Semigroup Forum (2022) 105:745–755
https://doi.org/10.1007/s00233-022-10307-0

RESEARCH ARTICLE

On commutative elemental annihilator monoids

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Received: 26 February 2022 / Accepted: 22 July 2022 / Published online: 12 October 2022

Abstract
In this paper we describe commutative monoids S containing a zero element in which every ideal is the annihilator of an element of S.

Keywords Annihilator · Ideal · Monoid · Semigroup

1 Introduction and motivation

For a nonempty subset X of a semigroup with a zero element, let $\mathfrak{A}_l(X)$, $\mathfrak{A}_r(X)$, and $\mathfrak{A}(X)$ denote the left annihilator, the right annihilator, and the annihilator of X. A semigroup S with a zero element is called a *dual semigroup* if $\mathfrak{A}_l(\mathfrak{A}_r(L)) = L$ and $\mathfrak{A}_r(\mathfrak{A}_l(R)) = R$ are satisfied for every left ideal L and every right ideal R of S. The notion of a dual semigroup was introduced by S. Schwarz in [6] motivated by Baer's notion of a dual ring [1]. By Lemma 2, $\mathfrak{A}_l(\mathfrak{A}_r(L)) = L$ is satisfied for a left ideal L of a semigroup S containing a zero element if and only if L is the left annihilator of a nonempty subset of S. This result and its dual imply that a semigroup S with a zero element is a dual semigroup if and only if every left ideal of S is the left annihilator of a nonempty subset of S, and every right ideal of S is the right annihilator of a nonempty subset of S (Corollary 3). Motivated by Yohe's notion of a left (resp., right) elemental annihilator ring, we introduce the notion of a left (resp., right) elemental annihilator semigroup. We say that a semigroup S with a zero element is a left elemental annihilator semigroup if every left ideal of S is the left annihilator of a one-element subset of S. A right elemental annihilator semigroup is defined analogously. In [7], C.R. Yohe proved that a commutative ring with a unit element is an elemental annihilator ring if and only if it is a direct sum of completely primary principal ideal rings. This result motivates us to investigate commutative elemental annihilator monoids. As a main result of

Communicated by Victoria Gould.

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our paper we prove that the following three conditions on a nontrivial commutative monoid S containing a zero element are equivalent: (1) S is an elemental annihilator semigroup; (2) the unique maximal ideal M_S of S is a nilpotent semigroup, and the orbits of M_S under the action by the unit group $S^\times$ of S form a cyclic nilsemigroup; (3) the factor semigroup $S/\mathcal{J}$ is a cyclic nilsemigroup with an identity adjoined, where $\mathcal{J}$ is Green's equivalence on S defined by $(a, b) \in \mathcal{J}$ if and only if a and b generate the same principal ideal of S.

2 Preliminaries

Let S be a multiplicative semigroup with a zero element 0. By the *left annihilator* of a nonempty subset X of S we mean the subset $\mathfrak{A}_l(X) = \{s \in S : sX = \{0\}\}$. The *right annihilator* $\mathfrak{A}_r(X)$ of X is defined analogously. It is clear that $\mathfrak{A}_l(X)$ (resp., $\mathfrak{A}_r(X)$) is a left (resp., right) ideal of S. If S is commutative, then $\mathfrak{A}_l(X) = \mathfrak{A}_r(X)$ is called the *annihilator* of X and denoted by $\mathfrak{A}(X)$.

The proofs of the following two lemmas are straightforward and we omit them.

Lemma 1 *For an arbitrary nonempty subset X of a semigroup S containing a zero element, the following two equations hold:*

(1) $\mathfrak{A}_l(\mathfrak{A}_r(\mathfrak{A}_l(X))) = \mathfrak{A}_l(X)$,
(2) $\mathfrak{A}_r(\mathfrak{A}_l(\mathfrak{A}_r(X))) = \mathfrak{A}_r(X)$.

Lemma 2 *The following two conditions on a left ideal L of a semigroup S containing a zero element are equivalent:*

(1) $\mathfrak{A}_l(\mathfrak{A}_r(L)) = L$,
(2) *L is the left annihilator of a nonempty subset of S.*

Corollary 3 *A semigroup containing a zero element is a dual semigroup if and only if every left ideal of S is the left annihilator of a nonempty subset of S, and every right ideal of S is the right annihilator of a nonempty subset of S.*

Proof By Lemma 2 and its dual, the assertion of the corollary is obvious. □

Lemma 4 *For an arbitrary element x of a semigroup S containing a zero element,* $\mathfrak{A}_l(x) = \mathfrak{A}_l(x \cup xS)$ *and* $\mathfrak{A}_r(x) = \mathfrak{A}_r(x \cup Sx)$.

