Semigroup Forum (2024) 108:734–758
https://doi.org/10.1007/s00233-024-10445-7

RESEARCH ARTICLE

Stability for coupled thermoelastic systems with nonlinear localized damping and Wentzell boundary conditions

André Vicente[1]

Received: 16 April 2024 / Accepted: 2 May 2024 / Published online: 18 June 2024

Abstract
This paper is concerning with the study of stability involving a thermoelastic system with internal nonlinear localized damping. The main novelty of the paper is to introduce to the study of thermoelastic system the general Wentzell boundary conditions associated to the internal heat equation. This boundary condition takes into account that there is a boundary source of heat which depends on the heat flow along the boundary, the heat flux across the boundary, and the temperature at the boundary. The tools are the use of multipliers with the construction of appropriate perturbed energy functionals.

Keywords Stability · Thermoelastic system · Wentzell boundary conditions · Existence and uniqueness of solution

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be an open, bounded and connected set, $N \geq 2$, with smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ such that $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$. In this paper we study the following problem

$$u_{tt} - c\Delta u + \operatorname{div}(\theta) + \rho(x)g(u_t) = 0 \ \text{ in } \Omega \times (0, \infty), \tag{1}$$

$$\theta_t - \Delta\theta + \operatorname{div}(u_t) = 0 \ \text{ in } \Omega \times (0, \infty), \tag{2}$$

$$u = 0 \ \text{ on } \Gamma \times (0, \infty), \tag{3}$$

$$\theta = 0 \ \text{ on } \Gamma_0 \times (0, \infty), \tag{4}$$

$$\theta_t - \alpha\theta - \beta\Delta_\Gamma\theta + \beta\frac{\partial\theta}{\partial\nu} = 0 \ \text{ on } \Gamma_1 \times (0, \infty), \tag{5}$$

Communicated by Jerome A. Goldstein.

Research of André Vicente is partially supported by the Conselho Nacional de Desenvolvimento Científico e Tecnológico—CNPq, Grant 306771/2023-3.

✉ André Vicente
andre.vicente@unioeste.br

1 CCET, Western Paraná State University, Rua Universitária, 2069, Cascavel, Paraná 85819-110, Brazil

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \theta(x,0) = \theta_0(x)\ \ x \in \Omega, \tag{6}$$

where Δ and Δ_Γ are, respectively, the Laplace and Laplace-Beltrami operators in the spatial variable; ν is the outward unit normal vector at Γ; c is a positive real number; $\alpha, \beta : \Gamma_1 \to \mathbb{R}$ are positive and continuous functions; u_0, u_1, and θ_0 are the initial data; and ρ is a nonnegative function responsible for the localized damping effect.

The problem (1, 2) is a n-dimensional version of the classical one-dimensional thermoelastic system

$$u_{tt} - u_{xx} + \gamma\theta_x = 0\ \text{ in } (0, L) \times (0, \infty), \tag{7}$$

$$\theta_t - k\theta_{xx} + \gamma\theta_{xt} = 0\ \text{ in } (0, L) \times (0, \infty), \tag{8}$$

where u is the displacement, θ is the temperature deviation from the reference temperature, and γ and k are positive constants depending on the material. The system (7)–(8) was studied for instance by Dafermos [17], Liu and Zheng [33, 34] and Rivera [35]. In [33, 35] the authors proved that, even in the absence of damping term, the energy associated to the problem decays. The n-dimensional case was studied by Clark, San Gil Jutuca, and Milla Miranda [12] and Apolaya, Clark, and Feitosa [1]. In [12] the authors proved the exponential stability with the damping term acting on a boundary portion of the domain. In [1] they studied the system without damping, the authors also considered a time-dependent coefficient multiplying the Laplace operator.

More recently Braz e Silva, Clark, and Frota [6] proved the existence, uniqueness, and asymptotic behavior of global solutions for the following thermoelastic system with nonlocal nonlinearities under the acoustic boundary conditions

$$u_{tt} - c\Delta u + \lambda|u|^\rho u + (a \cdot \nabla)\theta = 0\ \text{ in } \Omega \times (0, \infty), \tag{9}$$

$$\theta_t - \beta\left(\int_\Omega \theta\, dx\right)\Delta\theta + (a \cdot \nabla)u_t = 0\ \text{ in } \Omega \times (0, \infty), \tag{10}$$

$$u = 0\ \text{ on } \Gamma_0 \times (0, \infty), \tag{11}$$

$$u_t + f_1\delta_{tt} + f_2\delta_t + f_3\delta = 0\ \text{ on } \Gamma_0 \times (0, \infty), \tag{12}$$

$$\frac{\partial u}{\partial \nu} - \delta_t + \eta(\cdot, u_t) = 0\ \text{ on } \Gamma_1 \times (0, \infty), \tag{13}$$

where β, f_1, f_2, f_3, and η are known functions, c, ρ, and λ are constants, and a is a known vector of $\mathbb{R}^N$. In [6], to prove the existence and uniqueness of solutions the authors employed the Faedo-Galerkin method and the energy method, respectively, with no restrictions on the geometry of the domain. To prove that the energy associated to the problem is uniformly stable, the authors used some usual restrictions on the geometry of the domain. Problems with acoustic boundary conditions can be found in [4, 5, 8, 20–22, 25, 32, 39, 43] and references therein. We highlight the work of Frota and Goldstein [22] which was the pioneer paper studying nonlinear problems.

On the other hand, the boundary condition (5) is associated to the following equation

$$Au + \beta\partial_n^a u + \gamma u - q\beta\Delta_\Gamma u = 0\ \text{ on } \Gamma, \tag{14}$$

where A is a second order uniformly elliptic operator defined by

$$Au = \sum_{i,j=1}^{N} \partial_i (a_{ij}(x)\partial_j u) = \nabla \cdot (a(x)\nabla)u,$$

here $a = (a_{ij}(\cdot))_{1\le i,j\le N}$ and a_{ij} are real valued functions, $\partial_n^a u$ is the conormal derivative of u with respect to the matrix a, and γ and β are continuously differentiable functions. Problems with the boundary condition (14) has been studied by many authors. See [13–16, 18, 19] and references therein. They are called into the literature the general Wentzell boundary conditions (GWBC). Recently, Romanelli [36] called these the Goldstein–Wentzell boundary conditions (GWBC). See also the works of Cavalcanti, Lasiecka, and Toundykov [10, 11].

Concerning the GWBC we would like to cite the paper which is our main motivation to study the system (1–6). In [26], G. R. Goldstein gives new derivations of the heat and wave equations which incorporate the boundary conditions into the formulation of the problems. She makes several descriptions on classical boundary conditions as well as on the general Wentzell and dynamic boundary conditions. Our motivation is precisely Sect. 3 of the paper where she considered the heat equation and GWBC. For the reader's convenience we rewrite the ideas of G. R. Goldstein here. It is well known that the linear heat equation on a domain Ω is given by

$$(\rho c\theta)_t = \kappa\Delta\theta + s, \tag{15}$$

where $\theta(x, t)$ represents the temperature at position $x \in \overline{\Omega}$ at time $t \ge 0$; κ is the thermal conductivity constant, ρ is the density of the material, c is the heat capacity of the material, and s represents a heat source. We suppose that there exists a heat source acting on the boundary of the region Ω. Moreover, we suppose that the source should depend on the heat flow along the boundary, the heat flux across the boundary and the temperature at the boundary. If we take it into account, then the boundary condition becomes

$$\theta_t = a(x)\Delta_\Gamma\theta - b(x)\frac{\partial\theta}{\partial\nu} + c(x)\theta \;\text{ on } \Gamma, \tag{16}$$

where a, b, and c are known functions. The Laplace-Beltrami operator describes the heat flux along the boundary and, since $c > 0$, the term $c\theta$ represents a heat source on the boundary.

Therefore, observing (15, 16), the main goal of the present paper is to incorporate into the thermoelastic system the equation (5) which takes into account the heat flow along the boundary, the heat flux across the boundary, and the temperature at the boundary. The result extends the preview literature involving the thermoelastic system, because, to the best of our knowledge, it is the first concerning the GWBC associated to the heat equation. We would like to mention that the present paper extends the discussion started by Bras Silva, Clark, and Frota [6]. Indeed, [6] was the first paper concerning some dynamics on a boundary portion using the acoustic boundary conditions. But their boundary equation is associated to the internal wave equation (and

δ models the boundary behaviour) while in our manuscript the boundary equation is associated to the internal heat equation (and it models the temperature at the boundary). Our work also extends in some direction the paper of Kasri [30] where a thermoelastic system with static Wentzell boundary conditions was studied. We highlight that in [30] the boundary equation also is associated to the internal wave equation.

The tools of our work are the use of multipliers with the construction of appropriate perturbed energy functionals. We consider that the function g satisfies the assumptions introduced by Lasiecka and Tataru [31]. Due to the localized damping effect and the presence of nonhomogeneos boundary conditions, there are some technical difficulties to overcome.

