique morphism $\beta = (\beta_1, 1_V) : h \to g$ satisfying $\beta \circ \gamma = \alpha$. However, no such morphism exists since if $\beta_1(o_1) = y_1$, then $\beta_1 \circ h(o_1, v_2) = \beta_1(o_1) = y_1 \neq y_2 = g(\beta_1(o_1), v_2)$. Similarly, if $\beta_1(o_1) = y_2$ we have $\beta_1 \circ h(o_1, v_2) = \beta_1(o_1) = y_1 \neq y_2 = g(\beta_1(o_1), v_2)$. On the other hand it is not difficult to see that there exists a unique morphism β from g to h given by $\beta_1(y_1) = o_1 = \beta_1(y_2)$.

Control system h is thus over-constrained in the sense that too many states are mapped by the map γ_1 into the same state. It is in this sense that we can see the unique factorization requirement in Definition 4.1 as the specification of the least constrained control system equipped with a morphism from (f, r).

The unique factorization property of α_2-completions guarantees that (g, s) is unique up to isomorphism. Even though this is true in more general contexts it is illustrative to see how it works in this particular case. Let us assume that (g, s) and (h, t) are α_2-completions of (f, r). Then, since (g, s) is a α_2-completion there exists a unique morphism $\beta_{gh} : (g, s) \to (h, t)$. Since (h, t) is also a α_2-completion there exists a unique morphism $\beta_{hg} : (h, t) \to (g, s)$. Composing β_{gh} with β_{hg} we obtain a morphism $\beta_{gh} \circ \beta_{hg} : (g, s) \to (g, s)$ from (g, s) to (g, s). Furthermore, $1_{(g,s)} = (1_N, 1_V, 1_Q)$ is also a morphism from (g, s) to (g, s) and from uniqueness of β_{gh} and β_{hg} follows uniqueness of $\beta_{gh} \circ \beta_{hg}$ which implies the equality $\beta_{gh} \circ \beta_{hg} = 1_{(g,s)}$. The same argument applied to the composition $\beta_{hg} \circ \beta_{gh} : (h, t) \to (h, t)$ leads to $\beta_{hg} \circ \beta_{gh} = 1_{(h,t)}$ from which we conclude that β_{gh} is an isomorphism thus showing that (g, s) is unique up to isomorphism.

The following result shows that output maps have no contribution for the existence of α_2-completions. We shall therefore assume throughout this section that output sets equal the state sets and that output maps are the identity.

Proposition 4.2 *Consider control system $M \times U \xrightarrow{f} M \xrightarrow{1_M} M$ and the map $\alpha_2 : M \times U \to V$. If the α_2-completion of $(f, 1_M)$ exists, then the α_2-completion of $M \times U \xrightarrow{f} M \xrightarrow{r} P$ also exists for any output map $r : M \to P$.*

Proof Let $N \times V \xrightarrow{g} N \xrightarrow{s} Q$ be the α_2-completion of $(f, 1_M)$. We shall construct the α_2-completion of (f, r) from g. Consider the maps $r : M \to P$ and $\alpha_1 : M \to N$. There is a unique,[2] up to isomorphism, way of constructing $\alpha_3 : P \to R$ and $t : N \to R$ satisfying $\alpha_3 \circ r = t \circ \alpha_1$. Let $X = P + N$ (+ denotes disjoint union) and let S be the equivalence relation on X generated by the pairs $(\alpha_1(x), r(x))$ for every $x \in M$. Define $R = X/S$, $\alpha_3 = \pi_S \circ i_P$ and $t = \pi_R \circ i_N$ where $i_P : P \to X$ and $i_N : N \to X$ are the natural inclusions and $\pi_S : X \to R = X/S$ is the natural projection. The equality $\alpha_3 \circ r = t \circ \alpha_1$ follows immediately by construction of R, α_3 and t. Furthermore, maps α_3 and t have the following property. For any other maps $a : P \to T$ and $b : N \to T$ there exists one and only one map:

$$c : R \to T \tag{4.2}$$

satisfying:

$$a = c \circ \alpha_3 \tag{4.3}$$
$$b = c \circ t \tag{4.4}$$

We leave to the reader to verify that the desired unique map c is given by $c([p]) = a(p)$ and $c([n]) = b(n)$ for any $p \in P$ and $n \in N$.

We now claim that $((g, t), \alpha_1, \alpha_3)$ is the α_2-completion of (f, r). It is obvious that α is a morphism from (f, r) to (g, t) since (α_1, α_2) satisfy (2.5) and α_3 satisfies (2.6). Let now (g', s') be any other control system equipped with a morphism $(\alpha_1', \alpha_2, \alpha_3') : (f, r) \to (g', s')$. Since (g, s) is the α_2-completion of (f, r) there exists one and only

---

[2] What we are about to construct is the push-out of $r : M \to P$ and $\alpha_1 : M \to N$.

one map $\beta_1 : N \to N'$ making $(\beta_1, 1_V)$ a morphism from g to g' and satisfying $(\beta_1, 1_V) \circ (\alpha_1, \alpha_2) = (\alpha_1', \alpha_2)$. To conclude the proof we only need to show existence and uniqueness of $\beta_3 : R \to P'$ satisfying $\beta_3 \circ t = s' \circ \beta_1$ and $\beta_3 \circ \alpha_3 = \alpha_3'$. But this follows at once by taking $a = \alpha_3'$, $b = s' \circ \beta_1$ in (4.3) and (4.4), and $\beta_3 = c$. $\qquad\square$

We now arrive at one of the main contributions of this section.

Theorem 4.3 *The α_2-completion of a control system $M \times U \xrightarrow{\;f\;} M$ exists for any map $\alpha_2 : M \times U \to V$.*

Theorem 4.3 shows that bisimilar control systems induced by α_2 maps always exist in the class of control systems considered in this paper. Furthermore, when α_2 is fiber-surjective, the resulting α_2-completion is guaranteed to be bisimilar to (f, r).

Proof The proof of Theorem 4.3 makes essential use of the following lemma which can be seen as the key ingredient of the classical Myhill-Nerode canonical realization result. The same ideas have been used before to generalize realization theory to more abstract contexts [3,12].