Proof Let x be an arbitrary element of a semigroup S containing a zero element 0. If $t \in \mathfrak{A}_l(x)$, then $tx = 0$, hence $txS = \{0\}$. Thus $t \in \mathfrak{A}_l(x \cup xS)$ from which it follows that $\mathfrak{A}_l(x) \subseteq \mathfrak{A}_l(x \cup xS)$. It is obvious that $\mathfrak{A}_l(x \cup xS) \subseteq \mathfrak{A}_l(x)$. Hence $\mathfrak{A}_l(x) = \mathfrak{A}_l(x \cup xS)$. The equation $\mathfrak{A}_r(x) = \mathfrak{A}_r(x \cup Sx)$ can be proved in a similar way. □

We say that an equivalence relation σ on a semigroup S is a *left congruence* on S if $(a, b) \in \sigma$ implies $(sa, sb) \in \sigma$ for every $a, b, s \in S$. The notion of the *right congruence* on S is defined analogously.

Let S be a semigroup containing a zero element. Let $\alpha_{\mathfrak{A}_l}$ denote the equivalence relation on S defined as follows: $(a, b) \in \alpha_{\mathfrak{A}_l}$ if and only if $\mathfrak{A}_l(a) = \mathfrak{A}_l(b)$. The equivalence relation $\alpha_{\mathfrak{A}_r}$ on S is defined analogously.

Proposition 5 *Let S be a semigroup with a zero element. Then $\alpha_{\mathfrak{A}_l}$ is a left congruence on S, and $\alpha_{\mathfrak{A}_r}$ is a right congruence on S.*

Proof Assume $(a, b) \in \alpha_{\mathfrak{A}_l}$ for elements $a, b \in S$. Let $s \in S$ and $t \in \mathfrak{A}_l(sa)$ be arbitrary elements. Then $ts \in \mathfrak{A}_l(a) = \mathfrak{A}_l(b)$, and hence $t \in \mathfrak{A}_l(sb)$. Thus $\mathfrak{A}_l(sa) \subseteq \mathfrak{A}_l(sb)$. Similarly, $\mathfrak{A}_l(sb) \subseteq \mathfrak{A}_l(sa)$, and consequently $\mathfrak{A}_l(sa) = \mathfrak{A}_l(sb)$, that is, $(sa, sb) \in \alpha_{\mathfrak{A}_l}$. Hence $\alpha_{\mathfrak{A}_l}$ is a left congruence on S. The proof that $\alpha_{\mathfrak{A}_r}$ is a right congruence on S is similar. □

Two elements of a semigroup S are said to be $\mathcal{L}$-*equivalent* if they generate the same principal left ideal of S. The $\mathcal{R}$-*equivalence* is defined dually. $\mathcal{L}$ is a right congruence and $\mathcal{R}$ is a left congruence on an arbitrary semigroup.

Proposition 6 *If S is a semigroup containing a zero element, then $\mathcal{R} \subseteq \alpha_{\mathfrak{A}_l}$ and $\mathcal{L} \subseteq \alpha_{\mathfrak{A}_r}$.*

Proof Let S be a semigroup containing a zero element. If $(a, b) \in \mathcal{R}$ for elements $a, b \in S$, then $a \cup aS = b \cup bS$. Using Lemma 4, we get $\mathfrak{A}_l(a) = \mathfrak{A}_l(a \cup aS) = \mathfrak{A}_l(b \cup bS) = \mathfrak{A}_l(b)$, that is, $(a, b) \in \alpha_{\mathfrak{A}_l}$. Thus $\mathcal{R} \subseteq \alpha_{\mathfrak{A}_l}$. The proof of $\mathcal{L} \subseteq \alpha_{\mathfrak{A}_r}$ is similar. □

Definition 1 A semigroup S with a zero element is called a *left elemental annihilator semigroup* if, for every left ideal L of S, there exists an element $x \in S$ such that $L = \mathfrak{A}_l(x)$. The *right elemental annihilator semigroup* is defined analogously.

For notions and notations not defined but used in this paper, we refer to the books [2] and [4].

3 Left elemental annihilator semigroups

Proposition 7 *If S is a left elemental annihilator semigroup and L is a left ideal of S, then we have $\mathfrak{A}_l(\mathfrak{A}_r(L)) = L$.*

Proof By Lemma 2, it is obvious. □

By Proposition 7 and its dual, we have the following corollary.

Corollary 8 *If a semigroup is a left elemental annihilator semigroup and a right elemental annihilator semigroup, then it is a dual semigroup.*

Lemma 9 *If a semigroup S is a left elemental annihilator semigroup and a right elemental annihilator semigroup, then $x \in Sx$ and $x \in xS$ for every $x \in S$.*

Proof See Corollary 8 and [6, Lemma 1.6]. □

The next proposition is about the relationship between the right congruence $\alpha_{\mathfrak{A}_r}$ and Green's right congruence $\mathcal{L}$ on a left elemental annihilator semigroup.

