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RESEARCH ARTICLE

Some properties of monoids with infinity

Hamid Kulosman[1] · Alica Miller[1]

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Abstract
We introduce the notion of PC cancellative additive monoids with infinity and use it to characterize cancellative additive principal ideal domains with infinity. Our characterization improves various known characterizations from the literature, both, in the context of the commutative cancellative monoids, as well as in the context of the analogues of the statements from the commutative ring theory.

Keywords Cancellative additive monoid with infinity · Domain with infinity · PC domain with infinity · Atomic domain with infinity · Bézout · Prüfer and valuation monoid with infinity · Ideal theory of semigroups

1 Introduction

The ideal theory of commutative monoids and the ideal theory of commutative rings are two parallel theories with a lot of similarities. Commutative ring theory was first developed, but already in the 1920's and 1930's several papers appeared in which commutative semigroup theory was treated and the similarities between the ideal theories of the two were amazing. The 1938 paper [7] by Clifford is an excellent illustration. Many papers in the years after that just reiterated this observation over and over. Of course, there are significant differences as well. To quote from the excellent 1984 paper [2] by Anderson and Johnson (in which both, similarities and differences, were studied): "the simpler axioms for a semigroup allow much more freedom and hence generally only weaker structural results are possible." In the 1990's many semigroup papers by Matsuda appeared (for example [21]) and in them it was repeatedly conjectured that almost all propositions in multiplicative ideal theory for commutative rings hold for commutative semigroups.

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✉ Hamid Kulosman
hamid.kulosman@louisville.edu

Alica Miller
alica.miller@louisville.edu

1 Department of Mathematics, University of Louisville, Louisville, KY 40292, USA

In almost all books and papers about commutative semigroups the binary operation is multiplicative. In that way, especially when the ideal theory is the topic one investigates, one can think of the underlying semigroup of a commutative ring as of the most natural example of a commutative semigroup. However, there are some sources where the additive notation is used, for example the book [17] and all the semigroup papers by Matsuda. *In this manuscript we will also be using the additive notation.* It turns out that in that case the similarities between the two ideal theories are the most transparent. *Another feature of our manuscript is that we will always have the infinity element in our semigroups* (it corresponds to the zero element in the multiplicative notation). Having the infinity element, we don't need to omit the additive counterpart of the zero ideal, nor to assume that the empty set is an ideal.

In Sect. 2 we give an overview of the basic elements of the ideal theory of monoids, using the additive notation and with an infinity element in each monoid. *We are not aware of any other source offering both of these features.* In Sect. 3 we focus on cancellative monoids with infinity (we call them domains with infinity as they are analogues of the integral domains from commutative ring theory). We consider various types of domains with infinity and discuss the relations between them. This discussion is pretty much summarized in Sect. 4 in the form of two diagrams of implications and equivalences between various types of domains with infinity. These diagrams illustrate Matsuda's conjecture (almost all statements from the ideal theory of commutative rings can be proved in the context of commutative semigroups), but they also illustrate one of the points from the paper by D. D. Anderson and E. W. Johnson: the structural results are weaker as many types of domains with infinity (whose analogues in commutative ring theory are different notions) coincide in the commutative semigroup context. *The diagrams contain a new type of the domains (with infinity) that we introduced here and haven't met in the literature: the PC domains (with infinity).* This type is the most general on one of the diagrams. We also show that all implications on the diagrams are strict and that there are no other implications.

In Sect. 5 we consider various characterizations of principal ideal domains (with infinity) that appear in the commutative ring and the commutative semigroup literature. For those that appear in the commutative ring literature, we consider the analogues in the context of commutative monoids. The characterizations use one of the conditions from each of the diagrams. We show that it is sufficient to take the most general condition from each of the diagrams. (This thus includes our new type of domains with infinity.) *In that way many characterizations in the literature are corollaries of our characterization (9 corollaries in total).* (We have done a related investigation of just commutative ring theory conditions in our paper [6].)

For all the notions used but not defined in this article the reader can consult the book [14] by Grillet.

2 Monoids with infinity

Definition 2.1 We call a *monoid with infinity* a set T with a binary operation written additively, satisfying the following conditions:

(i) $(x + y) + z = x + (y + z)$ for every $x, y, z \in T$;
(ii) $x + y = y + x$ for every $x, y \in T$;
(iii) there is an element in T called *zero* and denoted by 0 such that $x + 0 = 0 + x = x$ for every $x \in T$;
(iv) there is an element in T called *infinity* and denoted by ∞ such that $x + \infty = \infty + x = \infty$ for every $x \in T$.

If all conditions except the condition (iv) are satisfied, then $(T, +)$ is a standard *additive monoid*. The multiplicative analogue of the additive monoid is a standard *multiplicative monoid*. If a multiplicative monoid has a *multiplicative zero* (i.e., an element 0 such that $x \cdot 0 = 0 \cdot x = 0$ for all $x \in T$), we will call it a *multiplicative monoid with zero*. The element 0 of a multiplicative monoid with zero corresponds to the element ∞ of a monoid with infinity.

Examples 2.2 (1) The *infinity monoid with infinity*, shortly the *infinity monoid*, is a monoid with infinity T in which $0 = \infty$. Then $x = x + 0 = x + \infty = \infty$, so ∞ is the only element of T.
(2) Additive monoids usually do not have an infinity element (for example, $\mathbb{N}_0$, $\mathbb{Q}_+$, $\mathbb{R}_+$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$). To each additive monoid T we can adjoin the infinity element ∞ and define $x + \infty = \infty + x = \infty$ for every $x \in T$, as well as $\infty + \infty = \infty$. In that way $T_\infty = T \cup \{\infty\}$ becomes a monoid with infinity. For example, we get in that way the monoids with infinity $\mathbb{N}_{0,\infty}$, $\mathbb{Q}_{+,\infty}$, $\mathbb{R}_{+,\infty}$, $\mathbb{Z}_\infty$, $\mathbb{Q}_\infty$, $\mathbb{R}_\infty$, $\mathbb{C}_\infty$.
(3) Let (T_α) be a family of monoids with infinity. Then the set $\prod T_\alpha$, equipped with the addition defined by

$$(x_\alpha) + (y_\alpha) = (x_\alpha + y_\alpha),$$

is a monoid with infinity. Its zero element is (0_α), where, for each α, 0_α is the zero of T_α. Its infinity element is (∞_α), where, for each α, ∞_α is the infinity of T_α.

Let T be a monoid with infinity, $x \in T$. We say that x is *invertible* if there exists an element $y \in T$ such that $x + y = 0$. If an element x of a monoid with infinity T is invertible, its inverse is unique. If x has an inverse, it is denoted by $-x$. Invertible elements are called *units*. The set of units of a monoid with infinity T is denoted by $U(T)$. The set $U(T)$ with the addition induced from T is a group. It is called the *group of units* of the monoid with infinity T. If x, y are the elements of T and y is a unit, then $x + (-y)$ is also written $x - y$. The sum $x + y$, with x a nonunit and y any element of T, is a nonunit. If $x + y$ is a unit, then both x and y are units.

Definition 2.3 Let T be a monoid with infinity. A *submonoid with infinity* of T is any subset T_1 of T satisfying the following conditions:

(i) $x, y \in T_1$ implies $x + y \in T_1$ for all $x, y \in T$;
(ii) $0 \in T_1$;
(iii) $\infty \in T_1$.