Lemma 4.4 *Let $\varkappa : (M \times V^*) \times V \to (M \times V^*)$ be the control system defined by $\varkappa((x, v_1), v_2) = (x, v_1 v_2)$. For any map $\gamma : M \to N$ and for any control system $N \times V \xrightarrow{\;g\;} N$ there exists a unique morphism $\delta = (\delta_1, 1_V) : \varkappa \to g$ such that $\delta_1(x, \varepsilon) = \gamma(x)$.*

Proof of Lemma 4.4 Map δ_1 is defined by $\delta_1(x, v) = g^*(\gamma(x), v)$. This choice results in a morphism since:

$$
\begin{aligned}
\delta_1 \circ \varkappa((x, v_1), v_2) &= \delta_1(x, v_1 v_2) && \text{by definition of } \varkappa \\
&= g^*(\gamma(x), v_1 v_2) && \text{by definition of } \delta_1 \\
&= g(g^*(\gamma(x), v_1), v_2) && \text{by definition of } g^* \\
&= g(\delta_1(x, v_1), 1_V(v_2)) && \text{by definition of } \delta_1
\end{aligned}
$$

Furthermore $\delta_1(x, \varepsilon) = g^*(\gamma(x), \varepsilon) = \gamma(x)$. To show uniqueness consider any other morphism $(\delta_1', 1_V) : \varkappa \to g$. We immediately have $\delta_1'(x, \varepsilon) = \gamma(x) = \delta_1(x, \varepsilon)$. For any $(x, v) \in M \times V^*$ equality between δ_1' and δ_1 follows from:

$$
\begin{aligned}
\delta_1'(x, v) &= \delta_1' \circ \varkappa((x, \varepsilon), v) && \text{by definition of } \varkappa \\
&= g(\delta_1'(x, \varepsilon), 1_V(v)) && \text{since } (\delta_1', 1_V) \text{ is a morphism} \\
&= g(\gamma(x), v) && \text{since } \delta_1'(x, \varepsilon) = \gamma(x) \\
&= g(\delta_1(x, \varepsilon), 1_V(v)) && \text{since } \delta_1(x, \varepsilon) = \gamma(x) \\
&= \delta_1 \circ \varkappa((x, \varepsilon), v) && \text{since } (\delta_1, 1_V) \text{ is a morphism} \\
&= \delta_1(x, v) && \text{by definition of } \varkappa
\end{aligned}
$$

$\qquad\square$

We now return to the proof of Theorem 4.3

The desired completion $N \times V \xrightarrow{\;g\;} N$ will be obtained from control system $\varkappa$ defined in Lemma 4.4. The state set N is defined as the quotient of $M \times V^*$ by the equivalence relation R generated by the pairs:

$$
((f(x, u), \varepsilon), (x, \alpha_2(x, u))) \tag{4.5}
$$

and invariant under the family of maps maps $\{\varkappa_v\}_{v \in V}$ (invariant under $\varkappa$ for short). Map α_1 is the composition $\pi_R \circ i$ where $i : M \to M \times V^*$ is the inclusion $i(x) = (x, \varepsilon)$. Finally, we define control system g by:

$$g([(x, v_1)], v_2) = [(x, v_1 v_2)] = [\varkappa((x, v_1), v_2)] \tag{4.6}$$

It follows from invariance of R under h that g is well defined. To show that α is indeed a morphism consider any $(x, u) \in M \times U$. The following sequence of identities:

$$
\begin{aligned}
\alpha_1 \circ f(x, u) &= \pi_R \circ i \circ f(x, u) \\
&= \pi_R(f(x, u), \varepsilon) \quad \text{by definition of } i \\
&= [(f(x, u), \varepsilon)] \quad \text{by definition of } \pi_R \\
&= [(x, \alpha_2(x, u))] \quad \text{by (4.5)} \\
&= [\varkappa((x, \varepsilon), \alpha_2(x, u))] \quad \text{by definition of } \varkappa \\
&= g([(x, \varepsilon)], \alpha_2(x, u)) \quad \text{by definition of } g \\
&= g(\pi_R \circ i(x), \alpha_2(x, u)) \quad \text{by definition of } i \text{ and } \pi_R \\
&= g(\alpha_1(x), \alpha_2(x, u)) \quad \text{by definition of } \alpha_1
\end{aligned}
$$

shows that α is a valid morphism.

Consider now any other control system $g' : N' \times V \to N'$ and any other map $\alpha'_1 : M \to N'$ making $\alpha' = (\alpha'_1, \alpha_2)$ a morphism from f to g'. Map α'_1 factors as $\alpha'_1 = \delta_1 \circ i$ where $i : M \to M \times V^*$ is the inclusion $i(x) = (x, \varepsilon)$ and δ_1 the unique map $\delta_1 : M \times V^* \to N'$ whose existence is asserted by Lemma 4.4. Map $\alpha_1 : M \to N$ also factors as $\alpha_1 = \pi_R \circ i$. Provided that we can show that:

$$\pi_R(x_1, v_1) = \pi_R(x_2, v_2) \quad \Rightarrow \quad \delta_1(x_1, v_1) = \delta_1(x_2, v_2) \tag{4.7}$$

we can define $\beta : N \to N'$ by $\beta([(x, v)]) = \delta_1(x, v)$. This is what we will now show by proving that the equivalence relation S on $M \times V^*$ defined by $((x_1, v_1), (x_2, v_2)) \in S$ when $\delta_1(x_1, v_1) = \delta_1(x_2, v_2)$ is invariant under $\varkappa$ and contains the pairs:

$$\big((f(x, u), \varepsilon), (x, \alpha_2(x, u))\big)$$

Invariance under $\varkappa$ follows immediately from the fact that $(\delta_1, 1_V)$ is a morphism from $\varkappa$ to g'. Consider now an arbitrary point $(x, u) \in M \times U$. From the fact that $\alpha' : f \to g'$ is a morphism we have:

$$\alpha'_1 \circ f(x, u) = g'(\alpha'_1(x), \alpha_2(x, u))$$

Since α' factors as $\alpha' = (\delta_1, 1_V) \circ (i, \alpha_2)$:

$$\delta_1 \circ i \circ f(x, u) = g'(\delta_1 \circ i(x), \alpha_2(x, u))$$

We now use the fact that $(\delta_1, 1_V)$ is a morphism from h to g' to conclude:

$$
\begin{aligned}
\delta_1 \, (f(x, u), \varepsilon) \\
&= \delta_1 \circ i \circ f(x, u) \quad \text{by definition of } i \\
&= g'(\delta_1 \circ i(x), 1_V \circ \alpha_2(x, u)) \quad \text{since } (\delta_1, 1_V) \circ (i, \alpha_2) = \alpha' \text{ is a morphism from } f \text{ to } g' \\
&= g'(\delta_1(x, \varepsilon), \alpha_2(x, u)) \quad \text{by definition of } i \\
&= \delta_1 \circ \varkappa((x, \varepsilon), \alpha_2(x, u)) \quad \text{since } (\delta_1, 1_V) \text{ is a morphism from } \varkappa \text{ to } g' \\
&= \delta_1(x, \alpha_2(x, u)) \quad \text{by definition of } \varkappa
\end{aligned}
$$

We thus conclude that S is invariant under $\varkappa$ and contains the pairs:

$$\big((f(x,u),\varepsilon),(x,\alpha_2(x,u))\big)$$

Since R is the coarsest equivalence relation satisfying these two properties it follows that (4.7) holds and $\beta_1 : N \to N'$ is well defined. The fact that $(\beta_1, 1_V)$ is a morphism and unique is inherited from the same properties of $(\delta_1, 1_V)$ proved in Lemma 4.4. The proof is therefore complete.   □

Throughout the remaining paper we will be especially interested in α_2-completions for fiber surjective maps α_2. In this case, α_2-completions admit the following simplified description that will later be used to establish existence of symbolic completions.