Finally, we cite the work of Frota, Medeiros, and Vicente [23] which studied problems with acoustic boundary conditions to non-locally reacting boundary. This boundary condition involves the Laplace–Beltrami operator and it is associated to the motion of the boundary. See also [2, 24, 27–29, 38, 40–42].

The paper is organized as follows. In Sect. 2 we present the notation and the existence theorem. In Sect. 3 we prove the stability result, the main theorem of the paper.

2 Notations and existence of solution

As described in the introduction, in this section we present the notations and an existence theorem. We suppose that the following assumptions hold.
Assumption 1. The function ρ satisfies

$$\rho(x) \geq \rho_0 > 0 \text{ a.e. in } \omega, \tag{17}$$

where ω is a neighborhood, in Ω, of Γ_1.
Assumption 2. The function g is continuous and monotone increasing such that

$$\begin{cases} g(s)s > 0 \text{ for all } s \neq 0, \\ c_1|s| \leq |g(s)| \leq c_2|s| \text{ for all } |s| \geq 1, \end{cases} \tag{18}$$

for some positive constants c_1, c_2.

We recall that Assumption 2 is the classical one introduced by Lasiecka and Tataru [31].

We denote by $\|\cdot\|_{L^2(\Omega)}$ the usual norm in the Hilbert space $L^2(\Omega)$ endowed with the inner product $(u, v)_{L^2(\Omega)} = \int_\Omega u(x)v(x)\,dx$. We consider $H_0^1(\Omega)$, which is a Hilbert space with the inner product

$$(u, v)_{H_0^1(\Omega)} = \int_\Omega c\nabla u \cdot \nabla v\,dx.$$

We also consider the subspace of $H^1(\Omega)$, denoted by V, as the closure of $C^1(\overline{\Omega})$ such that $u_{|\Gamma_0} = 0$ in the strong topology of $H^1(\Omega)$, i.e.,

$$V = \overline{\{u \in C^1(\overline{\Omega});\ u_{|\Gamma_0} = 0\}}^{H^1(\Omega)}.$$

We know that the Poincaré inequality holds in V, thus there exists a positive constant c_p such that

$$\int_\Omega u^2\,dx \le c_p \int_\Omega |\nabla u|^2\,dx,$$

for all $u \in V$. Therefore, the space V can be endowed with the norm, $\|\nabla \cdot\|_{L^2(\Omega)}$, induced by the inner product

$$(u, v)_V = \int_\Omega \nabla u \cdot \nabla v\,dx,$$

which is equivalent to usual norm of $H^1(\Omega)$. The dual space of V is denoted by V'.

Finally, we define

$$L^2_\beta(\Gamma_1) = \left\{ u : \Omega \to \mathbb{R};\ \int_{\Gamma_1} \frac{1}{\beta} u^2\,d\Gamma < \infty \right\},$$

which is endowed with the inner product

$$(u, v)_{L^2_\beta(\Gamma_1)} = \int_{\Gamma_1} \frac{1}{\beta} uv\,d\Gamma,$$

and norm

$$\|u\|_{L^2_\beta(\Gamma_1)} = \left(\int_{\Gamma_1} \frac{1}{\beta} u^2\,d\Gamma \right)^{1/2}.$$

We denote by $\gamma_0 : H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma)$ the trace map of order zero and by $\gamma_1 : H^1(\Omega) \to H^{-\frac{1}{2}}(\Gamma)$ the trace map of order one, i.e. $\gamma_1(\cdot) = \frac{\partial \cdot}{\partial \nu}$.

We define

$$\mathcal{H} = H^1_0(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2_\beta(\Gamma_1)$$

with the inner product and norm given by

$$((u, v, \theta, z), (r, s, \mu, p))_{\mathcal{H}} = (u, r)_{H_0^1(\Omega)} + (v, s)_{L^2(\Omega)} + (\theta, \mu)_{L^2(\Omega)} + (z, p)_{L^2_\beta(\Gamma_1)}$$

and

$$\|(u, v, \theta, z)\|^2_{\mathcal{H}} = \|u\|^2_{H_0^1(\Omega)} + \|v\|^2_{L^2(\Omega)} + \|\theta\|^2_{L^2(\Omega)} + \|z\|^2_{L^2_\beta(\Gamma_1)}.$$

Finally, we define the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ \theta \\ z \end{pmatrix} = \begin{pmatrix} v \\ c\Delta u - \operatorname{div}(\theta) - \rho(x)g(v) \\ \Delta\theta - \operatorname{div}(v) \\ \beta\Delta_\Gamma z + \alpha z - \beta\frac{\partial\theta}{\partial\nu} \end{pmatrix},$$

where

$$D(\mathcal{A}) = \Big\{(u, v, \theta, z) \in \mathcal{H};\ v \in H_0^1(\Omega),\ \theta \in V,\ c\Delta u - \operatorname{div}(\theta) - \rho(\cdot)g(v) \in L^2(\Omega),$$
$$\Delta\theta - \operatorname{div}(v) \in L^2(\Omega),\ \ \beta\Delta_\Gamma z + \alpha z - \beta\frac{\partial\theta}{\partial\nu} \in L^2_\beta(\Gamma_1),\ \gamma_0(\theta) = z\Big\}.$$

Therefore, the problem (1–6) can be written as

$$\frac{d}{dt}U(t) = \mathcal{A}U(t) \ \text{ in } (0, \infty), \tag{19}$$
$$U(0) = U_0, \tag{20}$$

where $U = (u, v, \theta, \gamma_0(\theta))^T$ and $U_0 = (u_0, u_1, \theta_0, \gamma_0(\theta_0))^T \in D(\mathcal{A})$. To prove that (19–20) has solution, it suffices to show that the operator $\mathcal{A} - \delta I$ is maximal dissipative on $\mathcal{H}$ for some positive real number δ. To prove that the operator is dissipative, we define $y_i = (u_i, v_i, \theta_i, z_i)^T \in D(\mathcal{A}), i = 1, 2, y = (u, v, \theta, z) = y_1 - y_2$. We observe that

$$\begin{aligned}
&(\mathcal{A}y_1 - \mathcal{A}y_2, y_1 - y_2)_{\mathcal{H}} - \delta(y_1 - y_2, y_1 - y_2)_{\mathcal{H}} \\
&= \int_\Omega c\nabla v \cdot \nabla v\, dx + \int_\Omega [c\Delta u - \operatorname{div}(\theta) - \rho(x)(g(v_1) - g(v_2))]v\, dx \\
&+ \int_\Omega (\Delta\theta - \operatorname{div}(v))\theta\, dx + \int_{\Gamma_1} \left(\beta\Delta_\Gamma\theta + \alpha\theta - \beta\frac{\partial\theta}{\partial\nu}\right)\theta\frac{1}{\beta}\, d\Gamma \\
&-\delta(y_1 - y_2, y_1 - y_2)_{\mathcal{H}} \le 0,
\end{aligned}$$

for δ large enough. Thus, the operator $\mathcal{A} - \delta I$ is dissipative.

To show that $\mathcal{A} - \delta I$ is maximal dissipative it is sufficient to prove that the operator $\lambda I - \mathcal{A}$ is onto $\mathcal{H}$ for some $\lambda > \delta$. Thus, let (x_1, x_2, x_3, x_4) be an arbitrary element of

$\mathcal{H}$. We are going to prove that there exists $(u, v, \theta, z) \in D(\mathcal{A})$ such that

$$(\lambda I - \mathcal{A}) \begin{pmatrix} u \\ v \\ \theta \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \tag{21}$$

for some positive λ. We define $A : \mathcal{D}(A) \subset L^2(\Omega) \to L^2(\Omega)$ by

$$Au = -c\Delta u,$$

where $\mathcal{D}(A) = H_0^1(\Omega) \cap H^2(\Omega)$. To deal with the heat equation, we define $B : \mathcal{D}(B) \subset L^2(\Omega) \to L^2(\Omega)$ by

$$B\theta = -\Delta\theta,$$

where

$$D(B) = \left\{ \psi \in L^2(\Omega);\ \Delta\psi \in L^2(\Omega),\ \psi_{|\Gamma_0} = 0,\ \frac{\partial \psi}{\partial \nu}_{|\Gamma_1} = 0 \right\}.$$

Let $\mathcal{N} : L^2_\beta(\Gamma_1) \to H^1(\Omega)$ be the Neumann operator such that $\phi \mapsto \mathcal{N}\phi$, where $\mathcal{N}\phi$ is the solution of

$$\begin{cases} \Delta\mathcal{N}\phi = 0 \ \text{ in } \ \Omega, \\ \dfrac{\partial \mathcal{N}\phi}{\partial \nu} = \phi \text{ on } \ \Gamma_1, \\ \mathcal{N}\phi = 0 \quad \text{ on } \ \Gamma_0. \end{cases}$$

