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RESEARCH ARTICLE

Structural Decomposition of Thermo-elastic Semigroups with Rotational Forces*

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Abstract

We consider a (linear) system of thermo-elastic plate equations which accounts for rotational forces, under all canonical boundary conditions (B.C.). These include cases of *coupled* B.C. such as: hinged mechanical/Neumann thermal B.C., and the most challenging case of all, the so-called case of free B.C. In all cases, the original thermo-elastic s.c. semigroup of contractions admits a structural decomposition, for all positive times, as the sum of a "non-compact semigroup" and a compact component. In all cases, save (at present) the case of free B.C., the "non-compact semigroup" component is actually a s.c. uniformly (exponentially) stable group, based only on the mechanical variables: as a consequence, a precise uniform (exponential) stability result of the original thermo-elastic semigroup is then obtained. For the free B.C. case, the "non-compact" semigroup corresponds to a simpler problem with *uncoupled* elastic equation and shear force B.C. The stated structural decomposition requires, for its proof, sharp/optimal regularity theory of the associated elastic Kirchoff equation; including two new such results as in [27, 28] for hinged/Neumann and for free B.C., respectively. The structural decomposition results of this paper for models that account for rotational forces are at striking contrast with the property of analyticity of the thermo-elastic semigroup, which characterizes, instead, models which do not account for rotational forces. Implications on exact controllability are noted.

1. Introduction. Problem statement. Main results

1.1. A coupled B.C. case: Hinged mechanical B.C. and Neumann (Robin) thermal B.C.

The partial differential equations (P.D.E.'s) of linear thermo-elastic plate equations on a bounded domain Ω of $\mathbf{R}^2$ are derived e.g. in Lagnese [18]. In general, a thermo-elastic system consists of an elastic equation in the vertical displacement w and a heat equation in the relative temperature θ about the stress-free state $\theta = 0$, which transfer mechanical and thermal energy through coupling. In the linear, homogeneous case, if one normalizes inessential constants and omits lower-order terms, these equations are

$$\begin{cases} w_{tt} - \gamma\Delta w_{tt} + \Delta^2 w + \Delta\theta = 0 \quad \text{in } (0,T] \times \Omega \equiv Q; & (1.1.1) \\ \theta_t - \Delta\theta - \Delta w_t = 0 \quad \text{in } Q; & (1.1.2) \\ w(0,\cdot) = w_0;\ w_t(0,\cdot) = w_1;\ \theta(0,\cdot) = \theta_0 \quad \text{in } \Omega, & (1.1.3) \end{cases}$$

to be augmented by boundary conditions (B.C.) on $\partial\Omega$, where throughout this paper, the constant γ is positive: $\gamma > 0$. We shall associate with the above equations several canonical B.C. We begin in this section with a challenging case of *coupled*

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B.C.: hinged mechanical B.C. and Neumann (Robin) thermal B.C., i.e., if $\nu =$ unit outward normal

$$w|_{\Sigma} \equiv 0, \;\; [\Delta w + \theta]_{\Sigma} \equiv 0; \;\; \left[\frac{\partial \theta}{\partial \nu} + b\theta\right]_{\Sigma} \equiv 0, \;\; b \geq 0, \;\; \Sigma = (0, T] \times \Gamma\,. \tag{1.1.4}$$

We note the *coupling between* w and θ *in the second* B.C. It is critical to distinguish between the case $\gamma = 0$ (whereby Eqn. (1.1.1) becomes the Euler-Bernoulli equation with infinite speed of propagation) and the case $\gamma > 0$ (whereby Eqn. (1.1.1) becomes the hyperbolic Kirchoff equation with finite speed of propagation. The constant γ accounts for rotational forces and is proportional to the square of the thickness of the plate in the two-dimensional case.

For $\gamma = 0$ in (1.1.1) it has been recently shown that under *all canonical* B.C., Eqns. (1.1.1) – (1.1.3) define a s.c. contraction semigroup: $[w_0, w_1, \theta_0] \rightarrow [w(t), w_t(t), \theta(t)]$ on a natural energy space, which, moreover, is *analytic*: see [32] for the first (technical) key contribution in the case of clamped mechanical B.C./Dirichlet thermal B.C., which spurred interest on this issue; and, subsequently, [21-23] for all canonical B.C.; see also [31] for some uncoupled cases by an indirect proof by contradiction. A recent review is given in [24]. As a consequence of analyticity, such contraction semigroups are, moreover, uniformly (exponentially) stable.

The present paper is entirely devoted to the case $\gamma > 0$ in (1.1.1): here, the corresponding s.c. contraction semigroup on a different natural energy space displays radically different structural properties, as the main results of this paper will show. However, the property of uniform exponential stability is preserved. See Literature below.

Abstract model and well-posedness. Our starting point is an abstract model for problem (1.1.1) – (1.1.4), which is readily seen to be (details e.g., in [3; 25, Chapter 3, Section 12]

$$\begin{cases} w_{tt} + \gamma \mathcal{A}_D w_{tt} + \mathcal{A}_D^2 w - \mathcal{A}_R \theta = \mathcal{A}_D D(\theta|_{\Gamma}); & (1.1.5) \\ \theta_t + \mathcal{A}_R \theta + \mathcal{A}_D w_t = 0, & (1.1.6) \end{cases}$$

$$\mathcal{A}_D f = -\Delta f, \; \mathcal{D}(\mathcal{A}_D) = H^2(\Omega) \cap H_0^1(\Omega); \; \mathcal{D}(\mathcal{A}_D^{\frac{1}{2}}) = H_0^1(\Omega); \tag{1.1.7}$$

$$\mathcal{A}_R f = -\Delta f, \; \mathcal{D}(\mathcal{A}_R) = \left\{ f \in H^2(\Omega) : \left[\frac{\partial f}{\partial \nu} + bf\right]_{\Gamma} = 0 \right\}; \; \mathcal{D}(\mathcal{A}_R^{\frac{1}{2}}) = H^1(\Omega); \tag{1.1.8}$$

$$\mathcal{A}_D^2 f = \Delta^2 f, \; \mathcal{D}(\mathcal{A}_D^2) = \{ f \in H^4(\Omega) : f|_{\Gamma} = \Delta f|_{\Gamma} = 0 \}; \tag{1.1.9}$$

$$\left.\begin{array}{l} h = Dg \Longleftrightarrow \{\Delta h = 0 \text{ in } \Omega : h|_{\Gamma} = g\}; \\ D: \text{ continuous } H^s(\Gamma) \rightarrow H^{s+\frac{1}{2}}(\Omega), \;\; s \in R; \end{array}\right\} \tag{1.1.10}$$

$$\text{thus } \; L_2(\Gamma) \rightarrow H^{\frac{1}{2}}(\Omega) \subset H^{\frac{1}{2}-2\epsilon}(\Omega) = \mathcal{D}(\mathcal{A}_D^{\frac{1}{4}-\epsilon}), \; \epsilon > 0. \tag{1.1.11}$$

We shall explicitly consider the case $b > 0$ throughout (Robin B.C.); for $b = 0$ (Neumann B.C.), one works on $L_2(\Omega)$ factored by the one-dimensional null space of $\mathcal{A}_R$. Setting throughout $y = [w, w_t, \theta]$, the first-order system corresponding to (1.1.5), (1.1.6) is

$$\dot{y} = \mathbb{A}_\gamma y;\ \mathbb{A}_\gamma : Y_\gamma \supset \mathcal{D}(\mathbb{A}_\gamma) \to Y_\gamma;\ Y_\gamma = \mathcal{D}(\mathcal{A}_D) \times \mathcal{D}(\mathcal{A}_{D,\gamma}^{\frac{1}{2}}) \times L_2(\Omega); \tag{1.1.12}$$

$$\mathcal{A}_{D,\gamma} = (I + \gamma\mathcal{A}_D);\ (x_1, x_2)_{\mathcal{D}(\mathcal{A}_{D,\gamma}^{\frac{1}{2}})} = ((I + \gamma\mathcal{A}_D)x_1, x_2)_{L_2(\Omega)}; \tag{1.1.13}$$

$$\mathbb{A}_\gamma = \begin{bmatrix} 0 & I & 0 \\ -\mathcal{A}_{D,\gamma}^{-1}\mathcal{A}_D^2 & 0 & \mathcal{A}_{D,\gamma}^{-1}[\mathcal{A}_R + \mathcal{A}_D D(\,\cdot\,|_\Gamma)] \\ 0 & -\mathcal{A}_D & -\mathcal{A}_R \end{bmatrix}; \tag{1.1.14a}$$

$$\mathcal{D}(\mathbb{A}_\gamma) = \left\{ w_1, w_2 \in \mathcal{D}(\mathcal{A}_D), \theta \in \mathcal{D}(\mathcal{A}_R) :\ [\mathcal{A}_D w_1 - D(\theta|_\Gamma)] \in \mathcal{D}(\mathcal{A}_D^{\frac{1}{2}}) \right\}. \tag{1.1.14b}$$

Proposition 1.1.1. (i) *The operator* $\mathbb{A}_\gamma$ *in* (1.1.14) *is dissipative and becomes skew-adjoint on* Y_γ, *if one removes the bottom-right corner element* $-\mathcal{A}_R$ *from* (1.1.14a):

$$\operatorname{Re}(\mathbb{A}_\gamma x, x)_{Y_\gamma} = -(\mathcal{A}_R x_3, x_3)_{L_2(\Omega)},\ x = [x_1, x_2, x_3] \in \mathcal{D}(\mathbb{A}_\gamma) \tag{1.1.15}$$

(ii) *In fact,* $\mathbb{A}_\gamma$ *is maximal dissipative and generates a s.c. contraction semigroup* $e^{\mathbb{A}_\gamma t}$ *on the space* Y_γ *defined by* (1.1.12): $[w(t), w_t(t), \theta(t)] = e^{\mathbb{A}_\gamma t}[w_0, w_1, \theta_0] \in C([0,T]; Y_\gamma)$.

(iii) *The resolvent* $R(\lambda, \mathbb{A}_\gamma)$ *is compact on* Y_γ. ■

For details of the proof via the Lumer-Phillip theorem we refer to [3], [25, Chapter 3, Section 12].

Main result: structural decomposition. Substituting $\mathcal{A}_R\theta$ from (1.1.6) into (1.1.5), yields the equation

$$w_{tt} + \gamma\mathcal{A}_D w_{tt} + \mathcal{A}_D^2 w + \mathcal{A}_D w_t = \mathcal{A}_D D(\theta|_\Gamma) - \theta_t\,. \tag{1.1.16}$$

We then introduce the damped Kirchoff equation corresponding to the left-hand side of Eqn. (1.1.16):

$$\phi_{tt} + \gamma\mathcal{A}_D\phi_{tt} + \mathcal{A}_D^2\phi + \mathcal{A}_D\phi_t = 0;\ \phi(0) = w_0, \phi_t(0) = w_1\,, \tag{1.1.17}$$

whose solution is (compare with (1.1.12) for Y_γ)

$$\begin{bmatrix} \phi(t) \\ \phi_t(t) \end{bmatrix} = e^{\mathbb{A}_{1,\gamma} t} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \in C([0,T]; Y_{1,\gamma});\ Y_{1,\gamma} \equiv \mathcal{D}(\mathcal{A}_D) \times \mathcal{D}(\mathcal{A}_{D,\gamma}^{\frac{1}{2}}), \tag{1.1.18}$$

$\mathcal{A}_{D,\gamma}$ defined in (1.1.13), where the operator $\mathbb{A}_{1,\gamma}$ is defined by

$$\mathbb{A}_{1,\gamma} = \begin{bmatrix} 0 & I \\ -\mathcal{A}_{D,\gamma}^{-1}\mathcal{A}_D^2 & -\mathcal{A}_{D,\gamma}^{-1}\mathcal{A}_D \end{bmatrix} :\ \mathcal{D}(\mathbb{A}_{1,\gamma}) = \mathcal{D}(\mathcal{A}_D^{\frac{3}{2}}) \times \mathcal{D}(\mathcal{A}_D); \tag{1.1.19}$$

and plainly generates a s.c. group $e^{\mathbb{A}_{1,\gamma} t}$ on $Y_{1,\gamma}$ (the entry $\mathcal{A}_{D,\gamma}^{-1}\mathcal{A}_D$ is a bounded perturbation on $\mathcal{D}(\mathcal{A}_{D,\gamma}^{\frac{1}{2}})$, see (1.1.13)). Moreover, $e^{\mathbb{A}_{1,\gamma} t}$ is uniformly (exponentially) stable on $Y_{1,\gamma}$: there exist constants $M \geq 1,\ \delta > 0$ such that, as is well known (and can be shown in a few ways)

$$\|e^{\mathbb{A}_{1,\gamma} t}\|_{\mathcal{L}(Y_{1,\gamma})} \leq M\, e^{-\delta t},\ t \geq 0,\ \delta > 0\,. \tag{1.1.20}$$

Our main result of the present subsection for problem (1.1.1) – (1.1.4) is the following structural decomposition theorem.

Theorem 1.1.2. *With reference to problem* (1.1.1) – (1.1.4) *and its corresponding s.c. contraction semigroup* $e^{\mathbb{A}_\gamma t}$ *guaranteed by Proposition* 1.1.1, *we have that the following decomposition holds true for all* $t > 0$:

$$\begin{bmatrix} w(t) \\ w_t(t) \\ \theta(t) \end{bmatrix} = e^{\mathbb{A}_\gamma t} \begin{bmatrix} w_0 \\ w_1 \\ \theta_0 \end{bmatrix} = \begin{bmatrix} \left[e^{\mathbb{A}_{1,\gamma} t} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right] \\ 0 \end{bmatrix} + \mathcal{K}_\gamma(t) \begin{bmatrix} w_0 \\ w_1 \\ \theta_0 \end{bmatrix}, \tag{1.1.21}$$

where $e^{\mathbb{A}_{1,\gamma} t}$ *is the s.c. contraction uniformly stable group defined in* (1.1.19) – (1.1.20), *and where the operator* $\mathcal{K}_\gamma(t)$ (*defined explicitly in* (2.29) *below*) *satisfies*

$$\mathcal{K}_\gamma(t): \ \textit{compact}\ Y_\gamma \to Y_\gamma. \qquad \blacksquare \tag{1.1.22}$$

Corollary 1.1.3. *The s.c. contraction semigroup* $e^{\mathbb{A}_\gamma t}$ *is uniformly* (*exponentially*) *stable on* Y_γ : *moreover,* $r_{\text{ess}}(e^{\mathbb{A}_\gamma t}) = r_{\text{ess}}(e^{\mathbb{A}_{1,\gamma} t}) < 1,\ t > 0,$ *where* r_{ess} *denotes the essential spectrum radius. Thus,* $e^{\mathbb{A}_{1,\gamma} t}$ *determines the decay of* $e^{\mathbb{A}_\gamma t}$, *except possibly for a finite-dimensional subspace.* ■

Remark 1.1.1. *A fortiori*, it follows from the decomposition in (1.1.21), the s.c. semigroup $e^{\mathbb{A}_\gamma t}$ cannot be compact for all $t > 0$ (for otherwise the group $e^{\mathbb{A}_{1,\gamma} t}$ would be compact for all $t > 0$, impossible!). Equivalently, since the resolvent $R(\lambda, \mathbb{A}_\gamma)$ of $\mathbb{A}_\gamma$ is compact by Proposition 1.1.1 (iii), then $e^{\mathbb{A}_\gamma t}$ cannot be continuous in the uniform operator topology of Y_γ for all $t > 0$ [33, p. 48-51]. These results for $\gamma > 0$ should be contrasted with the case $\gamma = 0$ where the corresponding s.c. semigroup is *analytic* on Y_γ, [21-24], in particular, [22] for the B.C. in (1.1.4). ■

1.2. An abstract model for uncoupled B.C.

In this subsection we introduce an abstract thermo-elastic model, which in particular will cover concrete thermo-elastic problems (1.1.1) – (1.1.3), possibly defined on a bounded domain $\Omega \subset R^n$, with *uncoupled* B.C. (see Section 4).

Assumptions and well-posedness. With motivation coming from Eqns. (1.1.1), (1.1.2), the abstract model considered in this subsection is

$$\begin{cases} w_{tt} + \gamma C w_{tt} + A w - B\theta = 0, \quad \gamma > 0; & (1.2.1) \\ \theta_t + B\theta + B w_t = 0; & (1.2.2) \\ w(0) = w_0 \in \mathcal{D}(A^{\frac{1}{2}});\ w_t(0) = w_1 \in \mathcal{D}(A^{\frac{1}{4}});\ \theta(0) = \theta_0 \in H, & (1.2.3) \end{cases}$$

with constant $\gamma > 0$ *throughout*, under the following standing assumptions:

(H.1) (i) A, B, C are (unbounded) positive, self-adjoint operators on the Hilbert space H and $\mathcal{D}(A^{\frac{1}{2}}) \cap \mathcal{D}(B)$ is dense in H; (ii) with compact resolvent (as in the motivating physical models).

(H.2)

$$\mathcal{D}(A^{\frac{1}{4}}) \subset \mathcal{D}(B^{\frac{1}{2}}) \iff B^{\frac{1}{2}} A^{-\frac{1}{4}} \in \mathcal{L}(H), \tag{1.2.4}$$

(H.3)

$$\mathcal{D}(A^{\frac{1}{4}}) = \mathcal{D}(C^{\frac{1}{2}}); \text{ so that } A^{\frac{1}{4}} C^{-\frac{1}{2}} \in \mathcal{L}(H) \text{ and } C^{\frac{1}{2}} A^{-\frac{1}{4}} \in \mathcal{L}(H). \tag{1.2.5}$$

Combining (H.3) = (1.2.5) with (H.2) = (1.2.4), we obtain that

$$\mathcal{D}(C^{\frac{1}{2}}) \subset \mathcal{D}(B^{\frac{1}{2}}) \iff B^{\frac{1}{2}} C^{-\frac{1}{2}} \in \mathcal{L}(H). \tag{1.2.6}$$

The first-order version corresponding to Eqns. (1.2.1), (1.2.2) is

$$\dot{y}(t) = \mathbb{A}_\gamma y(t), \; y(0) = y_0 = [w_0, w_1, \theta_0] \in Y_\gamma; \; y(t) = [w(t), w_t(t), \theta(t)]; \tag{1.2.7}$$

$$\mathbb{A}_\gamma = \begin{bmatrix} 0 & I & 0 \\ -C_\gamma^{-1} A & 0 & C_\gamma^{-1} B \\ 0 & -B & -B \end{bmatrix} : Y_\gamma \supset \mathcal{D}(\mathbb{A}_\gamma) \to Y_\gamma; \tag{1.2.8}$$

$$\left.\begin{aligned} &\mathcal{D}(\mathbb{A}_\gamma) = \mathcal{D}(A^{\frac{3}{4}}) \times [\mathcal{D}(A^{\frac{1}{2}}) \cap \mathcal{D}(B)] \times \mathcal{D}(B); \\ &Y_\gamma \equiv \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(C_\gamma^{\frac{1}{2}}) \times H \equiv Y_{1,\gamma} \times H\,; \end{aligned}\right\} \tag{1.2.9}$$

$$\left.\begin{aligned} &Y_{1,\gamma} \equiv \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(C_\gamma^{\frac{1}{2}}), \mathcal{D}(C_\gamma^{\frac{1}{2}}) = \mathcal{D}(A^{\frac{1}{4}}); \\ &C_\gamma = (I + \gamma C), \; (x_1, x_2)_{\mathcal{D}(C_\gamma^{\frac{1}{2}})} = (C_\gamma x_1, x_2)_H\,. \end{aligned}\right\} \tag{1.2.10}$$

Notice that at the level of writing $\mathcal{D}(\mathbb{A}_\gamma)$ in (1.2.9), as dense in Y_γ, we have used (H.1) as well as $\mathcal{D}(C^{\frac{1}{2}}) \subset \mathcal{D}(A^{\frac{1}{4}})$, contained in (H.3) = (1.2.5). By (1.2.10) (right)

$$\left.\begin{aligned} &\text{the operators } C_\gamma^{-1} A \text{ and } C_\gamma^{-1} B \text{ are} \\ &\text{positive self-adjoint on the space } \mathcal{D}(C_\gamma^{\frac{1}{2}}). \end{aligned}\right\} \tag{1.2.11}$$

The densely defined operator $\mathbb{A}_\gamma$ in (1.2.8), (1.2.9) is dissipative, hence closable [33, p. 15]. [In the applications of Section 4, we have $\mathcal{D}(A^{\frac{1}{2}}) \subset \mathcal{D}(B)$, and $\mathbb{A}_\gamma$ is closed.] Once closed, the operator obtained from $\mathbb{A}_\gamma$ in (1.2.8) by omitting the bottom-right corner element $(-B)$ is *skew-adjoint* on the space Y_γ defined by (1.2.10). Thus, the Lumer-Phillips theorem, or its corollary as in [33, p. 15], readily yields

Proposition 1.2.1. *Assume* (H.1) (i), (H.2), (H.3). *Then:*

(i)
$$\mathrm{Re}\,(\mathbb{A}_\gamma x, x)_{Y_\gamma} = -(Bx_3, x_3)_H,\ \ x = [x_1, x_2, x_3] \in \mathcal{D}(\mathbb{A}_\gamma)\,.$$

(ii) *Once closed, $\mathbb{A}_\gamma$ is maximal dissipative and hence is the generator of a s.c. contraction semigroup $e^{\mathbb{A}_\gamma t}$ on the space Y_γ in* (1.2.9)*: $y(t) = [w(t), w_t(t), \theta(t)] = e^{\mathbb{A}_\gamma t}[w_0, w_1, \theta_0] \in C([0,T]; Y_\gamma)$.*

(iii) *Assume further* (H.1) (ii). *Then, the resolvent $R(\lambda, \mathbb{A}_\gamma)$ of $\mathbb{A}_\gamma$ is compact on Y_γ.* ■

Main result: structural decomposition. In order to state our main result, Theorem 1.2.2 below, we need to introduce two (uncoupled) systems related to (1.2.1): the abstract undamped Kirchoff equation

$$\begin{cases} \psi_{tt} + \gamma C \psi_{tt} + A\psi = 0,\ \psi(0) = \psi_0;\ \psi_t(0) = \psi_1\,; & (1.2.12) \\ \{\psi_0, \psi_1\} \in Y_{1,\gamma} \equiv \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(C_\gamma^{\frac{1}{2}}),\ \mathcal{D}(C_\gamma^{\frac{1}{2}}) = \mathcal{D}(A^{\frac{1}{4}})\,. & (1.2.13) \end{cases}$$

as well as its damped version

$$\begin{cases} \phi_{tt} + \gamma C \phi_{tt} + A\phi + B\phi_t = 0,\ \phi(0) = \phi_0;\ \phi_t(0) = \phi_1\,; & (1.2.14) \\ \{\phi_0, \phi_1\} \in Y_{1,\gamma} = \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(C_\gamma^{\frac{1}{2}}),\ \mathcal{D}(C_\gamma^{\frac{1}{2}}) = \mathcal{D}(A^{\frac{1}{4}})\,. & (1.2.15) \end{cases}$$

Thus, we set, with $\mathcal{D}(\mathbb{A}_{1,\gamma}) = \mathcal{D}(\mathbb{A}_{0,\gamma}) = \mathcal{D}(A^{\frac{3}{4}}) \times \mathcal{D}(A^{\frac{1}{2}})$,

$$\left.\begin{aligned} \mathbb{A}_{0,\gamma} &= \begin{bmatrix} 0 & I \\ -C_\gamma^{-1}A & 0 \end{bmatrix}; \\ \mathbb{A}_{1,\gamma} &= \begin{bmatrix} 0 & I \\ -C_\gamma^{-1}A & -C_\gamma^{-1}B \end{bmatrix} = \mathbb{A}_{0,\gamma} + \begin{bmatrix} 0 & 0 \\ 0 & -C_\gamma^{-1}B \end{bmatrix} :\ \mathcal{D}(\mathbb{A}_{0,\gamma}) \to Y_{1,\gamma}. \end{aligned}\right\} \quad (1.2.16)$$

In (1.2.16), the perturbation $C_\gamma^{-1}B$ is *bounded* on the space $\mathcal{D}(C_\gamma^{\frac{1}{2}})$, which is the second component space of $Y_{1,\gamma}$: recalling (1.2.10) and the implication (1.2.6),

$$\|C_\gamma^{-1}Bx\|_{\mathcal{D}(C_\gamma^{\frac{1}{2}})} = \|C_\gamma^{-\frac{1}{2}}B^{\frac{1}{2}}B^{\frac{1}{2}}C_\gamma^{-\frac{1}{2}}C_\gamma^{\frac{1}{2}}x\|_H \le k\|C_\gamma^{\frac{1}{2}}x\|_H = k\|x\|_{\mathcal{D}(C_\gamma^{\frac{1}{2}})}. \qquad (1.2.17)$$

The operator $\mathbb{A}_{0,\gamma}$ in (1.2.16) is skew-adjoint on the space $Y_{1,\gamma}$ in (1.2.13) and thus generates a s.c. unitary group $e^{\mathbb{A}_{0,\gamma}t}$ on $Y_{1,\gamma}$. By (1.2.17), the operator $\mathbb{A}_{1,\gamma}$ in (1.2.16) generates likewise a s.c. contraction group $e^{\mathbb{A}_{1,\gamma}t}$ on $Y_{1,\gamma}$ which, moreover, is uniformly stable: there exist constants $M \ge 1$ and $\delta > 0$, such that

$$\|e^{\mathbb{A}_{1,\gamma}t}\|_{\mathcal{L}(Y_{1,\gamma})} \le M\, e^{-\delta t},\ \ t \ge 0,\ \ \delta > 0\,. \qquad (1.2.18)$$

Thus, the solutions $[\psi(t), \psi_t(t)]$, $[\phi(t), \phi_t(t)] \in C([0,T]; Y_{1,\gamma})$ of problems (1.2.12) and (1.2.14) are

$$\begin{bmatrix} \psi(t) \\ \psi_t(t) \end{bmatrix} = e^{\mathbb{A}_{0,\gamma}t} \begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix} \in C([0,T]; Y_{1,\gamma}),\ \psi(t; \phi_0, \phi_1) = C_{0,\gamma}(t)\psi_0 + S_{0,\gamma}(t)\psi_1, \quad (1.2.19)$$

$$\begin{bmatrix} \phi(t) \\ \phi_t(t) \end{bmatrix} = e^{\mathbb{A}_{1,\gamma} t} \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} = e^{\mathbb{A}_{0,\gamma} t} \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} + \int_0^t e^{\mathbb{A}_{0,\gamma}(t-\tau)} \begin{bmatrix} 0 & 0 \\ 0 & -C_\gamma^{-1} B \end{bmatrix} \begin{bmatrix} \phi(\tau) \\ \phi_t(\tau) \end{bmatrix} d\tau\,; \tag{1.2.20}$$

$$\phi(t) = \phi(t;\phi_0,\phi_1) = \psi(t;\phi_0,\phi_1) - \int_0^t S_{0,\gamma}(t-\tau) C_\gamma^{-1} B \phi_t(\tau)\, d\tau\,; \tag{1.2.21}$$

where $C_{0,\gamma}(t)$ is the cosine operator on $\mathcal{D}(C_\gamma^{\frac{1}{2}})$ corresponding to the negative self-adjoint operator $[-C_\gamma^{-1}A]$ on $\mathcal{D}(C_\gamma^{\frac{1}{2}})$, and $S_{0,\gamma}(t)$ its corresponding sine operator. We may now state our final assumption (which is satisfied for all B.C. of interest, either trivially (Eqn. (4.1.6) of Section 4.1) or via "sharp trace regularity theory" (Eqn. (4.2.7) of Section 4.2)).