If T_1 is a submonoid with infinity of T, then, with the addition induced form T, it is a monoid with infinity.

Examples 2.4 (1) If T is a non-infinity monoid with infinity, then $T_1 = \{0, \infty\}$ is the smallest submonoid with infinity of T. It is called the *trivial submonoid with infinity* of T.

(2) A *numerical monoid* is a submonoid of the additive monoid $\mathbb{N}_0$, generated by the elements $a_1, \ldots, a_n \in \mathbb{N}$ which satisfy the condition $\gcd(a_1, \ldots, a_n) = 1$. A *numerical monoid with infinity* is any submonoid with infinity $T \cup \{\infty\}$ of the monoid with infinity $\mathbb{N}_{0,\infty}$ such that T is a numerical submonoid of $\mathbb{N}_0$.

(3) Let T be a monoid with infinity and (T_α) a family of submonoids with infinity of T. It is immediate that $\cap T_\alpha$ is a submonoid with infinity of T. In particular, the intersection of all the submonoids with infinity of T containing a subset X of T is the smallest submonoid with infinity of T containing X. It is called the *submonoid with infinity of T generated by X* and is denoted by $\langle X \rangle$.

Definition 2.5 Let T be a monoid with infinity. A nonempty subset I of T is called an *ideal* if the relations $x \in I$, $t \in T$, imply $x + t \in I$.

In other words, I is an ideal of T if $I + T \subseteq I$, or, equivalently, $I + T = I$. Every ideal I contains the element ∞.

Examples 2.6 (1) The whole monoid with infinity T is an ideal of T. So is the set $\{\infty\}$. This ideal is called the *infinity ideal* and is also denoted by (∞) instead of $\{\infty\}$.

(2) For every element $x \in T$ the set $x + T$ of all translates $x + t$ of x by the elements of T is an ideal of the monoid with infinity T. This ideal is called the *principal ideal* generated by x and is denoted by (x).

(3) If (I_α) is a family of ideals of a monoid with infinity T, then the *union* $\cup I_\alpha$ is an ideal of T. By convention, the union of an empty family of ideals of T is the infinity ideal of T.

(4) If (I_α) is a family of ideals of a monoid with infinity T, then the *intersection* $\cap I_\alpha$ is an ideal of T. By convention, the intersection of an empty family of ideals of T is the ideal T of T. For every subset A of T there exists the smallest ideal of T containing A, namely the intersection of all the ideals of T containing A. This ideal is called the ideal *generated by* A and is denoted by (A). We have $(A) = \cup_{a \in A}(a)$. Thus $x \in (A)$ if and only if $x = a + t$ for some $a \in A$ and $t \in T$.

(5) If I, J are two ideals of a monoid with infinity T, then the set $I + J = \{x + y \mid x \in I, y \in J\}$ is an ideal of T. It is called the *sum* of the ideals I and J. We have $I + J \subseteq I \cap J$. If $J = (t)$, then $I + J = \{x + t \mid x \in I\}$, so instead of $I + (t)$ we often write $I + t$.

(6) If I, J are two ideals of a monoid with infinity T, then the set $I : J = \{t \in T \mid J + t \subseteq I\}$ is an ideal of T. It is called the *transporter* of J into I.

Let T be a monoid with infinity and x an element of T. Then $(x) = T$ if and only if x is a unit.

Definition 2.7 Let T, T' be two monoids with infinity. A *homomorphism* of T into T' is any map $f : T \to T'$ satisfying the following conditions:

(i) $f(x + y) = f(x) + f(y)$ for all $x, y \in T$;

(ii) $f(0) = 0$;
(iii) $f(\infty) = \infty$.

Let $f : T \to T'$ be a monoid with inifinity homomorphism. If x is a unit of T, then $f(x)$ is a unit of T' and $f(-x) = -f(x)$.

Definition 2.8 Let $f : T \to T'$ be a homomorphism of monoids with infinity. The set

$$\operatorname{Ker}(f) = \{x \in T \mid f(x) = \infty\}$$

is called the *kernel* of f. The set

$$\operatorname{Im}(f) = \{y \in T' \mid (\exists x \in T)\ y = f(x)\}$$

is called the *image* of f.

$\operatorname{Ker}(f)$ is an ideal of T, while $\operatorname{Im}(f)$ is a submonoid with infinity of T'. The homomorphism f is said to be *pure* if $\operatorname{Ker}(f) = (\infty)$. (This terminology is from [15, Definition 2.9].) Note that f pure does not imply f injective. An injective homomorphism is also called an *embedding*.

Let T be a monoid with infinity and I an ideal of T. Let

$$T/I = \{\{x\} \mid x \in T \setminus I\} \cup \{I\}.$$

On the set T/I we define addition in the following way:

$$\{x\} + \{y\} = \begin{cases} \{x+y\} & \text{if } x+y \notin I, \\ I & \text{if } x+y \in I, \end{cases}$$
$$\{x\} + I = I \text{for every } x \in T,$$
$$I + I = I.$$

Then T/I becomes a monoid with infinity, called the *quotient monoid with infinity* of T by I. The map $\pi : T \to T/I$, defined by

$$\pi(x) = \begin{cases} \{x\} & \text{if } x \notin I, \\ I & \text{if } x \in I, \end{cases}$$

is a homomorphism of monoids with infinity, called *canonical*. Its kernel is I.

Let T be a monoid. The set

$$T[X] = \{a + nX \mid a \in T,\ n \in \mathbb{N}_0\},$$

of formal expressions $a + nX$, where X is a variable, with addition defined by

$$(a + mX) + (b + nX) = (a + b) + (m + n)X,$$

is called the *polynomial monoid in the variable X over the monoid T*. If T is a monoid with infinity, $T[X]$ is a monoid with infinity, the infinity element of $T[X]$ being $\infty = \infty + 0X$ (which is equal to any $\infty + nX$). If $a \neq \infty$ or $b \neq \infty$, then the polynomials $a + mX$ and $b + nX$ are defined to be equal if $a = b$ and $m = n$. The map $a \mapsto a + 0X$ is the *canonical* embedding of T into $T[X]$.

Definition 2.9 Let T be a monoid with infinity. An ideal $\mathfrak{m}$ of T is said to be *maximal* if it is a maximal element of the set of all proper ideals of T ordered by inclusion.

In other words, $\mathfrak{m}$ is maximal if $\mathfrak{m} \neq T$ and the only ideal of T properly containing $\mathfrak{m}$ is T.

Every monoid with infinity $T \neq \{\infty\}$ has exactly one maximal ideal. It consists of all nonunits of T. It is denoted by $\mathfrak{m}_T$.

Definition 2.10 An element x of a monoid with infinity T is said to be *cancellable* if $x + y = x + z$ with $y, z \in T$ implies $y = z$. If an element is not cancellable, it is called *non-cancellable*.

The element ∞ is non-cancellable unless $T = \{\infty\}$.

We denote by $C(T)$ the set of cancellable and by $NC(T)$ the set of non-cancellable elements of a monoid with infinity T. The set of cancellable elements of T is a submonoid of T. If $T \neq \{\infty\}$, the set of non-cancellable elements of T is an ideal of T.

Definition 2.11 An element x of a monoid with infinity T is called an *infinity subtractor* if there is an element $y \neq \infty$ such that $x + y = \infty$. If x is not an infinity subtractor, it is called a *non-infinity-subtractor*.