Therefore, (21) becomes

$$\begin{aligned} &\lambda v + \tfrac{1}{\lambda} A v + \mathrm{div}(\theta) + \rho(x) g(v) = x_2 - \tfrac{1}{\lambda} A x_1 \\ &\lambda\theta + B\left(\theta + \mathcal{N}\left(\tfrac{\beta\Delta_\Gamma\theta + \alpha\theta - \lambda\theta}{\beta}\right)\right) + \mathrm{div}(v) = x_3 - B\mathcal{N}\left(\tfrac{x_4}{\beta}\right). \end{aligned} \tag{22}$$

Moreover, this problem can be written as

$$(F + C + M) \begin{pmatrix} v \\ \theta \end{pmatrix} = \begin{pmatrix} x_2 - \frac{1}{\lambda} A x_1 \\ x_3 - B\mathcal{N}\left(\frac{x_4}{\beta}\right) \end{pmatrix}, \tag{23}$$

where $F : H_0^1(\Omega) \times V \to H^{-1}(\Omega) \times V'$ is the duality mapping of $L^2(\Omega) \times L^2(\Omega)$ given by

$$F \begin{pmatrix} v \\ \theta \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} v \\ \theta \end{pmatrix},$$

$C : H_0^1(\Omega) \times V \to H^{-1}(\Omega) \times V'$ is the bounded, hemicontinuous and monotone operator defined by

$$C \begin{pmatrix} v \\ \theta \end{pmatrix} = \begin{pmatrix} \lambda & \text{div} \\ \text{div} & \lambda + \mathcal{N}\left(\frac{\beta \Delta_\Gamma \cdot + \alpha \cdot - \lambda \cdot}{\beta}\right) \end{pmatrix} \begin{pmatrix} v \\ \theta \end{pmatrix},$$

and $M : H_0^1(\Omega) \times V \to H^{-1}(\Omega) \times V'$ is the maximal monotone (see [7]) operator given by

$$M \begin{pmatrix} v \\ \theta \end{pmatrix} = \begin{pmatrix} \rho(x) g(v) \\ 0 \end{pmatrix}. \tag{24}$$

Since M is maximal monotone and C is monotone and hemicontinuous, we can use Corollary 1.3 of Barbu [3, page 48] to conclude that $C + M$ is maximal monotone. From this and as F is a duality mapping, we can use Theorem 1.2 of Barbu [3, page 39] to infer that $R(F + C + M)$ is all of $H^{-1}(\Omega) \times V'$. Thus, there exists $(v, \theta) \in H_0^1(\Omega) \times V$ such that (23) holds. Consequently, (22) also holds. Defining $u = \frac{x_1 + v}{\lambda}$ and $z = \gamma_0(\theta)$, we have that $(u, v, \theta, z) \in D(\mathcal{A})$ satisfies (21). Therefore, $\mathcal{A} - \delta I$ is maximal dissipative. From nonlinear semigroup theory, there exists a unique solution $U \in C([0, T]; D(\mathcal{A}))$ of (19)–(20) for any $T > 0$ finite (see Showalter [37]). Summarizing, we have the following result.

Theorem 2.1 (Existence and uniqueness) *Assume that Assumptions* 1 *and* 2 *hold. If* $(u_0, u_1, \theta_0, \gamma_0(\theta_0)) \in D(\mathcal{A})$*, then* (1–6) *has a unique solution* $(u, u_t, \theta, \gamma_0(\theta)) \in C([0, T]; D(\mathcal{A}))$*, for all* $T > 0$.

3 Stability

In this section, we prove the main result. We start by defining the energy associated to the problem (1–6) by

$$E(t) = \frac{1}{2}\left(\int_\Omega u_t^2\, dx + c \int_\Omega |\nabla u|^2\, dx + \int_\Omega \theta^2\, dx + \int_{\Gamma_1} \frac{1}{\beta}\theta^2\, d\Gamma\right). \tag{25}$$

To prove the stability it is necessary to make more two assumptions.

Assumption 3. Let x_0 be a fixed point of $\mathbb{R}^N$. We define

$$m(x) = (x - x_0) \cdot \nu,$$

for all $x \in \mathbb{R}^N$. We consider that the boundary Γ of Ω is given by

$$\Gamma_0 = \{x \in \Gamma;\ m \cdot \nu < 0\} \quad \text{and} \quad \Gamma_1 = \{x \in \Gamma;\ m \cdot \nu \geq 0\}.$$

Since the trace map γ_0 is continuous, there exists a positive constant c_{tr} such that

$$\int_{\Gamma_1} \theta^2 \, d\Gamma \leq c_{tr} \int_{\Omega} |\nabla\theta|^2 \, dx, \tag{26}$$

for all $\theta \in V$.

Assumption 4. We suppose that α and β satisfy

$$\max_{x\in\overline{\Gamma}_1} |\alpha(x)| \leq \frac{\min_{x\in\overline{\Gamma}_1} |\beta(x)|}{2c_{tr}}. \tag{27}$$

Lemma 3.1 *Suppose that Assumptions* 1, 2, *and* 3 *hold. Suppose that* $(u_0, u_1, \theta_0, \gamma_0(\theta_0)) \in D(\mathcal{A})$ *and let* $(u, u_t, \theta, \gamma_0(\theta))$ *be the solution of* (1–6) *given by Theorem* 2.1 *and* $E(t)$ *the energy defined in* (25). *Then, we have*

$$E'(t) + \int_{\Omega} |\nabla\theta|^2 \, dx + \int_{\Gamma_1} |\nabla_{\Gamma}\theta|^2 \, d\Gamma - \int_{\Gamma_1} \frac{\alpha}{\beta}\theta^2 \, d\Gamma + \int_{\Omega} \rho(x)g(u_t)u_t \, dx = 0, \tag{28}$$

for all $t \geq 0$. *Moreover, if Assumption* 4 *holds, then*

$$E'(t) \leq 0, \tag{29}$$

for all $t \geq 0$.

Proof Multiplying (1) by u_t and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\left(\int_{\Omega} u_t^2 \, dx + c\int_{\Omega} |\nabla u|^2 \, dx\right) + \int_{\Omega} u_t \text{div}(\theta) \, dx + \int_{\Omega} \rho(x)g(u_t)u_t \, dx = 0. \tag{30}$$

Multiplying (2) by θ and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} \theta^2 \, dx + \int_{\Omega} |\nabla\theta|^2 \, dx - \int_{\Gamma_1} \frac{\partial\theta}{\partial\nu}\theta \, d\Gamma + \int_{\Omega} \theta \text{div}(u_t) \, dx = 0. \tag{31}$$

From (5), we infer

$$\begin{aligned} -\int_{\Gamma_1} \frac{\partial\theta}{\partial\nu}\theta \, d\Gamma &= \int_{\Gamma_1} \frac{1}{\beta}\left(\theta_t - \beta\Delta_{\Gamma}\theta - \alpha\theta\right)\theta \, d\Gamma \\ &= \frac{1}{2}\frac{d}{dt}\int_{\Gamma_1} \frac{1}{\beta}\theta^2 \, d\Gamma + \int_{\Gamma_1} |\nabla_{\Gamma}\theta|^2 \, d\Gamma - \int_{\Gamma_1} \frac{\alpha}{\beta}\theta^2 \, d\Gamma. \end{aligned} \tag{32}$$

Moreover, we also have

$$\int_{\Omega} u_t \mathrm{div}(\theta)\, dx = -\int_{\Omega} \theta \mathrm{div}(u_t)\, dx. \tag{33}$$

Combining (30–33), we infer

$$E'(t) + \int_{\Omega} |\nabla\theta|^2\, dx + \int_{\Gamma_1} |\nabla_\Gamma \theta|^2\, d\Gamma - \int_{\Gamma_1} \frac{\alpha}{\beta}\theta^2\, d\Gamma + \int_{\Omega} \rho(x) g(u_t) u_t\, dx = 0. \tag{34}$$

Using the inequality (26), we have

$$\int_{\Gamma_1} \frac{\alpha}{\beta}\theta^2\, d\Gamma \leq \frac{c_{tr} \max\limits_{x\in\overline{\Gamma}_1} |\alpha(x)|}{\min\limits_{x\in\overline{\Gamma}_1} |\beta(x)|} \int_{\Gamma_1} |\nabla\theta|^2\, dx. \tag{35}$$

Thus

$$E'(t) + \left(\frac{1}{2} - \frac{c_{tr} \max\limits_{x\in\overline{\Gamma}_1} |\alpha(x)|}{\min\limits_{x\in\overline{\Gamma}_1} |\beta(x)|} \right) \int_{\Omega} |\nabla\theta|^2\, dx + \int_{\Gamma_1} |\nabla_\Gamma \theta|^2\, d\Gamma + \int_{\Omega} \rho(x) g(u_t) u_t\, dx \leq 0. \tag{36}$$