(H.4) With reference to the Kirchoff problem (1.2.12), (1.2.13), assume that the following regularity property holds true for one, hence any, T :

$$\{\psi_0,\psi_1\} \in Y_{1,\gamma} \Rightarrow \psi_{tt} \in L_2(0,T;H), \ \text{continuously.} \tag{1.2.22}$$

(For a dual version, see Remark 1.2.1 below.)

Our main result of the present subsection for problem (1.2.1), (1.2.2) is the following structural decomposition theorem, the counterpart of Theorem 1.1.2 of the preceding subsection.

Theorem 1.2.2. *Assume* (H.1), (H.2) = (1.2.4), (H.3) = (1.2.5), (H.4) = (1.2.22). *With reference to problem* (1.2.1), (1.2.2) *and its corresponding s.c. contraction semigroup* $e^{\mathbb{A}_\gamma t}$ *guaranteed by Proposition* 1.2.1, *we have that the following decomposition holds true for all* $t > 0$,

$$\begin{bmatrix} w(t) \\ w_t(t) \\ \theta(t) \end{bmatrix} = e^{\mathbb{A}_\gamma t} \begin{bmatrix} w_0 \\ w_1 \\ \theta_0 \end{bmatrix} = \begin{bmatrix} \left[e^{\mathbb{A}_{1,\gamma} t} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right] \\ 0 \end{bmatrix} + \mathcal{K}_\gamma(t) \begin{bmatrix} w_0 \\ w_1 \\ \theta_0 \end{bmatrix}, \tag{1.2.23}$$

where $e^{\mathbb{A}_{1,\gamma} t}$ *is the s.c. contraction uniformly stable group defined in* (1.2.16), (1.2.18), *and where the operator* $\mathcal{K}_\gamma(t)$ (*defined explicitly in* (3.35) *below*) *satisfies*

$$\mathcal{K}_\gamma(t) : \ \textit{compact}\ Y_\gamma \to Y_\gamma \ \text{(see (1.2.9))}. \quad \blacksquare \tag{1.2.24}$$

Corollary 1.2.3. *Assume* (H.1), (H.2) = (1.2.4), (H.3) = (1.2.5), (H.4) = (1.2.22). *Then the s.c. semigroup* $e^{\mathbb{A}_\gamma t}$ *is uniformly* (*exponentially*) *stable on* Y_γ : *moreover,* $r_{\text{ess}}(e^{\mathbb{A}_\gamma t}) = r_{\text{ess}}(e^{\mathbb{A}_{1,\gamma} t}) < 1$, $t > 0$, *where* r_{ess} *denotes the essential spectrum radius. Thus,* $e^{\mathbb{A}_{1,\gamma} t}$ *determines the decay of* $e^{\mathbb{A}_\gamma t}$, *except possibly for a finite-dimensional subspace. Moreover,* $e^{\mathbb{A}_\gamma t}$ *cannot be compact, or continuous in the uniform operator topology of* Y_γ, *for all* $t > 0$, *by the same reason as in Remark* 1.1.1. $\blacksquare$

Remark 1.2.1. A dual version of assumption (H.4) = (1.2.22) is as follows:

(H.4*)

$$f \to \int_0^t e^{\mathbb{A}_{0,\gamma}(t-\tau)} \begin{bmatrix} C_\gamma^{-1} f(\tau) \\ 0 \end{bmatrix} d\tau : \ \text{continuous}\ L_2(0,t;H) \to Y_{1,\gamma}. \tag{1.2.25}$$

In fact, pick $\{\psi_0, \psi_1\} \in Y_{1,\gamma}$, recall (below (1.2.17)) that the operator $\mathbb{A}_{0,\gamma}$ is skew-adjoint on $Y_{1,\gamma}$: $\mathbb{A}^*_{0,\gamma} = -\mathbb{A}_{0,\gamma}$, so that $e^{\mathbb{A}^*_{0,\gamma}(t-\tau)} = e^{\mathbb{A}_{0,\gamma}(\tau-t)}$ and compute via (1.2.12), (1.2.19)

$$\left(\int_0^t e^{\mathbb{A}_{0,\gamma}(t-\tau)}\begin{bmatrix} C_\gamma^{-1}f(\tau) \\ 0 \end{bmatrix} d\tau, \begin{bmatrix}\psi_0 \\ \psi_1\end{bmatrix}\right)_{Y_{1,\gamma}} = \int_0^t \left(\begin{bmatrix} C_\gamma^{-1}f(\tau) \\ 0 \end{bmatrix}, e^{\mathbb{A}_{0,\gamma}(\tau-t)}\begin{bmatrix}\psi_0 \\ \psi_1\end{bmatrix}\right)_{Y_{1,\gamma}} d\tau \quad (1.2.26)$$

$$= \int_0^t (C_\gamma^{-1}f(\tau), \psi(\tau-t;\psi_0,\psi_1))_{\mathcal{D}(A^{\frac{1}{2}})} d\tau \quad (1.2.27)$$

$$= (f, C_\gamma^{-1}A\psi(\,\cdot\,-t);\psi_0,\psi_1))_{L_2(0,T;H)} = (f, \psi_{tt}(\,\cdot\,-t);\ \psi_0,\psi_1))_{L_2(0,T;H)}, \quad (1.2.28)$$

where $\psi(\tau-t;\psi_0,\psi_1)$ solves Eqn. (1.2.12) with data $\{\psi_0,\psi_1\} \in Y_{1,\gamma}$ at the initial time t, backward in time (Eqn. (1.2.12) is time reversible). Thus, (1.2.28) shows that (H.4) = (1.2.22) holds true if and only if (H.4*) = (1.2.25) holds true. ■

Remark 1.2.2. Assumption (H.4) = (1.2.22) on the undamped Kirchoff problem (1.2.12), (1.2.13), will imply the same property for the damped Kirchoff problem (1.2.14), (1.2.15), and for the thermo-elastic problem (1.2.1) – (1.2.3): i.e.,

$$\phi_{tt} \in L_2(0,T;H) \text{ and } w_{tt} \in L_2(0,T;H) \text{ continuously,} \quad (1.2.29)$$

see Lemma 3.5, Eqn. (3.15), and Remark 3.3, Eqn. (3.36), respectively. Then, the property in (1.2.29) for w_{tt} will, in turn, imply the regularity

$$[w_0, w_1, 0] \in Y_\gamma \Rightarrow \theta_t \in L_2(0,T;H) \text{ and } \theta_{tt} \in L_2(0,T;[\mathcal{D}(B)]') \quad (1.2.30)$$

on the thermal component of the thermo-elastic problem (1.2.1) – (1.2.3) with $\theta_0 = 0$; see Remark 3.4. ■

1.3. The case of (coupled) free B.C.

In this subsection, we consider a thermo-elastic problem defined on a smooth bounded domain $\Omega \subset \mathbb{R}^2$ with boundary Γ, this time with free B.C. [18, p. 151],

$$\begin{cases} w_{tt} - \gamma\Delta w_{tt} + \Delta^2 w + \Delta\theta = 0 \quad \text{in } (0,T]\times\Omega = Q; & (1.3.1) \\ \theta_t - \Delta\theta - \Delta w_t = 0 \quad \text{in } Q; & (1.3.2) \\ w(0,\,\cdot\,) = w_0;\ w_t(0,\,\cdot\,) = w_1;\ \theta(0,\,\cdot\,) = \theta_0 \quad \text{in } \Omega; & (1.3.3) \\ \Delta w + B_1 w + \theta = 0 \quad \text{in } (0,T]\times\Gamma = \Sigma; & (1.3.4) \\ \dfrac{\partial\Delta w}{\partial\nu} + B_2 w - \gamma\dfrac{\partial w_{tt}}{\partial\nu} + \dfrac{\partial\theta}{\partial\nu} = 0 \quad \text{in } \Sigma; & (1.3.5) \\ \dfrac{\partial\theta}{\partial\nu} + b\theta = 0 \quad b \geq 0 \text{ in } \Sigma. & (1.3.6) \end{cases}$$

As in preceding cases, the constant γ is positive, $\gamma > 0$, throughout. The boundary operators B_1 and B_2 are defined by [18, p. 16],

$$B_1 w = -(1-\mu)[2\nu_1\nu_2 w_{xy} - \nu_1^2 w_{yy} - \nu_2^2 w_{xy}] \tag{1.3.7}$$

$$B_2 w = (1-\mu)\frac{\partial}{\partial\tau}[(\nu_1^2 - \nu_2^2)w_{xy} + \nu_1\nu_2(w_{yy} - w_{xx})], \tag{1.3.8}$$

where $0 < \mu < 1$ is the Poisson's modulus, $\nu = [\nu_1, \nu_2]$ is the unit outward normal and $\tau = [-\nu_2, \nu_1]$ is a tangential unit vector. We note the coupling between w and θ on the two boundary conditions (1.3.5) and (1.3.6).

Abstract model and well-posedness. Our starting point is the following abstract model for problem (1.3.1) – (1.3.6), for whose derivation we refer to [4] (however, a hint will be given below (1.3.17)):

$$\begin{cases} w_{tt} + \gamma\mathcal{A}_N w_{tt} + \mathcal{A}w - \mathcal{A}_R\theta = -\mathcal{A}G_1(\theta|_\Gamma) + b\mathcal{A}G_2(\theta|_\Gamma) \text{ in } [\mathcal{D}(\mathcal{A})]'; & (1.3.9) \\ \theta_t + \mathcal{A}_R\theta + \mathcal{A}_N w_t - \mathcal{A}_N N\dfrac{\partial w_t}{\partial\nu} = 0 \quad \text{in } [\mathcal{D}(\mathcal{A}_R)]', & (1.3.10) \end{cases}$$

where $\mathcal{A}_N$, $\mathcal{A}$, and $\mathcal{A}_R$ are the following positive, self-adjoint operators on $L_2(\Omega)$ (we take $b > 0$ for definiteness):

$$\mathcal{A}_N f = -\Delta f,\ \mathcal{D}(\mathcal{A}_N) = \left\{ f \in H^2(\Omega) : \frac{\partial f}{\partial\nu}\Big|_\Gamma = 0 \right\};\ \mathcal{D}(\mathcal{A}_N^{\frac{1}{2}}) \equiv H^1(\Omega); \tag{1.3.11}$$

$$\left.\begin{aligned} &\mathcal{A}f = \Delta^2 f,\ \mathcal{D}(\mathcal{A}) = \left\{ f \in H^4(\Omega) : \Delta f + B_1 f = 0,\ \frac{\partial\Delta f}{\partial\nu} + B_2 f = 0 \text{ on } \Gamma \right\}; \\ &\qquad \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega); \end{aligned}\right\} \tag{1.3.12}$$

$$\mathcal{A}_R f = -\Delta f,\ \mathcal{D}(\mathcal{A}_R) = \left\{ f \in H^2(\Omega) : \frac{\partial f}{\partial\nu} + bf = 0 \text{ in } \Gamma \right\};\ \mathcal{D}(\mathcal{A}_R^{\frac{1}{2}}) \equiv H^1(\Omega). \tag{1.3.13}$$

Moreover, the Neumann map N and Green maps G_1, G_2 are defined by (here $\epsilon > 0$):

$$\left.\begin{aligned} &h = Ng \iff \left\{ \Delta h = 0 \text{ in } \Omega;\ \frac{\partial h}{\partial\nu} = g \text{ on } \Gamma \right\};\ N : H^s(\Gamma) \to H^{s+\frac{3}{2}}(\Omega); \\ &\qquad \mathcal{A}_N^{\frac{3}{4}-\epsilon} N \in \mathcal{L}(L_2(\Gamma);\ L_2(\Omega)); \end{aligned}\right\} \tag{1.3.14}$$

$$h = G_1 g \iff \left\{ \Delta^2 h = 0 \text{ in } \Omega;\ \Delta h + B_1 h = g,\ \frac{\partial\Delta h}{\partial\nu} + B_2 h = 0 \text{ in } \Gamma \right\}; \tag{1.3.15}$$

$$h = G_2 g \iff \left\{ \Delta^2 h = 0 \text{ in } \Omega;\ \Delta h + B_1 h = 0,\ \frac{\partial\Delta h}{\partial\nu} + B_2 h = g \text{ in } \Gamma \right\}; \tag{1.3.16}$$

$$\left.\begin{aligned} &G_1 :\ H^s(\Gamma) \to H^{s+\frac{5}{2}}(\Omega); \\ &G_2 :\ H^s(\Gamma) \to H^{s+\frac{7}{2}}(\Omega), s \in \mathbb{R};\ \mathcal{A}^{\frac{5}{8}-\epsilon}G_2, \mathcal{A}^{\frac{7}{8}-\epsilon}G_2 \in \mathcal{L}(L_2(\Gamma); L_2(\Omega)). \end{aligned}\right\} \tag{1.3.17}$$

The following trace properties (proved by means of Green's second theorem) are known [25]:

$$\left.\begin{aligned} N^*\mathcal{A}_N f = f|_\Gamma,\ f\in H^1(\Omega);\ G_1^*\mathcal{A}f = \frac{\partial f}{\partial\nu},\ f\in H^2(\Omega);\\ G_2^*\mathcal{A}f = -f|_\Gamma,\ f\in H^1(\Omega). \end{aligned}\right\} \tag{1.3.18}$$

By way of a brief explanation, we recall that the derivation of the abstract Eqns. (1.3.9) and (1.3.10) from the original P.D.E. problem (1.3.1) – (1.3.6) proceeds by writing, via (1.3.14), (1.3.11),

$$\Delta f = \Delta\left[f - N\frac{\partial f}{\partial\nu}\right] = -\mathcal{A}_N\left[f - N\frac{\partial f}{\partial\nu}\right],\ f\in H^2(\Omega),$$

first for $f = w_{tt}$ with reference to Eqn. (1.3.1), whereby the terms $-\gamma\mathcal{A}_N N\frac{\partial w_{tt}}{\partial\nu}$ and $-\gamma\mathcal{A}G_2\frac{\partial w_{tt}}{\partial\nu}$, cancel out by (1.3.18); next, for $f = w_t$ with reference to Eqn. (1.3.2). Details are in [4]. Setting

$$\left.\begin{aligned} C_\gamma = (I+\gamma\mathcal{A}_N),\ \text{so that } C_\gamma^{-1}\mathcal{A}_N = \frac{1}{\gamma}(I - C_\gamma^{-1}) \in \mathcal{L}(L_2(\Omega)),\\ \mathcal{D}(C_\gamma^{\frac12}) = \mathcal{D}(\mathcal{A}_N^{\frac12}) = \mathcal{D}(\mathcal{A}_R^{\frac12}) = H^1(\Omega), \end{aligned}\right\} \tag{1.3.19}$$

we have that the abstract Eqns. (1.3.9), (1.3.10) may be rewritten as the first-order system

$$\dot y = \mathbb{A}_\gamma y;\ \mathbb{A}_\gamma:\ Y_\gamma\supset\mathcal{D}(\mathbb{A}_\gamma)\to Y_\gamma,\ y(t) = [w(t), w_t(t), \theta(t)]; \tag{1.3.20}$$

$$Y_\gamma = \mathcal{D}(\mathcal{A}^{\frac12})\times\mathcal{D}(C_\gamma^{\frac12})\times L_2(\Omega);\ (x_1,x_2)_{\mathcal{D}(C_\gamma^{\frac12})} = (C_\gamma^{\frac12}x_1, x_2)_{L_2(\Omega)}; \tag{1.3.21}$$

$$\mathbb{A}_\gamma = \begin{bmatrix} 0 & I & 0\\ -C_\gamma^{-1}\mathcal{A} & 0 & C_\gamma^{-1}[\mathcal{A}_R - \mathcal{A}G_1(\,\cdot\,|_\Gamma) + b\mathcal{A}G_2(\,\cdot\,|_\Gamma)]\\ 0 & -\mathcal{A}_N + \mathcal{A}_N N\dfrac{\partial}{\partial\nu} & -\mathcal{A}_R \end{bmatrix}; \tag{1.3.22a}$$

$$\left.\begin{aligned} \mathcal{D}(\mathbb{A}_\gamma) = \{x_1, x_2\in\mathcal{D}(\mathcal{A}^{\frac12}) = H^2(\Omega);\ x_3\in\mathcal{D}(\mathcal{A}_R)\subset H^2(\Omega):\\ [x_1 + G_1(x_3|_\Gamma)]\in\mathcal{D}(C_\gamma^{-\frac12}\mathcal{A}) = \mathcal{D}(\mathcal{A}^{\frac34})\} \end{aligned}\right\} \tag{1.3.22b}$$

since $[x_2 - N\frac{\partial x_2}{\partial\nu}]\in\mathcal{D}(\mathcal{A}_N)$ automatically, and since $G_2(x_2|_\Gamma)\in H^{\frac32+\frac72}(\Omega)$, by trace theory on x_3 and (1.3.17), and satisfies the first B.C. of $\mathcal{A}$ in (1.3.12), so that $G_2(x_3|_\Gamma)\in\mathcal{D}(\mathcal{A}^{\frac34})$ [10]. See also (5.1.2) below for $C_\gamma^{-1}\mathcal{A}_R(x_3|_\Gamma)\in\mathcal{D}(C_\gamma^{\frac12})$ automatically.

Proposition 1.3.1. (i) *The operator* $\mathbb{A}_\gamma$ *in* (1.3.22) *is dissipative and becomes skew-adjoint on* Y_γ, *if one removes the bottom-right corner element* $-\mathcal{A}_R$ *from* (1.3.22a):

$$\operatorname{Re}(\mathbb{A}_\gamma x, x)_{Y_\gamma} = -(\mathcal{A}_R x_3, x_3)_{L_2(\Omega)};\ x = [x_1,x_2,x_3]\in\mathcal{D}(\mathbb{A}_\gamma) \tag{1.3.23}$$

(ii) *In fact,* $\mathbb{A}_\gamma$ *is maximal dissipative and generates a s.c. contraction semigroup* $e^{\mathbb{A}_\gamma t}$ *on the space* Y_γ *defined by* (1.3.21): $[w(t), w_t(t), \theta(t)] = e^{\mathbb{A}_\gamma t}[w_0, w_1, \theta_0] \in C([0,T]; Y_\gamma)$.

(iii) *The resolvent* $R(\lambda, \mathbb{A}_\gamma)$ *is compact on* Y_γ. ∎

For details of the proof based on the Lumer-Phillips theorem, see [4].

Main results: Structural decomposition of $e^{\mathbb{A}_\gamma t}$. Substituting $\mathcal{A}_R\theta$ from (1.3.10) into (1.3.9) yields the equation

$$w_{tt}+\gamma\mathcal{A}_N w_{tt}+\mathcal{A}w+\mathcal{A}_N w_t-\mathcal{A}_N N\,\frac{\partial w_t}{\partial\nu} = -\theta_t-\mathcal{A}G_1(\theta|_\Gamma)+b\mathcal{A}G_2(\theta|_\Gamma)\,, \tag{1.3.24}$$

which, combined with Eqn. (1.3.10), allows us to express the semigroup solution $[w(t), w_t(t), \theta(t)]$ guaranteed by Proposition 1.3.1 as follows via (1.3.19),

$$\frac{d}{dt}\begin{bmatrix} w(t)\\ w_t(t)\\ \theta(t)\end{bmatrix} = A_{\gamma,s}\begin{bmatrix} w(t)\\ w_t(t)\\ \theta(t)\end{bmatrix} + \begin{bmatrix} 0\\ C_\gamma^{-1}[-\theta_t + b\mathcal{A}G_2(\theta|_\Gamma)]\\ 0\end{bmatrix}; \tag{1.3.25}$$

$$\left.\begin{aligned} A_{\gamma,s} &= \begin{bmatrix} 0 & I & 0\\ -C_\gamma^{-1}\mathcal{A} & -C_\gamma^{-1}\mathcal{A}_N + C_\gamma^{-1}\mathcal{A}_N N\,\dfrac{\partial}{\partial\nu} & -C_\gamma^{-1}\mathcal{A}G_1(\,\cdot\,|_\Gamma)\\ 0 & -\mathcal{A}_N + \mathcal{A}_N N\,\dfrac{\partial}{\partial\nu} & -\mathcal{A}_R \end{bmatrix};\\ &\qquad\qquad\qquad\qquad \mathcal{D}(A_{\gamma,s}) = \mathcal{D}(\mathbb{A}_\gamma)\,. \end{aligned}\right\} \tag{1.3.26}$$

Proposition 1.3.2. *The operator* $A_{\gamma,s}$ *defined by* (1.3.26) *generates a s.c. semigroup* $e^{A_{\gamma,s}t}$ *on* Y_γ*, see* (1.3.21). [*The subindex stands for 'stripped' or 'simplified' over* $\mathbb{A}_\gamma$ *in* (1.3.22).] ∎

The proof of Proposition 1.3.2, where $e^{A_{\gamma,s}t}$ is not claimed to be a contraction, is given in Section 5.2. Returning to Eqn. (1.3.25) and invoking the generation results of Proposition 1.3.1 (ii) for $\mathbb{A}_\gamma$, and of Proposition 1.3.2 for $A_{\gamma,s}$, we can write

$$\begin{bmatrix} w(t)\\ w_t(t)\\ \theta(t)\end{bmatrix} = e^{\mathbb{A}_\gamma t}\begin{bmatrix} w_0\\ w_1\\ \theta_0\end{bmatrix} = \begin{bmatrix} v(t)\\ v_t(t)\\ \phi(t)\end{bmatrix} + \begin{bmatrix} z(t)\\ z_t(t)\\ \psi(t)\end{bmatrix}; \begin{bmatrix} v(t)\\ v_t(t)\\ \phi(t)\end{bmatrix} = e^{A_{\gamma,s}t}\begin{bmatrix} w_0\\ w_1\\ \theta_0\end{bmatrix}; \tag{1.3.27}$$

$$\begin{bmatrix} z(t)\\ z_t(t)\\ \psi(t)\end{bmatrix} = \mathcal{K}_\gamma(t)\begin{bmatrix} w_0\\ w_1\\ \theta_0\end{bmatrix} = \int_0^t e^{A_{\gamma,s}(t-\tau)}\begin{bmatrix} 0\\ -C_\gamma^{-1}\theta_t(\tau) + bC_\gamma^{-1}\mathcal{A}G_2(\theta(\tau)|_\Gamma)]\\ 0\end{bmatrix} d\tau. \tag{1.3.28}$$

Thus, by (1.3.26), $[v(t), v_t(t), \phi(t)]$ in (1.3.27) solves problem

$$\begin{cases} \dfrac{d}{dt}\begin{bmatrix} v \\ v_t \\ \phi \end{bmatrix} = A_{\gamma,s}\begin{bmatrix} v \\ v_t \\ \phi \end{bmatrix}; \\ \text{i.e.} \begin{cases} v_{tt} + \gamma \mathcal{A}_N v_{tt} + \mathcal{A} v + \mathcal{A}_N v_t - \mathcal{A}_N N \dfrac{\partial v_t}{\partial \nu} = -\mathcal{A} G_1(\phi|_\Gamma); & (1.3.29) \\ \phi_t + \mathcal{A}_R \phi + \mathcal{A}_N v_t - \mathcal{A}_N N \dfrac{\partial v_t}{\partial \nu} = 0; & (1.3.30) \\ v(0) = w_0,\ v_t(0) = w_1,\ \phi(0) = \theta_0, \end{cases} \end{cases}$$

or, in explicit P.D.E. terms, as in (1.3.34) below. Instead, $[z(t), z_t(t), \psi(t)]$ in (1.3.28) solves problem

$$\frac{d}{dt}\begin{bmatrix} z \\ z_t \\ \psi \end{bmatrix} = A_{\gamma,s}\begin{bmatrix} z \\ z_t \\ \psi \end{bmatrix} + \begin{bmatrix} 0 \\ -C_\gamma^{-1}\theta_t + bC_\gamma^{-1}\mathcal{A} G_2(\theta|_\Gamma)) \\ 0 \end{bmatrix} \text{ i.e.,} \tag{1.3.31}$$

$$\begin{cases} z_{tt} + \gamma \mathcal{A}_N z_{tt} + \mathcal{A} z + \mathcal{A}_N z - \mathcal{A}_N N \dfrac{\partial z_t}{\partial \nu} = -\mathcal{A} G_1(\psi|_\Gamma) - \theta_t + b\mathcal{A} G_2(\theta|_\Gamma); & (1.3.32) \\ \psi_t + \mathcal{A}_R \psi + \mathcal{A}_N z_t - \mathcal{A}_N N \dfrac{\partial z_t}{\partial \nu} = 0;\ z(0) = z_t(0) = \psi(0) = 0. & (1.3.33) \end{cases}$$

In explicit P.D.E. terms, $[v, v_t, \phi]$ and $[z, z_t, \psi]$ solve (writing $v(0, \cdot) = v(0)$ for short):

$$\begin{cases} v_{tt} - \gamma \Delta v_{tt} + \Delta^2 v - \Delta v_t = 0 \\ \phi_t - \Delta \phi - \Delta v_t = 0 \\ v(0) = w_0;\ v_t(0) = w_1;\ \phi(0) = \theta_0 \\ \Delta v + B_1 v + \phi = 0 \\ \dfrac{\partial \Delta v}{\partial \nu} + B_2 v - \gamma \dfrac{\partial v_{tt}}{\partial \nu} = 0 \\ \dfrac{\partial \phi}{\partial \nu} + b\phi = 0 \end{cases} ; \begin{cases} z_{tt} - \gamma \Delta z_{tt} + \Delta^2 z - \Delta z_t = -\theta_t \text{ in } Q & (1.3.34) \\ \psi_t - \Delta \psi - \Delta z_t = 0 \quad \text{in } Q & (1.3.35) \\ z(0) = 0;\ z_t(0) = 0;\ \psi(0) = 0 \quad \text{in } \Omega & (1.3.36) \\ \Delta z + B_1 z + \psi = 0 \quad \text{in } \Sigma & (1.3.37) \\ \dfrac{\partial \Delta z}{\partial \nu} + B_2 z - \gamma \dfrac{\partial z_{tt}}{\partial \nu} + \dfrac{\partial \theta}{\partial \nu} = 0 \quad \text{in } \Sigma; & (1.3.38) \\ \dfrac{\partial \psi}{\partial \nu} + b\psi = 0 \quad \text{in } \Sigma. & (1.3.39) \end{cases}$$

(Notice that the $\{v, \phi\}$-problem has the v-equation and the second B.C. *uncoupled*, unlike the original $\{w, \theta\}$-problem.)