Proposition 10 *If S is a left elemental annihilator semigroup, then $\alpha_{\mathfrak{A}_r} = \mathcal{L}$.*

Proof Let S be a left elemental annihilator semigroup. Assume $(a, b) \in \alpha_{\mathfrak{A}_r}$ for elements $a, b \in S$. Then, by Lemma 4, $\mathfrak{A}_r(a \cup Sa) = \mathfrak{A}_r(a) = \mathfrak{A}_r(b) = \mathfrak{A}_r(b \cup Sb)$, and hence $\mathfrak{A}_l(\mathfrak{A}_r(a \cup Sa)) = \mathfrak{A}_l(\mathfrak{A}_r(b \cup Sb))$. By Proposition 7, we get $a \cup Sa = b \cup Sb$. Thus $(a, b) \in \mathcal{L}$, and consequently $\alpha_{\mathfrak{A}_r} \subseteq \mathcal{L}$. By Proposition 6, $\mathcal{L} \subseteq \alpha_{\mathfrak{A}_r}$ is satisfied in an arbitrary semigroup with a zero element. Thus we have $\alpha_{\mathfrak{A}_r} = \mathcal{L}$. □

For a congruence α on a semigroup S, let $[a]_\alpha$ denote the α-class of S containing the element a of S.

Proposition 11 *If α is a congruence on a left elemental annihilator semigroup S such that $[0]_\alpha = \{0\}$, then the factor semigroup S/α is a left elemental annihilator semigroup.*

Proof Let S be a left elemental annihilator semigroup. Assume that α is a congruence on S such that $[0]_\alpha = \{0\}$. Let L be a left ideal of the factor semigroup S/α. Then $L' = \{s \in S : [s]_\alpha \in L\}$ is a left ideal of S. Since S is a left elemental annihilator semigroup, there is an element $b \in S$ such that $\mathfrak{A}_l(b) = L'$. We show that $\mathfrak{A}_l([b]_\alpha) = L$ from which it already follows that S/α is a left elemental annihilator semigroup. If $[x]_\alpha \in L$, then $x \in L'$, and hence $[x]_\alpha[b]_\alpha = [xb]_\alpha = [0]_\alpha$. Thus $[x]_\alpha \in \mathfrak{A}_l([b]_\alpha)$, and consequently $L \subseteq \mathfrak{A}_l([b]_\alpha)$. To prove the converse inclusion, assume $[y]_\alpha \in \mathfrak{A}_l([b]_\alpha)$. Then $[yb]_\alpha = [y]_\alpha[b]_\alpha = [0]_\alpha$, and hence $yb = 0$ by hypothesis for $[0]_\alpha$. Thus $y \in \mathfrak{A}_l(b) = L'$ from which we get $[y]_\alpha \in L$. Hence $\mathfrak{A}_l([b]_\alpha) \subseteq L$. Consequently $\mathfrak{A}_l([b]_\alpha) = L$. □

Proposition 12 *Let α be a congruence on a semigroup S containing a zero element 0 such that $[0]_\alpha = \{0\}$ and the factor semigroup S/α is a left elemental annihilator semigroup. Then, for every left ideal L of S which is a union of α-classes of S, there is an element $b \in S$ such that $\mathfrak{A}_l(b) = L$.*

Proof Let L be a left ideal of S which is a union of α-classes of S. Let $L' = \{[s]_\alpha : s \in L\}$. Then L' is a left ideal of S/α. Since S/α is a left elemental annihilator semigroup, there is an element $[b]_\alpha \in S/\alpha$ such that $\mathfrak{A}_l([b]_\alpha) = L'$. We show that $\mathfrak{A}_l(b) = L$. Let $x \in \mathfrak{A}_l(b)$. Then $xb = 0$, and hence $[x]_\alpha[b]_\alpha = [xb]_\alpha = [0]_\alpha$ which means that $[x]_\alpha \in \mathfrak{A}_l([b]_\alpha) = L'$. Thus $x \in L$, and hence $\mathfrak{A}_l(b) \subseteq L$. To show the converse inclusion, assume that $y \in L$ be an arbitrary element. Then $[y]_\alpha \in L' = \mathfrak{A}_l([b]_\alpha)$ from which it follows that $[yb]_\alpha = [y]_\alpha[b]_\alpha = [0]_\alpha$. Thus $yb = 0$ by hypothesis for $[0]_\alpha$, that is, $y \in \mathfrak{A}_l(b)$, and hence $L \subseteq \mathfrak{A}_l(b)$. Consequently $L = \mathfrak{A}_l(b)$. □

Theorem 13 *Let S be a semigroup containing a zero element, and let α be a congruence on S such that every left ideal of S is the union of α-classes of S. Then S is a left elemental annihilator semigroup if and only if the factor semigroup S/α is a left elemental annihilator semigroup.*