From (34), (36), Assumptions 2 and 4, we conclude the proof. □

Now, for each $\varepsilon > 0$, we define the perturbed energy by

$$E_\varepsilon(t) = E(t) + \varepsilon\Psi(t), \tag{37}$$

where

$$\Psi(t) = 2\int_{\Omega} u_t m \cdot \nabla u\, dx + (N-1)\int_{\Omega} u_t u\, dx. \tag{38}$$

Our decay result follows the ideas of Lasiecka and Tataru [31] which gives us general decay rates. This idea was used by many authors, see for instance Cavalcanti, Domingos Cavalcanti and Lasiecka [9], where one can find examples of explicit decay rates. It is well known that, thanks to Assumption 2 it is possible to build a concave, strictly increasing function φ such that $\varphi(0) = 0$ and

$$\varphi(sg(s)) \geq s^2 + g^2(s), \text{ for } |s| < 1. \tag{39}$$

We define

$$\tilde{\varphi}(\cdot) = \varphi\Big(\frac{\cdot}{meas\,(\omega \times (0,T))}\Big). \tag{40}$$

Since $\tilde{\varphi}$ is monotone increasing, we have that $MI + \tilde{\varphi}$ is invertible for all $M \geq 0$. We define

$$p(x) = (MI + \tilde{\varphi})^{-1}\,(Lx)\,, \tag{41}$$

where $L = (C\,meas(\omega \times (0,T)))^{-1}$ and C is a specific positive constant. The function p is positive, continuous and strictly increasing with $p(0) = 0$. We also consider the function

$$q(x) = x - (I + p)^{-1}\,(x)\,.$$

Finally, let $S(t)$ be the solution of the following ordinary differential equation

$$\frac{d}{dt}S(t) + q(S(t)) = 0,\ S(0) = E(0). \tag{42}$$

Theorem 3.1 (Stability) *Assume that Assumptions* 1, 2, 3, *and* 4 *hold. Suppose that* $(u_0, u_1, \theta_0, \gamma_0(\theta_0)) \in D(\mathcal{A})$ *and let* $(u, u_t, \theta, \gamma_0(\theta))$ *be the solution of* (1–6) *given by Theorem* 2.1*, then there exists a* $T_0 > 0$ *such that for any* $T > T_0$ *the energy satisfies*

$$E(t) \leq S\Big(\frac{t}{T} - 1\Big),$$

for all $t > T$*, with* $\lim_{t\to\infty} S(t) = 0$*, decreasing monotonically (*$S(t)$ *is the solution of* (42)*).*

Proof Taking the derivative of E_ε, we have

$$\begin{aligned} E'_\varepsilon(t) \leq &-\frac{1}{2}\int_\Omega |\nabla\theta|^2\,dx - \int_\Omega \rho(x)g(u_t)u_t\,dx \\ &+ \varepsilon\int_\Omega u_{tt}[2m\cdot\nabla u + (N-1)u]\,dx \\ &+ \varepsilon\int_\Omega u_t[2m\cdot\nabla u_t + (N-1)u_t]\,dx. \end{aligned} \tag{43}$$

Since u is a solution of (1–6), we have

$$\int_\Omega u_{tt}u\,dx = \int_\Omega [c\Delta u - \operatorname{div}(\theta) - \rho(x)g(u_t)]u\,dx$$

$$= -c \int_{\Omega} |\nabla u|^2 \, dx - \int_{\Omega} u \, \mathrm{div}(\theta) \, dx - \int_{\Omega} \rho(x) g(u_t) u \, dx. \tag{44}$$

Moreover

$$2 \int_{\Omega} u_t m \cdot \nabla u_t \, dx = -N \int_{\Omega} u_t^2 \, dx + \int_{\Gamma} m \cdot \nu u_t^2 \, d\Gamma. \tag{45}$$

Observing the identity

$$2 \int_{\Omega} \Delta u \, m \cdot \nabla u \, dx = (N-2) \int_{\Omega} |\nabla u|^2 \, dx + 2 \int_{\Gamma} \frac{\partial u}{\partial \nu} m \cdot \nabla u \, d\Gamma - \int_{\Gamma} m \cdot \nu |\nabla u|^2 \, d\Gamma$$

and that u is a solution of (1–6), we infer

$$\begin{aligned} 2 \int_{\Omega} & u_{tt} \, m \cdot \nabla u \, dx \\ &= c(N-2) \int_{\Omega} |\nabla u|^2 \, dx + 2c \int_{\Gamma} \frac{\partial u}{\partial \nu} m \cdot \nabla u \, d\Gamma \\ &\quad - c \int_{\partial\Omega} m \cdot \nu |\nabla u|^2 \, d\Gamma - 2 \int_{\Omega} \mathrm{div}(\theta) \, m \cdot \nabla u \, dx - 2 \int_{\Omega} \rho(x) g(u_t) \, m \cdot \nabla u \, dx. \end{aligned}$$

As $u = 0$ on Γ, it holds

$$|\nabla u|^2 = \left| \frac{\partial u}{\partial \nu} \right|^2 \quad \text{and} \quad \frac{\partial u}{\partial \nu} m \cdot \nabla u = m \cdot \nu \left(\frac{\partial u}{\partial \nu} \right)^2 \text{ on } \Gamma.$$

Therefore

$$\begin{aligned} 2 \int_{\Omega} u_{tt} \, m \cdot \nabla u \, dx &= c(N-2) \int_{\Omega} |\nabla u|^2 \, dx + c \int_{\Gamma} m \cdot \nu \left(\frac{\partial u}{\partial \nu} \right)^2 \, d\Gamma \\ &\quad - 2 \int_{\Omega} \mathrm{div}(\theta) \, m \cdot \nabla u \, dx - 2 \int_{\Omega} \rho(x) g(u_t) \, m \cdot \nabla u \, dx. \end{aligned} \tag{46}$$

Using (44–46) into (43), we obtain

$$\begin{aligned} E_\varepsilon'(t) \le & -\frac{1}{2} \int_{\Omega} |\nabla \theta|^2 \, dx - \int_{\Omega} \rho(x) g(u_t) u_t \, dx \\ & + \varepsilon \left[c(N-2) \int_{\Omega} |\nabla u|^2 \, dx + c \int_{\Gamma} m \cdot \nu \left(\frac{\partial u}{\partial \nu} \right)^2 \, d\Gamma \right. \\ & - 2 \int_{\Omega} \mathrm{div}(\theta) \, m \cdot \nabla u \, dx - 2 \int_{\Omega} \rho(x) g(u_t) \, m \cdot \nabla u \, dx - c(N-1) \int_{\Omega} |\nabla u|^2 \, dx \end{aligned}$$

$$-(N-1)\int_\Omega u\operatorname{div}(\theta)\,dx-(N-1)\int_\Omega \rho(x)g(u_t)u\,dx-\int_\Omega u_t^2\,dx\Bigg]. \tag{47}$$

Now, we are going to estimate the term $c\int_\Gamma m\cdot\nu\left(\frac{\partial u}{\partial\nu}\right)^2\,d\Gamma$. We consider $\hat{\omega}$ a neighborhood of Γ_1 in Ω such that

$$\overline{\hat{\omega}}\cap\Omega\subset\omega.$$

Let $h\in(W^{1,\infty}(\Omega))^N$ be a vector field such that

$$\begin{cases} h=\nu \quad \text{on } \Gamma_1\\ h\cdot\nu\geq 0, \quad \text{a. e. on } \Gamma\\ h=0 \quad \text{in } \Omega\setminus\hat{\omega}. \end{cases}$$

We define

$$E_1(t)=2\int_\Omega u_t h\cdot\nabla u\,dx. \tag{48}$$

Thus

$$\begin{aligned} E_1'(t)&=2\int_\Omega u_t h\cdot\nabla u_t\,dx+2\int_\Omega u_{tt}h\cdot\nabla u\,dx\\ &=-\int_\Omega \operatorname{div}(h)\,u_t^2\,dx+\int_\Gamma h\cdot\nu\,u_t^2\,d\Gamma+2\int_\Omega[c\Delta u-\operatorname{div}(\theta)\\ &\quad-\rho(x)g(u_t)]h\cdot\nabla u\,dx. \end{aligned} \tag{49}$$