Equivalently, an element $x \in T$ is an non-infinity-subtractor if for every $y \neq \infty$, $x + y \neq \infty$.

The element ∞ is an infinity subtractor unless $T = \{\infty\}$.

We denote by $IS(T)$ the set of infinity subtractors and by $NIS(T)$ the set of non-infinity-subtractors of a monoid with infinity T. The set of non-infinity-subtractors of T is a submonoid of T. If $T \neq \{\infty\}$, the set of infinity subtractor of T is an ideal of T. We have:

(a) $U(T) \subseteq C(T) \subseteq NIS(T)$;
(b) if $T \neq \{\infty\}$, $IS(T) \subseteq NC(T) \subseteq \mathfrak{m}_T$.

Definition 2.12 A monoid with infinity G is called a *group with infinity* if it does not consist only of ∞ and every non-infinity element of G is a unit.

If G is a group, then G_∞ is a group with infinity. Conversely, if G is a group with infinity, then $G \setminus \{\infty\}$ is a subgroup of G and $G = (G \setminus \{\infty\})_\infty$.

Let T be a monoid with infinity. The following are equivalent:

(a) T is a group with infinity;
(b) $T \neq \{\infty\}$ and the only ideals of T are (∞) and T.

Definition 2.13 An ideal $\mathfrak{p}$ of a monoid with infinity T is said to be *prime* if the following two conditions hold:

(1) $\mathfrak{p} \neq T$;
(2) if x, y are two elements of T such that $x \notin \mathfrak{p}$ and $y \notin \mathfrak{p}$, then $x + y \notin \mathfrak{p}$.

The condition (2) can be expressed in the following way:

(2') if x, y are two elements of T such that $x + y \in \mathfrak{p}$, then $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.

If T is a monoid with infinity $\neq \{\infty\}$, then the unique maximal ideal $\mathfrak{m}$ of T is prime.

Let T be a monoid with infinity, $I_1, I_2, \ldots, I_n$ ideals of T and $\mathfrak{p}$ a prime ideal of T. If $\mathfrak{p}$ contains the sum $I_1 + I_2 + \cdots + I_n$, it contains at least one of the I_i.

Remark 2.14 According to Anderson and Johnson [2, page 135], "perhaps the greatest divergence between the ideal theory of semigroups and that of rings is due to the next simple proposition which stands in contrast to the following well-known and useful result from commutative ring theory: If I is an ideal of a commutative ring R and $I \subseteq I_1 \cup \cdots \cup I_n$ $(n \geq 2)$ where $I_1, \ldots, I_n$ are ideals of R with at least $n - 2$ of them prime, then $I \subseteq I_i$ for some i."

The next proposition is easy to prove (see [2, Proposition 3.1]).

Proposition 2.15 *Let T be a monoid with infinity and $(\mathfrak{p}_i)$ a nonempty family of prime ideals of T. Then $\cup\mathfrak{p}_i$ is a prime ideal of T.*

Definition 2.16 Let T be a monoid with infinity. A subset S of T is said to be *additively closed* if the following two conditions hold:

(1) $0 \in S$;
(2) the sum of any two elements of S belongs to S.

In other words, S is an additively closed subset of T if it is a submonoid of T.

Examples 2.17 (1) For any $x \in T$ the set $S_x = \{0, x, 2x, 3x, \ldots\}$ is an additively closed subset of T.
(2) Let $\mathfrak{p}$ be an ideal of T. For $T \setminus \mathfrak{p}$ to be an additively closed subset of T it is necessary and sufficient that $\mathfrak{p}$ is prime.
(3) The set $NIS(T)$ of all non-infinity-subtractors of T is an additively closed subset of T.
(4) The set $C(T)$ of all cancellable elements of T is in additively closed subset of T.

We will now describe the *construction of the monoid with infinity $T - S$ of differences of a monoid with infinity T with subtrahends in S.*

Let T be a monoid with infinity and S an additively closed subset of T. On the product monoid $T \times S$, we consider the relation $\equiv$ defined in the following way:

$$(x, p) \equiv (y, q) \text{ if there is an } s \in S \text{ such that } x + q + s = y + p + s.$$

It is easy to see that this relation is an equivalence relation. The equivalence class of (x, p) modulo $\equiv$ will be denoted by $x - p$ and called the *difference* of x and p. The quotient monoid $(T \times S)/\equiv$ will be denoted by $T - S$. The *operation on* $T - S$ is given by

$$(x - p) + (y - q) = (x + y) - (p + q).$$

Two elements $x - p$ and $y - q$ are *equal* if and only if $x + q + s = y + p + s$ for some $s \in S$. The monoid $T - S$ is in fact a monoid with infinity. The element $\infty - 0$ is the infinity element of $T - S$ (denoted simply by ∞). (Note that $\infty - 0 = \infty - p$ for any $p \in S$. Hence for any $x - p \in T - S$ we have $(\infty - 0) + (x - p) = (\infty + x) - (0 + p) = \infty - p = \infty - 0$.)

The monoid with infinity $T - S$ is called the *monoid with infinity of differences of the monoid with infinity* T *with subtrahends in* S. Its identity element is $0 - 0$, denoted simply by 0.

We define the map $\varepsilon : T \to T - S$ by $\varepsilon(t) = t - 0$. This map is a homomorphism of monoids with infinity, called *canonical*. Every element $s_1 - s_2$, where $s_1, s_2 \in S$, is a unit in $T - S$. Its inverse is $s_2 - s_1$. In particular, every element of $\varepsilon(S)$ is a unit in $T - S$. For every $t \in T$ and $s \in S$ we have $t - s = (t - 0) + (0 - s)$, so that

$$T - S = \langle \{t - 0 \mid t \in T\} \cup \{0 - s \mid s \in S\} \rangle = \langle \varepsilon(T) \cup (-\varepsilon(S)) \rangle .$$

Note that, in general, in $T - S$ there may exist units that cannot be expressed in the form $s_1 - s_2$, with $s_1, s_2 \in S$.

Keeping the above notation, the following statements hold:

(i) for any $t_1, t_2 \in T$, $\varepsilon(t_1) = \varepsilon(t_2)$ if and only if $t_1 + s = t_2 + s$ for some $s \in S$;
(ii) ε is injective if and only if $S \subseteq C(T)$;
(iii) ε is bijective if and only if $S \subseteq U(T)$;
(iv) $\mathrm{Ker}(\varepsilon) = \{t \in T \mid (\exists s \in S)\ t + s = \infty\}$;
(v) ε is pure if and only if $S \subseteq NIS(T)$;
(vi) $T - S = \{\infty\}$ if and only if $\infty \in S$.

Let T be a monoid with infinity. The monoid with infinity $\Psi(T) = T - NIS(T)$ is called the *total monoid with infinity of differences of* T. The canonical homomorphism $\varepsilon : T \to \Psi(T)$ is pure.

Definition 2.18 A monoid with infinity T is called a *quasi-domain with infinity* if $T \neq \{\infty\}$ and $NIS(T) = T \setminus \{\infty\}$.

Equivalently, T is a quasi-domain with infinity if $T \neq \{\infty\}$ and $x + y = \infty$ implies $x = \infty$ or $y = \infty$.