Proof Since $[0]_\alpha = \{0\}$ by hypothesis, the assertion of the theorem is an immediate consequence of Proposition 11 and Proposition 12. □

By a *monoid* we mean a semigroup containing an identity element. Let S be a semigroup and let 1 be a symbol not representing any element of S. Extend the operation on S to $S \cup 1$ such that $11 = 1$ and $x1 = 1x = x$ for every $x \in S$. Then $S \cup 1$ is a monoid in which 1 is the identity element. We say that this semigroup is obtained from the semigroup S by the adjunction of an identity element 1 to S. If S is a semigroup, then S^1 denotes the following monoid: $S^1 = S$ if $|S| \geq 2$ and S has an identity element; $S^1 = S \cup 1$ otherwise. Recall that if S is a one-element semigroup, then S^1 is a two-element monoid.

Proposition 14 *Let S be a semigroup with a zero element 0 such that $S \neq S^1$. Then the semigroup S^1 is a left elemental annihilator semigroup if and only if every nonzero left ideal of S is the left annihilator of a nonzero element of S.*

Proof Assume that S^1 is a left elemental annihilator semigroup. Let $L \neq \{0\}$ be a left ideal of S. Then L is a left ideal of S^1, and hence there is an element $x \in S^1$ such that $\mathfrak{A}_l(x) = L$ in S^1. Since $\mathfrak{A}_l(1) = \{0\}$ in S^1, we have $x \in S$. Since $\mathfrak{A}_l(0) = S^1$ in S^1, we have $x \neq 0$ and so $\mathfrak{A}_l(x) = L$ is also satisfied in S. Thus every nonzero left ideal of S is the left annihilator of a nonzero element of S.

Conversely, assume that every nonzero left ideal of S is the left annihilator of a nonzero element of S. It is clear that, in S^1, $\mathfrak{A}_l(1) = \{0\}$ and $\mathfrak{A}_l(0) = S^1$. Let L be a left ideal of S^1 with $L \neq \{0\}$ and $L \neq S^1$. Then L is a nonzero left ideal of S and hence there is a nonzero element $x \in S$ such that $\mathfrak{A}_l(x) = L$ in S. It is obvious that $\mathfrak{A}_l(x) = L$ is also satisfied in S^1, because $x \neq 0$. Consequently S^1 is a left elemental annihilator semigroup. □

4 Commutative elemental annihilator semigroups

A commutative semigroup S with a zero element is called an *elemental annihilator semigroup* if every ideal of S is the annihilator of an element of S.

For an element a of a semigroup S, let $J(a)$ denote the principal ideal of S generated by a. It is known that $J(a) = a \cup aS \cup Sa \cup SaS$. If S is commutative, then $J(a) = a \cup aS$. From Lemma 9 it follows that if S is a commutative elemental annihilator semigroup then $J(a) = aS$.

A semigroup S is called a *principal ideal semigroup* if every ideal I of S is principal, that is, $I = J(a)$ for some $a \in S$.

Theorem 15 *Every commutative elemental annihilator semigroup is a principal ideal semigroup.*

Proof Let A be an arbitrary ideal of a commutative elemental annihilator semigroup S. Since $\mathfrak{A}(A)$ is an ideal of S, there is an element $x \in S$ such that $\mathfrak{A}(A) = \mathfrak{A}(x)$. By Lemmas 4 and 9, $\mathfrak{A}(x) = \mathfrak{A}(xS)$. Using Proposition 7, we get $A = \mathfrak{A}(\mathfrak{A}(A)) = \mathfrak{A}(\mathfrak{A}(x)) = \mathfrak{A}(\mathfrak{A}(xS)) = xS$, because A and xS are ideals of S. Thus A is a principal ideal. Consequently S is a principal ideal semigroup. □

Proposition 16 *Let S be a commutative elemental annihilator semigroup. Then, for arbitrary elements a and b of S, $\mathfrak{A}(b) = aS$ if and only if $\mathfrak{A}(a) = bS$.*

Proof Assume $\mathfrak{A}(b) = aS$ for elements $a, b \in S$. By Proposition 7, Lemmas 4, and 9, $\mathfrak{A}(a) = \mathfrak{A}(aS) = \mathfrak{A}(\mathfrak{A}(b)) = \mathfrak{A}(\mathfrak{A}(bS)) = bS$. This proves the proposition. □

If S is a commutative semigroup, then Green's equivalences $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$ are congruences on S, and $\mathcal{L} = \mathcal{R} = \mathcal{J}$.