Using Gauss' theorem and observing the definition of the vector field h, we have

$$\begin{aligned} 2c\int_\Omega \Delta u\,h\cdot\nabla u\,dx&=c\int_\Omega \operatorname{div}(h)|\nabla u|^2\,dx+c\int_{\Gamma_1}\left(\frac{\partial u}{\partial\nu}\right)^2\,d\Gamma\\ &\quad-2c\int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial h_j}{\partial x_i}\frac{\partial u}{\partial x_j}\,dx, \end{aligned} \tag{50}$$

where $h=(h_1,h_2,\ldots,h_N)$. Combining (49) with (50), we infer

$$\begin{aligned} c\int_\Gamma m\cdot\nu\left(\frac{\partial u}{\partial\nu}\right)^2\,d\Gamma&\leq c\int_{\Gamma_1} m\cdot\nu\left(\frac{\partial u}{\partial\nu}\right)^2\,d\Gamma\leq cM_1\int_{\Gamma_1}\left(\frac{\partial u}{\partial\nu}\right)^2\,d\Gamma\\ &=M_1\Bigg[E_1'(t)+\int_\Omega \operatorname{div}(h)[u_t^2-c|\nabla u|^2]\,dx+2c\int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial h_j}{\partial x_i}\frac{\partial u}{\partial x_j}\,dx \end{aligned}$$

$$+2\int_\Omega \operatorname{div}(\theta)\, h\cdot\nabla u\, dx + 2\int_\Omega \rho(x)g(u_t)\, h\cdot\nabla u\, dx\Bigg], \tag{51}$$

where $M_1 = \max_{x\in\overline{\Omega}} |m(x)|$. From (47) and (51), we obtain

$$\begin{aligned}
E_\varepsilon'(t) \le & -\frac{1}{2}\int_\Omega |\nabla\theta|^2\, dx - \int_\Omega \rho(x)g(u_t)u_t\, dx + \varepsilon\Bigg\{c(N-2)\int_\Omega |\nabla u|^2\, dx \\
& + M_1\Bigg[E_1'(t) + \int_\Omega \operatorname{div}(h)[u_t^2 - c|\nabla u|^2]\, dx + 2c\int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial h_j}{\partial x_i}\frac{\partial u}{\partial x_j}\, dx \\
& +2\int_\Omega \operatorname{div}(\theta)\, h\cdot\nabla u\, dx + 2\int_\Omega \rho(x)g(u_t)\, h\cdot\nabla u\, dx\Bigg] \\
& - 2\int_\Omega \operatorname{div}(\theta)\, m\cdot\nabla u\, dx - 2\int_\Omega \rho(x)g(u_t)\, m\cdot\nabla u\, dx - c(N-1)\int_\Omega |\nabla u|^2\, dx \\
& -(N-1)\int_\Omega u\operatorname{div}(\theta)\, dx - (N-1)\int_\Omega \rho(x)g(u_t)u\, dx - \int_\Omega u_t^2\, dx\Bigg\}.
\end{aligned} \tag{52}$$

The next step is to estimate the term $\int_{\hat{\omega}} |\nabla u|^2\, dx$. Thus, we define a function $\eta : \overline{\Omega} \to \mathbb{R}$ such that

$$\begin{cases}
\eta = 0 & \text{a. e. in } \Omega\setminus\omega \\
\eta = 1 & \text{a. e. in } \hat{\omega} \\
0 \le \eta \le 1 & \\
\frac{|\nabla\eta|^2}{\eta} \in L^\infty(\omega). &
\end{cases}$$

We also define

$$E_2(t) = \int_\Omega \eta\, u_t u\, dx.$$

Taking the derivative of E_2, we have

$$\begin{aligned}
c\int_{\hat{\omega}} |\nabla u|^2\, dx \le & -2E_2'(t) + 2\int_\Omega \eta\, u_t^2\, dx + c\int_\Omega \frac{|\nabla\eta|^2}{\eta}u^2\, dx \\
& - 2\int_\Omega \eta\, u\operatorname{div}(\theta)\, dx + 2\int_\Omega \eta\, u\rho(x)g(u_t)\, dx.
\end{aligned}$$

For each $\lambda > 0$, we have

$$\int\limits_{\Omega} \eta\, u\, \mathrm{div}(\theta)\, dx \leq C(\lambda) \int_{\Omega} |\nabla\theta|^2\, dx + \lambda E(t).$$

Thus

$$c \int\limits_{\hat{\omega}} |\nabla u|^2\, dx \leq -2E_2'(t) + 2\int_{\Omega} \eta\, u_t^2\, dx + c\int_{\Omega} \frac{|\nabla\eta|^2}{\eta} u^2\, dx + 2\int\limits_{\Omega} \eta\, u\rho(x)g(u_t)\, dx$$
$$+ C(\lambda)\int\limits_{\Omega} |\nabla\theta|^2\, dx + \lambda E(t). \tag{53}$$

On the other hand, we have

$$2\int\limits_{\Omega} \mathrm{div}(\theta)\, h \cdot \nabla u\, dx + 2\int\limits_{\Omega} \mathrm{div}(\theta)\, h \cdot \nabla u\, dx - (N-1)\int\limits_{\Omega} u\, \mathrm{div}(\theta)\, dx$$
$$\leq C(\lambda)\int\limits_{\Omega} |\nabla\theta|^2\, dx + \lambda E(t) \tag{54}$$

and

$$2M_1\int\limits_{\Omega} \rho(x)g(u_t)\, h \cdot \nabla u\, dx + 2\int\limits_{\Omega} \rho(x)g(u_t)\, m \cdot \nabla u\, dx$$
$$+(N-1)\int\limits_{\Omega} \rho(x)g(u_t)u\, dx$$
$$\leq C(\lambda)\int\limits_{\Omega} \rho(x)g^2(u_t)\, dx + \lambda E(t). \tag{55}$$

Therefore, (52–55) give

$$E_\varepsilon'(t) \leq -\left(\frac{1}{2} - C(\lambda)\varepsilon\right)\int\limits_{\Omega} |\nabla\theta|^2\, dx - \int\limits_{\Omega} \rho(x)g(u_t)u_t\, dx - \varepsilon\,[2 - C\lambda]\, E(t)$$
$$+M_1\varepsilon\left[E_1'(t) - 2E_2'(t)\right] + 2M_0\varepsilon\int\limits_{\Omega} \eta u_t^2\, dx$$
$$+C\varepsilon\int\limits_{\Omega} u^2\, dx + \varepsilon C(\lambda)\int_{\Omega} \rho(x)g^2(u_t)\, dx. \tag{56}$$

Defining

$$\tilde{\Psi}(t) = \Psi(t) + M_1 E_1(t) - 2M_1 E_2(t), \tag{57}$$
$$\tilde{E}_\varepsilon(t) = E(t) + \varepsilon \tilde{\Psi}(t), \tag{58}$$

and choosing λ small enough, we infer

$$\begin{aligned}&\tilde{E}_\varepsilon'(t) + \varepsilon C_2 E(t)\\ &\quad \leq -\left(\frac{1}{2} - C_1\varepsilon\right) \int\limits_\Omega |\nabla\theta|^2\, dx + C\varepsilon \int\limits_\Omega u^2\, dx + \varepsilon C \int_\Omega \rho(x)[u_t^2 + g^2(u_t)]\, dx.\end{aligned} \tag{59}$$

It is not difficult to prove that there exists a positive constant $\tilde{C}$ such that

$$|\tilde{E}_\varepsilon(t) - E(t)| \leq \tilde{C}\varepsilon E(t), \tag{60}$$

for all $t \geq 0$ and for $\varepsilon > 0$ small enough.

Integrating (59) from 0 to T and observing (60), we have

$$\begin{aligned}&(1 - \tilde{C}\varepsilon)E(T) + \varepsilon C_2 \int_0^T E(t)\, dt\\ &\quad \leq (1 + \tilde{C}\varepsilon)E(0) - \left(\frac{1}{2} - C_1\varepsilon\right) \int\limits_0^T \int_\Omega |\nabla\theta|^2\, dx\, dt\\ &\quad\quad + C\varepsilon \int_0^T \int_\Omega u^2\, dx\, dt + \varepsilon C \int_0^T \int_\Omega \rho(x)[u_t^2 + g^2(u_t)]\, dx\, dt.\end{aligned} \tag{61}$$

Since $E(t)$ is decreasing, we have that

$$T E(T) \leq \int_0^T E(t)\, dt.$$

Thus, we infer

$$\begin{aligned}&(1 + \varepsilon(C_2 T - \tilde{C}))E(T)\\ &\quad \leq (1 + \tilde{C}\varepsilon)E(0) - \left(\frac{1}{2} - C_1\varepsilon\right) \int\limits_0^T \int\limits_\Omega |\nabla\theta|^2\, dx\, dt\\ &\quad\quad + C\varepsilon \int_0^T \int\limits_\Omega u^2\, dx\, dt + \varepsilon C \int_0^T \int\limits_\Omega \rho(x)[u_t^2 + g^2(u_t)]\, dx\, dt.\end{aligned} \tag{62}$$

On the other hand, Lemma 3.1 gives us

$$E(0) = E(T) + \int_0^T \int_\Omega |\nabla\theta|^2 \, dx \, dt + \int_0^T \int_{\Gamma_1} |\nabla_\Gamma \theta|^2 \, d\Gamma \, dt \\ - \int_0^T \int_{\Gamma_1} \frac{\alpha}{\beta}\theta^2 \, d\Gamma \, dt + \int_0^T \int_\Omega \rho(x) g(u_t) u_t \, dx \, dt. \tag{63}$$