Examples 2.19 (1) Let T be the monoid with infinity $[0, 1]$ with the operation $x + y = \sup\{x, y\}$, where the order on $[0, 1]$ is induced from $\mathbb{R}$. (The element 1 is the infinity element of T.) Then T is a quasi-domain with infinity.
(2) Every group with infinity is a quasi-domain with infinity.
(3) A submonoid with infinity of a quasi-domain with infinity is a quasi-domain with infinity. In particular, a submonoid with infinity of a group with infinity is a quasi-domain with infinity.

The first of the following three statements is obvious. The second and third statements are translations into the context of monoids with infinity of the well-known statements about the monoids of commutative rings.

A monoid with infinity T is a quasi-domain with infinity if and only if $T = S_\infty$ for some monoid S. A monoid with infinity T is a quasi-domain with infinity if and only if the ideal (∞) of T is prime. More generally, an ideal $\mathfrak{p}$ of a monoid with infinity T is prime if and only if $T/\mathfrak{p}$ is a quasi-domain with infinity.

If T is a quasi-domain with infinity, the total monoid with infinity $\Psi(T)$ of differences of T becomes a group with infinity $D(T) = T - (T \setminus \{\infty\})$. This group is called the *group with infinity of differences of the quasi-domain with infinity T*.

Let T be a monoid with infinity and $C(T)$ the set of cancellable elements of T. Let $\Phi(T) = T - C(T)$. Then:

(a) the canonical homomorphism $\varepsilon : T \to \Phi(T)$ is injective and we can identify T and $\varepsilon(T)$, so that T is a submonoid with infinity of $\Phi(T)$;
(b) every cancellable element of T has an inverse in $\Phi(T)$;
(c) every element of $\Phi(T)$ has the form $t-p, t \in T, p \in C(T)$; we have $t-p = t'-p'$ if and only if $t + p' = t' + p$;
(d) the units of $\Phi(T)$ are the differences p_1-p_2, where both p_1 and p_2 are cancellable, and $p_2 - p_1$ is the inverse of $p_1 - p_2$.

Let T be a monoid with infinity. The monoid with infinity $\Phi(T) = T - C(T)$ is called the *full monoid with infinity of differences of T*.

Definition 2.20 A monoid with infinity T is said to be *cancellative* if $T \neq \{\infty\}$ and $C(T) = T \setminus \{\infty\}$. A cancellative monoid with infinity is called a *domain with infinity*.

Examples 2.21 (1) The monoids with infinity $\mathbb{N}_{0,\infty}$ and $(\mathbb{N}_0^2)_\infty$ are domains with infinity. (However, $\mathbb{N}_{0,\infty}^2$ is not a domain with infinity since the elements (x, ∞) and (∞, y) with $x, y \in \mathbb{N}_0$ are not cancellable.)
(2) Every group with infinity is a domain with infinity.
(3) A submonoid with infinity of a domain with infinity is a domain with infinity. In particular, a submonoid with infinity of a group with infinity is a domain with infinity.

Every domain with infinity is a quasi-domain with infinity, but not vice-versa.

If T is a domain with infinity, the full monoid with infinity $\Phi(T)$ coincides with the total monoid with infinity $\Psi(T)$ and they become the group with infinity $D(T) = T - (T \setminus \{\infty\})$ of differences of T.

A monoid with infinity is a domain with infinity if and only if it is isomorphic to a submonoid with infinity of a group with infinity. (This corresponds to the embedding of an integral domain into its field of fractions in commutative ring theory.)

A monoid with infinity T is called *Noetherian* if it satisfies the ascending chain condition on ideals. It is easily seen that T is Noetherian if and only if any nonempty collection of ideals of T, ordered by inclusion, has a maximal element, or if and only if every ideal of T is finitely generated. This, further, is equivalent to every prime ideal being finitely generated (Cohen's Theorem for monoids with infinity).

Let Γ be a group with infinity. A total order $\leq$ on Γ is said to be *compatible with the operation on* Γ if the following conditions hold:

(1) $\alpha \leq \infty$ for every $\alpha \in \Gamma$;
(2) $\alpha \leq \beta$ implies $\alpha + \gamma \leq \beta + \gamma$ for every $\alpha, \beta, \gamma \in \Gamma$.

A group with infinity Γ equipped with a total order compatible with the operation on Γ is called a *totally ordered group with infinity*.

Definition 2.22 Let G be a group with infinity and Γ a totally ordered group with infinity. We call a *valuation* of G into Γ any homomorphism $v : G \to \Gamma$ of groups with infinity.

Thus v satisfies:
(VAL$_1$) $v(x + y) = v(x) + v(y)$ for all $x, y \in G$;
(VAL$_2$) $v(0) = 0$ and $v(\infty) = \infty$.
Let G be a group with infinity and $v : G \to \Gamma$ a valuation of G with values in Γ. Then:

(i) For every $x \neq \infty$ we have $v(x) \neq \infty$.
(ii) The set $T = \{x \in G \mid v(x) \geq 0\}$ is a submonoid with infinity of G. (It is called the *monoid with infinity of the valuation* v.)
(iii) For every $\alpha \geq 0$ in Γ, the set $V_\alpha = \{x \in T \mid v(x) \geq \alpha\}$ (resp. $V'_\alpha = \{x \in T \mid v(x) > \alpha\}$) is an ideal of T, and every ideal $\neq \infty$ of T contains one of $V_\alpha \neq \infty$.
(iv) The set $\mathfrak{m}_T = \{x \in T \mid v(x) > 0\}$ is the maximal ideal of T.
(v) For every $x \in G \setminus T$ we have $-x \in \mathfrak{m}_T$.

A domain with infinity T is called a *valuation monoid with infinity* (VM with infinity) if T is a the monoid with infinity of some valuation $v : G \to \Gamma$.

A surjective valuation $v : G \to \Gamma$ is called *discrete* if there exists a (necessarily unique) isomorphism of the ordered group with infinity Γ onto $\mathbb{Z}_\infty$.
A domain with infinity T is called a *discrete valuation monoid with infinity* (DVM with infinity) if T is a the monoid with infinity of a discrete valuation $v : G \to \Gamma$.

Let $T \subseteq T_1$ be two monoids with infinity. An element $x \in T_1$ is said to be *integral* over T if there exists $n \in \mathbb{N}$ such that $nx \in T$. If all elements of T_1 are integral over T, we say that T_1 is *integral* over T. The set T' of all elements of T_1 that are integral over T is a submonoid with infinity of T_1.

Let T be a domain with infinity. We call the *integral closure of* T the integral closure T' of T in its difference group $D(T)$. We say that a monoid with infinity T is *integrally closed* if it is a domain with infinity and equal to its integral closure.

3 Domains with infinity

Definition 3.1 A domain with infinity T is called a *principal ideal domain with infinity* (PID with infinity) if every ideal of T is principal.

Examples 3.2 (1) $T = G$, a group with infinity. The ideals of T are (∞) and (0).
(2) $T = G[X]$, where G is a group with inifinity. The ideals of T are (nX), $n \in \mathbb{N}_0$. Note that when $G = \{0, \infty\}$, then $G[X] \cong \mathbb{N}_{0,\infty}$.

Proposition 3.3 ([10], Theorem 12) *A monoid with infinity T is a principal ideal domain with infinity if and only if $T = G$ or $T = G[X]$, where G is a group with infinity.*

Proposition 3.4 ([21], Proposition 81) *A monoid with infinity T is a principal ideal domain with infinity if and only if it is a discrete valuation monoid with infinity.*

Definition 3.5 A monoid domain with infinity T is called a *Dedekind domain with infinity* if every ideal of T is a finite sum of prime ideals.