Proposition 17 *If S is a commutative elemental annihilator semigroup, then $\mathcal{J} = \alpha_{\mathfrak{A}}$, where $\alpha_{\mathfrak{A}}$ is the congruence on S defined by $(a, b) \in \alpha_{\mathfrak{A}}$ if and only if $\mathfrak{A}(a) = \mathfrak{A}(b)$.*

Proof It is an immediate consequence of Proposition 10. □

5 Commutative elemental annihilator monoids

By Corollary 8, every commutative elemental annihilator monoid is a dual semigroup. Results related to dual monoids can be found in Chapter 7 of [6], however these results cannot be used in our investigation. For example, it is assumed in [6, Theorem 7.2] that the examined dual monoid S contains a nilpotent radical, and it is proved (using further conditions) that the nilpotent radical is the unique maximal ideal of S such that the complement of the nilpotent radical in S is a subgroup of S. In our study, we start from the fact that every nontrivial commutative monoid S containing a zero element is a disjoint union $S = M_S \cup S^{\times}$, where $S^{\times}$ is the unit group of S and M_S is the unique maximal ideal of S. We prove in Theorem 18 that if S is a nontrivial commutative elemental annihilator monoid, then the maximal ideal M_S is nilpotent. Using this result, we give a characterization of nontrivial commutative elemental annihilator monoids in Theorem 23.

An element a of a semigroup S with a zero element 0 is said to be *nilpotent* if there is a positive integer n such that $a^n = 0$. A semigroup containing a zero element is called a *nilsemigroup* if all its elements are nilpotent. We say that a semigroup S containing a zero element 0 is a *nilpotent semigroup* if there is a positive integer n such that $S^n = \{0\}$.

Theorem 18 *If S is a nontrivial commutative elemental annihilator monoid, then the unique maximal ideal M_S of S is a nilpotent semigroup.*

Proof Let S be a nontrivial commutative elemental annihilator monoid. Then S is a principal ideal semigroup by Theorem 15, and hence the ideals of S form a chain with respect to inclusion by [5, 1.1. Theorem]. Let 0 denote the zero element of S, and let a be an arbitrary element of M_S. We show that a is nilpotent. Let $g \in S^{\times}$ be an arbitrary element. Then obviously $J(g) = S$. Since $J(a) \subseteq M_S$, we have $(a, g) \notin \mathcal{J}$. By Proposition 17, $\mathcal{J} = \alpha_{\mathfrak{A}}$. Thus $(a, g) \notin \alpha_{\mathfrak{A}}$. Since $\mathfrak{A}(g) = \{0\}$, we have $\mathfrak{A}(a) \neq \{0\}$. Then, by [5, 1.5. Theorem], a is a nilpotent element. Consequently M_S is a nilsemigroup. As every ideal of S is a principal ideal, there exists an element $b \in M_S$ such that $M_S = J(b) = bS$. Since b is nilpotent, there is a positive integer k such that $b^k = 0$. Then, for arbitrary $x_1, \ldots, x_k \in M_S$, $x_1 \cdots x_k \in b^k S = \{0\}$, and hence $(M_S)^k = \{0\}$. Consequently M_S is a nilpotent semigroup. □

A semigroup S is called a *cyclic semigroup* if S is generated by a single element of S. A semigroup S with a zero element is called a *cyclic nilsemigroup* if S is generated by a single nilpotent element. A semigroup S is called a cyclic nilsemigroup with an identity adjoined if S is the result of adjoining an identity to a cyclic nilsemigroup, i.e., S has an identity 1 and $S \setminus \{1\}$ is a cyclic nilsemigroup.

Proposition 19 *Let N be a commutative nilsemigroup. Then every nonzero ideal of N is the annihilator of a nonzero element of N if and only if N is a cyclic nilsemigroup.*

Proof The assertion of the proposition is trivial in that case when N contains one element. Thus we can suppose that N is nontrivial.

Let N be a nontrivial cyclic nilsemigroup. We show that every nonzero ideal of N is the annihilator of an element of N. Let $N = \{b, b^2, \ldots, b^{k-1}, 0\}$, where $k \geq 2$ is the least integer with the property $b^k = 0$. The ideals of N are $\{0\}$ and $N^t = J(b^t) = \{b^t, \ldots, b^{k-1}, 0\}$ $(t = 1, \ldots, k-1)$. Since $N^t = \mathfrak{A}(b^{k-t})$ $(t = 1, \ldots, k-1)$, every nonzero ideal of N is the annihilator of a nonzero element of N.