Proposition 4.5 *Given a control system $f : M \times U \to M$, if $\alpha_2 : M \times U \to V$ is fiber surjective, then the α_2-completion of f admits the following description:*

(1) $N = M/S$ *where S is the coarsest equivalence relation on M satisfying for any $x \in M$:*

$$\alpha_2^*(x,u_1) = \alpha_2^*(x,u_2) \quad \Rightarrow \quad \big(f^*(x,u_1),f^*(x,u_2)\big) \in S$$

(2) $\alpha_1 = \pi_S : M \to M/S = N;$
(3) g *is defined by* $g(y,v) = \alpha_1 \circ f(x,u)$ *for any* $(x,u) \in M \times U$ *such that* $\alpha(x,u) = (y,v)$.

Proof Consider the equivalence relation R on $M \times V^*$ defined by (4.5) in the proof of Theorem 4.3. The proof will consist in showing that every $(x,v) \in M \times V^*$ is equivalent under R to $(f^*(x,u),\varepsilon)$ for any $u \in U$ such that $\alpha_2^*(x,u) = v$. This equivalence allows to identify each equivalence class of R with an element of the form $(f^*(x,u),\varepsilon)$ thus showing that we can equivalently describe R by equivalence relation S. The proof will be done by induction on the length of v.

For $|v| = 1$ the result follows from (4.5).

Let now $|v| = l + 1$ and assume the result holds for all $v' \in V^l$. We will show that (x,v) is equivalent under R to $(f^*(x,u),\varepsilon)$. Consider $(x,v[l])$. Since $v[l] \in V^l$ we know that $((x,v[l]),(f^*(x,u'),\varepsilon)) \in R$ for any $u' \in U^l$ such that $\alpha_2^*(x,u') = v[l]$. We also know that S is invariant under h so that:

$$\big((x,v[l]v(l+1)),(f^*(x,u'),v(l+1))\big) \in R \tag{4.8}$$

Since $|v(l+1)| = 1$ it follows from (4.5) that:

$$\big((f^*(x,u'),v(l+1)),(f^*(x,u'u''),\varepsilon)\big) \in R \tag{4.9}$$

for any $u'' \in U$ such that $\alpha_2(f^*(x,u),u'') = v(l+1)$. Transitivity of R applied to (4.8) and (4.9) now implies that $(x,v) = (x,v[l]v(l+1))$ is equivalent under R to $(f^*(x,u),\varepsilon)$ where $u = u'u'' \in U^{l+1}$ is any element satisfying $\alpha_2^*(x,u) = v$.   □

4.2 Existence of input induced symbolic bisimulations

In this section we characterize which α_2-completions result in control systems with finite state set. The finiteness results developed in this section will require the notion of reachable control system.

Definition 4.6 A control system $M \times U \xrightarrow{\ f\ } M$ is said to be reachable from $x_0 \in M$ if $f_{x_0}^* : U^* \to M$ is surjective.

We will restrict our attention to maps $\alpha_2 : M \times U \to V$ that are fiber surjective onto a finite set V. Fiber surjectivity will imply bisimilarity while finiteness of V can be assumed without loss of generality since if $N \times V \xrightarrow{g} N$ is a control system with finite state set, then we need an input set V with no more than $|N|$ elements as there are no more than $|N|$ states that can be reached from any state in N.

Theorem 4.7 *Let $M \times U \xrightarrow{f} M$ be a control system reachable from $x_0 \in M$ and let $\alpha_2 : M \times U \to V$ be a fiber surjective map onto a finite set V. The α_2-completion of f has finite state set iff there exists $k \in \mathbb{N}$ such that $\alpha_1 \circ f_{x_0}^{\leq k} : U^{\leq k} \to N$ is surjective.*

Proof Consider the family of sets $\{E_v\}_{v \in V^{\leq k}}$ where each set E_v is defined by the points $x \in M$ for which the following conditions are satisfied:

(1) there exists a $u \in U^{\leq k}$ such that $f^{\leq k}(x_0, u) = x$;
(2) $\alpha_2^*(x_0, u) = v$.

We first note that by Proposition 4.5 each set E_v is totally contained in an equivalence class of S. We now claim that S has no more than $\sum_{i=1}^{k} |V|^i$ equivalence classes. This follows from the surjectivity assumption on $\alpha_1 \circ f_{x_0}^{\leq k} : U^{\leq k} \to N$ which implies that cardinality of N equals the cardinality of the image of $\alpha_1 \circ f_{x_0}^{\leq k}$ which, in view of $\alpha_1 \circ f_{x_0}^{\leq k} = g_{\alpha_1(x_0)}^{\leq k} \circ \alpha_2(x_0, \cdot)$, is bounded by $\sum_{i=1}^{k} |V|^i$.

Assume now that the α_2-completion g of f has finite state set, let $y_0 = \alpha_1(x_0)$ and let $y \in g_{y_0}^n(V^n)$ for any $n \in \mathbb{N}$. Fiber surjectivity of α_2 implies fiber surjectivity of α_2^n from which we conclude existence of $u \in U^n$ satisfying $\alpha_2^n(x_0, u) = v$ for any $v \in V^n$ satisfying $g_{y_0}^n(v) = y$. We now use the fact that $\alpha : f \to g$ is a morphism to conclude $\alpha_1 \circ f^n(x_0, u) = g^n(y_0, v) = g^n(\alpha_1(x_0), \alpha_2(x_0, u))$ which shows that $\alpha_1 \circ f_{x_0}^n(U^n) = g_{y_0}^n(V^n)$. To conclude the proof it suffices to show that $g_{y_0}^n(V^n) = g_{y_0}^{\leq k}(V^{\leq k})$ for some fixed $k \in \mathbb{N}$. But this follows at once from the observation that finiteness of the state set of g implies that y can be reached from y_0 in no more than $k = |N| - 1$ steps. □

Even though Theorem 4.7 provides us with a necessary and sufficient condition for finiteness of input completions, we cannot test surjectivity of $\alpha_1 \circ f_{x_0}^{\leq k}$ without determining α_1. The following corollary strengths surjectivity of $\alpha_1 \circ f_{x_0}^{\leq k}$ to a condition not requiring a priori knowledge of α_1.