Proposition 3.6 ([9], Theorem 4, or [10], Theorem 12) *A monoid with infinity T is a Dedekind domain with infinity if and only if it is a principal ideal domain with infinity.*

Proposition 3.7 ([18], Corollary 1.2) *A monoid with infinity T is a principal ideal domain with infinity if and only if every prime ideal of T is principal.*

Definition 3.8 Let T be a domain with infinity. If $x, y \in T$, we say that x *subtracts* y, and write $x \mid y$, if there exists an element $z \in T$ such that $x + z = y$. We then also say x is a *subtractor* or an *addend* of y. If x does not subtract y, we write $x \nmid y$. If $x \mid y$ and $y \mid x$, we say that x and y are associates and write $x \sim y$.

The relation $x \mid y$ is transitive and reflexive, but is not symmetric. An element u is a unit if and only if $u \mid 0$. The units are trivial addends since they are addends of every element, namely, $x = u + ((-u) + x)$. The relation $x \sim y$ is an equivalence relation. The element ∞ is associated only with itself, the set $U(T)$ is one of the equivalence classes. For $x, y \in T$ the following are equivalent: (a) $x \sim y$; (b) $x = y + u$, where u is a unit; (c) $(x) = (y)$. If $x \mid y$, but $y \nmid x$, we say that x is a *proper* addend of y. An addend of y is proper if and only if it is not an associate of y. The units of T do not have proper addends.

Definition 3.9 Let T be a domain with infinity. A non-infinity, non-unit element $x \in T$ is said to be *irreducible*, or an *atom*, if x has no proper addends other than units. Otherwise, we say that x is *reducible*.

A non-infinity, non-unit element $x \in T$ is irreducible if and only if $x = y + z$ implies that y or z is unit. x is reducible if and only if $x = y + z$, where each of y, z is a non-infinity, non-unit. If $x \sim y$, then x is irreducible if and only if y is irreducible.

A non-infinity, non-unit element x of a domain with infinity T is irreducible if and only if the ideal (x) is a maximal element of the set of all proper principal ideals of T, ordered by inclusion. (Indeed, suppose x is irreducible. Let $(x) \subseteq (y)$ for some non-infinity non-unit element y of T. Then $x = y + t$ for some $t \in T$. Since x is non-infinity, $t \neq \infty$. Since x is irreducible, t has to be a unit. Hence $(x) = (y)$. The converse is similar.)

Definition 3.10 A domain with infinity T is said to be *atomic* if every non-infinity, non-unit element of T can be written as finite sum of atoms.

Definition 3.11 A domain with infinity T is called an *ACCP domain with infinity* if every ascending chain $(x_1) \subseteq (x_2) \subseteq (x_3) \subseteq \dots$ of principal ideals of T stabilizes.

Proposition 3.12 ([16], page 143) *Every ACCP domain with infinity is atomic.*

Writing an element $x \in T$ as a sum $x = y_1 + \cdots + y_n$ is called an *addendization* of x. We say that two addendizations $x = y_1 + \cdots + y_m$ and $x = z_1 + \cdots + z_n$ of x into atoms are *essentially equal* if $m = n$ and $y_i = z_{\sigma(i)}$ for a suitable permutation σ of $\{1, \ldots, m\}$. We say that an addendization $x = x_1 + \cdots + x_n$ of x into atoms is *essentially unique* if any other addendization of x into atoms is essentially equal to this one.

Definition 3.13 Let T be a domain with infinity. We say T is a *unique addendization domain with infinity* (UAD with infinity) if every non-infinity, non-unit element x of T has one and essentially one addendization of x into atoms.

Examples 3.14 (1) The domain with infinity $T = \mathbb{N}_{0,\infty}$ is a UAD with infinity. The only atom in it is 1 and every non-infinity, non-unit element n has a unique addendization into atoms, namely $n = 1 + \cdots + 1$ (n addends). Note that this UAD is a PID with infinity.

(2) $T = \mathbb{N}_{0,\infty}[X]$ is a UAD with infinity. The only atoms are 1 and X and every non-infinity, non-unit element $m + nX$ has a unique addendization up to the order of addends, namely $m + nX = 1 + \cdots + 1 + X + \cdots + X$. Note that this UAD with infinity is not a PID with infinity as the maximal ideal $\mathfrak{m}_T$ is not principal.

Definition 3.15 Let T be a domain with infinity. A non-infinity, non-unit element $p \in T$ is said to be *prime* if $p \mid (x + y)$ implies $p \mid x$ or $p \mid y$.

In other words, p is a non-infinity, non-unit and the relations $p \nmid x$ and $p \nmid y$ imply $p \nmid (x + y)$.

Every prime element is an atom.

A non-infinity, non-unit element p of a domain with infinity T is prime if and only if the ideal (p) is prime.

Proposition 3.16 ([7], Theorem 5.1) *Every principal ideal domain with infinity is a unique addendization domain with infinity.*

Clearly every PID with infinity is Noetherian.

We will use the notation $\mathfrak{m}_T^{\omega} = \bigcap_{n\in\mathbb{N}} n\mathfrak{m}_T$.

Proposition 3.17 ([21], Proposition 66) *If T is a Noetherian domain with infinity, then* $\mathfrak{m}_T^{\omega} = (\infty)$.

The next notion, in the commutative ring version, was introduced by D. D. Anderson, D. F. Anderson and M. Zafrullah in [1].

Definition 3.18 A domain with infinity T is called a *bounded addendization domain with infinity* (BAD with infinity) if it is atomic and if for each non-infinity, non-unit element $x \in T$ there is a positive integer $N(x)$ such that whenever $x = x_1 + \cdots + x_n$ is an addendization of x into atoms, we have $n \leq N(x)$.

Proposition 3.19 *A domain with infinity T is a BAD with infinity if and only if* $\mathfrak{m}_T^{\omega} = (\infty)$.

Proof Assume that T is a BAD with infinity. We want to show that then $\mathfrak{m}_T^\omega = (\infty)$. Suppose to the contrary. Let $x \neq \infty$ be an element of $\mathfrak{m}_T^\omega$. Let $n > N(x)$. Since $x \in n\mathfrak{m}_T$, $x = x_1 + \cdots + x_n$ with each x_i from $\mathfrak{m}_T \setminus \{\infty\}$. Since T is atomic, each x_i can be written as $x_i = x_{i,1} + \cdots + x_{i,k_i}$, where each $x_{i,j}$ is an atom. Hence x can be written as a sum of more than $N(x)$ atoms, a contradiction.

Assume that $\mathfrak{m}_T^\omega = (\infty)$. We want to show that T is a BAD with infinity. Suppose to the contrary. Let $x \in \mathfrak{m}_T \setminus \{\infty\}$ be such that for any $k \in \mathbb{N}$ we can write $x = x_1 + \cdots + x_n$, where $n \geq k$ and each x_i is an atom. Hence $x \in n\mathfrak{m}_T$, whence $x \in \mathfrak{m}_T^\omega$, a contradiction. □

Proposition 3.20 *Every BAD with infinity is an ACCP domain with infinity.*

Proof Let T be a BAD with infinity. Suppose T is not ACCP. Then there is an increasing chain of non-infinity principal ideals $(x_1) \subset (x_2) \subset (x_3) \subset \ldots$ of T. Hence $x_1 = x_2 + y_2 = x_3 + y_3 + y_2 = x_4 + y_4 + y_3 + y_2 = \ldots$, where all x_i and all y_j are non-infinity, non-unit elements of T. Since each x_i and each y_j can be written as a sum of atoms, it follows that there is no bound $N(x_1)$ for the number of addends in the addendizations of x_1 into atoms, a contradiction. □

Clearly each UAD with infinity is a BAD with infinity.