Having established the decomposition of the original thermo-elastic semigroup solution $[w(t),\ w_t(t),\ \theta(t)]$ as in (1.3.27) – (1.3.39), we can now state our structural result in its first form.

Theorem 1.3.3. *With reference to* (1.3.27) – (1.3.39), *we have that the following decomposition of the original thermo-elastic semigroup of Proposition* 1.3.1 (ii) *holds true, for all* $t > 0$:

$$\begin{bmatrix} w(t) \\ w_t(t) \\ \theta(t) \end{bmatrix} = e^{\mathbb{A}_\gamma t} \begin{bmatrix} w_0 \\ w_1 \\ \theta_0 \end{bmatrix} = e^{A_{\gamma,s} t} \begin{bmatrix} w_0 \\ w_1 \\ \theta_0 \end{bmatrix} + \mathcal{K}_\gamma(t) \begin{bmatrix} w_0 \\ w_1 \\ \theta_0 \end{bmatrix}, \tag{1.3.40}$$

where the operator $\mathcal{K}_\gamma(t)$ *is explicitly defined in* (5.4.1), *via* (5.4.2) – (5.4.4) *below;*

(i)

$$\mathcal{K}_\gamma(t) : \text{ compact } Y_\gamma \to Y_\gamma ; \tag{1.3.41}$$

(ii) *the resolvent* $R(\lambda, A_{\gamma,s})$ *of the generator* $A_{\gamma,s}$ *satisfies the following property,*

$$\|R(\lambda = a + i\tau, A_{\gamma,s})\|_{\mathcal{L}(Y_\gamma)} \text{ does not tend to zero, as } \tau \to \infty , \tag{1.3.42}$$

on vertical lines of the complex plane, with $\operatorname{Re}\lambda = a$ *fixed and sufficiently large. This implies* [33, p. 50] *that the statement "the s.c. semigroup* $e^{\mathbb{A}_{\gamma,s} t}$ *is compact on* Y_γ *for all* $t > 0$*" is false.* ∎

Part (i), Eqn. (1.3.41), is proved in Section 5.4; while Part (ii), Eqn. (1.3.42), is proved in Section 5.3. A finer decomposition of the original thermo-elastic semigroup $e^{\mathbb{A}_\gamma t}$ than that in Theorem 1.3.3 is available, by extracting a larger "compact part". The price to pay, however, is a loss of the interpretation of the "non-compact semigroup," as arising precisely from a thermo-elastic problem. For this reason, we shall not pursue this.

1.4. Comments and literature

Main contributions of the present paper. The main contributions of our present paper over available literature [15] and [9] (of which we became aware only after submitting our first version of this paper) concern the cases of *coupled* B.C. of Subsection 1.1 (hinged/Neumann B.C.) and of Subsection 1.3 (free B.C.): these are entirely new, along with their critically supporting sharp regularity results [27, 28]. For completeness, we also provide, in Subsection 1.2, an abstract model, which includes various *uncoupled* B.C. cases. Our model of Subsection 1.2 is *not* included in the abstract setting of [15] — which is motivated by the *wave-like* system of n-dimensional elasticity, rather than by the thermo-elastic *plate* equations considered in this paper; see a more detailed comparison at the end of this subsection. [9] considers, specifically, a thermo-elastic plate system with clamped/Dirichlet B.C. and studies a controllability problem. Our Section 6 is useful here in re-obtaining this result via a general strategy. Our abstract structural decomposition result of Subsection 1.2, Theorem 1.2.2, when specialized (as in Subsection 4.2) to the specific case of clamped/Dirichlet B.C., improves upon [9, Theorem 2, p. 372], by further eliminating the thermal component from the "non-compact part" of the thermo-elastic semigroup, and thus reducing the "non-compact part" only to the mechanical damped Kirchoff equation. This way, we obtain also a precise uniform stability result as in Corollary 1.2.3.

Dynamical properties of thermo-elastic plate equations. The present paper intends to continue recent investigations on the *dynamical properties* of thermo-elastic *plate* equations, as carried out in two distinctive settings: (a) in the case of $\gamma = 0$ (no rotational forces), originally studied in the influential paper [32], in the special but not trivial case of uncoupled clamped/Dirichlet B.C., and subsequently in [21-24] for all canonical B.C., including coupled ones, and (b) in the case $\gamma > 0$, however specialized to hinged/Dirichlet B.C., studied in [6].

Case $\gamma = 0$. In the case $\gamma = 0$, the cited literature shows that *analyticity* of the s.c. contraction semigroup is the distinctive dynamical property under all canonical B.C.

Case $\gamma > 0$. (i) In addressing the question of dynamical properties for $\gamma > 0$, it is natural to begin with the canonical, and particularly attractive, case of *hinged* (or simply supported) Dirichlet B.C. This is done in [6]: for this special set of B.C. it is possible to give a very precise description of the resulting s.c. semigroup. While simplifying the analysis, the hinged B.C. also yield a richer theory, which may be summarized as follows: here, the corresponding s.c. contraction thermo-elastic semigroup admits a direct (non-orthogonal) sum decomposition of one analytic self-adjoint component and of one s.c. *group* (infinite-dimensional) component. This decomposition is established in [6] both directly, and via an associated bounded perturbation, all via spectral analysis. Some related results in the one-dimensional case are given in [14]. This spectral analysis expands on the results obtained in [13] in the case of a one-dimensional thermo-elastic *rod*, where the eigenvalues approach asymptotically a vertical line.

(ii) The present work deals with the case $\gamma > 0$ under all canonical B.C. In effect, the present paper contains three separate parallel treatments: (a) the case of a thermo-elastic plate with *coupled* hinged/Neumann B.C., which is given in Subsection 1.1 (statement of results) and Section 2 (proofs); (b) the case of (*coupled*) free B.C., which is given in Subsection 1.3 (Statement of results) and Section 5 (proofs); and, for completeness, (c) the case of an abstract model encompassing the cases of *uncoupled* B.C., which is given in Subsection 1.2 (statement of results), in Section 3 (proofs), and in Section 4 (illustrative examples including clamped/Dirichlet or clamped/Neumann B.C.).

For the first two cases of Subsections 1.1 and 1.2, the structural decompositions of Theorems 1.1.2, Eqn. (1.1.21), and of Theorem 1.2.2, Eqn. (1.2.23), are sharp. Concerning thermo-elastic dynamical properties, they strikingly emphasize the contrast between the "group-dominant case" of $\gamma > 0$, and the analytic case of $\gamma = 0$. The proofs of these results in Sections 2 and 3 combine energy estimates with the dominant idea of one of the two *positive* proofs of analyticity in [21], used, however, "in reverse", as we now explain. After a substitution, the elastic component of the thermo-elastic system may be rewritten with a damping term w_t as in (1.1.16), or (3.4), respectively, in the first two cases of Sections 1.1 and 1.2. In such form, the "driver" is the semigroup $e^{\mathbb{A}_{1,\gamma} t}$ corresponding to the mechanical variables $[w, w_t]$. Then:

(ii$_1$) If $\gamma > 0$, $e^{\mathbb{A}_{1,\gamma} t}$ is a group (negative "driver"), and we are then able to show that the remaining component is compact for all $t > 0$, by the key use of new, sharp regularity theory (see below).

(ii$_2$) By contrast, if $\gamma = 0$, then the semigroup $e^{\mathbb{A}_{1,\gamma=0} t}$ is, instead, *analytic*, by the theory of [7-8], and acts now as a positive "driver", which preserves analyticity in all variables $[w(t), w_t(t), \theta(t)]$, under coupling with θ, as proved in [21] at least in the uncoupled B.C. case.

The *coupled* case of free B.C. of Subsection 1.3 and Section 5 is, of course, more challenging: here, the structural decomposition results which we obtain in Theorem 1.3.3 (or in Theorem 1.3.5) are weaker than in previous cases, even though stronger efforts are needed to achieve them, including a new, non-trivial regularity result for the elastic Kirchoff equation (see below).

Uniform stability. As an unexpected bonus, our structural decomposition Theorems 1.1.2 and 1.2.2 yield a precise uniform (exponential) stability of the corresponding s.c. contraction, thermo-elastic semigroup, at least for the coupled case of hinged/Neumann B.C. of Subsection 1.1 and of the abstract model for uncoupled B.C. of Subsection 1.2. This way, we recover the uniform stability results of the literature obtained by other methods (a novel operator multiplier), with the added information that exponential decay of the thermo-elastic semigroup is controlled by the elastic damped group $e^{\mathbb{A}_{1,\gamma} t}$ modulo a finite-dimensional subspace. However, the stability results of [3] are uniform in $0 \leq \gamma \leq \gamma_0$, for some γ_0, a result that cannot follow from our decomposition. At present, the weaker structural decomposition results of Subsection 1.3 for the most demanding coupled case of free B.C. do not yield uniform (exponential) stability.

The role of sharp regularity theory. In achieving the structural decomposition results of the present paper, the importance and role of sharp (optimal) regularity results of corresponding elastic Kirchoff equations cannot be over-estimated. Two of such non-trivial regularity results are new [27, 28], and were established precisely to support the present paper. The sharp regularity results are of recent origin [26, Thm. 1.2], [19, p. 123]. More precisely: it is thanks to a new trace regularity result, Eqn. (2.23), of the thermo-elastic system [27], that we are able to show compactness of the thermal component (2.22) in Proposition 2.6, in the case of coupled hinged/Neumann B.C. of Subsection 1.1. Similarly, in the case of the abstract system for uncoupled B.C. of Subsection 1.2, the proof of compactness of the operator L_t in Proposition 3.8 rests (via Lemma 3.5) on the abstract regularity assumption (H.4) = (1.2.22) for the (undamped) Kirchoff equation (1.2.12): in the illustrations in Section 4 dealing with clamped/Dirichlet, or clamped/Neumann B.C., verification of such sharp regularity result ultimately hinges on the sharp trace regularity (4.2.7) [19] (undamped) Kirchoff problem (4.2.5). A similar, direct analysis for the clamped/Dirichlet B.C. case as in our Subsection 4.2, is given in [9, § 2.2].

Finally, in the case of coupled free B.C., the new, sharp regularity result [29] described by the Theorem of Remark 5.3.1 for the Kirchoff equation is what guarantees well-posedness (continuity) of the map (5.3.15). Its proof [29] requires micro-local analysis and pseudo-differential techniques to complement energy methods.

Comparison with [15] and [9]. Paper [15] considers an abstract setting, which is motivated by the system of n-dimensional elasticity: this is wave-like rather than plate-like. The main result of [15] — whose ultimate goal is a study of stability properties — is a decomposition theorem [15, Theorem 3 and Corollary 2] of the type of our Theorem 1.2.2. However, the abstract setting of [15], when tested for thermo-elastic *plate* equations such as (1.1.1), (1.1.2) plus B.C., applies successfully *only* in the most amenable case of hinged/Dirichlet B.C., where a precise and rich spectral theory is available [6]. In the next challenging case of clamped/Dirichlet B.C. (such as in Subsection 4.2), it is assumption (H.2) in [15, p. 67] that fails: translated into the notation of our Subsection 4.2, that assumption (H.2) in [15] would require that the operator $C_\gamma^{-1}A^{\frac{1}{2}}$ be bounded (have a bounded extension) on $L_2(\Omega)$: but this is *false* since, by (4.2.2), (4.2.4), $\mathcal{D}(A^{\frac{1}{2}}) = H_0^2(\Omega) \subset \mathcal{D}(C) = H^2(\Omega) \cap H_0^1(\Omega)$, and so $A^{\frac{1}{2}}C^{-1}$ is *not* bounded on $L_2(\Omega)$.

"An adaptation of the analogous result for the linear system of three-dimensional thermo-elasticity due to [15]" is given in [9, Theorem 2] for the specific case of a thermo-elastic *plate* system (1.1.1), (1.1.2) with (uncoupled) damped/Dirichelt B.C., as in our Subsection 4.2. However, as noted before, this result of [9] is further improved as in the specialization of our abstract Theorem 1.2.2 to Subsection 4.2, by further removing the thermal component from the "non-compact part" of the resulting thermo-elastic semigroup. As a result, our non-compact part is defined solely by the elastic, uniformly stable group $e^{\mathbb{A}_{1,\gamma}}$ on the mechanical variables with the added information that it is $e^{\mathbb{A}_{1,\gamma}t}$ that controls the uniform stability of the thermo-elastic semigroup $e^{\mathbb{A}_\gamma t}$ as in Corollary 1.2.3, modulo a finite-dimensional subspace.

2. Coupled hinged/Robin B.C.: Proof of Theorem 1.1.2

Step 1. Lemma 2.1. (a) *With reference to problem* (1.1.1) – (1.1.4), *the following energy identity holds true, where* $H = L_2(\Omega)$ *and* $y_0 = [w_0, w_1, \theta_0] \in Y_\gamma$:

$$E(t) + 2\int_0^t \|\mathcal{A}_R^{\frac{1}{2}}\theta(\tau)\|_H^2 d\tau = E(0)\,; \tag{2.1}$$

$$\|e^{\mathbb{A}_\gamma t}y_0\|_{Y_\gamma}^2 \equiv E(t) \equiv \|\mathcal{A}_D w(t)\|_H^2 + \|w_t(t)\|_H^2 + \gamma\|\mathcal{A}_D^{\frac{1}{2}}w_t(t)\|_H^2 + \|\theta(t)\|_H^2 \tag{2.2a}$$

$$= \|\mathcal{A}_D w(t)\|_H^2 + \|w_t(t)\|^2_{\mathcal{D}(\mathcal{A}_{D,\gamma}^{\frac{1}{2}})} + \|\theta(t)\|_H^2\,. \tag{2.2b}$$

(b) *Moreover,*

$$E(t) + \int_0^t \left[\|\mathcal{A}_R^{\frac{1}{2}}\theta(\tau)\|_H^2 + \|\mathcal{A}_D^{-\frac{1}{2}}\theta_t(\tau)\|_H^2\right] d\tau \le (1 + kT)E(0)\,. \tag{2.3}$$

In particular, via $\mathcal{D}(\mathcal{A}_R^{\frac{1}{2}}) = H^1(\Omega)$ *and* $\mathcal{D}(\mathcal{A}_D^{\frac{1}{2}}) = H_0^1(\Omega)$, *by* (1.1.7) *and* (1.1.8), *we have*

$$\theta \in L_2(0,T;H^1(\Omega)); \theta|_\Gamma \in L_2(0,T;H^{\frac{1}{2}}(\Gamma));\ \theta_t \in L_2(0,T;H^{-1}(\Omega))\,. \tag{2.4}$$

Proof. (a) Either we use the dissipativity equality (1.1.15) for $\frac{d}{dt}(e^{\mathbb{A}_\gamma t}y_0, e^{\mathbb{A}_\gamma t}y_0) = 2\,\mathrm{Re}\,(\mathbb{A}_\gamma e^{\mathbb{A}_\gamma t}y_0,\ e^{\mathbb{A}_\gamma t}y_0)$; or else multiply Eqn. (1.1.1) by w_t, and Eqn. (1.1.2) by θ, and integrate by parts using the B.C. (1.1.4). As to (b), from (1.1.6) we obtain

$$\mathcal{A}_D^{-\frac{1}{2}}\theta_t = -\mathcal{A}_D^{-\frac{1}{2}}\mathcal{A}_R^{\frac{1}{2}}\mathcal{A}_R^{\frac{1}{2}}\theta - \mathcal{A}_D^{\frac{1}{2}}w_t, \text{ hence } \|\mathcal{A}_D^{-\frac{1}{2}}\theta_t\|_H^2 \le k\|\mathcal{A}_R^{\frac{1}{2}}\theta\|_H^2 + \|\mathcal{A}_D^{\frac{1}{2}}w_t\|_H^2, \tag{2.5}$$

using $\mathcal{D}(\mathcal{A}_D^{\frac{1}{2}}) = H_0^1(\Omega) \subset H^1(\Omega) = \mathcal{D}(\mathcal{A}_R^{\frac{1}{2}})$, hence $\mathcal{A}_R^{\frac{1}{2}}\mathcal{A}_D^{-\frac{1}{2}} \in \mathcal{L}(H)$, $H = L_2(\Omega)$, so that $\mathcal{A}_D^{-\frac{1}{2}}\mathcal{A}_R^{\frac{1}{2}}$ has a bounded extension in $\mathcal{L}(H)$. Thus, (2.5) yields (2.3) by virtue of (2.1), (2.2). ∎

Step 2. We return to Eqn. (1.1.16): recalling from (1.1.16) – (1.1.20) the s.c. group $e^{\mathbb{A}_{1,\gamma}t}$ on $Y_{1,\gamma}$, we rewrite the solution of (1.1.16) as

$$\begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} = e^{\mathbb{A}_{1,\gamma}t}\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} + L_t\theta + \tilde{L}_t\theta,\ y_0 = [w_0, w_1, \theta_0] \in Y_\gamma\,; \tag{2.6}$$

$$L_t\theta = \int_0^t e^{\mathbb{A}_{1,\gamma}(t-\tau)}\begin{bmatrix} 0 \\ -\mathcal{A}_{D,\gamma}^{-1}\theta_t(\tau) \end{bmatrix} d\tau;\ \tilde{L}_t\theta = \int_0^t e^{\mathbb{A}_{1,\gamma}(t-\tau)}\begin{bmatrix} 0 \\ \mathcal{A}_{D,\gamma}^{-1}\mathcal{A}_D D(\theta(\tau)|_\Gamma) \end{bmatrix} d\tau. \tag{2.7}$$

From (2.3), or (2.4), for θ_t, it is clear that $L_t\theta \in Y_{1,\gamma}$. However, with $\theta|_\Gamma \in L_2(0,T;H^{\frac{1}{2}}(\Gamma))$ from (2.4), hence $D\theta|_\Gamma \in L_2(0,T;H^1(\Omega))$ by (1.1.11), it is not clear at first glance that $\tilde{L}_t\theta$ is well defined, let alone in $Y_{1,\gamma}$. In fact, we shall show below that L_t and $\tilde{L}_t$ are *compact* from the thermal variable, hence from the initial condition $y_0 \in Y_\gamma$, to Y_γ; see (2.14) and (2.17) below. With reference to the regularity obtained for the thermal variable θ, we introduce the space $X_{[0,t]}$ by setting, with $H = L_2(\Omega)$:

$$X_{[0,t]} \equiv C([0,t];H) \cap L_2(0,t;\mathcal{D}(\mathcal{A}_R^{\frac{1}{2}})) \cap H^1(0,t;[\mathcal{D}(\mathcal{A}_D^{\frac{1}{2}})]')\,. \tag{2.8}$$

Lemma 2.2. *With reference to* (2.8), *we have,* $\forall\ 0 < \epsilon \le \frac{1}{2}$,

$$y_0 = [w_0, w_1, \theta_0] \in Y_\gamma : \text{ continuous } \to \theta \in X_{[0,t]} \underset{\text{compact}}{\hookrightarrow} L_2(0,t;\mathcal{D}((\mathcal{A}_R^{\frac{1}{2}-\epsilon}))\,. \tag{2.9}$$

Proof. The first part of (2.9) (continuity) follows from Proposition 1.1.1 (ii), and Lemma 2.1, Eqn. (2.3), for θ. The compactness part follows from the definition (2.8) by a direct application of Aubin's Lemma [1, $p = 2$], since $\mathcal{A}_R$ has compact resolvent on $H = L_2(\Omega)$. ∎

Step 3. Lemma 2.3. *$L_t\theta$ in* (2.7) *may be rewritten as*

$$L_t\theta = K_{1,t}\theta + K_{2,t}\theta \in Y_{1,\gamma}\,, \tag{2.10}$$

$$
\left.\begin{aligned}
K_{1,t}\theta \;&\equiv\; \begin{bmatrix} 0 \\ \mathcal{A}_{D,\gamma}^{-1}\theta(t) \end{bmatrix} + e^{\mathbb{A}_{1,\gamma}t}\begin{bmatrix} 0 \\ \mathcal{A}_{D,\gamma}^{-1}\theta_0 \end{bmatrix} \\
&\quad + \int_0^t e^{\mathbb{A}_{1,\gamma}(t-\tau)}\begin{bmatrix} 0 \\ \mathcal{A}_{D,\gamma}^{-1}\mathcal{A}_D\mathcal{A}_{D,\gamma}^{-1}\mathcal{A}_R^{-\frac{1}{2}} \end{bmatrix}\mathcal{A}_R^{\frac{1}{2}}\theta(\tau)d\tau \in Y_{1,\gamma}
\end{aligned}\right\} \tag{2.11}
$$

$$
K_{2,t}\theta \;=\; \int_0^t e^{\mathbb{A}_{1,\gamma}(t-\tau)}\begin{bmatrix} -\mathcal{A}_{D,\gamma}^{-1}\mathcal{A}_R^{-\frac{1}{2}} \\ 0 \end{bmatrix}\mathcal{A}_R^{\frac{1}{2}}\theta(\tau)d\tau \in Y_{1,\gamma}\,. \tag{2.12}
$$

Proof. We integrate by parts $L_t\theta$ in (2.7), using from (1.1.19) that

$$
\left.\begin{aligned}
\mathbb{A}_{1,\gamma}\begin{bmatrix} 0 \\ \mathcal{A}_{D,\gamma}^{-1}\theta(\tau) \end{bmatrix} \;&=\; \begin{bmatrix} 0 & I \\ -\mathcal{A}_{D,\gamma}^{-1}\mathcal{A}_D^2 & -\mathcal{A}_{D,\gamma}^{-1}\mathcal{A}_D \end{bmatrix}\begin{bmatrix} 0 \\ \mathcal{A}_{D,\gamma}^{-1}\theta(\tau) \end{bmatrix} \\
&=\; \begin{bmatrix} \mathcal{A}_{D,\gamma}^{-1}\theta(\tau) \\ -\mathcal{A}_{D,\gamma}^{-1}\mathcal{A}_D\mathcal{A}_{D,\gamma}^{-1}\theta(\tau) \end{bmatrix}.
\end{aligned}\right\} \quad \blacksquare \tag{2.13}
$$

Lemma 2.4. *With reference to* (2.11) *and* (2.12), *we have*

$$
K_{1,t};\; K_{2,t};\; L_t:\; X_{[0,t]} \to Y_{1,\gamma} = \mathcal{D}(\mathcal{A}_D)\times\mathcal{D}(\mathcal{A}_{D,\gamma}^{\frac{1}{2}}) \text{ is compact.} \tag{2.14}
$$

$$
y_0 = [w_0, w_1, \theta_0] \in Y_\gamma \to \theta \in X_{[0,t]} \to L_t\theta \in Y_{1,\gamma} \text{ is compact.} \tag{2.15}
$$

Proof. (2.14) For $\theta \in X_{[0,t]}$ in (2.8), we have $\theta_0,\ \theta(t) \in L_2(\Omega)$. Moreover, $\mathcal{A}_{D,\gamma}^{-\frac{1}{2}}$ is compact on $L_2(\Omega)$. Thus, the first two terms of $K_{1,t}$ in (2.11) are compact in $Y_{1,\gamma}$. Next, the operator $\mathcal{A}_{D,\gamma}^{-\frac{1}{2}}\mathcal{A}_D\mathcal{A}_{D,\gamma}^{-1}\mathcal{A}_R^{-\frac{1}{2}}$ is compact on $L_2(\Omega)$; and the operator $\mathcal{A}_D\mathcal{A}_{D,\gamma}^{-1}\mathcal{A}_R^{-\frac{1}{2}}$ is compact on $L_2(\Omega)$. Thus, the integral term of $K_{1,t}$ in (2.11) and $K_{2,t}$ in (2.12) are compact on $Y_{1,\gamma}$, since $\mathcal{A}_R^{\frac{1}{2}}\theta \in L_2(0,T;L_2(\Omega))$: this follows, e.g., by a direct abstract proof in [36] using Mazur's Theorem that the convex hull of a compact set is compact. Thus, $K_{1,t}$ and $K_{2,t}$ are compact on $Y_{1,\gamma}$, and so is L_t by (2.10). Hence, (2.14) is proved. Then (2.15) follows via (2.9), (2.14). ∎

Step 4. We now handle the more difficult term $\tilde{L}_t\theta$ in (2.7).