Corollary 4.8 *Let $M \times U \xrightarrow{f} M$ be a control system reachable from $x_0 \in M$ and let $\alpha_2 : M \times U \to V$ be a fiber surjective map onto a finite set V. If there exists $k \in \mathbb{N}$ such that $f_{x_0}^{\leq k} : U^{\leq k} \to M$ is surjective then the α_2-completion of f has finite state set*

We can use Corollary 4.8 to identify a concrete class of control systems admitting finite bisimulations. The assumption we need is reachability from some point $x_0 \in M$ in a finite number k of steps and this can easily be characterized for linear control systems.

Corollary 4.9 *Let $M \times U \xrightarrow{f} M$ be a discrete-time controllable linear system, that is, $f(x, u) = Ax + Bu$ with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, A and B matrices of appropriate dimensions and the matrix $[A^{n-1}B | A^{n-2}B | \ldots | B]$ has rank n. Then, for any fiber surjective map $\alpha_2 : M \times U \to V$ onto a finite set V, the α_2-completion of f has finite state set.*

Proof The result follows from Corollary 4.8 once we establish the existence of $x_0 \in M$ and $k \in \mathbb{N}$ making $f_{x_0}^{\leq k} : U^{\leq k} \to \mathbb{R}^n$ surjective. However, this follows at once from standard results in linear systems theory [4,38] which guarantee surjectivity of $f_{x_0}^{\leq n} : U^{\leq n} \to \mathbb{R}^n$ for any $x_0 \in \mathbb{R}^n$ under the linearity and controllability assumptions.   □

We now illustrate Corollary 4.9 in the simplest discrete-time controllable linear system:

$$x_1(t + \tau) = x_2(t)$$
$$x_2(t + \tau) = u(t)$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ is the state, $u \in \mathbb{R}$ is the input and $\tau > 0$. Let α_2 be defined by:

$$\alpha_2(x, u) = \begin{cases} -1 & \text{if } u \leq -1 \\ 0 & \text{if } -1 < u < 1 \\ 1 & \text{if } u \geq 1 \end{cases} \tag{4.10}$$

For simplicity we will identify $-1, 0, 1$ with the symbols $-, 0, +$, respectively and we introduce the sets $U_- = \{u \in \mathbb{R} \mid \alpha_2(u, x) = -1\}$, $U_0 = \{u \in \mathbb{R} \mid \alpha_2(u, x) = 0\}$ and $U_+ = \{u \in \mathbb{R} \mid \alpha_2(u, x) = 1\}$. Starting at the origin and applying the inputs in the symbolic set $V = \{-, 0, +\}$ we construct the following sets:

$$S_j = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0 \wedge x_2 \in U_j\}, \quad j \in \{-, 0, +\} \tag{4.11}$$

representing points that are identified under equivalence relation S defined in Proposition 4.5. Applying inputs again we obtain a complete description of S given by the following sets defining its equivalence classes:

$$S_j^i = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in U_i \wedge x_2 \in U_j\}, \quad i, j \in \{-, 0, +\} \tag{4.12}$$

The resulting control system $N \times V \xrightarrow{g} N$ is defined by:

$$g(S_j^i, -) = S_-^j \qquad g(S_j^i, 0) = S_0^j \qquad g(S_j^i, +) = S_+^j, \quad i, j \in \{-, 0, +\}$$

where N is identified with $\cup_{i,j \in \{-,0,+\}} \{S_j^i\}$ which has nine elements.

5 Output induced bisimulations

A different kind of completion can be obtained by specifying a map redefining the output rather than redefining the input. This motivates the study of symbolic models induced by outputs. We can use again the block example to illustrate the need for this kind of construction. Recall that we are interested in developing a controller to be implemented in embedded hardware/software in order to force the block to a non-zero displacement. Since we do not know the initial displacement of the block, this controller will change the applied force u based on measurements of the block displacement. However, we only have a sensor providing displacement measurements and furthermore the controller implementation can only deal with finitely many values. We can thus assume, for example, that the state (v, d) is transformed into symbols to be used by the controller software/hardware according to the following rule:

$$\alpha_3(v, d) = \begin{cases} -1 & \text{if } d \leq -1 \\ 0 & \text{if } -1 < d < 1 \\ 1 & \text{if } d \geq 1 \end{cases} \tag{5.1}$$

It is therefore natural to ask for the existence of an equivalent but simpler model of the continuous dynamics describing the evolution of the symbols $-1, 0$ and 1 defining the new output set Q.

5.1 Existence of output induced bisimulations

In this section we consider output completions induced by a map $\alpha_3 : P \to Q$ and defined as follows:

Definition 5.1 Let $M \times U \xrightarrow{f} M \xrightarrow{r} P$ be a control system and let $\alpha_3 : P \to Q$ be a surjective map. The α_3-completion of (f, r) is a triple $((g, s), \alpha_1, \alpha_2)$ where $N \times V \xrightarrow{g} N \xrightarrow{s} Q$ is a control system and $\alpha_1 : M \to N$ and $\alpha_2 : M \times U \to V$ are maps making $\alpha = (\alpha_1, \alpha_2, \alpha_3) : M \times U \times P \to N \times V \times Q$ a surjective morphism from (f, r) to (g, s) with the following property:

For any control system $N' \times V' \xrightarrow{g'} N' \xrightarrow{s'} Q$ and surjective morphism $\alpha' = (\alpha'_1, \alpha'_2, \alpha_3) : (f, r) \to (g', s')$ there exists a unique surjective morphism $\beta = (\beta_1, \beta_2, 1_Q) : (g', s') \to (g, s)$ satisfying $\alpha = \beta \circ \alpha'$.