Definition 3.21 A domain with infinity T is called a *Bézout domain with infinity* if every finitely generated ideal of T is principal.

Definition 3.22 A domain with infinity T is called a *Prüfer domain with infinity* if for every domain with infinity T_1 such that $T \subseteq T_1 \subseteq D(T)$ (the difference group with infinity of T), T_1 is integrally closed.

Proposition 3.23 ([10], Theorem 7) *Let T be a domain with infinity. T is Prüfer if and only if the set of all ideals of T is totally ordered by inclusion, if and only if the set of all principal ideals of T is totally ordered by inclusion.*

Proposition 3.24 *A domain with infinity T is Prüfer if and only if it is Bézout.*

Proof Suppose T is Prüfer. Then by Proposition 3.23, the set of principal ideals of T is totally ordered by inclusion. This implies that every two-generated ideal of T is principal. (Indeed, if $(a) \subseteq (b)$, then $(a, b) = (a) \cup (b) = (b)$. Similarly if $(b) \subseteq (a)$.) Hence T is Bézout. Conversely, suppose T is Bézout. Let (a), (b) be two principal ideals of T. Then $(a, b) = (c)$ for some $c \in T$. Hence $(c) = (a) \cup (b)$, so that $c \in (a)$ or $c \in (b)$. Say $c \in (a)$. Then $(a) \cup (b) = (c) \subseteq (a)$, hence $(a) \cup (b) = (a)$, whence $(b) \subseteq (a)$. By Proposition 3.23 this implies that T is Prüfer. □

Proposition 3.25 ([21], Proposition 52) *Let T be a domain with infinity. Then T is a valuation monoid with infinity if and only if it is Bézout.*

4 Two implication diagrams

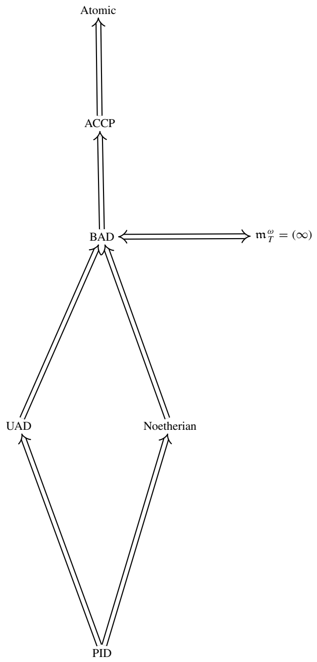
We now introduce a new type of domains with infinity that we have not met in the literature.

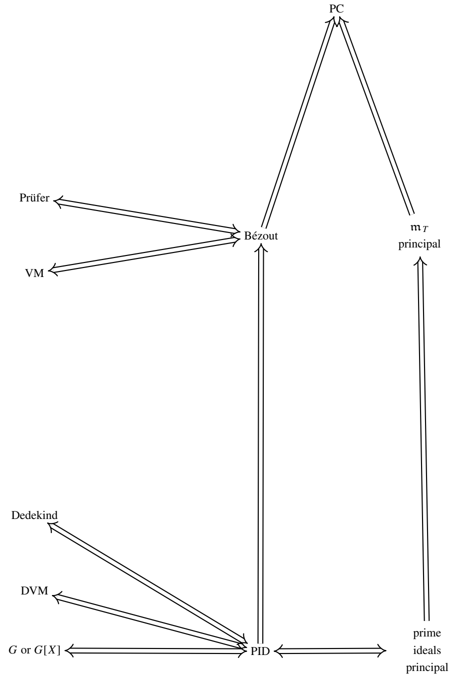
Definition 4.1 We say that a domain with infinity T is a *PC domain with infinity* if every proper finitely generated ideal of T is contained in a proper principal ideal of T.

In this definition PC stands for "principal containment."

Clearly if $\mathfrak{m}_T$ is principal or if T is Bézout, then T is a PC domain with infinity.

Consider the following two diagrams with implications and equivalences between various types of domains with infinity.





We will first justify these diagrams.

There are 7 equivalences in the diagrams, they are justified by Propositions 3.3, 3.4, 3.6, 3.7, 3.19, 3.24 and 3.25. There are 10 implications in the diagrams, all but four are obvious. Those four that are not obvious are justified by Propositions 3.12, 3.16, 3.17, and 3.20. It remains to show that all the implications are strict and that there are no other implications. This is done by the next two propositions.

Proposition 4.2 *(a) There exists an atomic domain with infinity which is neither an ACCP domain with infinity nor a PC domain with infinity.*

(b) There exists an ACCP domain with infinity which is not a BAD with infinity.

(c) There exists a BAD domain with infinity which is not Noetherian and is not a UAD with infinity.

(d) There exists a UAD with infinity which is not Noetherian.

(e) There exists a Noetherian domain with infinity which is not a UAD with infinity.

(f) There exists a Noetherian UAD with infinity, which is not a PID with infinity.

(g) There exists a domain with infinity which is neither atomic nor is a PC domain with infinity.

Proof (a) Let $T = \langle 1/3,\ 1/(2\cdot 5),\ \ldots, 1/(2^i \cdot p_i),\ \ldots\rangle_\infty$, a subdomain of $\mathbb{Q}_{+,\infty}$, where $p_0 = 3,\ \ p_1 = 5,\ \ p_2 = 7, \ldots$ is a sequence of prime numbers. (The monoid $T = \langle 1/3,\ 1/(2\cdot 5),\ \ldots, 1/(2^i \cdot p_i),\ \ldots\rangle$ was the starting point in [13, Section 1] for a construction of an atomic integral domain which is not an ACCP domain in commutative ring theory.) T is not an ACCP domain with infinity since the chain of principal ideals $(1/2) \subset (1/2^2) \subset (1/2^3) \subset \ldots$ does not stabilize. However, T is atomic. To prove that, it is enough to show that each $1/(2^k p_k)$ is an atom. Suppose to the contrary. Then $1/(2^k p_k) = n_1/3 + n_2/(2\cdot 5) + \cdots + n_t/(2^t p_t)$ ($t \geq 1, n_i \geq 0$). Here if $k \leq t$, then n_k must be 0. Since no fraction on the right hand side contains the prime number p_k in the denominator, the sum on the right hand side cannot have p_k in the denominator, so it cannot be equal to the left hand side, a contradiction.

Using similar reasoning one can see that no non-principal two-generated ideal of T is contained in any of the principal ideals $(1/(2^k p_k))$, $k \geq 0$, which implies that T is not a PC monoid with infinity.

(b) Let $T = \langle 1/2, 1/3, 1/5, \ldots\rangle_\infty$, a subdomain of $\mathbb{Q}_{+,\infty}$. We claim that the atoms of T are precisely the elements $1/2, 1/3, 1/5, \ldots$. Indeed, if $p, p_1, p_2, \ldots, p_t$ are different prime numbers, the relation $1/p = n_1/p_1 + \cdots + n_t/p_t$ cannot be satisfied for any $n_i \in \mathbb{N}_0$. So the elements $1/2, 1/3, 1/5, \ldots$ are atoms. As T is generated by these elements, T is atomic and no other element of T is an atom. Since $1 = p \cdot (1/p)$ for every prime number p, we see that T is not a BAD with infinity.