To prove the converse assertion, assume that N is a nontrivial commutative nilsemigroup having the property that every nonzero ideal of N is the annihilator of a nonzero element of N. Since $N \neq N^1$, Proposition 14 implies that N^1 is a commutative elemental annihilator monoid. Then, by Theorem 15, N^1 is a principal ideal semigroup. If I is an ideal of N, then I is an ideal of N^1, and hence there is an element $x \in N$ such that $I = xN^1 = x \cup xN$. Thus I is a principal ideal of N. Hence N is a principal ideal nilsemigroup. Let b be an element of N such that $N = b \cup bN$. Since $N \neq \{0\}$, we have $b \neq 0$. Let k be the least positive integer with the property $b^k = 0$. Then $k \geq 2$. We show that $N = \{b, b^2, \ldots, b^{k-1}, 0\}$. Let $t \in \{1, 2, \ldots, k-1\}$ be an arbitrary integer. Then $b^t \neq 0$ and $N^t \neq \{0\}$. Since $N^t = (b \cup bN)^t \subseteq b^t \cup b^t N \subseteq N^t$, we have

$$N^t = b^t \cup b^t N.$$

If $b^t = b^t x$ for some $x \in N$, then $b^t = b^t x^n$ for every positive integer n, from which it follows that $b^t = 0$. This is a contradiction. Thus

$$b^t \notin b^t N.$$

Since $N^{t+1} = N^t N = (b^t \cup b^t N)N = b^t N \cup b^t N^2 \subseteq b^t N \subseteq N^{t+1}$, we have

$$b^t N = N^{t+1}.$$

Thus

$$\begin{aligned} N &= b \cup bN = b \cup N^2 = b \cup (b^2 \cup b^2 N) = \\ &= \{b, b^2\} \cup b^2 N = \{b, b^2\} \cup N^3 = \cdots = \{b, b^2, \ldots, b^{k-1}, 0\}. \end{aligned}$$

□

Proposition 20 *For a commutative nilsemigroup N, the monoid N^1 is an elemental annihilator semigroup if and only if N is a cyclic nilsemigroup.*

Proof If N is a commutative nilsemigroup, then $N \neq N^1$. Thus, by Proposition 14, N^1 is an elemental annihilator semigroup if and only if every nonzero ideal of N is the annihilator of a nonzero element of N. By Proposition 19, every nonzero ideal of N is the annihilator of a nonzero element of N if and only if N is a cyclic nilsemigroup. □

It is obvious that if S is a nontrivial commutative monoid with a zero element, then the binary relation α_{orb} on S defined by $(a, b) \in \alpha_{orb}$ if and only if $aS^\times = bS^\times$ is a congruence on S. The α_{orb}-classes of S are precisely the orbits of S under the action by the unit group $S^\times$. The subgroup $S^\times$ is a single orbit, and the remaining orbits form a subsemigroup (in S/α_{orb}) denoted by M_S/α_{orb}. Thus $S/\alpha_{orb} = (M_S/\alpha_{orb})^1$.

Lemma 21 *If S is a nontrivial commutative monoid with a zero element, then every ideal of S is a union of α_{orb}-classes of S.*

Proof Let S be a nontrivial commutative monoid with a zero element. Let K be an arbitrary ideal of S. Assume $a \in K$ and $(a, b) \in \alpha_{orb}$ for elements $a, b \in S$. Since the identity element of $S^\times$ is the identity element of S, we have $b \in bS^\times = aS^\times \subseteq K$. Thus K is a union of α_{orb}-classes of S. □

Proposition 22 *A nontrivial commutative monoid S with a zero element is an elemental annihilator semigroup if and only if the factor semigroup S/α_{orb} is an elemental annihilator semigroup.*

Proof By Theorem 13 and Lemma 21, it is obvious. □

In the next theorem, we characterize nontrivial commutative elemental annihilator monoids.

Theorem 23 *The following three conditions on a nontrivial commutative monoid S containing a zero element are equivalent.*

(1) *S is an elemental annihilator semigroup.*
(2) *The unique maximal ideal M_S of S is a nilpotent semigroup, and the orbits of M_S under the action by the unit group $S^\times$ of S form a cyclic nilsemigroup.*
(3) *The factor semigroup $S/\mathcal{J}$ is a cyclic nilsemigroup with an identity adjoined.*

Proof (1) implies (2): Assume that S is an elemental annihilator semigroup. By Proposition 22, S/α_{orb} is an elemental annihilator monoid. By Theorem 18, the unique maximal ideal M_S of S is a nilpotent semigroup. Then the semigroup M_S/α_{orb} is a commutative nilsemigroup. Since $S/\alpha_{orb} = (M_S/\alpha_{orb})^1$, M_S/α_{orb} is a cyclic nilsemigroup by Proposition 20.

(2) implies (3): Assume that the unique maximal ideal M_S of S is a nilpotent semigroup and the orbits of M_S under the action by the unit group $S^\times$ of S form a cyclic nilsemigroup. Then S is an elementary semigroup [3], and hence $\alpha_{orb} = \mathcal{H}$ in S by [3, Proposition 5.1]. Since S is commutative, $\mathcal{H} = \mathcal{J}$. Thus $S/\mathcal{J} = S/\alpha_{orb} = (M_S/\alpha_{orb})^1$ is a cyclic nilsemigroup with an identity adjoined.