Note that although the definition of α_3-completion is very similar to the definition of α_2-completion, there is an important difference between the two: the direction of morphism β. Requiring existence of a morphism $\beta : (g, s) \to (g', s')$ would lead to the trivial solution $(g, s) = (f, r)$ and $\alpha = 1_{(f,r)}$. However, this reversal in the direction of morphism β will require additional assumptions to guarantee existence of α_3-completions. Part of these assumptions have already been built in the definition of α_3-completion through the use of the adjective surjective. In particular, requiring α to be surjective guarantees that α_3-completions are bisimilar to (f, r). The remaining assumptions require the following notion of indistinguishability.

Definition 5.2 Let $M \times U \xrightarrow{f} M \xrightarrow{r} P$ be a control system and let $\alpha_3 : P \to Q$ be a map. The indistinguishability relations $\{[[\alpha_3]]_i\}_{i \in \mathbb{N}}$ induced by α_3 are recursively defined by:

$$[[\alpha_3]]_0 = \left\{ (x_1, x_2) \in M \times M \mid \alpha_3 \circ r(x_1) = \alpha_3 \circ r(x_2) \right\}$$
$$[[\alpha_3]]_{i+1} = \left\{ (x_1, x_2) \in [[\alpha_3]]_0 \mid \forall u_1 \in U \; \exists u_2 \in U \quad (f(x_1, u_1), f(x_2, u_2)) \in [[\alpha_3]]_i \right.$$
$$\left. \wedge \; \forall u_2 \in U \; \exists u_1 \in U \quad (f(x_1, u_1), f(x_2, u_2)) \in [[\alpha_3]]_i \right\}$$

Two states $x_1, x_2 \in M$ are said to be distinguishable by α_3 in $n \in \mathbb{N}$ steps if $(x_1, x_2) \notin [[\alpha_3]]_n$ and they are said to be distinguishable if there exists $n \in \mathbb{N}$ such that they are distinguishable in n steps. Two states that are not distinguishable are said to be indistinguishable.

The indistinguishability relations introduced in Definition 5.2 are no more than a variation of the sequence of relations leading to the coarsest relation compatible with the transition relation defined by f [11,31]. The only difference is the fact that we allow the pairs $(f(x_1, u_1), f(x_2, u_2)) \in [[\alpha_3]]_i$ to have different inputs u_1 and u_2. This difference is a natural consequence of Definition 3.1 requiring matching of the outputs instead of matching of the labels.

Note that it follows from the definition of indistinguishability relation that if there exists a $k \in \mathbb{N}$ such that $[[\alpha_3]]_k = [[\alpha_3]]_{k+1}$ then $[[\alpha_3]]_k = [[\alpha_3]]_i$ for all $i \geq k$. In general, the sequence of indistinguishability relations need not stabilize after a finite

number of steps. In this case we shall work with the limit relation (see the Appendix for several notions related to relations):

$$[[\alpha_3]] = \lim_{i \to \infty} [[\alpha_3]]_i$$

which is in fact a bisimulation relation between (f, r) and (f, r). Since $[[\alpha_3]]_{i+1}$ refines $[[\alpha_3]]_i$ and $\pi_{[[\alpha_3]]_0} = \alpha_3 \circ r$, the map $\pi_{[[\alpha_3]][[\alpha_3]]_0} : M/[[\alpha_3]] \to Q$ satisfies:

$$\pi_{[[\alpha_3]][[\alpha_3]]_0} \circ \pi_{[[\alpha_3]]} = \alpha_3 \circ r \tag{5.2}$$

Existence of α_3-completions can now be characterized using $[[\alpha_3]]$.

Theorem 5.3 *Let $M \times U \xrightarrow{f} M \xrightarrow{r} P$ be a control system with a surjective output map and let $\alpha_3 : P \to Q$ be a surjective map. If for any $(x_1, x_2) \notin [[\alpha_3]]$ the following equality holds:*

$$\left| \pi_{[[\alpha_3]]} \circ f(x_1, U) \right| = \left| \pi_{[[\alpha_3]]} \circ f(x_2, U) \right| \tag{5.3}$$

then, the α_3-completion of (f, r) exists.

Proof We construct control system $N \times V \xrightarrow{g} N$ with $N = M/[[\alpha_3]]$ and V any set of cardinality $|\pi_{[[\alpha_3]]} \circ f(x, U)|$. Set V is independent of the point $x \in M$ since by assumption $(x_1, x_2) \notin [[\alpha_3]]$ implies $|\pi_{[[\alpha_3]]} \circ f(x_1, U)| = |\pi_{[[\alpha_3]]} \circ f(x_2, U)|$ and for $(x_1, x_2) \in [[\alpha_3]]$ it follows from the definition of $[[\alpha_3]]$ that $|\pi_{[[\alpha_3]]} \circ f(x_1, U)| = |\pi_{[[\alpha_3]]} \circ f(x_2, U)|$. Control system g is defined by letting $g_y : V \to N$ be any isomorphism between V and $\pi_{[[\alpha_3]]} \circ f(x, U)$ where x is any point satisfying $\pi_{[[\alpha_3]]}(x) = y$. By definition of $[[\alpha_3]]$, for any x_1 and x_2 such that $\pi_{[[\alpha_3]]}(x_1) = y = \pi_{[[\alpha_3]]}(x_2)$ we have $\pi_{[[\alpha_3]]} \circ f(x_1, U) = \pi_{[[\alpha_3]]} \circ f(x_2, U)$ thus showing that g_y does not depend on the choice of x. Furthermore, an isomorphism between V and $\pi_{[[\alpha_3]]} \circ f(x, U)$ always exists since these sets have the same cardinality. If we now denote $\pi_{[[\alpha_3]]}$ by α_1 it follows by construction of g that $\alpha_1 \circ f(x, U) \subseteq g(\alpha_1(x), V)$ and that g_y is injective. Therefore, by Proposition 2.2 there exists a unique map α_2 making (α_1, α_2) a morphism from f to g. Note that α_2 is necessarily fiber surjective since $\alpha_1 \circ f(x, U) = g(\alpha_1(x), V)$ and $g_{\alpha_1(x)}$ is a bijection between V and $g(\alpha_1(x), V)$. In view of (5.2) we equip g with the output map $s = \pi_{[[\alpha_3]][[\alpha_3]]_0}$. This results in the desired surjective morphism $\alpha = (\alpha_1, \alpha_2, \alpha_3) : (f, r) \to (g, s)$ where surjectivity of s follows from surjectivity of r and α_3.