However, T is ACCP. (For the details about the reasoning that follows see either [1, Example 2.1] or [12, Section 2].) Suppose to the contrary. Then there is a strictly increasing chain $(x_1) \subset (x_2) \subset (x_3) \subset \ldots$ of principal ideals of T. Note that each element x of T can be written in a unique way as $x = n + n_1/p_1 + \cdots + n_t/p_t$, where $p_1 = 2$, $p_2 = 3$, ..., $n \geq 0$ is an integer, $t \geq 0$, $0 \leq n_i < p_i$ for each $i = 1, \ldots, t$, and $n_t \neq 0$. The number n is called the *integer part* of x and the sum $n_1/p_1 + \cdots + n_t/p_t$ the *fractional part* of x. Since $x_1 > x_2 > x_3 > \ldots$ and $x_i = x_{i+1} + y_{i+1}$ for some $y_{i+1} \in T$ $(i = 1, 2, 3, \ldots)$, the integer parts of the x_i are decreasing and one can conclude that only finitely many consecutive elements of the sequence $x_1, x_2, x_3, \ldots$ can have the same integer part. Once the integer part becomes 0, say in some x_r, then x_{r+1} has the same denominators as x_r in the fractional part, with the corresponding numerators smaller, at least one strictly. The same holds for x_{r+2} with respect to x_{r+1}, etc. Hence there can only be finitely many terms in the sequence $x_r > x_{r+1} > x_{r+2} > \ldots$ Thus in total we have only finitely many elements in the sequence $x_1, x_2, x_3, \ldots$, a contradiction.

(c) Let $T = (\{0\} \cup [1, \rightarrow))_\infty$, a submonoid with infinity of $\mathbb{R}_{+,\infty}$. (Here $[1, \rightarrow)$ denotes the set of all real numbers that are ≥ 1.) The set $[1, 2)$ is the set of all atoms of T. Since $\mathfrak{m}_T$ is not finitely generated, T is not Noetherian. Since, for example, $2.2 = 1.2 + 1$ and $2.2 = 1.1 + 1.1$, T is not a UAD with infinity. However, T is a BAD with infinity, as it is clearly atomic and, as each atom is ≥ 1, the number of atoms in any addendization of a non-infinity, non-unit $x \in T$ into atoms is $\leq \lfloor x \rfloor$. (The monoid $T = \{0\} \cup [1, \rightarrow)$ was used in several examples in [1] in the context of the commutative ring theory.)

(d) Let $T = G[X_1, X_2, X_3, \ldots]$, where G is a group with infinity. This is a polynomial monoid with infinity in the variables $X_1, X_2, X_3, \ldots$ over G. The elements of T are formal expressions $a + n_1 X_{i_1} + \cdots + n_t X_{i_t}$, where $a \in G$ and $n_i \in \mathbb{N}_0$. The addition is defined in the natural way. The atoms of T are all the elements of the form $a + X_i$, where $a \in G \setminus \{\infty\}$. It is easy to see that T is a UAD with infinity. It is not Noetherian as the maximal ideal $\mathfrak{m}_T = (X_1, X_2, X_3, \ldots)$ is not finitely generated.

(e) Let $T = \langle 2, 3 \rangle_\infty$, a submonoid with infinity of $\mathbb{N}_{0,\infty}$. is atomic. The atoms of T are 2 and 3. Every element can be written as a sum of atoms, however not in an essentially unique way. For example, $6 = 2 + 2 + 2 = 3 + 3$. So T is not a UAD with infinity. However, T is Noetherian as every ideal of T can be generated by at most two elements.

(f) Let $T = G[X, Y]$, where G is a group with infinity. This is a polynomial monoid with infinity in the variables X, Y over G. The elements of T are formal expressions $a + mX + nY$, where $a \in G$ and $m, n \in \mathbb{N}_0$. The addition is defined in the natural way. The atoms of T are all the elements of the form $a + X$ or $a + Y$, where $a \in G \setminus \{\infty\}$. It is easy to see that T is a UAD with infinity. T is Noetherian as the polynomial monoid in finitely many variables over a Noetherian monoid with infinity is a Noetherian monoid with infinity ([21, Theorem 59]). T is not a PID with infinity since the maximal ideal $\mathfrak{m}_T = (X, Y)$ is not principal.

(g) The domain with infinity $T = (\mathbb{R}^2_+)_\infty$ has no atoms, in particular, it is not atomic. If $a > 0$ and $b > 0$, the proper 2-generated ideal $((a, 0), (0, b))$ is not contained in any proper principal ideal, hence T is not a PC domain with infinity.

□

Proposition 4.3 *(a) There exists a Bézout domain with infinity whose maximal ideal $\mathfrak{m}_T$ is principal, but which is not a PID domain with infinity.*

(b) There exists a domain with infinity T whose maximal ideal $\mathfrak{m}_T$ is principal and which is not Bézout.

(c) There exists a Bézout domain with infinity in which the maximal ideal $\mathfrak{m}_T$ is not principal.

(d) There exists a PC domain with infinity which is not Bézout and in which the maximal ideal $\mathfrak{m}_T$ is not principal.

Proof (a) Let $T = \mathbb{N}_0 \cup (X + \mathbb{Z}_\infty[X])$, a subdomain with infinity of the polynomial domain $\mathbb{Z}_\infty[X]$, consisting of all the elements of $\mathbb{Z}_\infty[X]$ except the elements $k < 0$. (This is the same as the domain with infinity $(\mathbb{Z} \times \mathbb{N}_0)_\infty$ with the elements $(k, 0)$ $(k < 0)$ excluded.) We have $\mathfrak{m}_T = (1)$. Indeed, any element $k \geq 1$ can be obtained as $1 + (k - 1)$ and any element $k + lX$ with $l \geq 1$ and $k \in \mathbb{Z}$ can be obtained as $1 + (k - 1 + lX)$. Thus $\mathfrak{m}_T$ is principal.

The set $\mathfrak{p} = \{k + lX \mid l \geq 1, k \in \mathbb{Z}_\infty\}$ is a prime ideal of T as the sum of any two elements outside of $\mathfrak{p}$ is outside of $\mathfrak{p}$. The ideal $\mathfrak{p}$ is not principal. Indeed, any element $k + lX$ with $k \geq 2$ is not a generator of $\mathfrak{p}$ as $(k + lX) + (m + nX)$ has the coefficient of X greater than or equal to l, which is ≥ 2. Any element $k + X$ is not a generator of $\mathfrak{p}$ as $k - 1 + X$ cannot be obtained in the form $(k + X) + (m + nX)$ since n has to be 0, and then $m \geq 0$. Thus T is not a PID with infinity.

Finally, T is Bézout since a two generated ideal $(k_1 + l_1 X,\ k_2 + l_2 X)$ with either (a) $l_1 < l_2$, or (b) $l_1 = l_2$ and $k_1 < k_2$, is equal to the principal ideal $(k_1 + l_1 X)$. In the cases (c) $l_2 < l_1$ and (d) $l_1 = l_2$ and $k_2 < k_1$ the two generated ideal $(k_1 + l_1 X,\ k_2 + l_2 X)$ is equal to the principal ideal $(k_2 + l_2 X)$.