Proposition 2.5. *With* $y_0 = [w_0, w_1, \theta_0] \in Y_\gamma$, *for* $\tilde{L}_t\theta$ *in* (2.7) *we have*

$$
y_0 \in Y_\gamma = \mathcal{D}(\mathcal{A}_D)\times\mathcal{D}(\mathcal{A}_{D,\gamma}^{\frac{1}{2}})\times L_2(\Omega) \overset{\text{compact}}{\longrightarrow} \theta|_\Gamma \in L_2(0,t;L_2(\Gamma)) \tag{2.16}
$$

$$
\overset{\text{continuous}}{\longrightarrow} \tilde{L}_t\theta \in Y_{1,\gamma} = \mathcal{D}(\mathcal{A}_D)\times\mathcal{D}(\mathcal{A}_{D,\gamma}^{\frac{1}{2}}). \tag{2.17}
$$

Proof. First, writing via (2.7),

$$
\begin{bmatrix} v(t) \\ v_t(t) \end{bmatrix} \equiv \tilde{L}_t\theta = \int_0^t e^{\mathbf{A}_{1,\gamma}(t-\tau)}\begin{bmatrix} 0 \\ (I+\gamma\mathcal{A}_D)^{-1}\mathcal{A}_D D(\theta(\tau)|_\Gamma) \end{bmatrix} d\tau, \tag{2.18}
$$

we recognize, as in (1.1.5), (1.1.6), that v solves the following mixed problem for the Kirchoff equation

$$\begin{cases} v_{tt} - \gamma\Delta v_{tt} + \Delta^2 v - \Delta v_t = 0 & \text{in } (0,T] \times \Omega = Q; \quad (2.19a) \\ v|_\Sigma \equiv 0; \ \ \Delta v|_\Sigma = -\theta & \text{in } (0,T) \times \Gamma = \Sigma; \quad (2.19b) \\ v(0,\,\cdot\,) = v_0 = 0; \ \ v_t(0,\,\cdot\,) = v_1 = 0 & \text{in } \Omega. \quad (2.19c) \end{cases}$$

Then, the validity of the second regularity claim in (2.17)

$$\theta|_\Gamma \in L_2(0,t;L_2(\Gamma)) \to \{v, v_t\} \equiv \tilde{L}_t\theta \in Y_{1,\gamma} = [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega) \qquad (2.20)$$

(norm equivalence) is precisely the sharp regularity result proved in [26, Theorem 1.2]. Next, it remains to show the first regularity claim in (2.16). But this follows readily from (2.9) of Lemma 2.2, with $\mathcal{D}(\mathcal{A}_R^{\frac{1}{2}-\epsilon}) = H^{1-2\epsilon}(\Omega)$, and trace theory. ∎

Step 5. Thus, at this stage, we have obtained the following decomposition, from (2.6) and (1.1.6),

$$\begin{bmatrix} w(t) \\ w_t(t) \\ \theta(t) \end{bmatrix} = e^{\mathbb{A}_\gamma t} \begin{bmatrix} w_0 \\ w_1 \\ \theta_0 \end{bmatrix} = \begin{bmatrix} \left[e^{\mathbb{A}_{1,\gamma} t} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} + L_t\theta + \tilde{L}_t\theta \right] \\ e^{-\mathcal{A}_R t}\theta_0 - \int_0^t e^{-\mathcal{A}_R(t-\tau)}\mathcal{A}_D w_t(\tau)d\tau \end{bmatrix}, \qquad (2.21)$$

where the map $y_0 \to [L_t\theta + \tilde{L}_t\theta, e^{-\mathcal{A}_R t}\theta_0]$ is compact $Y_\gamma \to Y_\gamma$ for all $t > 0$. At a first glance, it is not clear that the integral defining $\theta(t)$ in (2.21) is well defined, let alone in $L_2(\Omega)$, just by using $\mathcal{A}_D^{\frac{1}{2}} w_t \in C([0,T]; L_2(\Omega))$ from Proposition 1.1.1 (ii), since $\mathcal{A}_D$ and $\mathcal{A}_R$ do not commute. But, in fact, we have

Proposition 2.6. *With reference to* (2.21) *we have*

$$\left.\begin{aligned} y_0 \in Y_\gamma \to M_t y_0 \ &\equiv \ \int_0^t e^{-\mathcal{A}_R(t-\tau)}\mathcal{A}_D w_t(\tau)d\tau \in \mathcal{D}(\mathcal{A}_R^{\frac{1}{4}-\epsilon}) \\ &= \ H^{\frac{1}{2}-2\epsilon}(\Omega) \text{ compact } \hookrightarrow L_2(\Omega). \end{aligned}\right\} \qquad (2.22)$$

Proof. The proof uses critically the non-trivial regularity result

$$\frac{\partial w_t}{\partial \nu} \in L_2(0,T;L_2(\Gamma)) \equiv L_2(\Sigma) \qquad (2.23)$$

for problem (1.1.1) – (1.1.4). The validity of (2.23) is proved in [27]. Here we shall invoke (2.23) to prove (2.22). Let $f \in [\mathcal{D}(\mathcal{A}_R^{\frac{1}{4}-\epsilon})]'$ and compute recalling $w_t|_\Sigma \equiv 0$ from (1.1.4), whereby $-\mathcal{A}_D w_t = \Delta w_t$ by (1.1.7) by use of Green's first theorem:

$$\left.\begin{aligned} -(M_t y_0, f)_{L_2(\Omega)} &= -\left(\int_0^t e^{-\mathcal{A}_R(t-\tau)}\mathcal{A}_D w_t(\tau)d\tau, f\right)_{L_2(\Omega)} \\ &= \int_0^t \left(\Delta w_t(\tau), e^{-\mathcal{A}_R(t-\tau)}f\right)_{L_2(\Omega)} d\tau \end{aligned}\right\} \tag{2.24}$$

$$\left.\begin{aligned} &= \int_0^t \left(\frac{\partial w_t(\tau)}{\partial\nu}, [e^{-\mathcal{A}_R(t-\tau)}f]_\Gamma\right)_{L_2(\Gamma)} d\tau \\ &- \int_0^t \left(\nabla w_t(\tau), \nabla(e^{-\mathcal{A}_R(t-\tau)}f)\right)_{L_2(\Omega)} d\tau. \end{aligned}\right\} \tag{2.25}$$

By assumption, $\mathcal{A}_R^{-\frac{1}{4}+\epsilon}f \in L_2(\Omega)$, and thus by analyticity of the self-adjoint semigroup $e^{-\mathcal{A}_R t}$, we have (in τ):

$$e^{-\mathcal{A}_R(t-\tau)}f = \mathcal{A}_R^{\frac{1}{4}-\epsilon}e^{-\mathcal{A}_R(t-\tau)}\mathcal{A}_R^{-\frac{1}{4}+\epsilon}f \in L_2(0,t;\mathcal{D}(\mathcal{A}_R^{\frac{1}{4}+\epsilon}) \equiv H^{\frac{1}{2}+2\epsilon}(\Omega)); \tag{2.26}$$

$$\begin{aligned} \left\||\nabla(e^{-\mathcal{A}_R(t-\tau)}f)|\right\|_{L_2(\Omega)} &\leq c\left\|\mathcal{A}_R^{\frac{1}{2}}e^{-\mathcal{A}_R(t-\tau)}f\right\|_{L_2(\Omega)} \\ &= c\left\|\mathcal{A}_R^{\frac{3}{4}-\epsilon}e^{-\mathcal{A}_R(t-\tau)}\mathcal{A}^{-\frac{1}{4}+\epsilon}f\right\|_{L_2(\Omega)} \leq \frac{c}{(t-\tau)^{\frac{3}{4}}}, \end{aligned} \tag{2.27}$$

by analyticity of the semigroup. Thus, by trace theory on (2.26), and by (2.27), we obtain (in τ)

$$\left.\begin{aligned} &[e^{-\mathcal{A}_R(t-\tau)}f]_\Gamma \in L_2(0,t;H^{2\epsilon}(\Gamma));\ |\nabla(e^{-\mathcal{A}_R(t-\tau)}f)| \in L_1(0,t;L_2(\Omega)); \\ &|\nabla w_t| \in C([0,T];L_2(\Omega)), \end{aligned}\right\} \tag{2.28}$$

recalling also Proposition 1.1.1 (ii) for $w_t \in C[0,T];H_0^1(\Omega))$. Thus, (2.28) combined with (2.23) guarantee that both integrals terms in (2.25) are well defined and continuous with respect to $y_0 \in Y_\gamma$ and $f \in [\mathcal{D}(\mathcal{A}_R^{\frac{1}{4}-\epsilon})]'$. Thus, $M_t y_0 \in \mathcal{D}(\mathcal{A}_R^{\frac{1}{4}-\epsilon})$, and (2.22) is proved. ∎

Step 6. With reference to (2.21), we then have

$$\mathcal{K}_\gamma(t)\begin{bmatrix} w_0 \\ w_1 \\ \theta_0 \end{bmatrix} = \begin{bmatrix} L_t\theta + \tilde{L}_t\theta \\ e^{-\mathcal{A}_R t}\theta_0 - M_t y_0 \end{bmatrix}, \text{ and } \mathcal{K}_\gamma(t)\text{: compact on } Y_\gamma, \tag{2.29}$$

by Proposition 2.6 on M_t, Proposition 2.5 on $\tilde{L}_t\theta$, and Lemma 2.4 on $L_t\theta$. Theorem 1.1.2 is proved.

Proof of Corollary 1.1.3. This is standard and included for completeness. The decomposition in Theorem 1.1.2 with $\mathcal{K}_\gamma(t)$ compact yields that $r_{\text{ess}}(e^{\mathbb{A}_\gamma t}) = r_{\text{ess}}(e^{\mathbb{A}_{1,\gamma} t}) < 1$, since $e^{\mathbb{A}_{1,\gamma} t}$ is uniformly (exponentially) stable. But, as one verifies directly, the operator $\mathbb{A}_\gamma$ does not have any eigenvalues on the imaginary axis. Thus, $r(e^{\mathbb{A}_\gamma t}) = r_{\text{ess}}(e^{\mathbb{A}_\gamma t}) < 1$, where $r(\cdot)$ denotes the spectral radius. Thus, $e^{\mathbb{A}_\gamma t}$ is uniformly (exponentially) stable. ∎

3. Abstract system: Proof of Theorem 1.2.2

The present proof follows closely the pattern of the proof of Section 2, with some marked simplifications (due to the absence of the highly unbounded boundary coupling term $\mathcal{A}_D D(\,\cdot\,|_\Gamma)$ in the model, where, moreover, the coupling operators $\mathcal{A}_R\theta$ and $\mathcal{A}_D w_t$ reduce now to the same operator B); but also with some new, additional difficulties, not present in Section 2, where the feature $C = \mathcal{A}_D$, $A = \mathcal{A}_D^2$ of the abstract model there helped. Accordingly, the present proof will dwell mostly on the new difficulties.

Step 1. Lemma 3.1. (a) *Assume* (H.1) (i), (H.2) = (1.2.4), (H.3) = (1.2.5). *Let* $\{w_0, w_1, \theta_0\} \in Y_\gamma$, *in* (1.2.9). *Then, the following energy identity holds true:*

$$E(t) + 2\int_0^t \|B^{\frac{1}{2}}\theta(\tau)\|_H^2 d\tau = E(0)\,; \tag{3.1}$$

$$\|e^{\mathbb{A}_\gamma t}y_0\|_{Y_\gamma}^2 = E(t) \;\equiv\; \|A^{\frac{1}{2}}w(t)\|_H^2 + \|w_t(t)\|^2_{\mathcal{D}(C_\gamma^{\frac{1}{2}})} + \|\theta(t)\|_H^2 \tag{3.2a}$$

$$(\text{by } (1.2.10)) \qquad = \|A^{\frac{1}{2}}w(t)\|_H^2 + \|w_t(t)\|_H^2 + \gamma\|C^{\frac{1}{2}}w_t(t)\|_H^2 + \|\theta(t)\|_H^2. \tag{3.2b}$$

(b) *Moreover,*

$$E(t) + \int_0^t [\|B^{\frac{1}{2}}\theta(\tau)\|_H^2 + \|B^{-\frac{1}{2}}\theta_t(\tau)\|_H^2]d\tau \le (2+kT)E(0),\ 0 \le t \le T. \tag{3.3}$$

Step 2. Substituting $B\theta$ from (1.2.2) into (1.2.1) and recalling C_γ in (1.2.10) yields

$$w_{tt} + \gamma C w_{tt} + Aw + Bw_t = -\theta_t;\ w_{tt} + C_\gamma^{-1}Aw + C_\gamma^{-1}Bw_t = -C_\gamma^{-1}\theta_t\,. \tag{3.4}$$

Recalling the s.c. group $e^{\mathbb{A}_{1,\gamma}t}$ on $Y_{1,\gamma}$ from (1.2.16), we can write the solution of (3.4) as

$$\begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} = e^{\mathbb{A}_{1,\gamma}t}\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} + L_t\theta, \quad L_t\theta = \int_0^t e^{\mathbb{A}_{1,\gamma}(t-\tau)}\begin{bmatrix} 0 \\ -C_\gamma^{-1}\theta_t(\tau) \end{bmatrix} d\tau \in Y_{1,\gamma}\,. \tag{3.5}$$

Since $B^{-\frac{1}{2}}\theta_t \in L_2(0,T;H)$ by (3.3) and $C_\gamma^{-\frac{1}{2}}B^{\frac{1}{2}}$ admits a bounded extension on H by the implication (1.2.6), we readily obtain by (1.2.10) that $C_\gamma^{-1}\theta_t \in L_2(0,T;\mathcal{D}(C_\gamma^{\frac{1}{2}}))$ and so $L_t\theta \in Y_{1,\gamma}$. With reference to the regularity obtained for the thermal variable θ, we introduce the space $X_{[0,t]}$ by setting

$$X_{[0,t]} \equiv C([0,t];H) \cap L_2(0,t;\mathcal{D}(B^{\frac{1}{2}})) \cap H^1(0,t;[\mathcal{D}(B^{\frac{1}{2}})]')\,. \tag{3.6}$$

Lemma 3.2. *Assume* (H.1), (H.2) = (1.2.4), *and* (H.3) = (1.2.5), *let* $\{w_0, w_1, \theta_0\} \in Y_\gamma$ *in* (1.2.9). *Then, with reference to problems* (1.2.1) – (1.2.3), *we have that the thermal component* $\theta(t)$ *satisfies the following regularity properties, continuously with respect to* Y_γ :

$$y_0 \equiv [w_0, w_1, \theta_0] \in Y_\gamma \to \theta \in X_{[0,t]} \underset{\text{compact}}{\hookrightarrow} L_2(0,t;\mathcal{D}(B^{\frac{1}{2}-\epsilon})),\ \forall\ 0 < \epsilon \le \frac{1}{2}\,. \tag{3.7}$$

Proof. Via Aubin's Lemma, as for Lemma 2.2, using B^{-1} compact on H. ■

Step 3. Lemma 3.3. *Assume* (H.1) (i), (H.2) = (1.2.4), *and* (H.3) = (1.2.5), *so that, for* $\{w_0, w_1, \theta_0\} \in Y_\gamma$, *then* $\theta \in X_{[0,t]}$ *as in Lemma* 2.2, *Eqn.* (3.7). *Then,* $L_t\theta \in Y_{1,\gamma}$ *in* (3.5) *may be rewritten as*

$$L_t\theta = K_{1,t}\theta + K_{2,t}\theta \in Y_{1,\gamma}\,, \tag{3.8}$$

where

$$\left.\begin{aligned} K_{1,t}\theta \;\equiv\; & \begin{bmatrix} 0 \\ -C_\gamma^{-1}\theta(t) \end{bmatrix} + e^{\mathbf{A}_{1,\gamma}t}\begin{bmatrix} 0 \\ C_\gamma^{-1}\theta_0 \end{bmatrix} \\ & + \int_0^t e^{\mathbf{A}_{1,\gamma}(t-\tau)}\begin{bmatrix} 0 \\ C_\gamma^{-1}BC_\gamma^{-1}B^{-\frac{1}{2}} \end{bmatrix} B^{\frac{1}{2}}\theta(\tau)d\tau; \end{aligned}\right\} \tag{3.9}$$

$$K_{2,t}\theta \;=\; \int_0^t e^{\mathbf{A}_{1,\gamma}(t-\tau)}\begin{bmatrix} -C_\gamma^{-1}B^{-\frac{1}{2}} \\ 0 \end{bmatrix} B^{\frac{1}{2}}\theta(\tau)d\tau \in Y_{1,\gamma}\,. \tag{3.10}$$

Proof. We integrate by parts $L_t\theta$ in (3.5), using via (1.2.16) that

$$\mathbf{A}_{1,\gamma}\begin{bmatrix} 0 \\ C_\gamma^{-1}\theta(\tau) \end{bmatrix} = \begin{bmatrix} 0 & I \\ -C_\gamma^{-1}A & -C_\gamma^{-1}B \end{bmatrix}\begin{bmatrix} 0 \\ C_\gamma^{-1}\theta(\tau) \end{bmatrix} = \begin{bmatrix} C_\gamma^{-1}\theta(\tau) \\ -C_\gamma^{-1}BC_\gamma^{-1}\theta(\tau) \end{bmatrix}. \quad ■ \tag{3.11}$$

Lemma 3.4. *Assume* (H.1), (H.2) = (1.2.4), (H.3) = (1.2.5). *Then, with reference to* (3.9), *we have*

$$K_{1,t}:\; X_{[0,t]} \to Y_{1,\gamma} = \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(C_\gamma^{\frac{1}{2}}) \text{ is compact.} \tag{3.12}$$

Proof. As in the proof of Lemma 2.4, after noticing that

$$C_\gamma^{-1}BC_\gamma^{-1}B^{-\frac{1}{2}} = [C_\gamma^{-1}B^{\frac{1}{2}}][B^{\frac{1}{2}}C_\gamma^{-\frac{1}{2}}][C_\gamma^{-\frac{1}{2}}B^{-\frac{1}{2}}] \text{ is compact } H \to \mathcal{D}(C_\gamma^{\frac{1}{2}}), \tag{3.13}$$

by use of implication (1.2.6), and compactness of B^{-1}, or C^{-1}, in H. ■

Step 4. So far, the proof of the present subsection has been essentially contained in that of Section 2, *mutatis mutandis*. However, in tackling the second term $K_{2,t}\theta$ in (3.10) of $L_t\theta$ in (3.8), we face additional difficulties over the same point in Section 2. For one, with $\theta \in X_{[0,t]}$, it is not clear from (3.10) *per se* that $K_{2,t}\theta$ is well defined, let alone in $Y_{1,\gamma}$, as we cannot assert that $A^{\frac{1}{2}}C_\gamma^{-1}B^{-\frac{1}{2}} \in \mathcal{L}(H)$, as desirable. In the case of the term $K_{2,t}\theta$ in (2.12) of Section 2, the special property valid there that $C = A^{\frac{1}{2}} = \mathcal{A}_D$ helped (indeed, $\mathcal{D}(C) \subset \mathcal{D}(A^{\frac{1}{2}})$ would suffice, but this is not assumed in the present subsection either). Moreover, we actually seek to establish that $K_{2,t}$: compact $X_{[0,t]} \to Y_{1,\gamma}$. To this end we shall invoke, for the first time, assumption (H.4) = (1.2.22).

Lemma 3.5. *Assume* (H.1)(i), (H.2) = (1.2.4), (H.3) = (1.2.5). *Moreover, with reference to the undamped Kirchoff problem* (1.2.12), (1.2.13), *assume hypothesis* (H.4) = (1.2.22); *i.e., that*

$$\{\psi_0, \psi_1\} \in Y_{1,\gamma} \Rightarrow \psi_{tt} = -C_\gamma^{-1} A\psi \in L_2(0,T;H) \text{ continuously.} \tag{3.14}$$

Then, with reference to the damped Kirchoff problem (1.2.14), (1.2.15), *with the same initial conditions* $\{\phi_0 = \psi_0,\ \phi_1 = \psi_1\}$ *as the* ψ*-problem, it follows that, in fact,*

$$\{\psi_0, \psi_1\} \in Y_{1,\gamma} \Rightarrow C_\gamma^{-1} A\phi, \phi_{tt} = -C_\gamma^{-1}A\phi - C_\gamma^{-1}B\phi_t \in L_2(0,T;H) \text{ continuously.} \tag{3.15}$$

Proof. By the *a-priori* group regularity, see (1.2.20), we have $\phi_t \in C([0,T];\mathcal{D}(C_\gamma^{\frac{1}{2}}))$. Then, by the implication (1.2.6), it follows that

$$C_\gamma^{-1}B\phi_t = C_\gamma^{-\frac{1}{2}}[C_\gamma^{-\frac{1}{2}}B^{\frac{1}{2}}][B^{\frac{1}{2}}C_\gamma^{-\frac{1}{2}}]C_\gamma^{\frac{1}{2}}\phi_t \in C([0,T];\mathcal{D}(C_\gamma^{\frac{1}{2}})). \tag{3.16}$$

Thus, with reference to (3.15) as well as to Eqn. (1.2.21) relating ϕ to ψ, it remains to show that

$$C_\gamma^{-1}A\phi = C_\gamma^{-1}A\psi - (C_\gamma^{-1}A)\int_0^t S_{0,\gamma}(t-\tau)C_\gamma^{-1}B\phi_t(\tau)d\tau \in L_2(0,T;H), \tag{3.17}$$

where, as noted below (1.2.21), $S_{0,\gamma}(\,\cdot\,)$ is the sine operator corresponding to the cosine operator $C_{0,\gamma}(\,\cdot\,)$ on the space $Z_\gamma \equiv \mathcal{D}(C_\gamma^{\frac{1}{2}})$, generated by the negative self-adjoint operator $(-C_\gamma^{-1}A)$ on $Z_\gamma \equiv \mathcal{D}(C_\gamma^{\frac{1}{2}})$. For the first term on the right-hand side of (3.17), we invoke assumption (3.14). To show that the second term on the right-hand side of (3.17) is also in $L_2(0,T;H)$, we provide two proofs, one here below and one (purely operator-theoretic) in Appendix A. Recalling (1.2.19) we set

$$S_{0,\gamma}(t-\tau)C_\gamma^{-1}B\phi_t(\tau) = \psi(t-\tau;\bar\psi_0;\bar\psi_1(\tau));\bar\psi_0 = 0,\ \bar\psi_1(\tau) = C_\gamma^{-1}B\phi_t(\tau), \tag{3.18}$$

for the solution at time $(t-\tau)$ of problem (1.2.12), with initial position $\bar\psi_0$, and initial velocity $\bar\psi_1(\tau)$. Next, we take any $f \in L_2(0,T;H)$ and compute via (3.18)

$$\left|\left(f, C_\gamma^{-1}A\int_0^{\cdot} S_{0,\gamma}(\,\cdot\,-\tau)C_\gamma^{-1}B\phi_t(\tau)d\tau\right)_{L_2(0,T;H)}\right|$$

$$= \left|\int_0^T\int_0^t (f(t), C_\gamma^{-1}A\psi(t-\tau;\bar\psi_0,\bar\psi_1(\tau))_H d\tau\, dt\right| \tag{3.19}$$

(interchanging the order of integration)

$$= \left|\int_0^T\int_\tau^T (f(t), C_\gamma^{-1}A\psi(t-\tau;\bar\psi_0,\bar\psi_1(\tau))_H dt\, d\tau\right| \tag{3.20}$$

$$\leq \|f\|_{L_2(0,T;H)}\int_0^T \|C_\gamma^{-1}A\psi(\,\cdot\,;\bar\psi_0,\bar\psi_1(\tau)\|_{L_2(0,T;H)}d\tau \tag{3.21}$$

$$\text{(by (3.14))} \quad \leq c_T\|f\|_{L_2(0,T;H)}\int_0^T \|\bar\psi_1(\tau)\|_{\mathcal{D}(C_\gamma^{\frac{1}{2}})}d\tau \tag{3.22}$$

$$\left.\begin{aligned} \text{(by (3.18))} \quad &= c_T\|f\|_{L_2(0,T;H)}\int_0^T \|C_\gamma^{-1}B\phi_t(\tau)\|_{\mathcal{D}(C_\gamma^{\frac{1}{2}})}d\tau \\ &\leq \text{const}_T\|f\|_{L_2(0,T;H)}\|\{\psi_0,\psi_1\}\|_{Y_{1,\gamma}}. \end{aligned}\right\} \tag{3.23}$$

We have used: the Schwarz inequality from (3.20) to (3.21); assumption (3.14) from (3.21) to (3.22); the definition of $\bar{\psi}_1(\tau)$ in (3.18) from (3.22) to (3.23); finally, (3.16) in the last step of (3.23). Then, (3.23) shows that continuously,

$$\{\phi_0 = \psi_0,\ \phi_1 = \psi_1\} \in Y_{1,\gamma} \to C_\gamma^{-1} A \int_0^{\cdot} S_{0,\gamma}(\,\cdot\, - \tau) C_\gamma^{-1} B \phi_t(\tau) d\tau \in L_2(0,T;H), \quad (3.24)$$

as desired. Thus, (3.17), and hence (3.15), have been proved. ∎

Remark 3.1. A similar argument shows that (3.15) ⇒ (3.14). ∎

Step 5. A duality argument as in Remark 1.2.1 shows that

Lemma 3.6. (i) *With reference to* (3.10), *we have that:*

$$-K_{2,t} f = \int_0^t e^{\mathbb{A}_{1,\gamma}(t-\tau)} \begin{bmatrix} C_\gamma^{-1} f(\tau) \\ 0 \end{bmatrix} d\tau : \text{ continuous } L_2(0,t;H) \to Y_{1,\gamma}, \quad (3.25)$$

if and only if, with reference to problem (1.2.14) *or* (1.2.20),

$$\{\phi_0, \phi_1\} \in Y_{1,\gamma} \Rightarrow C_\gamma^{-1} A \phi \in L_2(0,t;H). \quad (3.26)$$

(ii) *Thus, under assumptions* (H.1) (i), (H.2) = (1.2.4), (H.3) = (1.2.5), (H.4) = (1.2.22), *the regularity* (3.25) *for* $K_{2,t}$ *holds true.*

Proof. (i) As in Remark 1.2.1 we compute

$$\left(\int_0^t e^{\mathbb{A}_{1,\gamma}(t-\tau)} \begin{bmatrix} C_\gamma^{-1} f(\tau) \\ 0 \end{bmatrix} d\tau, \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} \right)_{Y_{1,\gamma}} = \int_0^t \left(\begin{bmatrix} C_\gamma^{-1} f(\tau) \\ 0 \end{bmatrix}, e^{\mathbb{A}^*_{1,\gamma}(t-\tau)} \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} \right)_{Y_{1,\gamma}} d\tau \quad (3.27)$$

$$= \int_0^t (C_\gamma^{-1} f(\tau), \phi(t-\tau; \phi_0, \phi_1))_{\mathcal{D}(A^{\frac{1}{2}})} d\tau = (f, C_\gamma^{-1} A \phi(t - \,\cdot\,; \phi_0, \phi_1))_{L_2(0,t;H)}. \quad (3.28)$$

$\mathbb{A}^*_{1,\gamma}$ is readily computed from (1.2.16), using properties (1.2.11).