(3) implies (1): Assume that the factor semigroup $S/\mathcal{J}$ is a cyclic nilsemigroup with an identity adjoined. Then, by Proposition 20, $S/\mathcal{J}$ is an elemental annihilator semigroup. As every ideal of S is the union of $\mathcal{J}$-classes of S, Theorem 13 implies that S is an elemental annihilator semigroup. □

Next, we give a construction that can be used to obtain a nontrivial commutative elemental annihilator monoid whose factor semigroup modulo α_{orb} is isomorphic to a given cyclic nilsemigroup with an identity adjoined.

If I is an ideal of a semigroup S, then the relation ϱ_I on S defined by $(a, b) \in \varrho_I$ if and only if $a = b$ or $a, b \in I$ is a congruence on S which is called the *Rees congruence on S determined by I*. The equivalence classes of S mod ϱ_I are I itself and every one-element set $\{a\}$ with $a \in S \setminus I$. The factor semigroup S/ϱ_I is called the *Rees factor semigroup of S modulo I*. We shall write S/I instead of S/ϱ_I. We may describe S/I as the result of collapsing I into a single (zero) element, while the elements of S outside of I retain their identity.

Let G be a group and H be a semigroup with a zero element 0. Then the direct product $G \times \{0\}$ is an ideal of the direct product $G \times H$. Let $G \triangle H$ denote the Rees factor semigroup $(G \times H)/(G \times \{0\})$.

Theorem 24 *Let G be a commutative group and N^1 be a cyclic nilsemigroup with an identity adjoined. Then $G \triangle N^1$ is a nontrivial commutative elemental annihilator monoid such that the factor semigroup $(G \triangle N^1)/\alpha_{orb}$ is isomorphic to N^1.*

Proof Let J be an ideal of the direct product $G \times N^1$. Let I be the set of all elements $x \in N^1$ with the property that $(g, x) \in J$ for some $g \in G$. Then $J \subseteq G \times I$. Let e denote the identity element of G. If $x \in I$, that is, $(g, x) \in J$ for some $g \in G$, then $(g, xy) = (g, x)(e, y) \in J$ for all $y \in N^1$, which implies that I is an ideal of N^1. It is clear that $G \times I$ is an ideal of $G \times N^1$. We show that $J = G \times I$. By the above inclusion, it is sufficient to show that $G \times I \subseteq J$. Let $(g, x) \in G \times I$ be an arbitrary element. Since $x \in I$, there is an element $h \in G$ such that $(h, x) \in J$. Since G is a group, there is an element $\xi \in G$ such that $g = h\xi$. Thus $(g, x) = (h\xi, x) = (h, x)(\xi, 1) \in J$, where 1 is the identity element of N^1. Consequently $G \times I \subseteq J$, and hence $J = G \times I$. Thus, for every ideal J of $G \times N^1$, there is an ideal I of N^1 such that $J = G \times I$. It is clear that $G \triangle N^1 = (G \times N^1)/(G \times \{0\})$ is a commutative monoid containing a zero element. Let $0_\triangle$ denote the zero of $G \triangle N^1$. Recall that $G \triangle N^1$ can be considered as the result of collapsing $G \times \{0\}$ into the element $0_\triangle$, while the elements of $G \times N^1$ outside of $G \times \{0\}$ retain their identity. We show that $G \triangle N^1$ is an elemental annihilator semigroup. Let J be an arbitrary ideal of $G \triangle N^1$. Since $\mathfrak{A}(0_\triangle) = G \triangle N^1$, we can suppose that $J \neq G \triangle N^1$. By the above, there is an ideal I of N^1 with $I \neq N^1$ such that $J = ((G \times I) \setminus (G \times \{0\}) \cup \{0_\triangle\}$. Since N^1 is an elemental annihilator semigroup, there is an element $x \in N^1$ such that $\mathfrak{A}(x) = I$ in N^1. Because of $I \neq N^1$, we get $x \neq 0$. We show that $\mathfrak{A}((g, x)) = J$ for an arbitrary $g \in G$. Since $x \neq 0$, we have $(g, x) \neq 0_\triangle$. Let (a, h) be an arbitrary nonzero element of $G \triangle N^1$. If $(a, h) \in J$, then $0 \neq h \in I$, and hence $(g, x)(a, h) = (ga, xh) = 0_\triangle$, because $xh = 0$ by $h \in \mathfrak{A}(x)$. If $(a, h) \notin J$, then $h \notin I$. Thus $(g, x)(a, h) = (ga, xh) \neq 0_\triangle$, because $xh \neq 0$ by $h \notin \mathfrak{A}(x)$. Consequently $\mathfrak{A}((g, x)) = J$. Thus $G \triangle N^1$ is a commutative elemental annihilator monoid. The unit group of $G \triangle N^1$ is $G \times \{1\}$. The orbits of $G \triangle N^1$ under the action by its unit group are $\{0_\triangle\}$ and the subsets $G \times \{h\}$ where h is an arbitray nonzero element of N^1. Let ϕ denote the mapping of N^1 onto $(G \triangle N^1)/\alpha_{orb}$ defined in the following way: $\varphi(h) = G \times \{h\}$ if $h \in N^1 \setminus \{0\}$; $\varphi(h) = 0_\triangle$ if $h = 0$. It is clear that ϕ is bijective. Let $x, y \in N^1$ be arbitrary elements. If one of them is 0, then one of $\phi(x)$ and $\phi(y)$ is $0_\triangle$, and hence $\phi(xy) = \phi(0) = 0_\triangle = \phi(x)\phi(y)$. If $xy = 0$ and