Substituting (63) into (62) and choosing ε small enough, we obtain

$$\varepsilon(C_2 T - 2\tilde{C}) E(T) \le (1 + \tilde{C}\varepsilon)\Big[\int_0^T \int_\Omega |\nabla\theta|^2 \, dx \, dt + \int_0^T \int_{\Gamma_1} |\nabla_\Gamma \theta|^2 \, d\Gamma \, dt \\ + \int_0^T \int_\Omega \rho(x) g(u_t) u_t \, dx \, dt - \int_0^T \int_{\Gamma_1} \frac{\alpha}{\beta}\theta^2 \, d\Gamma \, dt\Big] \\ + C\varepsilon \int_0^T \int_\Omega u^2 \, dx \, dt + \varepsilon C \int_0^T \int_\Omega \rho(x)[u_t^2 + g^2(u_t)] \, dx \, dt. \tag{64}$$

Choosing $T > 0$ such that $C_2 T - 2\tilde{C} > 0$ and using the continuity of the trace map, we have

$$E(T) \le C \left[\int_0^T \int_\Omega |\nabla\theta|^2 \, dx \, dt + \int_0^T \int_{\Gamma_1} |\nabla_\Gamma \theta|^2 \, d\Gamma \, dt \right. \\ \left. + \varepsilon \int_0^T \int_\Omega u^2 \, dx \, dt + \varepsilon \int_0^T \int_\Omega \rho(x)[u_t^2 + g^2(u_t)] \, dx \, dt \right], \tag{65}$$

for $\varepsilon > 0$ small enough.

Now, we are going to estimate the low order term $\int_0^T \int_\Omega u^2 \, dx \, dt$. We claim that there exists a positive constant C such that

$$\int_0^T \int_\Omega u^2 \, dx \, dt \\ \le C \left[\int_0^T \int_\Omega \{\rho(x)[u_t^2 + g^2(u_t)] + |\nabla\theta|^2\} \, dx \, dt + \int_0^T \int_{\Gamma_1} |\nabla_\Gamma \theta|^2 \, d\Gamma \, dt \right]. \tag{66}$$

Indeed, suppose that (66) does not hold. Let $(u_{0k}, u_{1k}, \theta_{0k}, \gamma_0(\theta_{0k}))_{k\in\mathbb{N}}$ be a sequence of initial data and $(u_k, u_k', \theta_k, \gamma_0(\theta_k))_{k\in\mathbb{N}}$ the corresponding solutions of (1–6) such that

$$E_k(0) \leq C, \tag{67}$$

for all $k \in \mathbb{N}$, and one has

$$\lim_{k\to\infty} \frac{\int_0^T \int_\Omega u_k^2\, dx\, dt}{\int_0^T \int_\Omega \{\rho(x)[(u_k')^2 + g^2(u_k')] + |\nabla\theta_k|^2\}\, dx\, dt + \int_0^T \int_{\Gamma_1} |\nabla_\Gamma \theta_k|^2\, d\Gamma\, dt} = \infty, \tag{68}$$

where $'$ denotes the derivative with respect to the variable t and

$$E_k(t) = \frac{1}{2}\left(\int_\Omega (u_k')^2\, dx + c\int_\Omega |\nabla u_k|^2\, dx + \int_\Omega \theta_k^2\, dx + \int_{\Gamma_1} \frac{1}{\beta}\theta_k^2\, d\Gamma\right). \tag{69}$$

From (67) and (68), we have

$$\int_0^T \int_\Omega \{\rho(x)[(u_k')^2 + g^2(u_k')] + |\nabla\theta_k|^2\}\, dx\, dt + \int_0^T \int_{\Gamma_1} |\nabla_\Gamma\theta_k|^2\, d\Gamma\, dt \to 0, \tag{70}$$

as $k \to \infty$. Observing (65), (67), and (70), we infer

$$E_k(t) \leq C, \tag{71}$$

for all $k \in \mathbb{N}$ and for all $t \geq 0$. Estimating (71) yields subsequences of $(u_k)_{k\in\mathbb{N}}$ and $(\theta_k)_{k\in\mathbb{N}}$, that we still denote in the same way, and functions (u, θ), such that

$$u_k \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0, T; H_0^1(\Omega)), \tag{72}$$

$$u_k' \overset{*}{\rightharpoonup} u' \text{ in } L^\infty(0, T; L^2(\Omega)), \tag{73}$$

$$\theta_k \overset{*}{\rightharpoonup} \theta \text{ in } L^\infty(0, T; L^2(\Omega)), \tag{74}$$

as $k \to \infty$. Since $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, from the Aubin–Lions Theorem, we have

$$u_k \to u \text{ in } L^2(0, T; L^2(\Omega)), \tag{75}$$

as $k \to \infty$. On the other hand, from (70) we have

$$\theta_k \to 0 \text{ in } L^2(0, T; V), \tag{76}$$

as $k \to \infty$. Thus (74) and (76) allow us to conclude that $\theta = 0$.

At this point we are going to separate the proof into two cases.

Case $u \neq 0$.

For each $k \in \mathbb{N}$, (u_k, θ_k) is a solution of

$$u_k'' - c\Delta u_k + \operatorname{div}(\theta_k) + \rho(x)g(u_k') = 0 \ \text{ in } \Omega \times (0, T), \tag{77}$$
$$\theta_k' - \Delta\theta_k + \operatorname{div}(u_k') = 0 \ \text{ in } \Omega \times (0, T), \tag{78}$$
$$u_k = 0 \ \text{ on } \Gamma \times (0, T), \tag{79}$$
$$\theta_k = 0 \ \text{ on } \Gamma_0 \times (0, T), \tag{80}$$
$$\theta_k' - \beta\Delta_\Gamma\theta_k + \beta\frac{\partial\theta_k}{\partial\nu} - \alpha\theta_k = 0 \ \text{ on } \Gamma_1 \times (0, T). \tag{81}$$

Taking to the limit, as $k \to \infty$, and observing (70) and (76), we obtain

$$u'' - c\Delta u = 0 \ \text{ in } \Omega \times (0, T), \tag{82}$$
$$u = 0 \ \text{ on } \Gamma \times (0, T), \tag{83}$$
$$u' = 0 \ \text{ on } \omega \times (0, T). \tag{84}$$

Taking the derivative, with respect to t, and denoting by $v = u'$, we have

$$v'' - c\Delta v = 0 \ \text{ in } \Omega \times (0, T), \tag{85}$$
$$v = 0 \ \text{ on } \Gamma \times (0, T), \tag{86}$$
$$v = 0 \ \text{ on } \omega \times (0, T). \tag{87}$$

Therefore uniqueness arguments give us that $v = u' = 0$ in $\Omega \times (0, T)$. Thus $u'' = 0$ in $\Omega \times (0, T)$. Consequently (82)–(83) becomes

$$-\Delta u = 0 \ \text{ in } \Omega \times (0, T), \tag{88}$$
$$u = 0 \ \text{ on } \Gamma \times (0, T). \tag{89}$$

This allows us to conclude that $u = 0$, which is a contradiction.

Case $u = 0$. For each $k \in \mathbb{N}$, we define

$$c_k = \left(\int_0^T \int_\Omega u_k^2 \, dx \, dt + \int_0^T \int_\Omega |\nabla\theta_k|^2 \, dx \, dt + \int_0^T \int_{\Gamma_1} |\nabla_\Gamma\theta_k|^2 \, d\Gamma \, dt\right)^{\frac{1}{2}}, \tag{90}$$
$$\tilde{u}_k = \frac{u_k}{c_k}, \quad \text{and} \quad \tilde{\theta}_k = \frac{\theta_k}{c_k}. \tag{91}$$

From (75) and (76), we infer

$$c_k \to 0, \tag{92}$$

as $k \to \infty$. Moreover, we also have

$$\int_0^T \int_\Omega \tilde{u}_k^2 \, dx \, dt + \int_0^T \int_\Omega |\nabla \tilde{\theta}_k|^2 \, dx \, dt + \int_0^T \int_{\Gamma_1} |\nabla_\Gamma \tilde{\theta}_k|^2 \, d\Gamma \, dt = 1, \tag{93}$$

for all $k \in \mathbb{N}$.

The convergence (70) gives us

$$\int_0^T \int_\Omega \{\rho(x)[(\tilde{u}_k')^2 + \frac{g^2(u_k')}{c_k}] + |\nabla \tilde{\theta}_k|^2\} \, dx \, dt + \int_0^T \int_{\Gamma_1} |\nabla_\Gamma \tilde{\theta}_k|^2 \, d\Gamma \, dt \to 0, \tag{94}$$

as $k \to \infty$. Therefore

$$\sqrt{\rho}\tilde{u}_k' \to 0 \text{ in } L^2(0, T; L^2(\Omega)), \tag{95}$$

$$\sqrt{\rho}\frac{g(u_k')}{\sqrt{c_k}} \to 0 \text{ in } L^2(0, T; L^2(\Omega)), \tag{96}$$

$$\tilde{\theta}_k \to 0 \text{ in } L^2(0, T; V), \tag{97}$$

as $k \to \infty$.