The conclusion of part (i) follows now from (3.28). For part (ii) we invoke Lemma 3.5. ∎

Remark 3.2. Lemma 3.5, Remark 3.1, and Lemma 3.6 show that the regularity (1.2.25) for $\mathbb{A}_{0,\gamma}$ is equivalent to the regularity (3.25) for $\mathbb{A}_{1,\gamma}$. ∎

Step 6. Proposition 3.7. *Assume* (H.1) (i); *compactness of the resolvent of* B *in* (H.1) (ii); (H.2) = (1.2.4); (H.3) = (1.2.5) *and* (H.4) = (1.2.22). *Then, with reference to the operator* $K_{2,t}$ *in* (3.10) *or* (3.25), *we have*

$$K_{2,t} : \text{ compact } X_{[0,t]} \to Y_{1,\gamma} = \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(C_\gamma^{\frac{1}{2}})\,. \quad (3.29)$$

Proof. The validity of (3.29) follows from

$$\theta \in X_{[0,t]} \underset{\text{by (3.7)}}{\overset{\text{compact}}{\hookrightarrow}} \theta \in L_2(0,t;H) \underset{\text{by (3.25)}}{\overset{\text{bounded}}{\longrightarrow}} K_{2,t}\theta \in Y_{1,\gamma}\,, \tag{3.30}$$

via the compact embedding (with $\epsilon = \frac{1}{2}$) in (3.7) and the boundedness of $K_{2,t}$ as in (3.25). ∎

Step 7. Proposition 3.8. *Assume* (H.1) *through* (H.4). *Then,*

$$\left.\begin{aligned} &L_t:\ X_{[0,t]} \to Y_{1,\gamma} \text{ is compact;} \\ &y_0 = [w_0, w_1, \theta_0] \in Y_\gamma \to \theta \in X_{[0,t]} \to L_t\theta \in Y_{1,\gamma} \text{ is compact.} \end{aligned}\right\} \tag{3.31}$$

Proof. We return to identity (3.8), where $K_{1,t}$ is compact as in (3.12), and $K_{2,t}$ is compact as in (3.29). Finally, we recall (3.7). ∎

Step 8. Thus, at this stage, we have obtained the following decomposition, from (3.5) and (1.2.2),

$$\begin{bmatrix} w(t) \\ w_t(t) \\ \theta(t) \end{bmatrix} = e^{\mathbb{A}_\gamma t} \begin{bmatrix} w_0 \\ w_1 \\ \theta_0 \end{bmatrix} = \begin{bmatrix} \left[e^{\mathbb{A}_{1,\gamma} t} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} + L_t\theta \right] \\ e^{-Bt}\theta_0 - \displaystyle\int_0^t e^{-B(t-\tau)} B w_t(\tau)\, d\tau \end{bmatrix}. \tag{3.32}$$

where the map $y_0 \to [L_t\theta, e^{-Bt}\theta_0]$ is compact $Y_\gamma \to Y_\gamma$ for all $t > 0$. We further notice that the integral defining $\theta(t)$ is well defined and, in fact, compact on $Y_{1,\gamma}$ at each $t > 0$,

$$\int_0^t e^{-B(t-\tau)} B w_t(\tau)\, d\tau = \int_0^t B^{\frac{1}{2}} e^{-B(t-\tau)} (B^{\frac{1}{2}} C_\gamma^{-\frac{1}{2}}) C_\gamma^{\frac{1}{2}} w_t(\tau)\, d\tau \in \mathcal{D}(B^{\frac{1}{2}-\epsilon}) \underset{\text{compact}}{\hookrightarrow} H, \tag{3.33}$$

since $B^{\frac{1}{2}} C_\gamma^{-\frac{1}{2}} \in \mathcal{L}(H)$ by implication (1.2.6), $C_\gamma^{\frac{1}{2}} w_t \in C([0,T];H)$ by the *a-priori* semigroup regularity in Proposition 1.2.1 (ii); moreover,

$$\|B^{1-\epsilon} e^{-Bt}\|_{\mathcal{L}(H)} \le \frac{\text{const}}{t^{1-\epsilon}},\ \|B^{1-\epsilon} e^{-Bt}\|_{\mathcal{L}(H)} \in L_1(0,T)\,, \tag{3.34}$$

by analyticity of the (self-adjoint) semigroup e^{-Bt} on H, so that the integral in (3.33) is the convolution of an L_1-function with a C-function; finally, B^{-1} is compact on H by assumption (H.1) (ii). Thus, the operator $\mathcal{K}_\gamma(t)$,

$$\mathcal{K}_\gamma(t) \begin{bmatrix} w_0 \\ w_1 \\ \theta_0 \end{bmatrix} \overset{\text{def}}{=} \begin{bmatrix} L_t\theta \\ e^{-Bt}\theta_0 + \displaystyle\int_0^t e^{-B(t-\tau)} B w_t(\tau)\, d\tau \end{bmatrix} \text{is compact } Y_\gamma \to Y_\gamma\,, \tag{3.35}$$

for all $t > 0$, by Proposition 3.8 on $L_t\theta$ and (3.33). Theorem 1.2.2 is proved.

We conclude this section with two remarks which provide further relevant regularity theory on the thermo-elastic problem (1.2.1) – (1.2.3) of the present setting.

Remark 3.3. Under assumptions (H.1) – (H.4), the following regularity holds true for the mechanical component of problem (1.2.1) – (1.2.3):

$$y_0 = [w_0, w_1, \theta_0] \in Y_\gamma \to w_{tt} \in L_2(0,T;H) \text{ continuously.} \tag{3.36}$$

Proof. With reference to Eqn. (1.2.1), we have as in (1.2.20), (1.2.21), with $\psi(0) = w_0,\ \psi_t(0) = w_1$:

$$\begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} = \begin{bmatrix} \psi(t) \\ \psi_t(t) \end{bmatrix} + \int_0^t e^{\mathbb{A}_{0,\gamma}(t-\tau)} \begin{bmatrix} 0 \\ C_\gamma^{-1} B\theta(\tau) \end{bmatrix} d\tau\,; \tag{3.37}$$

$$w_t(t) = \psi_t(t) + \int_0^t C_{0,\gamma}(t-\tau) C_\gamma^{-1} B\theta(\tau) d\tau\,; \tag{3.38}$$

$$w_{tt}(t) = \psi_{tt}(t) + C_\gamma^{-1} B\theta(t) - C_\gamma^{-1} A \int_0^t S_{0,\gamma}(t-\tau) C_\gamma^{-1} B\theta(\tau) d\tau\,, \tag{3.39}$$

where $C_{0,\gamma}(t)$ is the cosine operator on $\mathcal{D}(C_\gamma^{\frac{1}{2}})$ corresponding to the negative self-adjoint operator $[-C_\gamma^{-1}A]$ on $\mathcal{D}(C_\gamma^{\frac{1}{2}})$, and $S_{0,\gamma}(t)$ its corresponding sine operator; see also (3.17). By the *a-priori* regularity of $B^{\frac{1}{2}}\theta$ in (3.1), and by (1.2.6), we have

$$C_\gamma^{-1} B\theta = C_\gamma^{-\frac{1}{2}} [C_\gamma^{-\frac{1}{2}} B^{\frac{1}{2}}] B^{\frac{1}{2}} \theta \in L_2(0,T;\mathcal{D}(C_\gamma^{\frac{1}{2}}))\,. \tag{3.40}$$

This regularity in (3.40) — though weaker in time than the one in (3.16) — is sufficient to carry out the argument below (3.17) (or in Appendix A), applied to the integral term in (3.39), to conclude that such integral term is in $L_2(0,T;H)$ (as in (3.24)). Thus, by assumption (H.4) = (1.2.22), we conclude from (3.39), (3.40), that (3.36) holds true. ∎

Remark 3.4. We now show that, under assumptions (H.1) – (H.4), the following regularity holds true for the thermal component of problem (1.2.1) – (1.2.3); with $\theta_0 = 0$:

$$y_0 = [w_0, w_1, 0] \in Y_\gamma \Rightarrow \theta_t \in L_2(0,T;H) \text{ and } \theta_{tt} \in L_2(0,T;[\mathcal{D}(B)]') \text{ continuously.} \tag{3.41}$$

Proof. By Eqn. (1.2.2), $(\theta_t)_t = -B\theta_t - Bw_{tt},\ \theta_t(0) = -Bw_t(0)$ with $\theta_0 = 0$, and

$$\theta_t(t) = -B^{\frac{1}{2}} e^{-Bt} [B^{\frac{1}{2}} C_\gamma^{-\frac{1}{2}}] C_\gamma^{\frac{1}{2}} w_t(0) - B \int_0^t e^{-B(t-\tau)} w_{tt}(\tau) d\tau\,, \tag{3.42}$$

where $B^{\frac{1}{2}} w_t(0) \in H$ by (1.2.6), since $w_t(0) = w_1 \in \mathcal{D}(C_\gamma^{\frac{1}{2}})$. The analyticity of the s.c. semigroup e^{-Bt} generated by the self-adjoint operator $(-B)$ then yields conclusion (3.41) for θ_t from (3.42) by use also of the regularity (3.36) of w_{tt}. Differentiating (3.42) yields readily the regularity (3.41) for θ_{tt}, again by the regularity (3.36) of w_{tt}. ∎

4. Examples of thermo-elastic plates fitting into the abstract model

In this section we consider three thermo-elastic plate equations (1.1.1), (1.1.2), supplemented by sets of physical B.C., which fit into the abstract model (1.2.1), (1.2.2) of Section 1.2. In all cases, we then verify all required assumptions (H.1), (H.2) = (1.2.4), (H.3) = (1.2.5), and (H.4) = (1.2.22). Unless otherwise stated, Eqn. (1.1.1), (1.1.2) will be defined on a smooth bounded domain Ω of R^n, n arbitrary.

4.1. Hinged mechanical B.C. and Dirichlet thermal B.C.

This is the simplest canonical case. We supplement Eqns. (1.1.1), (1.1.2) with the following (hinged/Dirichlet) B.C.

$$w|_\Sigma \equiv 0, \quad \Delta w|_\Sigma \equiv 0; \quad \theta|_\Sigma \equiv 0; \quad \Sigma = (0,T] \times \Gamma\,. \tag{4.1.1}$$

Then, the resulting abstract system for problem (1.1.1), (1.1.2), (4.1.1) is given by (1.2.1) – (1.2.3), where

$$B = C = A^{\frac{1}{2}};\ H = L_2(\Omega);\ Bf = -\Delta f,\ \mathcal{D}(B) = H^2(\Omega) \cap H_0^1(\Omega)\,; \tag{4.1.2}$$

$$Af = \Delta^2 f,\ \mathcal{D}(A) = \{f \in H^4(\Omega) :\ f|_\Gamma = \Delta f|_\Gamma = 0\}; \mathcal{D}(A^{\frac{1}{4}}) = H_0^1(\Omega) = \mathcal{D}(C^{\frac{1}{2}}); \tag{4.1.3}$$

$$\|f\|^2_{\mathcal{D}((I+\gamma C)^{\frac{1}{2}})} = \|f\|_H^2 + \gamma \|C^{\frac{1}{2}} f\|_H^2 = \int_\Omega [|f|^2 + \gamma |\nabla f|^2] d\Omega\,. \tag{4.1.4}$$

Assumptions (H.1) through (H.3) are trivially satisfied. So is (H.4), in fact, in the following stronger form: the solution of the Kirchoff problem:

$$\psi_{tt} + \gamma A^{\frac{1}{2}} \psi_{tt} + A\psi \equiv 0,\ \{\psi_0, \psi_1\} \in \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(A^{\frac{1}{4}}) \tag{4.1.5}$$

satisfies the regularity properties (see (1.2.9), (1.2.13))

$$\{\psi, \psi_t\} \in C([0,T]; \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(A^{\frac{1}{4}})), \psi_{tt} = -(I + \gamma A^{\frac{1}{2}})^{-1} A\psi \in C([0,T]; L_2(\Omega)). \tag{4.1.6}$$

For this special canonical case, a *precise description* of the corresponding s.c. semigroup $e^{\mathbf{A}_\gamma t}$ is given in [6] by spectral analysis.

4.2. Clamped mechanical B.C. and Dirichlet thermal B.C.

We supplement Eqns. (1.1.1), (1.1.2) with the following (clamped/Dirichlet) B.C.

$$w|_\Sigma \equiv 0,\ \frac{\partial w}{\partial \nu}\Big|_\Sigma \equiv 0;\ \theta|_\Sigma \equiv 0,\ \Sigma = (0,T] \times \Gamma\,. \tag{4.2.1}$$

Then, the resulting abstract system for problem (1.1.1), (1.1.2), (4.2.1) is given by (1.2.1) – (1.2.3), where

$$H = L_2(\Omega);\ B = C,\ Bf = -\Delta f,\ \mathcal{D}(B) = H^2(\Omega) \cap H_0^1(\Omega)\,; \tag{4.2.2}$$

$$Af = \Delta^2 f,\ \mathcal{D}(A) = \left\{ f \in H^4(\Omega) :\ f|_\Gamma = \frac{\partial f}{\partial \nu}\Big|_\Gamma = 0 \right\}; \tag{4.2.3}$$

$$\mathcal{D}(B^{\frac{1}{2}}) = \mathcal{D}(C^{\frac{1}{2}}) = \mathcal{D}(A^{\frac{1}{4}}) = H_0^1(\Omega); \quad \mathcal{D}(A^{\frac{1}{2}}) = H_0^2(\Omega) \subset \mathcal{D}(B) = \mathcal{D}(C)\,. \tag{4.2.4}$$

Eqn. (4.1.4) still holds true. Assumptions (H.1) through (H.3) are satisfied.

Assumption (H.4). Verification of assumption (H.4) = (1.2.22) is, however, non-trivial and requires a *sharp trace regularity* result for the Kirchoff equation

$$\left.\begin{array}{rcl}\psi_{tt}-\gamma\Delta\psi_{tt}+\Delta^2\psi & = & 0 \text{ in } Q\\ \psi|_\Sigma=\left.\dfrac{\partial\psi}{\partial\nu}\right|_\Sigma & \equiv & 0 \text{ in } \Sigma\end{array};\ \text{or } \psi_{tt}+\gamma C\psi_{tt}+A\psi=0\,;\right\} \tag{4.2.5}$$

$$\psi(0)=\psi_0,\ \psi_1(0)=\psi_1,\ \{\psi_0,\psi_1\}\in\mathcal{D}(A^{\frac12})\times\mathcal{D}(A^{\frac14})=H_0^2(\Omega)\times H_0^1(\Omega)\,, \tag{4.2.6}$$

namely [19, p. 123],

$$\int_0^t\int_\Gamma|\Delta\psi|^2d\Gamma\,dt\le C_t\|\{\psi_0,\psi_1\}\|_{H_0^2(\Omega)\times H_0^1(\Omega)}\,, \tag{4.2.7}$$

which is "$\frac12$" sharper in the space variable, than the result that one would get by a *formal* application of trace theory on the optimal *interior* regularity result $\psi\in C([0,T];\mathcal{D}(A^{\frac12})=H_0^2(\Omega))$ of (4.2.5). This fact is distinctive of many hyperbolic/Petrowski-type P.D.E.'s. With reference to the operator $\mathbf{A}_{0,\gamma}$ in the present case, see (1.2.16), and to $Y_{1,\gamma}$ which is topologically equivalent to $\mathcal{D}(A^{\frac12})\times\mathcal{D}(A^{\frac14})$, we have that

$$\mathcal{D}(\mathbf{A}_{0,\gamma}^2)=\mathcal{D}(A^{\frac12}C_\gamma^{-1}A)\times\mathcal{D}(A^{\frac34})\subset\mathcal{D}(A)\times\mathcal{D}(A^{\frac34})\,, \tag{4.2.8}$$

where the inclusion in (4.2.8) follows, since $A^{\frac12}C_\gamma^{-1}Ax_1\in L_2(\Omega)$ implies $Ax_1\in L_2(\Omega)$, as $C_\gamma A^{-\frac12}\in\mathcal{L}(L_2(\Omega))$ by (4.2.4). Thus, let $\{\psi_0,\psi_1\}\in\mathcal{D}(\mathbf{A}_{0,\gamma}^2)$, dense in $Y_{1,\gamma}$. The corresponding solution ψ of (4.2.5) then satisfies $\{\psi,\psi_t\}\in C([0,T];\mathcal{D}(A_{0,\gamma}^2))$, and thus $\psi\in C([0,T];\mathcal{D}(A))$ by (4.2.8). Hence, $A\psi=\Delta^2\psi$ and for $f\in L_2(0,T;L_2(\Omega))$ we compute with $H=L_2(\Omega)$, via Green's second theorem:

$$-((I+\gamma C)^{-1}f,A\psi)_H=((I+\gamma C)^{-1}f,\Delta^2\psi)_{L_2(\Omega)}=(\Delta(I+\gamma C)^{-1}f,\Delta\psi)_{L_2(\Omega)} \tag{4.2.9}$$

$$+\left((I+\gamma C)^{-1}f,\cancel{\frac{\partial\Delta\psi}{\partial\nu}}\right)_{L_2(\Gamma)}-\left(\frac{\partial(I+\gamma C)^{-1}f}{\partial\nu},\Delta\psi\right)_{L_2(\Gamma)}. \tag{4.2.10}$$

But, with $f\in L_2(\Omega)$ a.e. then $(I+\gamma C)^{-1}f\in\mathcal{D}(C)=\mathcal{D}(B)=H^2(\Omega)\cap H_0^1(\Omega)$ a.e. by (4.2.2) and so $[(I+\gamma C)^{-1}f]_\Gamma=0$ a.e. and the second term on the right-hand side of (4.2.10) vanishes. Next, $\Delta(I+\gamma C)^{-1}f\in L_2(\Omega)$ a.e. and then

$$\left|(\Delta(I+\gamma C)^{-1}f,\Delta\psi)_{L_2(\Omega)}\right|\le c\|\psi\|_{H_0^2(\Omega)}\|f\|_{L_2(\Omega)}\ \text{ a.e.} \tag{4.2.11}$$

Moreover, $\frac{\partial(I+\gamma C)^{-1}f}{\partial\nu}\in H^{\frac12}(\Gamma)$ a.e., and then

$$\left|\left(\frac{\partial(I+\gamma C)^{-1}f}{\partial\nu},\ \Delta\psi\right)_{L_2(\Gamma)}\right|\le c\|f\|_{L_2(\Omega)}\|\Delta\psi\|_{H^{-\frac12}(\Gamma)}\ \text{ a.e.} \tag{4.2.12}$$

Using (4.2.11), (4.2.12) in (4.2.10) yields

$$|((I+\gamma C)^{-1}f,A\psi)_H|\le c\|f\|_{L_2(\Omega)}[\|\psi\|_{H_0^2(\Omega)}+\|\Delta\psi\|_{H^{-\frac12}(\Gamma)}]\ \text{ a.e.} \tag{4.2.13}$$

Then, since $\{\psi_0, \psi_1\} \in H_0^2(\Omega) \times H_0^1(\Omega) = Y_{1,\gamma}$ implies $\psi \in C([0,T]; H_0^2(\Omega))$ by the semigroup estimate, integrating (4.2.13) in time yields by Schwarz inequality

$$\int_0^t |((I+\gamma C)^{-1} f(\tau), A\psi(\tau))_H| d\tau \le c_t \|f\|_{L_2(0,t;L_2(\Omega))} \|\{\psi_0, \psi_1\}\|_{H_0^2(\Omega)\times H_0^1(\Omega)} \quad (4.2.14)$$

a fortiori by use of the sharp estimate (4.2.7). Inequality (4.2.14), which is obtained at first for $\{\psi_0, \psi_1\} \in \mathcal{D}(\mathbf{A}_{0,\gamma}^2)$, is next extended to all of $\{\psi_0, \psi_1\} \in H_0^2(\Omega) \times H_0^1(\Omega) = \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(A^{\frac{1}{4}}) = Y_{1,\gamma}$ (topologically). Thus assumption (H.4) = (1.2.22) is verified, since $\psi_{tt} = -C_\gamma^{-1} A\psi$ and the results of Section 1.2 apply to the present case.

4.3. Clamped mechanical B.C. with Neumann (Robin) Thermal B.C.

We supplement Eqns. (1.1.1), (1.1.2) with the following (clamped/Neumann-Robin) B.C.

$$w|_\Sigma \equiv 0, \ \frac{\partial w}{\partial \nu}\Big|_\Sigma \equiv 0; \ \left[\frac{\partial \theta}{\partial \nu} + b\theta\right]_\Sigma \equiv 0, \ b \ge 0 \,. \quad (4.3.1)$$

Then, the resulting abstract system for problem (1.1.1), (1.1.2), (4.3.1) is given by (1.2.1) – (1.2.3), where for $b > 0$:

$$H = L_2(\Omega); \ Bf = -\Delta f, \ \mathcal{D}(B) = \left\{ f \in H^2(\Omega) : \ \left[\frac{\partial f}{\partial \nu} + bf\right]_\Gamma = 0 \right\}; \quad (4.3.2)$$

$$Cf = -\Delta f, \ \mathcal{D}(C) = H^2(\Omega) \cap H_0^1(\Omega)\,; \quad (4.3.3)$$

$$Af = \Delta^2 f, \ \mathcal{D}(A) = \left\{ f \in H^4(\Omega) : \ f|_\Gamma = \frac{\partial f}{\partial \nu}\Big|_\Gamma = 0 \right\}; \quad (4.3.4)$$

$$\mathcal{D}(A^{\frac{1}{2}}) = H_0^2(\Omega) \subset \mathcal{D}(B); \ \mathcal{D}(C^{\frac{1}{2}}) = H_0^1(\Omega) = \mathcal{D}(A^{\frac{1}{4}})\,. \quad (4.3.5)$$

[For $b = 0$, we take $H = L_2(\Omega)/\mathcal{N}(B)$, $\mathcal{N}(B)$ = one-dimensional null space of B.] Moreover, (4.1.4) still holds true. Assumptions (H.1) through (H.3) are trivially satisfied. Since A and C are the same as in Section 4.2, then assumption (H.4) = (1.2.22) likewise holds true, as seen there. Thus, the results of Section 1.2 apply to the present case.

5. Analysis of the free B.C. case of Section 1.3

5.1. Preliminary energy estimates. The auxiliary operator $A_{\gamma,0}$

We begin by collecting some formulas. If $f \in \mathcal{D}(\mathcal{A}_R)$, see (1.3.13), then by (1.3.14),

$$\mathcal{A}_R f = -\Delta f = -\Delta\left(f - N\frac{\partial f}{\partial \nu}\right) = \mathcal{A}_N\left(f - N\frac{\partial f}{\partial \nu}\right) = \mathcal{A}_N(f + bN(f|_\Gamma)), \quad (5.1.1)$$

since $\left[f - N\frac{\partial f}{\partial \nu}\right] \in \mathcal{D}(\mathcal{A}_N)$, see (1.3.11). From (5.1.1), recalling (1.3.19), we obtain

$$\left.\begin{aligned} \mathcal{A}_N^{-1}\mathcal{A}_R f &= f + bN(f|_\Gamma); \ C_\gamma^{-1}\mathcal{A}_R f \\ &= \frac{1}{\gamma}[f + bN(f|_\Gamma)] - \frac{1}{\gamma} C_\gamma^{-1}[f + bN(f|_\Gamma)], f \in \mathcal{D}(\mathcal{A}_R); \end{aligned}\right\} \quad (5.1.2)$$

$$\left.\begin{aligned} &\mathcal{A}^{\frac{1}{2}}C_\gamma^{-1} \in \mathcal{L}(L_2(\Omega)),\ \mathcal{A}^{\frac{1}{4}}C_\gamma^{-\frac{1}{2}} \in \mathcal{L}(L_2(\Omega)),\ \text{hence} \\ &C_\gamma^{-1}\mathcal{A}^{\frac{1}{2}} \in \mathcal{L}(L_2(\Omega));\ C_\gamma^{-\frac{1}{2}}\mathcal{A}_R^{\frac{1}{2}} \in \mathcal{L}(L_2(\Omega)), \end{aligned}\right\} \tag{5.1.3}$$

recalling (1.3.19), (1.3.12). The second statement in (5.1.3) means that, by duality on the first, $C_\gamma^{-1}\mathcal{A}^{\frac{1}{2}}$ has a bounded extension in $\mathcal{L}(L_2(\Omega))$.

We now present the counterpart of Lemma 2.1 and Lemma 3.1 of preceding cases (of Sections 1.1 and 1.2).

Lemma 5.1.1. (i) *With reference to problem* (1.3.1) – (1.3.6), *the following identity holds true, where* $H = L_2(\Omega)$ *and* $y_0 = [w_0, w_1, \theta_0] \in Y_\gamma$ *:*

$$E(t) + 2\int_0^t \left\| \mathcal{A}_R^{\frac{1}{2}}\theta(\tau) \right\|_H^2 d\tau \equiv E(0)\,; \tag{5.1.4}$$

$$\left\| e^{\mathbb{A}_\gamma t} y_0 \right\|_{Y_\gamma}^2 \equiv E(t) = \|\mathcal{A}^{\frac{1}{2}}w(t)\|_H^2 + \|w_t(t)\|_H^2 + \gamma\|\mathcal{A}_N^{\frac{1}{2}}w_t(t)\|_H^2 + \|\theta(t)\|_H^2 \tag{5.1.5a}$$

$$(\text{by (1.3.21)}) = \|\mathcal{A}^{\frac{1}{2}}w(t)\|_H^2 + \|w_t(t)\|^2_{\mathcal{D}(C_\gamma^{\frac{1}{2}})} + \|\theta(t)\|_H^2. \tag{5.1.5b}$$

(ii) *Moreover,*

$$y_0 \in Y_\gamma \Rightarrow \theta|_\Gamma \in L_2(0,T; H^{\frac{1}{2}}(\Gamma)) \text{ and } w_{tt} \in L_2(0,T; L_2(\Omega)) \text{ continuously.} \tag{5.1.6}$$

Proof. (i) The proof of part (i) is the same as in Lemma 2.1 or Lemma 3.1: either it relies on Proposition 1.3.1, or else one multiplies Eqn. (1.3.1) by w_t, (1.3.2) by θ, and integrates by parts. The use of Green's formula as in [18, p. 68] provides the desired cancellation of the boundary terms.