Let now $N' \times V' \xrightarrow{g'} N' \xrightarrow{s'} Q$ be any other control system equipped with a surjective morphism $\alpha' = (\alpha_1', \alpha_2', \alpha_3) : (f, r) \to (g', s')$. Since α_2' is fiber surjective and α_1 surjective, the equivalence relation $S \subseteq M \times M$ defined by $(x_1, x_2) \in S$ if $\alpha_1'(x_1) = \alpha_1'(x_2)$ is a bisimulation relation. Furthermore, $\alpha_3 \circ r$ factors through $\pi_S = \alpha_1'$ since $\alpha_3 \circ r = s' \circ \alpha_1'$. However, $[[\alpha_3]]$ is the coarsest equivalence relation satisfying these properties which implies existence of a surjective map $\pi_{S[[\alpha_3]]} = \beta_1 : N' \to N$ satisfying $\beta_1 \circ \pi_S = \pi_{[[\alpha_3]]}$. We also have:

$$\beta_1 \circ \pi_S \circ f(x, u) = \pi_{[[\alpha_3]]} \circ f(x, u)$$
$$\Rightarrow \beta_1 \circ g'(\alpha_1'(x), \alpha_2'(x, u)) = g(\alpha_1(x), \alpha_2(x, v)) \quad \text{since } \alpha_1 \text{ and } \alpha_1' \text{ are morphisms}$$
$$\Rightarrow \beta_1 \circ g'(\alpha_1'(x), V') \subseteq g(\alpha_1(x), V) = g(\beta_1 \circ \alpha_1'(x), V) \quad \text{since } S \text{ refines } [[\alpha_3]]$$

It now follows from injectivity of g_y and Proposition 2.2 the existence of a unique map β_2 making (β_1, β_2) a morphism from g' to g. Fiber urjectivity of β_2 can be shown by the same argument used to show fiber surjectivity of α_2 in the first part of the

proof and it follows from $\beta_1 \circ \alpha'_1 = \alpha_1$ that $(\beta_1, \beta_2) \circ (\alpha'_1, \alpha'_2) = (\alpha_1, \alpha_2)$. Furthermore, (β_1, β_2) is unique, up to isomorphism, since β_1 is uniquely determined by α_1 and α'_1 while β_2 is uniquely determined by β_1 and g, and g is unique up to the isomorphisms $g_y : V \rightarrow N$. To conclude the proof we only need to show that (2.6) is satisfied, that is, that $s' = s \circ \beta_1$ holds. But this follows from $s \circ \alpha_1 = \alpha_3 \circ r = s' \circ \alpha'_1$ combined with: $\beta_1 \circ \alpha'_1 = \alpha_1$ which results in $s \circ \beta_1 \circ \alpha'_1 = s' \circ \alpha'_1$. Surjectivity of α'_1 allows to simplify the previous expression and to obtain the desired equality $s' = s \circ \beta_1$.   □

The assumption expressed by equality (5.3) cannot be dropped since this can prevent the unique factorization property enjoyed by α_3-completions. This fact is illustrated in the following example. Consider control system $M \times U \xrightarrow{f} M$ with $M = \{x_1, x_2, x_3\}$, $U = \{u_1, u_2\}$, $Q = \{q_1, q_2\}$ and defined in Table 3. In this case we have $[[\alpha_3]]_0 = [[\alpha_3]]_1 = [[\alpha_3]]$. In particular $\pi_{[[\alpha_3]]} = \alpha_3$. Condition (5.3) is violated since:

$$\left| \alpha_3 \circ f(x_1, U) \right| = 2 \neq 1 = \left| \alpha_3 \circ f(x_3, U) \right|$$

Furthermore, the cardinality of the input set V of g has to be 2 since $|\alpha_3 \circ f(x_1, U)| = 2$ and for simplicity we will take $V = U$. We will now show that no control system with state set $N = M/[[\alpha_3]]$ and input set U satisfies the conclusions of Theorem 5.3. Existence of a morphism from f to g, implies that g satisfies:

$$\alpha_3 \circ f(x_1, U) = \{[x_1], [x_3]\} \subseteq g([x_1], V), \quad \alpha_3 \circ f(x_3, U) = \{[x_3]\} \subseteq g([x_3], V)$$

At state $[x_1]$, g is uniquely determined (up to isomorphism) by the fact that $\alpha_3 \circ f(x_1, u_1) = \alpha_3 \circ f(x_2, u_1)$ and $\alpha_3 \circ f(x_1, u_2) = \alpha_3 \circ f(x_2, u_2)$. We can thus define $g([x_1], u_1) = [x_1] = \alpha_3 \circ f(x_1, u_1) = \alpha_3 \circ f(x_2, u_1)$ and $g([x_1], u_2) = [x_3] = \alpha_3 \circ f(x_1, u_2) = \alpha_3 \circ f(x_2, u_2)$. At state $[x_3]$ we must have $g([x_3], u_1) = g([x_3], u_2) = [x_3]$. We will now show that there exists no fiber surjective map $\alpha_2 : M \times U \rightarrow V$ making $\alpha = (\alpha_1, \alpha_2) : f \rightarrow g$ a morphism such that for any other morphism $\alpha' : f \rightarrow g'$ there exists a unique morphism $\beta : g' \rightarrow g$ satisfying $\beta \circ \alpha' = \alpha$. For any fiber surjective map α_2 we must have $\alpha_2(x_3, u_1) \neq \alpha_2(x_3, u_2)$. Let us assume that $\alpha_2(x_3, u_1) = u_1$ and $\alpha_2(x_3, u_2) = u_2$ since the same argument works for $\alpha_2(x_3, u_1) = u_2$ and $\alpha_2(x_3, u_2) = u_1$. Consider now control system $g' = g$ and map $\alpha'_2 : M \times U \rightarrow V$ defined by:

$$\alpha'_2(x_3, u_1) = u_1, \quad \alpha'_2(x_3, u_2) = u_2, \quad \alpha'_2(x_4, u_1) = \alpha_2(x_4, u_2), \quad \alpha'_2(x_4, u_2) = \alpha_2(x_4, u_1)$$

and inducing morphism $\alpha' = (\alpha_1, \alpha'_2) : f \rightarrow g'$. Existence of a morphism $\beta : g' \rightarrow g$ satisfying $\beta \circ \alpha' = \alpha$ implies that:

$$\beta_2([x_3], a) = \beta_2 \circ (\alpha_1, \alpha'_2)(x_3, a) = \alpha_2(x_3, a) = a \tag{5.4}$$

$$\beta_2([x_3], u_2) = \beta_2 \circ (\alpha_1, \alpha'_2)(x_3, u_2) = \alpha_2(x_3, u_2) = u_2 \tag{5.5}$$