Adapting the proof of Lemma 3.1, it is possible to verify that

$$\begin{aligned} E_k(0) = E_k(T) &- \int_0^T \int_\Omega |\nabla \theta_k|^2 \, dx \, dt - \int_0^T \int_{\Gamma_1} |\nabla_\Gamma \theta_k|^2 \, d\Gamma \, dt \\ &+ \int_0^T \int_{\Gamma_1} \frac{\alpha}{\beta} \theta_k^2 \, d\Gamma \, dt - \int_0^T \int_\Omega \rho(x) g(u_k') u_k' \, dx \, dt. \end{aligned} \tag{98}$$

On the other hand, analogously to (65), we infer

$$\begin{aligned} E_k(T) \le C \Bigg[&\int_0^T \int_\Omega u_k^2 \, dx \, dt + \int_0^T \int_\Omega \rho(x)[(u_k')^2 + g^2(u_k')] \, dx \, dt \\ &+ \int_0^T \int_{\Gamma_1} |\nabla_\Gamma \theta_k|^2 \, d\Gamma \, dt + \int_0^T \int_\Omega |\nabla \theta_k|^2 \, dx \, dt \Bigg]. \end{aligned} \tag{99}$$

Now for each $k \in \mathbb{N}$, we define

$$\tilde{E}_k(t) = \frac{E_k(t)}{c_k}.$$

Thus since $E_k(t)$ is decreasing and observing (98) and (99), we have

$$\tilde{E}_k(t) \le \tilde{E}_k(0) \le C + C \int_0^T \int_\Omega \rho(x) \left((\tilde{u}_k')^2 + \frac{g^2(u_k')}{c_k} \right) dx \, dt, \tag{100}$$

for all $t \in [0, T]$. From (94) and (100), we conclude that

$$\tilde{E}_k(t) \leq C, \tag{101}$$

for all $k \in \mathbb{N}$ and $t \in [0, T]$.

Therefore, the estimate (101) yields subsequences of $(\tilde{u}_k)_{k\in\mathbb{N}}$ and $(\tilde{\theta}_k)_{k\in\mathbb{N}}$, that we still denote in the same way, and functions $(\tilde{u}, \tilde{\theta})$, such that

$$\tilde{u}_k \overset{*}{\rightharpoonup} \tilde{u} \text{ in } L^\infty(0, T; H_0^1(\Omega)), \tag{102}$$

$$\tilde{u}_k' \overset{*}{\rightharpoonup} \tilde{u}' \text{ in } L^\infty(0, T; L^2(\Omega)), \tag{103}$$

$$\tilde{\theta}_k \overset{*}{\rightharpoonup} \tilde{\theta} \text{ in } L^\infty(0, T; L^2(\Omega)), \tag{104}$$

as $k \to \infty$. Since $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, from the Aubin-Lions Theorem, we have

$$\tilde{u}_k \to \tilde{u} \text{ in } L^2(0, T; L^2(\Omega)), \tag{105}$$

as $k \to \infty$. From (97) and (104), we conclude that

$$\tilde{\theta} = 0. \tag{106}$$

For each $k \in \mathbb{N}$, $(\tilde{u}_k, \tilde{\theta}_k)$ is a solution of

$$\tilde{u}_k'' - c\Delta\tilde{u}_k + \operatorname{div}(\tilde{\theta}_k) + \rho(x)\frac{g(u_k')}{c_k} = 0 \text{ in } \Omega \times (0, T), \tag{107}$$

$$\tilde{\theta}_k' - \Delta\tilde{\theta}_k + \operatorname{div}(\tilde{u}_k') = 0 \text{ in } \Omega \times (0, T), \tag{108}$$

$$\tilde{u}_k = 0 \text{ on } \Gamma \times (0, T), \tag{109}$$

$$\tilde{\theta}_k = 0 \text{ on } \Gamma_0 \times (0, T), \tag{110}$$

$$\tilde{\theta}_k' - \beta\Delta_\Gamma\tilde{\theta}_k + \beta\frac{\partial\tilde{\theta}_k}{\partial\nu} - \alpha\tilde{\theta}_k = 0 \text{ on } \Gamma_1 \times (0, T). \tag{111}$$

Taking to the limit, as $k \to \infty$, and observing (95)–(97), and (102)–(106), we obtain

$$\tilde{u}'' - c\Delta\tilde{u} = 0 \text{ in } \Omega \times (0, T), \tag{112}$$

$$\tilde{u} = 0 \text{ on } \Gamma \times (0, T), \tag{113}$$

$$\tilde{u}' = 0 \text{ on } \omega \times (0, T). \tag{114}$$

Thus, we can use the same arguments of the case $u \neq 0$ and to conclude that $\tilde{u} = 0$. This and (106) give a contradiction with (93).

Therefore the claim (66) is proved. Combining (65) with (66), we obtain

$$E(T) \leq C\Big[\int_0^T \int_\Omega |\nabla\theta|^2 \, dx \, dt$$

$$+\int_0^T\int_{\Gamma_1}|\nabla_\Gamma\theta|^2\,d\Gamma\,dt+\int_0^T\int_\Omega\rho(x)[u_t^2+g^2(u_t)]\,dx\,dt\Big]. \tag{115}$$

Define

$$\omega_A=\{(x,t)\in\omega\times(0,T);\ |u_t(x,t)|>1\}$$

and

$$\omega_B=(\omega\times(0,T))\setminus\omega_A.$$

Using Assumptions 1 and 2, we obtain

$$\int_{\omega_A}(u_t^2+g^2(u_t))\,dx\,dt\le\left(\frac{c_1^{-1}+c_2}{\rho_0}\right)\int_0^T\int_\Omega\rho(x)g(u_t)u_t\,dx\,dt.$$

From (39), we have

$$\int_{\omega_B}(u_t^2+g^2(u_t))\,dx\,dt\le\int_{\omega_B}\varphi(g(u_t)u_t)\,dx\,dt.$$

Using Jensen's inequality, we obtain

$$\begin{aligned}&\int_{\omega_B}(u_t^2+g^2(u_t))\,dx\,dt\\&\le\text{meas}(\omega\times(0,T))\varphi\left(\frac{1}{\text{meas}(\omega\times(0,T))}\int_0^T\int_\omega\rho(x)g(u_t)u_t\,dx\,dt\right)\\&\le\text{meas}(\omega\times(0,T))\tilde{\varphi}\left(\int_0^T\int_\omega\rho(x)g(u_t)u_t\,dx\,dt\right).\end{aligned}$$

Thus

$$\begin{aligned}\int_\omega(u_t^2+g^2(u_t))\,dx\,dt&\le\left(\frac{c_1^{-1}+c_2}{\rho_0}\right)\int_0^T\int_\Omega\rho(x)g(u_t)u_t\,dx\,dt\\&+\text{meas}(\omega\times(0,T))\tilde{\varphi}\left(\int_0^T\int_\omega\rho(x)g(u_t)u_t\,dx\,dt\right).\end{aligned}$$

Since $\tilde{\varphi}$ is increasing and

$$\int_0^T\int_\Omega|\nabla\theta|^2\,dx\,dt+\int_0^T\int_{\Gamma_1}|\nabla_\Gamma\theta|^2\,d\Gamma\,dt-\int_0^T\int_{\Gamma_1}\frac{\alpha}{\beta}\theta^2\,dx\,dt\ge 0$$

we infer

$$\int_{\omega} (u_t^2 + g^2(u_t))\, dx\, dt \leq \left(\frac{c_1^{-1} + c_2}{\rho_0} \right) \Lambda + \text{meas}(\omega \times (0,T)) \tilde{\varphi}(\Lambda), \qquad (116)$$

where

$$\Lambda = \int_0^T \int_{\Omega} \rho(x) g(u_t) u_t \, dx\, dt + \int_0^T \int_{\Omega} |\nabla \theta|^2 \, dx\, dt + \int_0^T \int_{\Gamma_1} |\nabla_{\Gamma} \theta|^2 \, d\Gamma\, dt - \int_0^T \int_{\Gamma_1} \frac{\alpha}{\beta} \theta^2 \, dx\, dt.$$

Therefore, (115) and (116) give us that

$$E(T) \leq C \left(\frac{c_1^{-1} + c_2}{\rho_0} \right) \Lambda + C\text{meas}(\omega \times (0,T)) \tilde{\varphi}(\Lambda). \qquad (117)$$