(ii) The statement in (5.1.6) about $\theta|_\Gamma$ follows by trace theory on (5.1.4), i.e., $\theta \in L_2(0,T; H^1(\Omega))$.

Next, we return to Eqn. (1.3.9), which by means of (1.3.19), we rewrite as

$$w_{tt} = -C_\gamma^{-1}\mathcal{A}w + C_\gamma^{-1}\mathcal{A}_R\theta - C_\gamma^{-1}\mathcal{A}G_1(\theta|_\Gamma) + bC_\gamma^{-1}\mathcal{A}G_2(\theta|_\Gamma). \tag{5.1.7}$$

The *a-priori* regularity $\mathcal{A}^{\frac{1}{2}}w \in C([0,T]; L_2(\Omega))$ from Proposition 1.3.1 (ii), $\mathcal{A}_R^{\frac{1}{2}}\theta \in L_2(0,T; L_2(\Omega))$ from (5.1.4), along with that of $\theta|_\Gamma$ in (5.1.6), is combined with (5.1.3) and (1.3.15), (1.3.16), (1.3.17), (1.3.13) and readily yield (5.1.6) on w_{tt}. ∎

Next, we introduce the (auxiliary) operator $A_{\gamma,0}$ (compare with $A_{\gamma,s}$ in (1.3.26) and see (5.1.2)):

$$A_{\gamma,0} = \begin{bmatrix} 0 & I & 0 \\ -C_\gamma^{-1}\mathcal{A} & -C_\gamma^{-1}\mathcal{A}_N & -C_\gamma^{-1}\mathcal{A}G_1(\,\cdot\,|_\Gamma) \\ 0 & \mathcal{A}_N N\dfrac{\partial}{\partial\nu} & -\mathcal{A}_R \end{bmatrix} : Y_\gamma \supset \mathcal{D}(A_{\gamma,0}) \to Y_\gamma; \tag{5.1.8a}$$

$$\mathcal{D}(A_{\gamma,0}) = \left\{ \begin{array}{l} \Big\{ x_1, x_2 \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = H^2(\Omega);\ x_3 \in \mathcal{D}(\mathcal{A}_R) : \\ \qquad \big[N \tfrac{\partial x_2}{\partial \nu} - \mathcal{A}_N^{-1}\mathcal{A}_R x_3\big] \in \mathcal{D}(\mathcal{A}_N)\Big\} \end{array} \right\} \tag{5.1.8b}$$

Proposition 5.1.2. *The operator* $A_{\gamma,0}$ *in* (5.1.8) *is dissipative*

$$\operatorname{Re}(A_{\gamma,0}x, x)_{Y_\gamma} = -(\mathcal{A}_N x_2, x_2)_{L_2(\Omega)} - (\mathcal{A}_R x_3, x_3)_{L_2(\Omega)},\ x = [x_1, x_2, x_3] \in \mathcal{D}(A_{\gamma,0}), \tag{5.1.9}$$

in fact, maximal dissipative, and thus generates a s.c. contraction semigroup $e^{A_{\gamma,0}t}$ *on* Y_γ.

Proof. As the first 2×2 block of $A_{\gamma,0}$ is plainly dissipative on $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times \mathcal{D}(C_\gamma^{\frac{1}{2}})$, we limit ourselves to observe that

$$\operatorname{Re}\left(\begin{bmatrix} 0 & -C_\gamma^{-1}\mathcal{A}G_1(\,\cdot\,|_\Gamma) \\ \mathcal{A}_N N \dfrac{\partial}{\partial \nu} & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \right)_{\mathcal{D}(C_\gamma^{\frac{1}{2}}) \times L_2(\Omega)}$$

$$\left. \begin{aligned} = \ & -(\mathcal{A}G_1(x_3|_\Gamma), x_2)_{L_2(\Omega)} + (\mathcal{A}_N N \frac{\partial x_2}{\partial \nu}, x_3)_{L_2(\Omega)} \\ & -(x_3|_\Gamma, G_1^*\mathcal{A}x_2)_{L_2(\Gamma)} + \left(\frac{\partial x_2}{\partial \nu}, N^*\mathcal{A}_N x_3 \right)_{L_2(\Gamma)} \end{aligned} \right\} \tag{5.1.10}$$

$$= -\left(x_3|_\Gamma, \frac{\partial x_2}{\partial \nu} \right)_{L_2(\Gamma)} + \left(\frac{\partial x_2}{\partial \nu}, x_3|_\Gamma \right)_{L_2(\Gamma)} = 0, \tag{5.1.11}$$

recalling the trace results in (1.3.18). Maximal dissipativity is proved directly. Then, one invokes the Lumer-Phillips theorem. ∎

The point of the next result is a *gain* of regularity "of the order of $\mathcal{A}_R^{\frac{1}{2}}$" with respect to the third coordinate ("hidden regularity"): from $L_2(\Omega)$ to $[\mathcal{D}(\mathcal{A}_R^{\frac{1}{2}})]' = [H^1(\Omega)]' = [\mathcal{D}(C_\gamma^{\frac{1}{2}})]'$, see (1.3.11), (1.3.13).

Proposition 5.1.3. *With reference to the s.c. semigroup* $e^{A_{\gamma,0}t}$ *guaranteed by Proposition* 5.1.2, *we have*

$$\left. \begin{aligned} & y(t) = \int_0^t e^{A_{\gamma,0}(t-\tau)} \begin{bmatrix} 0 \\ 0 \\ f_3(\tau) \end{bmatrix} d\tau : \\ & \text{continuous } \ L_2(0,T;[\mathcal{D}(\mathcal{A}_R^{\frac{1}{2}})]' = [H^1(\Omega)]') \to C([0,T];Y_\gamma) \\ & \text{and } \ y_3 \in L_2(0,T;H^1(\Omega)), \end{aligned} \right\} \tag{5.1.12}$$

Proof. The function $y(t) = [y_1(t), y_2(t), y_3(t)]$ in (5.1.12) solves $\dot{y}(t) = A_{\gamma,0}y(t) + f(t)$, $y(0) = 0$, where $f(t) = [0, 0, f_3(t)]$. Taking the Y_γ-inner product of this equation with $y(t)$, where $f(t)$ is taken, at first, in $L_2(0,t;\mathcal{D}(A_{\gamma,0}))$, so that $y(t) \in C([0,T];\mathcal{D}(A_{\gamma,0}))$, we obtain by virtue of the dissipative identity (5.1.9):

$$\frac{1}{2}\|y(t)\|^2_{Y_\gamma} = (A_{\gamma,0}y(t), y(t))_{Y_\gamma} + (f(t), y(t))_{Y_\gamma} \tag{5.1.13}$$

$$= -\|\mathcal{A}_N^{\frac{1}{2}}y_2(t)\|^2_{L_2(\Omega)} - \|\mathcal{A}_R^{\frac{1}{2}}y_3(t)\|^2_{L_2(\Omega)} + (\mathcal{A}_R^{-\frac{1}{2}}f_3(t), \mathcal{A}_R^{\frac{1}{2}}y_3(t))_{L_2(\Omega)}, \tag{5.1.14}$$

a.e. in t. Integrating (5.1.14) yields since $y(0) = 0$,

$$\left.\begin{aligned} \|y(t)\|^2_{Y_\gamma} \ + \ 2\int_0^t \|\mathcal{A}_N^{\frac{1}{2}}y_2(\tau)\|^2_{L_2(\Omega)}d\tau + 2\int_0^t \|\mathcal{A}_R^{\frac{1}{2}}y_3(\tau)\|^2_{L_2(\Omega)}d\tau \\ = \ \epsilon\int_0^t \|\mathcal{A}_R^{\frac{1}{2}}y_3(\tau)\|^2_{L_2(\Omega)}d\tau + \frac{1}{\epsilon}\int_0^t \|\mathcal{A}_R^{-\frac{1}{2}}f_3(\tau)\|^2_{L_2(\Omega)}d\tau, \end{aligned}\right\} \tag{5.1.15}$$

from which we obtain

$$\left.\begin{aligned} \|y\|^2_{L_\infty(0,T;Y_\gamma)} + 2\|\mathcal{A}_N^{\frac{1}{2}}y_2\|^2_{L_2(0,T;L_2(\Omega))} \ + \ 2\|\mathcal{A}_R^{\frac{1}{2}}y_3\|^2_{L_2(0,T;L_2(\Omega))} \\ \leq \ \frac{1}{\epsilon}\,\|f_3\|^2_{L_2(0,T;[\mathcal{D}(\mathcal{A}_R^{\frac{1}{2}})]')}, \end{aligned}\right\} \tag{5.1.16}$$

first for f_3 smooth as above. Next f_3 is extended to all of $L_2(0,T;[\mathcal{D}(\mathcal{A}_R^{\frac{1}{2}})]')$. Finally, a standard approximation argument boosts $L_\infty(0,T;\,\cdot)$ to $C([0,T];\,\cdot)$ for y and (5.1.16) yields (5.1.11). ∎

5.2. The operator A_γ is a semigroup generator. Proof of Proposition 1.3.2

Step 1. In this subsection we prove Proposition 1.3.2: i.e., that the operator $A_{\gamma,s}$ defined by (1.3.26) is the generator of a s.c. semigroup $e^{A_{\gamma,s}t}$ on Y_γ. To this end, we shall employ the auxiliary semigroup generator $A_{\gamma,0}$ in (5.1.8) as guaranteed by Proposition 5.1.2, and write (recall (1.3.27))

$$A_{\gamma,s} = A_{\gamma,0} + \begin{bmatrix} 0 & I & 0 \\ 0 & C_\gamma^{-1}\mathcal{A}_N N\dfrac{\partial}{\partial\nu} & 0 \\ 0 & -\mathcal{A}_N & 0 \end{bmatrix}, \quad \dot{x} = A_{\gamma,s}x = A_{\gamma,0}x + \begin{bmatrix} 0 \\ C_\gamma^{-1}\mathcal{A}_N N\dfrac{\partial v_t}{\partial\nu} \\ -\mathcal{A}_N v_t \end{bmatrix} \tag{5.2.1}$$

for $x(t) = [v(t), v_t, \phi(t)] >$ Equivalently, we shall establish that the corresponding integral equation

$$x(t) = e^{A_{\gamma,0}t}x_0 + \int_0^t e^{A_{\gamma,0}(t-\tau)}\begin{bmatrix} 0 \\ C_\gamma^{-1}\mathcal{A}_N N\dfrac{\partial v_t}{\partial\nu}(\tau) \\ -\mathcal{A}_N v_t(\tau) \end{bmatrix} d\tau \tag{5.2.2}$$

has a unique semigroup solution $x \in C([0,T];Y_\gamma)$, so that $x(t) = e^{A_{\gamma,s}t}x_0$.

Step 2. We integrate by parts the second entry in (5.2.2) and obtain

$$\int_0^t e^{A_{\gamma,0}(t-\tau)} \begin{bmatrix} 0 \\ C_\gamma^{-1}\mathcal{A}_N \dfrac{\partial v_t}{\partial \nu}(\tau) \\ -\mathcal{A}_N v_t(\tau) \end{bmatrix} d\tau = (\mathcal{B}x)(t) + \mathcal{Q}x(t) - e^{A_{\gamma,0}t} \begin{bmatrix} 0 \\ C_\gamma^{-1}\mathcal{A}_N N \dfrac{\partial v(0)}{\partial \nu} \\ 0 \end{bmatrix}, \quad (5.2.3)$$

where we have defined the operator $\mathcal{B}$ by

$$(\mathcal{B}f)(t) = \int_0^t e^{A_{\gamma,0}(t-\tau)} \left\{ A_{\gamma,0} \begin{bmatrix} 0 \\ C_\gamma^{-1}\mathcal{A}_N N \dfrac{\partial f_1}{\partial \nu}(\tau) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\mathcal{A}_N f_2(\tau) \end{bmatrix} \right\} d\tau \quad (5.2.4)$$

$$= \int_0^t e^{A_{\gamma,0}(t-\tau)} \begin{bmatrix} C_\gamma^{-1}\mathcal{A}_N N \dfrac{\partial f_1}{\partial \nu}(\tau) \\ -C_\gamma^{-1}\mathcal{A}_N C_\gamma^{-1}\mathcal{A}_N N \dfrac{\partial f_1(\tau)}{\partial \nu} \\ \mathcal{A}_N N \dfrac{\partial}{\partial \nu}\left(C_\gamma^{-1}\mathcal{A}_N N \dfrac{\partial f_1(\tau)}{\partial \nu}\right) - \mathcal{A}_N f_2(\tau) \end{bmatrix} d\tau, \quad (5.2.5)$$

for $f = [f_1, f_2, f_3]$, while for $h = [h_1, h_2, h_3] \in Y_\gamma$, $x(0) = x_0 \in Y_\gamma$ we have set

$$\mathcal{Q}h = \begin{bmatrix} 0 \\ C_\gamma^{-1}\mathcal{A}_N N \dfrac{\partial}{\partial \nu} h_1 \\ 0 \end{bmatrix}; \ \hat{x}_0 = [I - \mathcal{Q}]x_0 = x_0 - \begin{bmatrix} 0 \\ C_\gamma^{-1}\mathcal{A}_N N \dfrac{\partial v(0)}{\partial \nu} \\ 0 \end{bmatrix}. \quad (5.2.6)$$

With $\hat{x}_0 \in Y_\gamma$ as in (5.2.6), using (5.2.3) into the right-hand side of (5.2.2), we see then that we seek a semigroup solution $x(t) \in C([0,T]; Y_\gamma)$ of

$$x(t) = e^{A_{\gamma,0}t}\hat{x}_0 + (\mathcal{B}x)(t) + \mathcal{Q}x(t). \quad (5.2.7)$$

Step 3. Proposition 5.2.1. *With reference to the operators $\mathcal{B}$ and $\mathcal{Q}$ in* (5.2.4) *and* (5.2.6)*, we have*

(i)
$$\mathcal{B}: \text{ continuous } C([0,T]; Y_\gamma) \to C([0,T]; Y_\gamma). \quad (5.2.8)$$

More precisely, with $f = [f_1, f_2, f_3] \in C([0,T]; Y_\gamma)$, we have

$$\|\mathcal{B}f\|^2_{C([0,T];Y_\gamma)} \le c_T \left\{ \|f_1\|^2_{C([0,T];H^2(\Omega))} + \|f_2\|^2_{C([0,T];H^1(\Omega))} \right\}, \quad (5.2.9)$$

where $c_T \searrow 0$ as $T \searrow 0$.

(ii)
$$\mathcal{Q}: \text{ compact } Y_\gamma \to Y_\gamma; \ [I - \mathcal{Q}] \text{ is boundedly invertible on } Y_\gamma; \quad (5.2.10)$$

(iii) *the operator*

$$\{I - [I - \mathcal{Q}]^{-1}\mathcal{B}\}: \text{ continuous } C([0,T];Y_\gamma) \to C([0,T];Y_\gamma) \tag{5.2.11}$$

is boundedly invertible.

Proof. (i) We return to the integral (5.2.5) defining $\mathcal{B}f$ with $f = [f_1, f_2, f_3] \in C([0,T];Y_\gamma)$, i.e., $f_1 \in C[0,T];H^2(\Omega))$, $f_2 \in C([0,T];H^1(\Omega))$. Then, regarding the three entries, we have via (1.3.19), $\mathcal{A}^{\frac{1}{2}}C_\gamma^{-1} \in \mathcal{L}(L_2(\Omega))$ (by (5.1.3), and the regularity of N in (1.3.14):

$$C_\gamma^{-1}\mathcal{A}_N N\frac{\partial f_1}{\partial \nu} = \frac{1}{\gamma}N\frac{\partial f_1}{\partial \nu} - \frac{1}{\gamma}C_\gamma^{-1}N\frac{\partial f_1}{\partial \nu} \in C([0,T];H^2(\Omega) = \mathcal{D}(\mathcal{A}^{\frac{1}{2}})); \tag{5.2.12}$$

$$\left.\begin{aligned} &N\frac{\partial}{\partial \nu}\left(C_\gamma^{-1}\mathcal{A}_N N\frac{\partial f_1}{\partial \nu}\right) \in C([0,T];H^2(\Omega));\\ &C_\gamma^{-1}\mathcal{A}_N C_\gamma^{-1}\mathcal{A}_N N\frac{\partial f_1}{\partial \nu} \in C([0,T];H^2(\Omega)); \end{aligned}\right\} \tag{5.2.13}$$

$$\left\{\begin{aligned} &\left.\begin{aligned} \mathcal{A}_N^{\frac{1}{4}+\epsilon}(\mathcal{A}_N^{\frac{3}{4}-\epsilon}N)\frac{\partial}{\partial \nu}\left(C_\gamma^{-1}\mathcal{A}_N N\frac{\partial f_1}{\partial \nu}\right) &\in C([0,T];[\mathcal{D}(\mathcal{A}_N^{\frac{1}{4}+\epsilon})]')\\ &\subset C([0,T];[\mathcal{D}(\mathcal{A}_N^{\frac{1}{2}})]'); \end{aligned}\right\} \quad (5.2.14)\\ &\mathcal{A}_N f_2 = \mathcal{A}_N^{\frac{1}{2}}\mathcal{A}_N^{\frac{1}{2}}f_2 \in C([0,T];[\mathcal{D}(\mathcal{A}_N^{\frac{1}{2}})]' = [H^1(\Omega)]'). \quad (5.2.15) \end{aligned}\right.$$

Thus, recalling the hidden regularity of Proposition 5.1.3 on the third entry in (5.2.5) via (5.2.14), (5.2.15), as well as (5.2.12), (5.2.13) for the first two entries, we readily obtain via the generation result of Proposition 5.1.2,

$$\|\mathcal{B}f\|^2_{C([0,T];Y_\gamma)} \le k_T\left\{\|f_1\|^2_{L_2(0,T;H^2(\Omega))} + \|f_2\|^2_{L_2(0,T;H^1(\Omega))}\right\} \tag{5.2.16}$$

$$\le Tk_T\left\{\|f\|^2_{C([0,T];H^2(\Omega))} + \|f_2\|^2_{C([0,T];H^1(\Omega))}\right\}, \tag{5.2.17}$$

and $c_T = Tk_T \searrow 0$ as $T \searrow 0$ since k_T is decreasing in T.

(ii) The second entry of $\mathcal{Q}h$ is compact $H^2(\Omega) \to H^1(\Omega) = \mathcal{D}(C_\gamma^{\frac{1}{2}})$, via (5.2.12). Moreover, the operator $[I - \mathcal{Q}]$ is plainly injective on Y_γ by its definition in (5.2.6). Thus, $[I - \mathcal{Q}]$ is boundedly invertible on Y_γ.

(iii) For $g \in C([0,T];Y_\gamma)$, we seek to solve uniquely

$$\{I - [I - \mathcal{Q}]^{-1}\mathcal{B}\}f = g, \text{ i.e., } f = g + [I - \mathcal{Q}]^{-1}\mathcal{B}f \tag{5.2.18}$$

for $f \in C([0,T];Y_\gamma)$. Since $c_T \searrow 0$ in (5.2.9), this can be done at least initially for T small, by the contraction mapping principle. A finite number of iterations allows extension to any T finite. ∎

Step 4. Corollary 5.2.2. *The integral equation* (5.2.2), *equivalently* (5.2.7), *has a unique semigroup solution* $x(t) \in C([0,T];Y_\gamma)$. *This is, in fact,* $x(t) = e^{A_\gamma t}x_0$, *and* A_γ *generates a s.c. semigroup* $e^{A_\gamma t}$ *on* Y_γ. *Moreover, the following representation formula holds true:*

$$x(\,\cdot\,) = e^{A_{\gamma,s}\,\cdot}x_0 = [I - [I-\mathcal{Q}]^{-1}\mathcal{B}]^{-1}\left[(I-\mathcal{Q})^{-1}e^{A_{\gamma,0}\,\cdot}\hat{x}_0\right], \tag{5.2.19}$$

with $\mathcal{Q}$, $\mathcal{B}$, $\hat{x}_0$ *as in* (5.2.6), (5.2.4).

Proof. We return to (5.2.7) where $e^{A_{\gamma,0}t}\hat{x}_0 \in C([0,T];Y_\gamma)$ by Proposition 5.1.2. Then (5.2.7) is rewritten as

$$[I-\mathcal{Q}]x(t) = e^{A_{\gamma,0}t}\hat{x}_0 + (\mathcal{B}x)(t), \;\; \{I-[I-\mathcal{Q}]^{-1}\mathcal{B}\}x(\,\cdot\,) = (I-\mathcal{Q})^{-1}e^{A_{\gamma,0}\,\cdot}\hat{x}_0, \tag{5.2.20}$$

and then the unique solution $x(\,\cdot\,)$ as in (5.2.19) is obtained by Proposition 5.2.1 (iii). The semigroup property of $x(t)$ is a standard consequence of being a solution in $C([0,T];Y_\gamma)$ of (5.2.2). ■

5.3. Analysis of the $\{v,\phi\}$-problem (1.3.29), (1.3.30): Proof of Theorem 1.3.3(ii)

In this subsection we return to the operator $A_{\gamma,s}$ in (1.3.25), which was established in Subsection 5.3, to be a generator of a s.c. semigroup. Such semigroup describes the $\{v,\phi\}$-dynamics in (1.3.29) – (1.3.30), or (1.3.34) – (1.3.39) (left). Our goal here is to show the following result contained in Theorem 1.3.3 (ii).

Theorem 5.3.1. *The statement "the s.c. semigroup* $e^{A_{\gamma,s}t}$ *is compact for all* $t>0$*" is false: this is implied* [33, *p.* 50] *by the fact that the resolvent* $R(\lambda, A_{\gamma,s})$ *possesses the following property:*

$$\left.\begin{array}{r}\|R(\lambda = a+i\tau, A_{\gamma,s})\|_{\mathcal{L}(Y_\gamma)} \text{ does not tend to zero as } \tau\to\infty, \text{ with}\\ \text{Re } \lambda = a \;\text{ fixed and sufficiently large.}\end{array}\right\} \quad ■ \tag{5.3.1}$$

Convention. In this subsection, when we write "$\tau\to\infty$" we mean, for short, the full statement in (5.3.1), unless otherwise noted.

We begin by showing the same property for the (simplified) generator $A_{\gamma,0}$ in (5.1.8) of Proposition 5.1.2.