Table 3 Definition of control system f and map α_3

x	u	$f(x,u)$	$\alpha_3(x)$
x_1	u_1	x_1	q_1
x_1	u_2	x_3	q_1
x_2	u_1	x_2	q_1
x_2	u_2	x_4	q_1
x_3	u_1	x_3	q_2
x_3	u_2	x_4	q_2
x_4	u_1	x_4	q_2
x_4	u_2	x_4	q_2

However, at x_4 we have:

$$\beta_2([x_4], \alpha_2'(x_4, a)) = \beta_2([x_3], \alpha_2'(x_4, a)) \quad \text{since } \alpha_3(x_4) = \alpha_3(x_3)$$
$$= \beta_2([x_3], \alpha_2(x_4, u_2)) \quad \text{by definition of } \alpha_2'$$
$$= \alpha_2(x_4, u_2) \quad \text{since by (5.4) and (5.5), } \beta_{2[x_3]} = 1_U$$

We thus see that $\beta_2([x_4], \alpha_2'(x_4, a)) = \alpha_2(x_4, u_2) \neq \alpha_2(x_4, a)$ thus violating the desired equality $\beta \circ \alpha' = \alpha$. Therefore, no fiber surjective morphism $\beta : g' \to g$ satisfying $\beta \circ \alpha' = \alpha$ can exist. Furthermore, the unique factorization property of α_3-completions is violated even if we try to define g on a larger set N since in this case there would be several choices of maps $\beta_1 : M/[[\alpha_3]] \to N$ that could be completed to a morphism from g' to g.

5.2 Existence of output induced symbolic α_2-completions

We now turn our attention to existence of symbolic α_3-completions. Since $[[\alpha_3]]$ is the coarsest bisimulation relation between (f, r) and (f, r) satisfying (5.2) the following corollary of Theorem 5.3 is immediate:

Corollary 5.4 *Let $M \times U \xrightarrow{f} M \xrightarrow{r} P$ be a control system with surjective output map and let $\alpha_3 : P \to Q$ be a surjective map onto a finite set Q for which the α_3-completion (g, s) of (f, r) exists. Control system (g, s) has finite state set iff there exists a $k \in \mathbb{N}$ such that any pair of distinguishable states by α_3 is distinguishable in k steps.*

Corollary 5.4, by equating finiteness of α_3-completions to finite distinguishability, suggest that we should identify the class of control systems generating the same outputs after a certain finite number of steps since this guarantees stabilization of the indistinguishability relations after a finite number of steps. We formalize this notion in the next result.

Corollary 5.5 *Let $M \times U \xrightarrow{f} M \xrightarrow{r} P$ be a control system with surjective output map and let $\alpha_3 : P \to Q$ be a surjective map onto a finite set Q. If $[[\alpha_3]]$ satisfies (5.3) and for any $x_1, x_2 \in M$ and $u \in U^*$ there exists a $k \in \mathbb{N}$ for which the following holds:*

$$r \circ f^*(x_1, u[i]) = r \circ f^*(x_2, u[i]) \quad \forall i \geq k$$

then the α_3-completion of (f, r) exists and has finite state set.

We can use Corollary 5.5 to identify concrete classes of control systems admitting finite bisimulations.

Corollary 5.6 *Let $M \times U \xrightarrow{f} M \xrightarrow{r} P$ be a control system and let $\alpha_3 : P \to Q$ be a surjective map satisfying any of the following assumptions:*

(1) *f is a discrete-time controllable linear system, that is, $f(x, u) = Ax + Bu$ with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, A and B are matrices of appropriate dimensions and the matrix $[A^{n-1}B | A^{n-2}B | \dots | B]$ has rank n;*

(2) *f is a discrete-time linear system, α_3 is a linear map and (f, α_3) is output-controllable, that is, $f(x, u) = Ax + Bu$ with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, A and B matrices of appropriate dimensions, $\alpha_3(x) = Cx \in \mathbb{R}^o$ with C a matrix of appropriate dimensions, and the matrix $[CA^{n-1}B | CA^{n-2}B | \dots | CB]$ has rank o.*

If $[[\alpha_3]]$ satisfies (5.3) then the α_3-completion of f exists and has finite state set.

Note that even if $[[\alpha_3]]$ fails to satisfy (5.3) a finite state bisimilar control system still exists and this is sufficient for most of the applications.

Proof Let us first assume that $f(x, u) = Ax + Bu$ is in Brunovsky normal form. For $m = 1$ this normal form is given by:

$$
\begin{aligned}
y_1(t + \tau) &= y_2(t) \\
y_2(t + \tau) &= y_3(t) \\
&\vdots \\
y_n(t + \tau) &= v(t)
\end{aligned}
\tag{5.6}
$$

By inspecting (5.6) we immediately conclude that $f^{j+n}(y, v) = (v(j), v(j+1), \dots, v(j+n))$ for $j \geq 0$ which implies $r \circ f^*(y, v[i]) = r \circ f^*(y', v[i])$ for $i \geq n$ and for any map $r : \mathbb{R}^n \to P$. The assumptions of Corollary 5.5 are thus satisfied. For $m > 1$ a similar argument applies since in this case the Brunovsky normal form is given by several blocks of the form (5.6). For general controllable linear systems the result follows from the previous argument and from the fact [4,38] that any controllable linear system is isomorphic to a linear system in Brunovsky normal form.

By Kalman's decomposition Theorem [4,38] any linear control system can be decomposed as:

$$
\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t)
\tag{5.7}
$$

where $x_1(t+1) = A_{11}x_1(t) + A_{12}x_2(t) + B_1 u(t)$ is controllable. It then follows from the previous argument that any linear control system equipped with an output map $r : M = \mathbb{R}^n \to P$ admits a finite bisimulation provided that r factors as $r = s \circ \pi$ for any map $s : \mathbb{R}^o \to P$ and for the projection $\pi(x) = x_1$. This factorization condition can be checked by determining if the matrix $[CB|CAC|\dots|CA^{n-1}B]$ defines a surjective linear transformation when r is a linear transformation and C is its matrix representation. $\qquad\square$

Let us consider again the simplest discrete-time controllable linear system:

$$
\begin{aligned}
x_1(t + \tau) &= x_2(t) \\
x_2(t + \tau) &= u(t)
\end{aligned}
$$

with $r(x_1, x_2) = (x_1, x_2)$ and let α_3 be given by:

$$
\alpha_3(x) = \begin{cases} -1 & \text{if } x_1 \leq -1 \\ 0 & \text{if } -1 < x_1 < 1 \\ 1 & \text{if } x_1 \geq 1 \end{cases}
\tag{5.8}
$$