Since $L = \frac{1}{C\text{meas}(\omega \times (0,T))}$ and $M = \frac{a_1^{-1} + a_2}{\rho_0 \text{meas}(\omega \times (0,T))}$, we have

$$E(T) \leq \frac{M}{L} \Lambda + \frac{1}{L} \tilde{\varphi}(\Lambda).$$

Since p, defined in (41), is increasing, we obtain

$$p(E(T)) \leq \Lambda.$$

This and Lemma 3.1 give us that

$$p(E(T)) + E(T) \leq E(0).$$

This inequality and Lemma 3.3 of Lasiecka and Tataru [31] give us the result. □

References

1. Apolaya, R.F., Clark, H.R., Feitosa, A.J.: On a nonlinear coupled system with internal damping. Electron. J. Diff. Equ. **2000**(64), 1–17 (2000)
2. Barbieri, A., Vitillaro, E.: The damped wave equation with acoustic boundary conditions and non-locally reacting surfaces. Semigroup Forum **105**, 646–679 (2022)
3. Barbu, V.: Nonlinear Semigroups and Differential Equations in Banach Spaces. Springer, Dordrecht (1976)
4. Beale, J.T., Rosencrans, S.I.: Acoustic boundary conditions. Bull. Am. Math. Soc. **80**, 1276–1278 (1974)
5. Boukhatem, Y., Benabderrahmane, B.: Asymptotic behavior for a past history viscoelastic problem with acoustic boundary conditions. Appl. Anal. **99**, 249–269 (2020)
6. Braz e Silva, P., Clark, H.R., Frota, C.: L,: On a nonlinear coupled system of thermoelastic type with acoustic boundary conditions. Comput. Appl. Math. **36**, 397–414 (2017)

7. Cavalcanti, M.M., Dias Silva, F.R., Domingos Cavalcanti, V.N.: Uniform decay rates for the wave equation with nonlinear damping locally distributed in unbounded domains with finite measure. SIAM J. Control. Optim. **52**, 545–580 (2014)
8. Cavalcanti, M.M., Domingos Cavalcanti, V.N., Frota, C.L., Vicente, A.: Stability for semilinear wave equation in an inhomogeneous medium with frictional localized damping and acoustic boundary conditions. SIAM J. Control. Optim. **58**, 2411–2445 (2020)
9. Cavalcanti, M.M., Domingos Cavalcanti, V.D., Lasiecka, I.: Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping-source interaction. J. Diff. Equ. **236**, 407–459 (2007)
10. Cavalcanti, M.M., Lasiecka, I., Toundykov, D.: Wave equation with damping affecting only a subset of static Wentzell boundary is uniformly stable. Trans. Am. Math. Soc. **364**, 5693–5713 (2012)
11. Cavalcanti, M.M., Lasiecka, I., Toundykov, D.: Geometrically constrained stabilization of wave equations with Wentzell boundary conditions. Appl. Anal. **91**, 1427–1452 (2012)
12. Clark, H.R., San-Gil-Jutuca, L.P., Milla-Miranda, M.: On a mixed problem for a linear coupled system with variable coefficients. Electron. J. Diff. Equ. **4**, 1–20 (1998)
13. Clendenen, R., Goldstein, G.R., Goldstein, J.A.: Degenerate flux for dynamic boundary conditions in parabolic and hyperbolic equations. Discrete Contin. Dyn. Syst. Ser. S **9**, 651–660 (2016)
14. Coclite, G.M., Goldstein, G.R., Goldstein, J.A.: Stability estimates for parabolic problems with Wentzell boundary conditions. J. Diff. Equ. **245**, 2595–2626 (2008)
15. Coclite, G.M., Goldstein, G.R., Goldstein, J.A.: Stability of parabolic problems with nonlinear Wentzell boundary conditions. J. Diff. Equ. **246**, 2434–2447 (2009)
16. Coclite, G.M., Goldstein, G.R., Goldstein, J.A.: Stability estimates for nonlinear hyperbolic problems with nonlinear Wentzell boundary conditions. Z. Angew. Math. Phys. **64**, 733–753 (2013)
17. Dafermos, C.M.: On the existence and the asymptotic stability of solutions to the equations of linear thermoelasticity. Arch. Rational Mech. Analysis **29**, 241–271 (1968)
18. Favini, A., Goldstein, G.R., Goldstein, J.A., Romanelli, S.: The heat equation with nonlinear general Wentzell boundary condition. Adv. Diff. Equ. **11**, 481–510 (2006)
19. Favini, A., Goldstein, G.R., Goldstein, J.A., Romanelli, S.: The heat equation with generalized Wentzell boundary condition. J. Evol. Equ. **2**, 1–19 (2002)
20. Frota, C.L., Cousin, A.T., Larkin, N.A.: On a system of Klein-Gordon type equations with acoustic boundary conditions. J. Math. Anal. Appl. **293**, 293–309 (2004)
21. Frota, C.L., Cousin, A.T., Larkin, N.A.: Global solvability and asymptotic behaviour of a hyperbolic problem with acoustic boundary conditions. Funkcialaj Ekvacioj **44**, 471–485 (2001)
22. Frota, C.L., Goldstein, J.A.: Some nonlinear wave equations with acoustic boundary conditions. J. Diff. Equ. **164**, 92–109 (2000)
23. Frota, C.L., Medeiros, L.A., Vicente, A.: Wave equation in domains with non-locally reacting boundary. Diff. Integral Equ. **24**, 1001–1020 (2011)
24. Frota, C.L., Vicente, A.: Uniform stabilization of wave equation with localized internal damping and acoustic boundary condition with viscoelastic damping. Z. Angew. Math. Phys. **69**, 85 (2018)
25. Gao, Y., Liang, J., Xiao, T.J.: A new method to obtain uniform decay rates for multidimentional wave equations with nonlinear acoustic boundary conditions. SIAM J. Control. Optim. **56**, 1303–1320 (2018)
26. Goldstein, G.R.: Derivation and physical interpretation of general boundary conditions. Adv. Diff. Equ. **11**, 457–480 (2006)
27. Ha, T.G.: On the viscoelastic equation with Balakrishnan-Taylor damping and acoustic boundary conditions. Evol. Equ. Control Theory **7**, 281–291 (2018)
28. Hipp, D., Kovács, B.: Finite element error analysis of wave equations with dynamic boundary conditions: L^2 estimates. IMA J. Numer. Anal. **41**, 683–728 (2021)
29. Jeong, J.M., Park, J.Y., Kang, Y.H.: Global nonexistence of solutions for a nonlinear wave equation with time delay and acoustic boundary conditions. Comput. Math. Appl. **76**, 661–671 (2018)
30. Kasri, H.: Uniform stabilisation of a thermoelastic system with static Wentzell conditions. Int. J. Control **96**(4), 949–962 (2023)
31. Lasiecka, I., Tataru, D.: Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping. Diff. Integral Equ. **6**, 507–533 (1993)
32. Limam, A., Boukhatem, Y., Benabderrahmane, B.: New general stability for a variable coefficient thermo-viscoelastic-coupled system of second sound with acoustic boundary conditions. Comput. Appl. Math. **40**, 88 (2021)

33. Liu, Z., Zheng, S.: Exponential stability of the semigroup associated with a thermoelastic system. Quart. Appl. Math. **51**, 535–545 (1993)
34. Liu, Z., Zheng, S.: Uniform exponential stability and approximation in control of a thermoelastic system. SIAM J. Control. Optim. **32**, 1226–1246 (1994)
35. Muñoz Rivera, J.E.: Energy decay rates in linear thermoelasticity. Funkcialaj Ekvacioj **35**, 19–30 (1992)
36. Romanelli, S.: Goldstein-Wentzell boundary conditions: Recent results with Jerry and Gisèle Goldstein. Discrete Contin. Dyn. Syst. **34**, 749–760 (2014)
37. Showalter, R.: Monotone Operators in Banach Spaces and Nonlinear Partial Differential Equations. American Mathematical Society, Providence (1997)
38. Vicente, A.: Blow-up of solution of wave equation with internal and boundary source term and non-porous viscoelastic acoustic boundary conditions. Math. Nachr. **292**, 645–660 (2019)
39. Vicente, A.: Well-posedness and stability for Kirchhoff equation with non-porous acoustic boundary conditions. J. Diff. Equ. **313**, 314–335 (2022)
40. Vicente, A., Frota, C.L.: Nonlinear wave equation with weak dissipative term in domains with non-locally reacting boundary. Wave Motion **50**, 162–169 (2013)
41. Vicente, A., Frota, C.L.: Uniform stabilization of wave equation with localized damping and acoustic boundary condition. J. Math. Anal. Appl. **436**, 639–660 (2016)
42. Vicente, A., Frota, C.L.: General decay of solutions of a wave equation with memory term and acoustic boundary condition. Math. Methods Appl. Sci. **40**, 2140–2152 (2017)
43. Wu, J.: Uniform energy decay of a variable coefficient wave equation with nonlinear acoustic boundary conditions. J. Math. Anal. Appl. **399**, 369–377 (2013)