Theorem 5.3.2. *The statement "the s.c. semigroup* $e^{A_{\gamma,0}t}$ *is compact for all* $t>0$*" is false: this is implied* [P.1, *p.* 50] *by the fact that the resolvent* $R(\lambda, A_{\gamma,0})$ *possesses the following property:*

$$\left.\begin{array}{r}\|R(\lambda = a+i\tau, A_{\gamma,0})\|_{\mathcal{L}(Y_\gamma)} \text{ does not tend to zero as } \tau\to\infty, \text{ with}\\ \text{Re } \lambda = a \;\text{ fixed and sufficiently large.}\end{array}\right\} \tag{5.3.2}$$

Proof. Step 1. For a s.c. semigroup the property "compactness for all $t > 0$" is invariant under a bounded perturbation of the generator [33, p. 79]. Hence, we may further simplify the analysis by removing from the generator $A_{\gamma,0}$ the second row-second column entry $[-C_\gamma^{-1}\mathcal{A}_N]$, which is a bounded operator on $\mathcal{D}(C_\gamma^{\frac{1}{2}})$. We then consider the following new generators on Y_γ:

$$\bar{A}_{\gamma,0} = \begin{bmatrix} 0 & I & 0 \\ -C_\gamma^{-1}\mathcal{A} & 0 & -C_\gamma^{-1}\mathcal{A}G_1(\,\cdot\,|_\Gamma) \\ 0 & \mathcal{A}_N N\dfrac{\partial}{\partial\nu} & -\mathcal{A}_R \end{bmatrix};\ A_d = \begin{bmatrix} 0 & I & 0 \\ -C_\gamma^{-1}\mathcal{A} & 0 & 0 \\ 0 & 0 & -\mathcal{A}_R \end{bmatrix}, \tag{5.3.3}$$

with the obvious domains, in particular $\mathcal{D}(\bar{A}_{\gamma,0}) = \mathcal{D}(A_{\gamma,0})$ in (5.1.8b). Both operators in (5.3.3) are generators of s.c. semigroups on Y_γ; indeed, $e^{A_d t}$ is a contraction which decomposes as a direct sum of the unitary group $e^{\mathbb{A}_{\gamma,0}t}$ on $\mathcal{D}(\mathcal{A}^{\frac{1}{2}})\times\mathcal{D}(C_\gamma^{\frac{1}{2}})$ with generator $\mathbb{A}_{\gamma,0} = \begin{bmatrix} 0 & I \\ -C_\gamma^{-1}\mathcal{A} & 0 \end{bmatrix}$, and the analytic semigroup $e^{-\mathcal{A}_R t}$ on $L_2(\Omega)$. Instead of (5.3.2), we thus prove equivalently that

$$\left.\begin{array}{r} \|R(\lambda = a + i\tau, \bar{A}_{\gamma,0})\|_{\mathcal{L}(Y_\gamma)} \text{ does not tend to zero as } \tau\to\infty, \text{ with} \\ \text{Re } \lambda = a \text{ fixed and sufficiently large.} \end{array}\right\} \tag{5.3.4}$$

To this end, we assume by contradiction that with Re $\lambda = a$ fixed and sufficiently large,

$$\|R(\lambda = a + i\tau, \bar{A}_{\gamma,0})\|_{\mathcal{L}(Y_\gamma)} \to 0 \text{ as } \tau\to\infty \tag{5.3.5}$$

Let $f = [f_1, f_2, f_3] \in Y_\gamma$, $\|f\|_{Y_\gamma} = 1$ and set with $v(\lambda) = [v_1(\lambda), v_2(\lambda), v_3(\lambda)]$,

$$v(\lambda) = R(\lambda, \bar{A}_{\gamma,0})f; \text{ or } (\lambda I - \bar{A}_{\gamma,0})v(\lambda) = f;\ \lambda v_1(\lambda) - v_2(\lambda) = f_1 \tag{5.3.6}$$

(we have chosen to write explicitly only the first row, to be invoked below), where $f_1 \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$, $f_2 \in \mathcal{D}(C_\gamma^{\frac{1}{2}})$ and $f_3 \in L_2(\Omega)$. We have by (5.3.6) and (5.3.5),

$$\sup_{\|f\|=1}\|v(\lambda)\|_{Y_\gamma} = \sup_{\|f\|=1}\|R(\lambda, \bar{A}_{\gamma,0})f\|_{Y_\gamma} = \|R(\lambda, \bar{A}_{\gamma,0})\|_{\mathcal{L}(Y_\gamma)} \to 0 \text{ as } \tau\to\infty, \tag{5.3.7}$$

so that, explicitly, via the definition of Y_γ in (1.3.21) we have

$$\left.\begin{array}{l} \sup\limits_{\|f\|=1}\|v_1(\lambda)\|_{H^2(\Omega)} \to 0;\ \sup\limits_{\|f\|=1}\|v_2(\lambda)\|_{H^1(\Omega)} \to 0; \\ \sup\limits_{\|f\|=1}\|v_3(\lambda)\|_{L_2(\Omega)} \to 0, \text{ as } \tau\to\infty. \end{array}\right\} \tag{5.3.8}$$

Recalling the generators $\bar{A}_{\gamma,0}$ and A_d in (5.3.3), we rewrite (5.3.6) as

$$f = (\lambda I - \bar{A}_{\gamma,0})v(\lambda) = (\lambda - A_d)v(\lambda) + \begin{bmatrix} 0 \\ C_\gamma^{-1}\mathcal{A}G_1(v_3(\lambda)|_\Gamma) \\ \mathcal{A}_N N\dfrac{\partial v_2(\lambda)}{\partial\nu} \end{bmatrix}, \tag{5.3.9}$$

from which we obtain by applying $R(\lambda, A_d)$:

$$R(\lambda, A_d)f = v(\lambda) + R(\lambda, A_d)\begin{bmatrix} 0 \\ C_\gamma^{-1}\mathcal{A}G_1(v_3(\lambda)|_\Gamma) \\ \mathcal{A}_N N\dfrac{\partial v_2(\lambda)}{\partial \nu} \end{bmatrix}. \tag{5.3.10}$$

As $v(\lambda) \to 0$ in Y_γ as $\tau \to \infty$ by (5.3.7), (5.3.8), uniformly in f in the unit sphere of Y_γ, our next goal is to show that

$$R(\lambda, A_d)\begin{bmatrix} 0 \\ C_\gamma^{-1}\mathcal{A}G_1(v_3(\lambda)|_\Gamma) \\ \mathcal{A}_N N\dfrac{\partial v_2(\lambda)}{\partial \nu} \end{bmatrix} = \begin{bmatrix} R(\lambda, \mathbb{A}_{\gamma,0})\begin{bmatrix} 0 \\ C_\gamma^{-1}\mathcal{A}G_1(v_3(\lambda)|_\Gamma)\end{bmatrix} \\ R(\lambda, -\mathcal{A}_R)\mathcal{A}_N N\dfrac{\partial v_2(\lambda)}{\partial \nu} \end{bmatrix} \to 0 \text{ as } \tau \to \infty, \tag{5.3.11}$$

in Y_γ, uniformly in f running over the unit sphere of Y_γ, where $\mathbb{A}_{\gamma,0} = \begin{bmatrix} 0 & I \\ -C_\gamma^{-1}\mathcal{A} & 0 \end{bmatrix}$ as below (5.3.3) and will see that this will lead to a contradiction.

Step 2. We need to boost (5.3.8) for v_3, as in the t-domain (see Lemma 5.1.1 (i)).

Lemma 5.3.3. *We have*

$$\sup_{\|f\|_{Y_\gamma}=1} \|v_3(\lambda)\|_{H^1(\Omega)} \to 0 \text{ as } \tau \to \infty \text{ as } \operatorname{Re}\lambda = a \text{ fixed and large enough.} \tag{5.3.12}$$

Proof. We first recall from Proposition 5.1.2 for $A_{\gamma,0}$ adapted to the simpler operator $\bar{A}_{\gamma,0}$, that $\bar{A}_{\gamma,0}$ in (5.3.3) becomes skew-adjoint if we remove the bottom right corner element $-\mathcal{A}_R$. Thus, from (5.3.6) we obtain

$$\left.\begin{aligned} &\operatorname{Re}((\lambda I - \bar{A}_{\gamma,0})v(\lambda), v(\lambda))_{Y_\gamma} \\ &= (\operatorname{Re}\lambda)\|v(\lambda)\|^2_{Y_\gamma} + (\mathcal{A}_R v_3(\lambda), v_3(\lambda))_{L_2(\Omega)} = (f, v(\lambda))_{Y_\gamma}, \end{aligned}\right\} \tag{5.3.13}$$

from which

$$\|v_3(\lambda)\|_{H^1(\Omega)} \doteq \|\mathcal{A}_R^{\frac{1}{2}} v_3(\lambda)\|_{L_2(\Omega)} \leq |\operatorname{Re}\lambda| \|v(\lambda)\|^2_{Y_\gamma} + \|f\|_{Y_\gamma}\|v_3(\lambda)\|_{Y_\gamma}. \tag{5.3.14}$$

Then (5.3.12) follows from (5.3.14), via (5.3.7), or (5.3.8). ∎

Step 3. We begin with the first term in (5.3.11). To this end, we invoke from Remark 5.3.1 (at the end of this subsection) the following sharp regularity result of the Kirchoff problem corresponding to the original thermo-elastic problem (1.3.1) – (1.3.6): the map

$$\left.\begin{aligned} v_3 \in L_2(0,T; H^{\frac{1}{2}}(\Gamma)) \to \int_0^t e^{\mathbb{A}_{\gamma,0}(t-\tau)}\begin{bmatrix} 0 \\ C_\gamma^{-1}\mathcal{A}G_1(v_3(\tau)|_\Gamma)\end{bmatrix} d\tau \\ \in C([0,T]; \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times \mathcal{D}(C_\gamma^{\frac{1}{2}})) \end{aligned}\right\} \tag{5.3.15}$$

is continuous. For more details see Remark 5.3.1 below, Eqn. (5.3.44). It then follows readily, via Laplace transform, see Remark 5.3.2 below, that in the norm of $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times \mathcal{D}(C_\gamma^{\frac{1}{2}})$, we have:

$$\left.\begin{aligned} &\sup_{\|f\|=1} \left\| R(\lambda, \mathbb{A}_{\gamma,0}) \begin{bmatrix} 0 \\ C_\gamma^{-1} \mathcal{A} G_1(v_3(\tau)|_\Gamma) \end{bmatrix} \right\|_{Y_\gamma} \\ &\leq \frac{\text{const}}{\sqrt{\operatorname{Re}\lambda}} \sup_{\|f\|=1} \|v_3(\lambda)|_\Gamma\|_{H^{\frac{1}{2}}(\Gamma)} \end{aligned}\right\} \tag{5.3.16}$$

$$\leq \frac{\text{const}}{\sqrt{\operatorname{Re}\lambda}} \sup_{\|f\|=1} \|v_3(\lambda)\|_{H^1(\Omega)} \to 0 \text{ as } \tau \to \infty, \tag{5.3.17}$$

where in going from (5.3.13) to (5.3.14) we have used trace theory. In (5.3.17), we have recalled (5.3.12) for v_3, as Re $\lambda = a$ is fixed.

Step 4. We next show likewise that for the second term in (5.3.11) we have

$$\sup_{\|f\|=1} \left\| R(\lambda, -\mathcal{A}_R) \mathcal{A}_N N \frac{\partial v_2(\lambda)}{\partial \nu} \right\|_{L_2(\Omega)} \to 0 \text{ as } \tau \to \infty. \tag{5.3.18}$$

This will be done by duality. Let $x \in L_2(\Omega)$ and compute

$$\left| \left(R(\lambda, -\mathcal{A}_R) \mathcal{A}_N N \frac{\partial v_2(\lambda)}{\partial \nu}, x \right)_{L_2(\Omega)} \right| = \left| \left(\frac{\partial v_2(\lambda)}{\partial \nu}, N^* \mathcal{A}_N R(\bar{\lambda}, -\mathcal{A}_R) \right)_{L_2(\Gamma)} \right| \tag{5.3.19}$$

$$\text{(by (1.3.18))} \qquad = \left| \left(\frac{\partial v_2(\lambda)}{\partial \nu}, [R(\bar{\lambda}, -\mathcal{A}_R) x]_\Gamma \right)_{L_2(\Gamma)} \right|$$

$$\leq \left\| \frac{\partial v_2(\lambda)}{\partial \nu} \right\|_{L_2(\Gamma)} \left\| [R(\bar{\lambda}, -\mathcal{A}_R) x]_\Gamma \right\|_{L_2(\Gamma)}. \tag{5.3.20}$$

By trace estimates [5, Thm. 1.6.6, p. 37], since $v_2(\lambda) = \lambda v_1(\lambda) - f_1$ from (5.3.6):

$$\left\| \frac{\partial v_2(\lambda)}{\partial \nu} \right\|^2_{L_2(\Gamma)} \leq c \|v_2(\lambda)\|_{H^2(\Omega)} \|v_2(\lambda)\|_{H^1(\Omega)} = c \|\lambda v_1(\lambda) - f_1\|_{H^2(\Omega)} \|v_2(\lambda)\|_{H^1(\Omega)} \tag{5.3.21}$$

$$\leq c \left\{ |\lambda| \|v_1(\lambda)\|_{H^2(\Omega)} \|v_2(\lambda)\|_{H^1(\Omega)} + \|f_1\|_{H^2(\Omega)} \|v_2(\lambda)\|_{H^1(\Omega)} \right\}. \tag{5.3.22}$$

Similarly, by trace estimates [5, Thm. 1.6.6, p. 37]

$$\|[R(\bar{\lambda}, -\mathcal{A}_R) x]_\Gamma\|^2_{L_2(\Gamma)} \leq c \|R(\bar{\lambda}, -\mathcal{A}_R) x\|_{H^1(\Omega)} \|R(\bar{\lambda}, -\mathcal{A}_R) x\|_{L_2(\Omega)} \tag{5.3.23}$$

$$\text{(by (1.3.13))} \qquad \leq c \|\mathcal{A}_R^{\frac{1}{2}} R(\bar{\lambda}, -\mathcal{A}_R) x\|_{L_2(\Omega)} \|R(\bar{\lambda}, -\mathcal{A}_R) x\|_{L_2(\Omega)} \tag{5.3.24}$$

$$\leq c \frac{1}{|\lambda|^{\frac{1}{2}}} \|x\|_{L_2(\Omega)} \frac{1}{|\lambda|} \|x\|_{L_2(\Omega)} = c \frac{1}{|\lambda|^{\frac{3}{2}}} \|x\|^2_{L_2(\Omega)}, \tag{5.3.25}$$

recalling the standard estimates for the resolvent of the analytic semigroup generator $-\mathcal{A}_R$. Combining estimates (5.3.22) and (5.3.25) and using $\sqrt{a^2+b^2} \le a+b$ for a, b positive, we obtain

$$\left\| \frac{\partial v_2(\lambda)}{\partial \nu} \right\|_{L_2(\Gamma)} \| [R(\bar{\lambda}, -\mathcal{A}_R)x]_\Gamma \|_{L_2(\Gamma)}$$

$$\le c \left\{ |\lambda|^{\frac{1}{2}} \|v_1(\lambda)\|^{\frac{1}{2}}_{H^2(\Omega)} \|v_2(\lambda)\|^{\frac{1}{2}}_{H^1(\Omega)} + \|f_1\|^{\frac{1}{2}}_{H^2(\Omega)} \|v_2(\lambda)\|^{\frac{1}{2}}_{H^1(\Omega)} \right\} \frac{1}{|\lambda|^{\frac{3}{4}}} \|x\|_{L_2(\Omega)} \quad (5.3.26)$$

$$\left. \begin{aligned} \le c \left\{ \frac{1}{|\lambda|^{\frac{1}{4}}} \|v_1(\lambda)\|^{\frac{1}{2}}_{H^2(\Omega)} \|v_2(\lambda)\|^{\frac{1}{2}}_{H^1(\Omega)} + \frac{1}{|\lambda|^{\frac{3}{4}}} \|v_2(\lambda)\|^{\frac{1}{2}}_{H^1(\Omega)} \|f_1\|^{\frac{1}{2}}_{H^2(\Omega)} \right\} \times \\ \times \|x\|_{L_2(\Omega)} \to 0 \text{ as } \tau \to \infty, \end{aligned} \right\} \quad (5.3.27)$$

uniformly in f running over the unit sphere of Y_γ, recalling (5.3.8) for v_1 and v_2. Using (5.3.27) on the right-hand side of (5.3.20), we obtain

$$\sup_{\|f\|=1} \left| \left(R(\lambda, -\mathcal{A}_R) \mathcal{A}_N N \frac{\partial v_2(\lambda)}{\partial \nu}, x \right)_{L_2(\Omega)} \right| \to 0 \text{ as } \tau \to \infty, \ \forall\, x \in L_2(\Omega), \quad (5.3.28)$$

and (5.3.18) is proved as desired.

Step 5. Thus (5.3.17) and (5.3.18), combined show (5.3.11), uniformly on all f running over the unit sphere of Y_γ, which was our objective. But then (5.3.11) and (5.3.7), or (5.3.8), used in (5.3.10) yield, with Re $\lambda = a$ fixed and sufficiently large:

$$\sup_{\|f\|=1} \|R(\lambda, A_d) f\|_{Y_\gamma} = \|R(\lambda, A_D)\|_{\mathcal{L}(Y_\gamma)} \to 0 \text{ as } \tau \to \infty. \quad (5.3.29)$$

However, since A_d in (5.3.3) splits as $\mathbb{A}_{\gamma,0} \oplus (-\mathcal{A}_R)$, see below (5.3.11), and $\mathbb{A}_{\gamma,0}$ generates a (unitary) *group*, (5.3.29) is impossible, see Remark 5.3.3 below: a contradiction. Thus, (5.3.5) is false, and (5.3.4) and (5.3.2) hold true. Theorem 5.3.2 is proved.

Completion of the proof of Theorem 5.3.1. We shall use Theorem 5.3.2.

Step 1. We return to (5.2.6) and identity (5.2.7), where $x(t) = e^{A_\gamma t} x_0$ by Corollary 5.2.2, rewritten now as

$$\left. \begin{aligned} e^{A_{\gamma,0} t} \hat{x}_0 &= e^{A_\gamma t} x_0 - (\mathcal{B}x)(t) - \mathcal{Q}x(t), \\ & x(t; x_0) = x(t) = e^{A_\gamma t} x_0, \ \hat{x}_0 = (I - \mathcal{Q}) x_0; \end{aligned} \right\} \quad (5.3.30)$$

$$R(\lambda, A_{\gamma,0}) \hat{x}_0 = R(\lambda, A_\gamma) x_0 - (\mathcal{B}x)(\lambda) - \mathcal{Q}x(\lambda), \ \ x(\lambda) = R(\lambda, A_\gamma) x_0. \quad (5.3.31)$$

Assume by contradiction that for $x_0 \in Y_\gamma$, with $\|x_0\| = 1$ in the Y_γ-norm:

$$\|R(\lambda, A_{\gamma,s})\|_{\mathcal{L}(Y_\gamma)} = \sup_{\|x_0\|=1} \|R(\lambda, A_{\gamma,s}) x_0\|_{Y_\gamma} = \sup_{\|x_0\|=1} \|x(\lambda)\|_{Y_\gamma} \to 0 \text{ as } \tau \to \infty, \quad (5.3.32)$$

in particular,

$$\sup_{\|x_0\|=1} \|x_1(\lambda)\|_{H^2(\Omega)} \to 0, \quad \sup_{\|x_0\|=1} \|x_2(\lambda)\|_{H^1(\Omega)} \to 0, \text{ as } \tau \to \infty, \tag{5.3.33}$$

for Re $\lambda = a$ fixed and large enough where we have written $x(\lambda) = [x_1(\lambda), x_2(\lambda), x_3(\lambda)]$. The goal of this proof is to show that then

$$\sup_{\|x_0\|=1} \|(\mathcal{B}x)(\lambda)\|_{Y_\gamma} \to 0 \text{ as } \tau \to \infty, \tag{5.3.34}$$

after which we obtain from (5.3.32), (5.3.34), used in (5.3.31):

$$\sup_{\|x_0\|=1} \|R(\lambda, A_{\gamma,0})\hat{x}_0\|_{Y_\gamma} \to 0 \text{ as } \tau \to \infty. \tag{5.3.35}$$

Since $\hat{x}_0 = [I - \mathcal{Q}]x_0$ and $(I - \mathcal{Q})$ is an isomorphism on Y_γ (by Proposition 5.2.1 (ii), Eqn. (5.2.10)), (5.3.35) is equivalent to

$$\sup_{\|x_0\|=1} \|R(\lambda, A_{\gamma,0})x_0\|_{Y_\gamma} = \|R(\lambda, A_{\gamma,0})\|_{\mathcal{L}(Y_\gamma)} \to 0 \text{ as } \tau \to \infty, \tag{5.3.36}$$

which is the sought-after contradiction by Theorem 5.3.2.

Step 2. Proof of (5.3.34). We return to the definition of $\mathcal{B}$ given by (5.2.5), where by Proposition 5.2.1, $\mathcal{B}$: continuous $C([0,T]; Y_\gamma) \to$ itself. Thus, by Laplace transform, see Remark 5.3.2 below, we deduce via (5.2.5) that

$$\left.\begin{aligned} &\sup_{\|x_0\|=1} \|(\mathcal{B}x)(\lambda)\|_{Y_\gamma} \\ &\le \frac{\text{const}}{\sqrt{\text{Re}\lambda}} \left\{ \sup_{\|x_0\|=1} \|x_1(\lambda)\|_{H^2(\Omega)} + \sup_{\|x_0\|=1} \|x_2(\lambda)\|_{H^1(\Omega)} \right\} \to 0 \text{ as } \tau \to \infty, \end{aligned}\right\} \tag{5.3.37}$$

recalling (5.3.33). Thus, assumption (5.3.32) applied to (5.3.37) yields (5.3.34) as desired. The proof of Theorem 5.3.1 is complete. ■

In the next three remarks, we provide further explanation on three critical issues used in the above proof.

Remark 5.3.1. (sharp interior regularity of the Kirchoff problem) We consider the following Kirchoff elastic problem corresponding to (1.3.1) – (1.3.6):

$$\begin{cases} u_{tt} - \gamma\Delta u_{tt} + \Delta^2 u = 0 & \text{in } Q = (0,T] \times \Omega; & (5.3.38) \\ u(0,\cdot) = u_0, \ u_t(0,\cdot) = u_1 & \text{in } \Omega; & (5.3.39) \\ \Delta u + B_1 u = g & \text{in } \Sigma = (0,T] \times \Gamma; & (5.3.40) \\ \dfrac{\partial \Delta u}{\partial \nu} + B_2 u - \gamma \dfrac{\partial u_{tt}}{\partial \nu} = 0 & \text{in } \Sigma. & (5.3.41) \end{cases}$$

The following result is proved in [28]: the proof requires micro-local analysis and pseudo-differential operator techniques. ■

Theorem. *With reference to problem* (5.3.38) – (5.3.41), *the following regularity result holds true: the map*

$$\left.\begin{aligned} &\{u_0, u_1, g\} \in H^2(\Omega) \times H^1(\Omega) \times L_2(0,T; H^{\frac{1}{2}}(\Gamma)) \\ &\qquad \to \{u, u_t\} \in C([0,T]; H^2(\Omega) \times H^1(\Omega)) \text{ is continuous.} \end{aligned}\right\} \tag{5.3.42}$$

The abstract version of problem (5.3.38) – (5.3.41) *is*

$$u_{tt} + \gamma \mathcal{A}_N u_{tt} + \mathcal{A} u = \mathcal{A} G_1(g|_\Gamma); \; \frac{d}{dt}\begin{bmatrix} u \\ u_t \end{bmatrix} = \mathbb{A}_{\gamma,0} \begin{bmatrix} u \\ u_t \end{bmatrix} \tag{5.3.43}$$

(*compare with* (1.3.10)) *with* $\mathbb{A}_{\gamma,0} = \begin{bmatrix} 0 & I \\ -C_\gamma^{-1}\mathcal{A} & 0 \end{bmatrix}$. *Hence*

$$\begin{bmatrix} u(t) \\ u_t(t) \end{bmatrix} = e^{\mathbb{A}_{\gamma,0} t} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} + \int_0^t e^{\mathbb{A}_{\gamma,0}(t-\tau)} \begin{bmatrix} 0 \\ \mathcal{A} G_1(g(\tau)|_\Gamma) \end{bmatrix} d\tau. \tag{5.3.44}$$

In particular, for $u_0 = u_1 = 0$, *the regularity result* (5.3.42) *may be rewritten as*

$$\left.\begin{aligned} g \to \int_0^t e^{\mathbb{A}_{\gamma,0}(t-\tau)} \begin{bmatrix} 0 \\ \mathcal{A} G_1(g(\tau)|_\Gamma) \end{bmatrix} d\tau : \; &\text{continuous } L_2(0,T; H^{\frac{1}{2}}(\Gamma)) \\ &\to C([0,T]; H^2(\Omega) \times H^1(\Omega)). \end{aligned}\right\} \tag{5.3.45}$$

It is in the form (5.3.44) *that this result is used in* (5.3.15) *above.* ■

Remark 5.3.2. In this remark, we justify the step from (5.3.15) to (5.3.16); and likewise that the operator $\mathcal{B}$ in (5.2.5) satisfies (5.3.37), as a consequence of its continuity property (5.2.8). We are using the following known result, a ready consequence of [12]. Let, as in [12; 25, Chapter 7], U and X be two Hilbert spaces; A be the generator of a s.c. semigroup e^{At} on X, $\|e^{At}\| \leq M e^{\omega t}$, $t \geq 0$; $B : U \to [\mathcal{D}(A^*)]'$ be such that

$$u \to \int_0^t e^{A(t-\tau)} B u(\tau) d\tau : \; L_2(0,T; U) \to C([0,T]; X) \tag{5.3.46}$$

continuously for some, hence any, $T > 0$; equivalently [12; Appendix A],

$$\int_0^T \|B^* e^{A^* t} x\|_U^2 dt \leq c_T \|x\|_X^2, \quad x \in X. \tag{5.3.47}$$

Then, for all $\lambda \in C$ such that Re $\lambda > \omega$, we have [25, Chapter 7],

$$\|R(\lambda, A) B\|_{\mathcal{L}(U;X)} = \|B^* R(\bar{\lambda}, A^*)\|_{\mathcal{L}(X;U)} \leq \frac{\text{const}}{\sqrt{\operatorname{Re}\lambda}}. \tag{5.3.48}$$

For completeness, we give a proof. First, as in [12; 25, Chapter 7], assumptions (5.3.47) implies

$$\int_0^\infty e^{(\operatorname{Re}\lambda)t} \|B^* e^{A^* t} x\|_U^2 dt \leq c_\omega \|x\|^2, \text{for Re } \lambda > \omega. \tag{5.3.49}$$

This follows by splitting $\int_0^\infty = \sum_n \int_{nT}^{(n+1)T}$, details in [12]. Next, at first for $x \in \mathcal{D}(A^*)$, we compute via Schwarz inequality

$$\|B^* R(\bar\lambda, A^*)x\|_U = \left\| \int_0^\infty e^{-\bar\lambda t} B^* e^{A^* t} x\, dt \right\|_U \tag{5.3.50}$$

$$\leq \left\{ \int_0^\infty e^{-(\mathrm{Re}\,\lambda)t} dt \right\}^{\frac12} \left\{ \int_0^\infty e^{-(\mathrm{Re}\,\lambda)t} \|B^* e^{A^* t} x\|_U^2 dt \right\}^{\frac12} \tag{5.3.51}$$

$$\text{(by (5.3.49))} \quad \leq \frac{c}{\sqrt{\mathrm{Re}\,\lambda}} \|x\|, \quad \mathrm{Re}\,\lambda > \omega, \tag{5.3.52}$$

and (5.3.52) is then extended to all $x \in X$, thus proving (5.3.48). ■

Remark 5.3.3. Here we justify the last statement of Step 5, just below Eqn. (5.3.29). Let A be the generator of a s.c. group e^{At} on the Banach space X. Assume that A has compact resolvent, so that its (point) spectrum $\{\mu_n\}_{n=1}^\infty$ is contained in a vertical strip $-d \leq \mathrm{Re}\,\lambda \leq d$ of the complex plane, $0 \leq d < \infty$: $\mu_n = a_n + i\omega_n$, $|a_n| \leq d$. Then we have that with $\mathrm{Re}\,\lambda = a$ fixed and $|a| > d$

$$\|R(\lambda = a + i\tau, A)\|_{\mathcal{L}(X)} \text{ does not tend to zero as } \tau \to \infty. \tag{5.3.53}$$

Indeed, let $\{e_n\}_{n=1}^\infty$ be the corresponding normalized eigenvectors. Then for $\lambda = a + iv$, a, v real, $|a| > d$ we obtain

$$R(\lambda, A)e_n = \frac{e_n}{\lambda - \lambda_n}, \ \|R(\lambda = a + iv, A)e_n\|_X^2 = \frac{1}{(a - a_n)^2 + (v - \omega_n)^2}. \tag{5.3.54}$$

Selecting the points $\lambda_n = a + i\omega_n$, with $|a| - d \geq \alpha > 0$,

$$\left.\begin{aligned} \|R(\lambda_n = a + i\omega_n, A)\|_{\mathcal{L}(Y)} &\geq \|R(\lambda_n = a + i\omega_n, A)e_n\|_X \\ &= \frac{1}{|a - a_n|^2} \geq \frac{1}{\alpha^2} > 0, \ n = 1, 2, \dots \end{aligned}\right\} \tag{5.3.55}$$

and (5.3.55) implies (5.3.53). ■

5.4. Analysis of the $\{z, \psi\}$-problem: compactness of $\mathcal{K}_\gamma$ in (1.3.28). Proof of Theorem 1.3.3 (i)

The goal of the present subsection is to show the following result contained in the statement of Theorem 1.3.3 (i).