The relation $[[\alpha_3]]_0$ is defined the following three equivalence classes:

$$
S_j = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in X_j\}, \quad j \in \{-, 0, +\}
$$

where $X_- =]-\infty, -1]$, $X_0 =]-1, 1[$ and $X_+ = [1, \infty[$. Relation $[[\alpha_3]]_1$ satisfies $[[\alpha_3]]_1 = [[\alpha_3]]_2$ and is defined by the equivalence classes:

$$
S_j^i = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in X_i \wedge x_2 \in X_j\}, \quad j \in \{-, 0, +\}
$$

The α_3-completion is now given by:

$$g(S_j^i, X_-) = S_-^j \quad g(S_j^i, X_0) = S_0^j \qquad g(S_j^i, X_+) = S_+^j, \quad i, j \in \{-, 0, +\}$$

where V is identified with $\cup_{i \in \{-, 0, +\}} \{X_i\}$ and N identified with $\cup_{i, j \in \{-, 0, +\}} \{S_j^i\}$. The fact that the α_3 completion is the same as the α_2-completion presented in Sect. 4 for the same control system is due to the fact that we have chosen a control system of a very special form (Brunovsky normal form). In general, α_2-completions and α_3-completions are not isomorphic.

6 Discussion

In this paper we have presented two very general constructions on the class of control systems: α_2-completions and α_3-completions. The first class of completions is mathematically very natural since no conditions are necessary for its existence. This implies that Theorem 4.3 is valid in a variety of different contexts obtained by imposing different structure on control systems (linearity, smoothness, analyticity, etc). This is in sharp contrast with α_3-completions which exist under stricter conditions. One of those conditions is surjectivity of the involved morphisms which has to be considerably strengthened when more structure is imposed such as smoothness for example. This is a consequence of Theorem 5.3 being based on the fact that every surjective map $\phi : A \to B$ can be seen as the projection from A to A/S where S is the equivalence relation on A identifying points a_1 and a_2 when $\phi(a_1) = \phi(a_2)$.

A third type of completion could also be considered if one starts with a control system and a map $\alpha_1 : M \to N$ between state sets. However, since our objective is to obtain bisimilar models, either the graph of α_1 already defines a bisimulation or there is no possibility of obtaining a bisimilar model by constructing a control system with state set N. This fact justifies why α_1-completions have not been considered in this paper.

Even though we make a distinction between inputs and outputs this distinction is not natural in many systems and it would be interesting to extend the results of this paper by dropping this distinction. Such behavioral approach [34] could help understand how to relate the sufficient conditions for the existence of symbolic models provided by Corollaires 4.8 and 5.5 and lead to a deeper understanding of which control systems are essentially finite.

Ongoing research is focusing on extending and characterizing new classes of systems satisfying the sufficient conditions for the existence of finite completions. While Corollary 4.8 requires reachability in a finite number of steps, Corollary 5.5 requires a finite output controllability or tracking assumption ensuring that the effect of the initial state will only influence the output in a finite number of steps. Even though the conditions required by Corollary 5.5 appear to be stronger than those required by Corollary 4.8, the possibility of choosing the output map appears to provide greater flexibility in the use of Corollary 5.5.

Appendix

Recall that an equivalence relation R on a set A is a subset $R \subseteq A \times A$ satisfying:

(1) Reflexivity: $\forall a \in M, (a, a) \in R$;
(2) Symmetry: $\forall a_1, a_2 \in M, (a_1, a_2) \in R$ implies $(a_2, a_1) \in R$;
(3) Transitivity: $(a_1, a_2) \in R$ and $(a_2, a_3) \in R$ implies $(a_1, a_3) \in R$.

Given an equivalence relation $R \subseteq A \times A$, we denote by $[a]$ the equivalence class containing $a \in A$ and by $\pi_R : A \to A/R$ the (surjective) natural projection map taking each point $a \in A$ to its equivalence class $[a]$. Conversely, any surjective map $\phi : A \to B$ defines an equivalence relation on A whose equivalence classes are given by $\phi^{-1}(b)$ for each $b \in B$.

An equivalence relation $R_2 \subseteq A \times A$ is said to refine equivalence relation $R_1 \subseteq A \times A$ if for every equivalence class E_2 of R_2 there is an equivalence class E_1 of R_1 such that $E_2 \subseteq E_1$. Equivalently, R_2 refines R_1 if there exists a map $\pi_{R_2 R_1} : A/R_2 \to A/R_1$ satisfying $\pi_{R_2 R_1} \circ \pi_{R_2} = \pi_{R_1}$. Map $\pi_{R_2 R_1}$ takes an equivalence class E_2 of R_2 into the unique equivalence class $\pi_{R_2 R_1}(E_2)$ of R_1 containing E_2.

Any arbitrary relation $R \subseteq A \times A$ can be completed to an equivalence relation by adding the identity relation $I = \{(a_1, a_2) \in A \times A \mid a_1 = a_2\}$ to R, resulting in $R \cup I$, and by closing $R \cup I$ under reflexivity and transitivity. This closure $\overline{R}$ is called the equivalence relation generated by R and it is the coarsest equivalence relation containing the pairs $(a_1, a_2) \in R$ in the sense that any other equivalence relation containing the pairs $(a_1, a_2) \in R$ is a refinement of $\overline{R}$.

Consider now an equivalence relation $R \subseteq A \times A$ and a map $f : A \to A$. Relation R is said to be invariant under map f when $(a_1, a_2) \in R$ implies $(f(a_1), f(a_2)) \in R$. In this case f induces a map $f/R : A/R \to A/R$ defined by $\big(f/R\big)([a]) = [f(a)]$. Map f/R is well define since for any $a_2, a_3 \in [a_1]$ we have $(f(a_2), f(a_1)) \in R$ and $(f(a_3), f(a_1)) \in R$. Consequently, $[f(a_2)] = [f(a_3)] = [f(a_1)] = \big(f/R\big)([a_1])$. If an equivalence relation R is not invariant under a map f, we can close it under f resulting in the coarsest equivalence relation $\overline{R}$ containing the pairs $(a_1, a_2) \in R$ and invariant under f, that is, any other equivalence relation containing the pairs $(a_1, a_2) \in R$ and invariant under f is a refinement of $\overline{R}$.

Consider now a sequence of equivalence relations $\{R_i\}_{i \in \mathbb{N}}$ on a set A where for each $i \in \mathbb{N}$, R_{i+1} is a refinement of R_i. The limit of this sequence is an equivalence relation $R = \lim_{i \to \infty} R_i$ refining every equivalence relation R_i and such that for any other equivalence relation S refining every equivalence relation R_i, S is a refinement of R.

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