(b) Let $T = (\mathbb{N}_0 \times \{0\}) \cup (X + (\mathbb{Z} \times \mathbb{Z})_\infty[X])$, a subdomain of the polynomial domain $(\mathbb{Z} \times \mathbb{Z})_\infty[X]$. We have $\mathfrak{m}_T = ((1, 0))$. Indeed, any element $(k, 0)$, $k \geq 1$, can be obtained as $(1, 0) + (k - 1, 0)$. Also any element $(k, l) + mX$ with $m \geq 1$ and $(k, l) \in \mathbb{Z} \times \mathbb{Z}$ can be obtained as $(1, 0) + ((k-1, l) + mX)$. Thus m_T is principal. The two-generated ideal $I = ((1, 0) + X, (0, 1) + X)$ is not principal. Indeed, the elements in I with the coefficient by X equal to 1 are $(1, 0) + X, (2, 0) + X, (3, 0) + X, \ldots, (0, 1) + X, (1, 1) + X, (2, 1) + X, \ldots$ and none of them could be a single generator of I as any $(k, 0) + X$ could not generate the elements $(l, 1) + X$ and any $(l, 1) + X$ could not generate the elements $(k, 0) + X$. Also any $(k, l) + mX$ with $m \geq 2$ could not be a single generator of I as it could not generate the elements with the coefficient by X equal to 1. Thus T is not Bézout.

(c) Let $T = \mathbb{R}_{+,\infty}$. The ideals of T are all $[a, \rightarrow)$ and $(a, \rightarrow)$, where $a \in T$. (The set of all ideals is clearly totally ordered by inclusion.) Since $(a_1, \ldots, a_n) = (\inf\{a_1, \ldots, a_n\})$, T is a Bézout domain with infinity. The maximal ideal $\mathfrak{m}_T = (0, \rightarrow)$ is not finitely generated, in particular T is not a PID with infinity.

(d) Let $T = \mathbb{R}_+ \cup (\mathbb{R}_+ + X) \cup (2X + \mathbb{R}_\infty[X])$, a subdomain with infinity of the polynomial domain $\mathbb{R}_\infty[X]$. It is clear that its maximal ideal $\mathfrak{m}_T = T \setminus \{0\}$ is not finitely generated. Note that for any $r + kX \in T$ the ideal $(r + kX)$ is given by

$$
\begin{aligned}
(r + kX) = \{s + kX \mid s \in [r, \rightarrow)\} &\cup \{s + (k + 1)X \mid s \in [r, \rightarrow)\} \\
&\cup \{s + kX \mid s \in \mathbb{R}, l \geq k + 2\} \cup \{\infty\}.
\end{aligned}
$$

Consider a two-generated ideal $I = (r_1 + k_1 X, r_2 + k_2 X)$ with $k_1 \leq k_2$. If $k_1 = k_2$ and $r_1 < r_2$, then $I = (r_1 + k_1 X)$. Suppose $k_1 < k_2$. If $r_1 \leq r_2$, then $I = (r_1 + k_1 X)$. If $r_1 > r_2$ and $l_2 \geq k_1 + 2$, then $I = (r_1 + k_1 X)$. Finally, if $r_1 > r_2$ and $k_2 = k_1 + 1$, the ideal I is not principal. However, I is then contained in the principal ideal $J = (r_2 + k_1 X)$. Thus T is not Bézout, but is a PC domain with infinity. □

5 A new characterization of principal ideal domains with infinity

One can try to characterize PIDs with infinity by combining one condition (i.e., one type of the domains with infinity) from the left diagram with one condition from the right diagram. Several theorems of that type have appeared in the literature for both: PIDs in the theory of commutative monoids, as well as PIDs in the theory of commutative rings (with analogous conditions). They are given as Corollaries 5.2, 5.3, 5.4, 5.5, 5.6, 5.7, 5.8, 5.9, and 5.10 of our Proposition 5.1. This proposition improves all those statements by weakenning the conditions used in them.

Proposition 5.1 *A domain with infinity is a principal ideal domain with infinity if and only if it is an atomic PC domain with infinity.*

Proof Suppose T is a PID with infinity. Then $\mathfrak{m}_T$ is principal, hence T is a PC domain with infinity. Also, by Proposition 3.16, T is an UAD with infinity, hence T is atomic.

Conversely, suppose that T is an atomic PC domain with infinity. By Proposition 3.7, it is enough to prove that every prime ideal of T is principal. Let $\mathfrak{p} \neq (\infty)$ be a prime ideal of T and let $x \neq \infty$ be an element of $\mathfrak{p}$. Since T is atomic, the element x has an addendization $x = p_1 + \cdots + p_n$ into atoms. Since $\mathfrak{p}$ is prime, one of these atoms, say p_1, is in $\mathfrak{p}$. We claim that $\mathfrak{p} = (p_1)$. Let $y \in \mathfrak{p}$. Since T is a PC domain with infinity, we have $(p_1, y) \subseteq (c)$ for some proper principal ideal (c). From $p_1 \in (c)$ we get $p_1 = c + t$ for some $t \in T$. As p_1 is an atom and c is a non-unit, we conclude that t is a unit. Now from $y \in (c)$ we get $y = c + r = p_1 + (-t) + r$ for some $r \in T$, hence $y \in (p_1)$. Thus $\mathfrak{p} = (p_1)$. □

Now we give 9 corollaries, all of them follow immediately from the previous proposition and the two diagrams.

Corollary 5.2 ([4], Ch. VI, no. 3, Proposition 9 (commutative ring version)) *A domain with infinity T is a principal ideal domain with infinity if and only if T is a Noetherian valuation monoid with infinity.*

Corollary 5.3 ([11], Theorem 5) *A domain with infinity T is a principal ideal domain with infinity if and only if T is a Prüfer unique addendization domain with infinity.*

Corollary 5.4 ([3], Theorem 5) *A domain with infinity T is a principal ideal domain with infinity if and only if T is a Noetherian domain with infinity whose set of ideals is totally ordered by inclusion.*

Proof This corollary follows from Propositions 5.1 and 3.23, and the diagrams. □

Corollary 5.5 ([7], Proposition 1.2 (commutative ring version)) *A domain with infinity T is a principal ideal domain with infinity if and only if T is an ACCP Bézout domain with infinity.*

Corollary 5.6 *A domain with infinity T is a principal ideal domain with infinity if and only if T is an atomic Bézout domain with infinity.*

Corollary 5.7 ([19], Theorem 3.5) *A domain with infinity T is a principal ideal domain with infinity if and only if its maximal ideal $\mathfrak{m}_T$ is principal and T is Noetherian.*

Corollary 5.8 ([5], Corollary 1.2 (commutative ring version)) *A domain with infinity T is a principal ideal domain with infinity if and only if T is a unique addendization domain with infinity, whose maximal ideal $\mathfrak{m}_T$ is principal.*

Corollary 5.9 ([4], Ch. VI, no. 3, Proposition 9 (commutative ring version)) *A domain with infinity T is a principal ideal domain with infinity if and only if its maximal ideal $\mathfrak{m}_T$ is principal and $\mathfrak{m}_T^{\omega} = \{\infty\}$.*

A version of the previous corollary for not necessarily cancellative monoids is given in [20, Theorem 3].

Corollary 5.10 *A domain with infinity T is a principal ideal domain with infinity if and only if it is an atomic domain with infinity whose maximal ideal $\mathfrak{m}_T$ is principal.*

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