Theorem 5.4.1. *With reference to the operator $\mathcal{K}_\gamma$ in* (1.3.28), *we have for* $[w_0, w_1, \theta_0] = y_0 \in Y_\gamma$,

$$[z(t), z_t(t), \psi(t)] = \mathcal{K}_\gamma(t) y_0 = [bK_1(t) + K_2(t) + K_3(t)] y_0 : \text{ compact } Y_\gamma \to Y_\gamma \tag{5.4.1}$$

(*dependence of* K_i *on* γ *is suppressed*) *where, explicitly, for each* $t > 0$,

$$K_1(t)y_0 = \int_0^t e^{A_{\gamma,s}(t-\tau)} \begin{bmatrix} 0 \\ C_\gamma^{-1}\mathcal{A}G_2(\theta(\tau)|_\Gamma) \\ 0 \end{bmatrix} d\tau : \text{ compact } Y_\gamma \to Y_\gamma; \tag{5.4.2}$$

$$K_2(t)y_0 = \int_0^t e^{A_{\gamma,s}(t-\tau)} \begin{bmatrix} -C_\gamma^{-1}\theta(\tau) \\ -C_\gamma^{-1}\Delta C_\gamma^{-1}\theta(\tau) \\ -\Delta C_\gamma^{-1}\theta(\tau) \end{bmatrix} d\tau : \text{ compact } Y_\gamma \to Y_\gamma; \tag{5.4.3}$$

$$K_3(t)y_0 = \begin{bmatrix} 0 \\ -C_\gamma^{-1}\theta(t) \\ 0 \end{bmatrix} + e^{A_{\gamma,s}t} \begin{bmatrix} 0 \\ C_\gamma^{-1}\theta_0 \\ 0 \end{bmatrix} : \text{ compact } Y_\gamma \to Y_\gamma. \tag{5.4.4}$$

Proof. We return to (1.3.28) and obtain by integrating by parts in t

$$\left.\begin{aligned} &\int_0^t e^{A_{\gamma,s}(t-\tau)} \begin{bmatrix} 0 \\ C_\gamma^{-1}\theta_t(\tau) \\ 0 \end{bmatrix} d\tau \\ &= \left\{ e^{A_{\gamma,s}(t-\tau)} \begin{bmatrix} 0 \\ -C_\gamma^{-1}\theta(\tau) \\ 0 \end{bmatrix} \right\}_{\tau=0}^{\tau=t} - \int_0^t e^{A_{\gamma,s}(t-\tau)} A_\gamma \begin{bmatrix} 0 \\ C_\gamma^{-1}\theta(\tau) \\ 0 \end{bmatrix} d\tau. \end{aligned}\right\} \tag{5.4.5}$$

Invoking the form of A_γ in (1.3.26), where we may replace $-\mathcal{A}_N\left(\cdot - \frac{\partial}{\partial\nu}\right)$ by Δ (recall the statement below (1.3.18)), we see that (5.4.5) yields the decomposition (5.4.1), with K_i defined as in (5.4.2) – (5.4.4). Thus, it remains to show compactness of each term.

$K_3(t)$: $K_3(t)$ in (5.4.3) is plainly compact on Y_γ, see (1.3.21), since $\theta(t) \in L_2(\Omega)$ for each $t \geq 0$ for $y_0 \in Y_\gamma$ by the semigroup generation of the original thermoelastic problem in Proposition 1.3.1 (ii), and $C_\gamma^{-1} : L_2(\Omega) \to \mathcal{D}(C_\gamma) = H^2(\Omega) \hookrightarrow$ compact $H^1(\Omega) = \mathcal{D}(C_\gamma^{\frac{1}{2}})$.

$K_2(t)$: We analyze the three terms in $Y_\gamma = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times \mathcal{D}(C_\gamma^{\frac{1}{2}}) \times L_2(\Omega)$:

$$\left.\begin{aligned} &\mathcal{A}^{\frac{1}{2}}C_\gamma^{-1}\theta(\tau) = (\mathcal{A}^{\frac{1}{2}}C_\gamma^{-1})\mathcal{A}_R^{-\frac{1}{2}}(\mathcal{A}_R^{\frac{1}{2}}\theta(\tau)); \\ &C_\gamma^{-\frac{1}{2}}\Delta C_\gamma^{-1}\theta(\tau) = (C_\gamma^{-\frac{1}{2}}\Delta C_\gamma^{-1})\mathcal{A}_R^{-\frac{1}{2}}(\mathcal{A}_R^{\frac{1}{2}}\theta(\tau)); \\ &\Delta C_\gamma^{-1}\theta(\tau) = (\Delta C_\gamma^{-1})\mathcal{A}_R^{-\frac{1}{2}}(\mathcal{A}_R^{\frac{1}{2}}\theta(\tau)); \end{aligned}\right\} \tag{5.4.6}$$

$$\left.\begin{aligned} &\mathcal{A}^{\frac{1}{2}}C_\gamma^{-1} \in \mathcal{L}(L_2(\Omega)) \text{ by (5.1.3); } \mathcal{A}_R^{\frac{1}{2}}\theta \in L_2(0,T;L_2(\Omega)) \text{ by (5.1.4);} \\ &\qquad \mathcal{A}_R^{-\frac{1}{2}} \text{ compact on } L_2(\Omega). \end{aligned}\right\} \tag{5.4.7}$$

Thus, we conclude by (5.4.6), (5.4.7), that

$$\begin{bmatrix} C_\gamma^{-1} \\ C_\gamma^{-1}\Delta C_\gamma^{-1} \\ \Delta C_\gamma^{-1} \end{bmatrix} : \text{ compact } H^1(\Omega) = \mathcal{D}(\mathcal{A}_R^{\frac{1}{2}}) \to Y_\gamma. \tag{5.4.8}$$

Hence, (5.4.8) used in (5.4.3), with $y_0 \to \mathcal{A}_R^{\frac{1}{2}}\theta$ continuous $Y_\gamma \to L_2(0,T;L_2(\Omega))$, shows that $K_2(t)$ is compact on Y_γ by invoking the proof of [36] based on Mazur's theorem.

$K_1(t)$: By (5.1.6), we have $\theta|_\Gamma \in L_2(0,T;H^{\frac{1}{2}}(\Gamma))$, hence by (1.3.17), $\mathcal{A}^{\frac{7}{8}-\epsilon}G_2(\theta|_\Gamma) \in L_2(0,T;L_2(\Omega))$ continuously in $y_0 \in Y_\gamma$; moreover, $[C_\gamma^{-\frac{1}{2}}\mathcal{A}^{\frac{1}{4}}] \in \mathcal{L}(L_2(\Omega))$ (has a bounded extension in $L_2(\Omega)$) by (5.1.3). Thus, we see that

$$\left.\begin{aligned} C_\gamma^{-1}\mathcal{A}G_2(\,\cdot\,|_\Gamma) = C_\gamma^{-\frac{1}{2}}(C_\gamma^{-\frac{1}{2}}\mathcal{A}^{\frac{1}{4}})\mathcal{A}^{-\frac{1}{8}+\epsilon}(\mathcal{A}^{\frac{7}{8}-\epsilon}G_2)(\,\cdot\,|_\Gamma) : \\ \text{compact } H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma) \to \mathcal{D}(C_\gamma^{\frac{1}{2}}), \end{aligned}\right\} \tag{5.4.9}$$

as $\mathcal{A}^{-\frac{1}{8}+\epsilon}$ is compact on $L_2(\Omega)$. Using (5.4.9) in (5.4.2), we conclude again as in the proof of [36], based on Mazur's theorem, that $K_2(t)$ is compact on Y_γ. Theorem 5.4.1 is proved. ∎

6. Implications of the structural decomposition on exact controllability

The structural decomposition of Theorem 1.1.2 and Theorem 1.2.2 has additional implications also on the issue of exact controllability of a thermo-elastic dynamics, which is subject to a control action. This was already exploited in the specific case of clamped/Dirichlet B.C. with interior control in [9]. Our considerations here are abstract and of a general nature and encompass the case of [9]. For lack of space, we only limit ourselves to a brief, incomplete sketch here.

6.1. From approximate controllability to exact controllability

In this subsection, we let $\mathcal{U}$ (control space) and X (state space) be a Hilbert and a Banach space, respectively. We shall follow into the following abstract setting, already essentially contained in, say, [30, p. 119-120].

Proposition 6.1.1. *Let $J = S + Q$, where:*

(i) *J is a closed operator $\mathcal{U} \subset \mathcal{D}(J) \to X$ with dense range $\overline{\mathcal{R}(J)} = X$ (approximate controllability); equivalently, with trivial null space of the adjoint J^* : $\mathcal{N}(J^*) = \{0\}$;*

(ii) *S is a closed, surjective operator: $\mathcal{U} \subset \mathcal{D}(S)$ onto X, where $\mathcal{D}(S) = \mathcal{D}(J)$;*

(iii) *Q is a compact operator: $\mathcal{U} \to X$.*

Then, J is surjective $\mathcal{U} \subset \mathcal{D}(J)$ onto X (exact controllability).

Proof. We may restrict to the Hilbert space $\tilde{\mathcal{U}} = [\mathcal{N}(S)]^{\perp}$, the orthogonal complement in $\mathcal{U}$ of the null space $\mathcal{N}(S)$ of S, so that S is injective and surjective: $\tilde{\mathcal{U}} \cap \mathcal{D}(S)$ onto X, with bounded inverse S^{-1} from X onto $\tilde{\mathcal{U}} \cap \mathcal{D}(S)$ (by the open mapping theorem). Write

$$Ju = Su + Qu = [I + QS^{-1}]Su, \quad u \in \tilde{\mathcal{U}} \cap \mathcal{D}(S). \tag{6.1.1}$$

Since $S[\tilde{\mathcal{U}} \cap \mathcal{D}(S)] = X$ by hypothesis (ii), we see by (6.1.1) that in order to show surjectivity of $J : \tilde{\mathcal{U}} \cap \mathcal{D}(J)$ onto X, it suffices to establish surjectivity of the bounded operator $V \equiv [I + QS^{-1}] : X$ onto X, i.e., equivalently [34, p. 235], that $V^* = I + S^{*-1}Q^*$ is bounded below

$$\|V^*x\| = \|[I + S^{*-1}Q^*]x\| \geq c\|x\|, \quad c > 0, \ \forall\, x \in X. \tag{6.1.2}$$

We now prove (6.1.2). First we notice that V^* is injective $X \to X : V^*x = x + S^{*-1}Q^*x = 0$, $x \in X$, $\Rightarrow x = 0$, since equivalently $J^*x = S^*x + Q^*x = 0, x \in \mathcal{D}(J^*) \Rightarrow x = 0$, which holds true by hypothesis (i). This says that (-1) is not an eigenvalue of the compact operator $S^{*-1}Q^*$ (by (iii)); thus (-1) belongs to the resolvent set of $S^{*-1}Q^*$ and the inverse $[I + S^{*-1}Q^*]^{-1}$ exists as a bounded operator $X \to X$. This, then, establishes (6.1.2), as desired. ■

6.2. Applications to thermo-elastic problems

Distributed control (bounded control operator). We return to the thermo-elastic dynamics of Sections 1.1 and 1.2. Set $\mathcal{U} = L_2(0, T; U)$, U a Hilbert (control) space, and $X = Y_{1,\gamma}$ the mechanical state space defined in Eqn. (1.1.18) and Eqn. (1.2.10), respectively. Let $\mathbb{A}_\gamma$ be the thermo-elastic generator defined by (1.1.14) and (1.2.8), respectively; and let $\mathbb{A}_{1,\gamma}$ be the corresponding operator (1.1.19), respectively (1.2.16) of the associated damped Kirchoff equation (1.1.17), respectively (1.2.14). We presently take a bounded control operator $\mathcal{B} \in \mathcal{L}(U; Y_\gamma)$, Y_γ as in (1.1.12) or (1.2.9), respectively. According to the structural decomposition result of Theorem 1.1.2, Eqn. (1.1.21) and Theorem 1.2.2, Eqn. (1.2.23), we take

$$Ju \equiv \Pi_m \int_0^T e^{\mathbb{A}_\gamma (T-t)} \mathcal{B}u(t)dt; \quad Su \equiv \int_0^T e^{\mathbb{A}_{1,\gamma}(T-t)} \Pi_m \mathcal{B}u(t)dt; \tag{6.2.1}$$

$$Qu = \Pi_m \int_0^T \mathcal{K}_\gamma(T-t)\mathcal{B}u(t)dt, \tag{6.2.2}$$

where Π_m is the orthogonal projection $Y_\gamma \to Y_{1,\gamma} : [w_0, w_1, \theta_0] \to [w_0, w_1]$ onto the mechanical state space, and where $\mathcal{K}_\gamma(t)$ is the compact operator, for all $t > 0$, $Y_\gamma \to Y_\gamma$ in Eqn. (1.1.22), (2.29), and, respectively, in (1.2.24), (3.35). Thus, $\Pi_m\mathcal{K}_\gamma$ is the first (mechanical) component in (2.29) or (3.35), respectively. Since $\mathcal{B}$ is bounded, it follows readily that $Q : \mathcal{U} \to Y_{1,\gamma} = X$ is compact, and hypothesis (iii) in Proposition 6.1.1 is verified. Moreover, under several choices of the operator $\mathcal{B}$, e.g., $\mathcal{B} = [0, I, 0]$, $U = L_2(\Omega)$, or $U = L_2(\omega)$, $\omega =$ a boundary layer of Ω as in [9] (mechanical distributed control), we have both that: (a) S is surjective $\tilde{\mathcal{U}}$ onto $X = Y_{1,\gamma}$ (i.e., the damped Kirchoff problem is exactly controllable), and J has dense range in $X = Y_{1,\gamma}$ (i.e., the thermo-elastic problem is approximately controllable in the mechanical variables). Thus, assumptions (i) and (ii) of Proposition 6.1.1 hold true in these cases. Then, Proposition 6.1.1 yields, in fact, an exact controllability result in the mechanical variables. A rather routine soft argument, e.g., [2], based also on the original approximate controllability property of the thermo-elastic problem on all variables, then permits to conclude with exact controllability in the mechanical variables $\{w, w_t\}$ and simultaneous approximate controllability of the thermal variable θ. In the case of Section 4.2 (clamped/Dirichlet B.C.) this exact/approximate controllability conclusion was obtained in [9], in the interior control case, through more ad hoc arguments.

Unbounded (boundary) control operator: hinged mechanical/Dirichlet thermal B.C. (Section 4.1), and clamped mechanical/Dirichlet thermal B.C. (Section 4.2). We now consider the thermo-elastic dynamics of Section 4.1 and Section 4.2, subject to a control operator such as it arises in the modeling of many boundary/point control problems. As before, let $\Pi_m[v_1, v_2, v_3] = [v_1, v_2]$ be the projection $Y_\gamma \to Y_{1,\gamma}$ and let $\Pi_m^*[v_1, v_2] = [v_1, v_2, 0]$ be its adjoint $Y_{1,\gamma} \to Y_\gamma$. Let $\mathcal{B}_m$: continuous $U \to [\mathcal{D}(\mathbb{A}_{1,\gamma})]'$, duality with respect to $Y_{1,\gamma}$, satisfy the abstract trace condition [29],

$$\mathcal{B}_m^* e^{\mathbb{A}_{1,\gamma} t} : \text{ continuous } Y_{1,\gamma} \to L_2(0, T; U), \tag{6.2.3}$$

equivalently, [29],

$$u \to \int_0^t e^{\mathbb{A}_{1,\gamma}^* (t-\tau)} \mathcal{B}_m u(\tau) d\tau : \text{ continuous } L_2(0, T; U) \to Y_{1,\gamma}. \tag{6.2.4}$$

Applying the structural decomposition Theorem 1.2.2, Eqn. (1.2.23), we can write with $y_0 = [w_0, w_1, 0]$, in the above notation:

$$\mathcal{B}_m^* \Pi_m e^{\mathbb{A}_\gamma t} y_0 = \mathcal{B}_m^* e^{\mathbb{A}_{1,\gamma} t} \Pi_m y_0 + \mathcal{B}_m^* \Pi_m \mathcal{K}_\gamma(t) y_0, \; t > 0, \tag{6.2.5}$$

where by (3.35) on $\mathcal{K}_\gamma$ and (3.5) on L_t,

$$\Pi_m \mathcal{K}_\gamma(t) y_0 = L_t \theta; \; \mathcal{B}_m^* \Pi_m \mathcal{K}_\gamma(t) y_0 = \mathcal{B}_m^* L_t \theta = \mathcal{B}_m^* \int_0^t e^{\mathbb{A}_{1,\gamma}(t-\tau)} \begin{bmatrix} 0 \\ -C_\gamma^{-1} \theta_t(\tau) \end{bmatrix} d\tau. \tag{6.2.6}$$

By duality on (6.2.5), we introduce the operators J, S and Q of Proposition 6.1.1

$$Ju = \Pi_m \int_0^T e^{\mathbb{A}_\gamma^* (T-t)} \Pi_m^* \mathcal{B}_m u(t) dt, \quad Su = \int_0^T e^{\mathbb{A}_{1,\gamma}^* (T-t)} \mathcal{B}_m u(t) dt; \tag{6.2.7}$$

$$Qu = \Pi_m \int_0^T \mathcal{K}_\gamma^*(T-t) \Pi_m^* \mathcal{B}_m u(t) dt, \tag{6.2.8}$$

where we have chosen to consider the dynamics of the adjoint thermo-elastic semigroup. We next show that Q is compact: $L_2(0, T; U) \to Y_{1,\gamma}$. This will be more conveniently established by duality.

Proposition 6.2.1. *For the settings of Section* 4.1 *and Section* 4.2, *under assumption* (6.2.3), *we have that*

$$Q^* \equiv \mathcal{B}_m^* \Pi_m \mathcal{K}_\gamma(t) : \text{ compact } Y_{1,\gamma} \times 0 \to L_2(0,T;U). \tag{6.2.9}$$

Proof. Step 1. With $H = L_2(\Omega)$ and $y_0 = [w_0, w_1, 0] \in Y_\gamma$ we have under assumption (6.2.3),

$$\int_0^T \|\mathcal{B}_m^* \Pi_m \mathcal{K}_\gamma(t) y_0\|_U^2 dt = \int_0^T \|\mathcal{B}_m^* L_t \theta\|_U^2 dt \le C_T \int_0^T \|C_\gamma^{-\frac{1}{2}} \theta_t(\tau)\|_H^2 d\tau. \tag{6.2.10}$$

The proof uses (6.2.6), a change in the order of integration, and hypothesis (6.2.3). Details are omitted.

Step 2. The map

$$y_0 = [w_0, w_1, 0] \in Y_\gamma \to C_\gamma^{-\frac{1}{2}} \theta_t \in L_2(0,T;H) \text{ is compact.} \tag{6.2.11}$$

To show (6.2.11), we recall that in the abstract setting of Section 1.2 and Section 3 with $y_0 = [w_0, w_1, 0] \in Y_\gamma$, we have, see (3.41):

$$\theta_t \in L_2(0,T;H) \;\text{ and }\; \theta_{tt} \in L_2(0,T;[\mathcal{D}(B)]'); \tag{6.2.12}$$

$$C_\gamma^{-\frac{1}{2}} \theta_t \in L_2(0,T;\mathcal{D}(C_\gamma^{\frac{1}{2}})), \text{ injection } \mathcal{D}(C_\gamma^{\frac{1}{2}}) \to H \text{ is compact,} \tag{6.2.13}$$

while, in the present setting of Section 4.1 and Section 4.2, we have, moreover, that $B = C$ (see (4.1.2) and (4.2.2)). Thus, the right-hand side of (6.2.12) yields

$$C^{-1} \theta_{tt} \in L_2(0,T;H) \;\text{ and }\; C_\gamma^{-\frac{1}{2}} \theta_{tt} \in L_2(0,T;[\mathcal{D}(C^{\frac{1}{2}})]'). \tag{6.2.14}$$

Then, Aubin's Lemma [1, $p = 2$] yields (6.2.11) from (6.2.13), (6.2.14).

Step 3. Using (6.2.11) in (6.2.10) yields (6.2.9), as desired. ∎

Proposition 6.2.1 verifies assumption (iii) in Proposition 6.1.1. As to the other two assumptions, the following considerations are relevant. In the two cases in question — with mechanical boundary conditions which are either hinged or clamped, as in Section 4.1 and Section 4.2 — there are known cases of boundary exact controllability of conservative Kirchoff equations in natural state spaces [19, 26], whose proof readily extends ([37] and references therein) to cover damped Kirchoff equations. This then provides specific boundary operators $\mathcal{B}_m$ for which the operator S in (6.2.7) is surjective onto a natural state space. Thus, assumption (ii) of Proposition 6.1.1 holds true in these cases. Finally, for assumption (i) of Proposition 6.1.1 — approximate controllability of the corresponding thermo-elastic problems — the recent unique continuation results for thermo-elastic equations in [16] would, by duality, provide relevant information on this issue. In the hinged/Dirichlet case of Section 4.1, the precise spectral decomposition of the thermo-elastic operator [6] is also useful here for backward continuation.

Finally, when the boundary operator $\mathcal{B}_m$ does not satisfy condition (6.2.3) — e.g., the case of thermo-elastic equations with so-called free B.C. — the present argument does not seem applicable. However, in the specific case of thermo-elastic equations with free B.C., a direct approach succeeds [2] in achieving the relevant continuous observability inequality, which by duality is equivalent to exact controllability. Prior work in [18] had obtained, in this case, exact controllability of the mechanical variables in the case of 'small' coupling parameter. Null boundary exact controllability results with clamped/hinged mechanical B.C. are given in [14].

Appendix A: Second proof of (3.24)

As to the second term in (3.17), since any sine operator absorbs the positive square root of the negative of its generator, we have $(C_\gamma^{-1}A)^{\frac{1}{2}}S_{0,\gamma}(\,\cdot\,)$: continuous $Z_\gamma \to C([0,T];Z)$, and hence, recalling (3.16)

$$z(t) = \int_0^t (C_\gamma^{-1}A)^{\frac{1}{2}} S_{0,\gamma}(t-\tau) C_\gamma^{-1} B\phi_t(\tau)d\tau \in C([0,T]; Z_\gamma = \mathcal{D}(C_\gamma^{\frac{1}{2}})). \tag{A.1}$$

Thus, invoking (H.3) = (1.2.5), if we can show that

$$(C_\gamma^{-1}A)^{\frac{1}{2}} : \text{ continuous } Z_\gamma \equiv \mathcal{D}(C_\gamma^{\frac{1}{2}}) = \mathcal{D}(A^{\frac{1}{2}}) \to H, \tag{A.2}$$

then it follows from (A.1), (A.2) that

$$(C_\gamma^{-1}A)\int_0^t S_{0,\gamma}(t-\tau)C_\gamma^{-1}B\phi_t(\tau)d\tau \in C([0,T];H). \tag{A.3}$$

In this case, (3.17), and hence (3.15), are then proved, as desired. We finally show (A.2). First, we notice that as an operator on $Z_\gamma \equiv \mathcal{D}(C_\gamma^{\frac{1}{2}}) = \mathcal{D}(A^{\frac{1}{4}})$, see (A.2), the positive self-adjoint cosine operator $C_\gamma^{-1}A$ has domain $\mathcal{D}_{Z_\gamma}(C_\gamma^{-1}A)$ given by

$$\left.\begin{aligned} \mathcal{D}_{Z_\gamma}(C_\gamma^{-1}A) &= \{x \in Z_\gamma : C_\gamma^{-1}Ax \in \mathcal{D}(C_\gamma^{\frac{1}{2}})\} \\ &= \{x \in Z_\gamma : C_\gamma^{-\frac{1}{2}}A^{\frac{1}{4}}A^{\frac{3}{4}}x \in H\} = \mathcal{D}(A^{\frac{3}{4}}), \end{aligned}\right\} \tag{A.4}$$

with $\mathcal{D}(A^{\frac{3}{4}})$ with respect to H, as usual, since $C_\gamma^{-1}A^{\frac{1}{4}}$ is an isomorphism on H by (H.3) = (1.2.5), and $Z_\gamma = \mathcal{D}(A^{\frac{1}{4}}) \supset \mathcal{D}(A^{\frac{3}{4}})$. Thus, by (3.21), $\mathcal{D}_{Z_\gamma}(C_\gamma^{-1}A)$ is $A^{\frac{1}{2}}$-smoother then $Z = \mathcal{D}(A^{\frac{1}{4}}) : \mathcal{D}_{Z_\gamma}(C_\gamma^{-1}A) = A^{-\frac{1}{2}}Z_\gamma$. It follows that $\mathcal{D}_{Z_\gamma}((C_\gamma^{-1}A)^{\frac{1}{2}})$, with respect to Z_γ, is $\mathcal{D}_{Z_\gamma}((C_\gamma^{-1}A)^{\frac{1}{2}}) = A^{-\frac{1}{4}}Z_\gamma = \mathcal{D}(A^{\frac{1}{2}})$. Then, the dual space $\left[\mathcal{D}_{Z_\gamma}((C_\gamma^{-1}A)^{\frac{1}{2}})\right]'_{Z_\gamma}$, duality with respect to the pivot space $Z_\gamma = \mathcal{D}(A^{\frac{1}{4}})$, is $\left[\mathcal{D}_{Z_\gamma}((C_\gamma^{-1}A)^{\frac{1}{2}})\right]'_{Z_\gamma} = H$, as desired. Therefore, if $z \in Z_\gamma$, then $(C_\gamma^{-1}A)^{\frac{1}{2}}z \in H$, continuously, and (3.17) is proved.

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