$0 \notin \{x, y\}$, then $\phi(xy) = \phi(0) = 0_\triangle = (G \times \{x\})(G \times \{y\}) = \phi(x)\phi(y)$. If $xy \neq 0$, then $\phi(xy) = G \times \{xy\} = (G \times \{x\})(G \times \{y\}) = \phi(x)\phi(y)$. Consequently ϕ is a homomorphism. Hence ϕ is an isomorphism of N^1 onto $(G\triangle N^1)/\alpha_{orb}$. □

Example 1 Let S be a semigroup, where $S = \{1, a, b, c, 0\}$, and the operation on S is defined by the following Cayley table:

	1	a	b	c	0
1	1	a	b	c	0
a	a	1	c	b	0
b	b	c	0	0	0
c	c	b	0	0	0
0	0	0	0	0	0

It is a matter of checking to see that S is isomorphic to the semigroup $G\triangle N^1$, where G is a two-element group and N is a two-element cyclic nilsemigroup. Then, by Theorem 24, S is a commutative elemental annihilator monoid. The unit group of S is $\{1, a\}$, and the α_{orb}-classes of S are $\{0\}$, $\{b, c\}$ and $\{1, a\}$. Thus $S/\alpha_{orb} \cong N^1$, as Theorem 24 states.

6 Appendix

If S is a commutative elemental annihilator semigroup, then every principal ideal of S is the annihilator of an element of S. The next example shows that the converse is not true, in general. We show an example of a commutative monoid S with a zero element in which every principal ideal is the annihilator of an element of S, but S is not an elemental annihilator semigroup.

Example 2 Let $A = \{1, a_1, a_2, \dots\}$ and $B = \{0, b_{-1}, b_{-2}, \dots\}$ be sets such that $A \cap B = \emptyset$. We define an operation on $S = A \cup B$ as follows:

$$1s = s1 = s \quad \text{and} \quad 0s = s0 = 0 \quad \text{for every } s \in S,$$
$$a_i a_j = a_{i+j} \quad \text{and} \quad b_{-i} b_{-j} = 0 \quad \text{for every positive integers } i \text{ and } j,$$
$$a_i b_{-j} = b_{-j} a_i = \begin{cases} 0, & \text{if } i - j \geq 0, \\ b_{i-j}, & \text{if } i - j < 0. \end{cases}$$

It is a matter of checking to see that this operation is associative. We present a case as an example. If $i - j + k \geq 0$, then

$$a_i b_{-j} = \begin{cases} 0, & \text{if } i - j \geq 0, \\ b_{i-j}, & \text{if } i - j < 0 \end{cases} \quad \text{and} \quad b_{-j} a_k = \begin{cases} 0, & \text{if } -j + k \geq 0, \\ b_{-j+k}, & \text{if } -j + k < 0. \end{cases}$$

Thus

$$(a_i b_{-j}) a_k = 0 = a_i (b_{-j} a_k).$$

If $i-j+k<0$, then $i-j<0$ and $-j+k<0$ from which we get

$$(a_i b_{-j})a_k = b_{i-j}a_k = b_{i-j+k} = a_i(b_{-j+k}) = a_i(b_{-j}a_k).$$

The operation is also commutative. 1 is the identity element and 0 is the zero element of S. Thus S is a commutative monoid with a zero element. It is easy to see that

$$1S = S = \mathfrak{A}(0), \quad 0S = \{0\} = \mathfrak{A}(1)$$

and, for every positive integer i,

$$a_i S = \{a_i, a_{i+1}, \dots, ; 0; b_{-1}, b_{-2}, \dots\} = \mathfrak{A}(b_{-i}),$$
$$b_{-i} S = \{b_{-i}, b_{-(i-1)}, \dots b_{-1}, 0\} = \mathfrak{A}(a_i).$$

Thus every principal ideal of S is the annihilator of an element of S. The ideal B of S is not the annihilator of an element of S. Hence S is not an elemental annihilator semigroup.

Funding Open access funding provided by Budapest University of Technology and Economics.

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