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RESEARCH ARTICLE

Well-posedness and stability results for some nonautonomous abstract linear hyperbolic equations with memory

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Abstract

In this paper, we study a class of second-order abstract linear hyperbolic equations with infinite memory that involve time-dependent unbounded linear operators. We obtain the well-posedness and stability of solutions to those nonautonomous second-order evolution equations under some appropriate assumptions. Our results generalize a number of previously known results in the autonomous case. Some specific examples are given to illustrate our abstract results, such as the nonautonomous Petrovsky type and wave equations.

Keywords Well-posedness · Asymptotic behavior · Self-adjoint · Infinite memory · Evolution families · Nonautonomous evolution equations · Nonautonomous petrovsky type equation · Nonautonomous wave equation

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1 Introduction

Fix a Hilbert space H once and for all, where $\langle\cdot,\cdot\rangle$, and $\|\cdot\|$ are the inner product and norm associated with it, respectively. We consider the following families of time-dependent linear operators, $A(t) : D(A(t)) \subset H \to H$ and $B(t) : D(B(t)) \subset H \to H$, where $D(A(t))$ and $D(B(t))$ are the domains of the linear operators $A(t)$ and $B(t)$ and let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a given function.

Consider the following class of nonautonomous second-order hyperbolic equations with infinite memory:

$$u_{tt}(t) + A(t)u(t) - \int_0^{\infty} g(s)B(t)u(t-s)ds = 0, \quad \forall t \in \mathbb{R}_+^* := (0,\infty), \tag{1.1}$$

equipped with the following initial conditions

$$\begin{cases} u(-t) = u_0(t), \ \forall t \in \mathbb{R}_+, \\ u_t(0) = u_1, \qquad \forall t \in \mathbb{R}_+, \end{cases} \tag{1.2}$$

where u_{tt} and u_t denote the second and first derivatives of u with respect to time t, the pair (u_0, u_1) is the initial data belonging to a suitable space, and $u : \mathbb{R}_+ \to H$ is the unknown of the system (1.1)–(1.2).

The main goal of this paper is to investigate the well-posedness and asymptotic stability of the solutions to the system (1.1)–(1.2) as time t approaches infinity, under some appropriate assumptions on the family of time-dependent linear operators $A(t)$ and $B(t)$, as well as the relaxation (kernel) function g.

In recent years, many mathematicians have been drawn to the problem of the well-posedness and stability (respectively, instability) of solutions for evolution equations with delay (respectively, memory), see, for example, [25–28]. Let us recall some works that are relevant to the issue under consideration in this paper. Indeed, a large literature exists in the case where the operators $A(t)$ and $B(t)$ are not time-dependent (autonomous case), addressing the issues of existence, uniqueness, and asymptotic behavior in time; see, for example, [1, 2], [9–15, 21, 24, 33, 34]. Depending on the growth of g at infinity, different decay estimates (exponential, polynomial, or others) have been obtained. Furthermore, in the case where the infinite memory is replaced with a finite one and $A(t)$ and $B(t)$ are not time-dependent, numerous papers on this topic are available in the literature, see, for example, [5, 6, 8, 18–20, 29–32, 36, 38–46], and the references therein. See, for example, [3, 4, 16, 17, 22, 23, 37], and the references therein in the autonomous case where a discrete or distributed time delay is added to (1.1).

In this paper, it goes back to investigating the nonautonomous case, that is, the case which involves time-dependent linear operators, $A(t)$ and $B(t)$. For more on time-dependent linear operators, evolution families, and evolution equations and their applications, we refer the reader to for instance [11, 25].

The following is the structure of the paper: in Sect. 2, we present our assumptions on $A(t)$, $B(t)$, and g, as well as state and prove the well-posedness of (1.1)–(1.2) (Theorem 2.1). Section 3 contains a statement and proof of the asymptotic stability of solutions to (1.1)–(1.2) under some additional assumptions on $A(t)$, $B(t)$, and g (Theorem 3.1). Finally, in Sects. 4 and 5, we present some examples as well as discuss some general remarks and open problems.

2 Well-posedness

In this section, we state our assumptions on $A(t)$, $B(t)$, and g, as well as establish the well-posedness of the system (1.1)–(1.2). In the sequel, this setting requires the following assumptions:

(A0) $A(t)$ and $B(t)$ are time-dependent positive definite self-adjoint linear operators that satisfy,

$$D(A(t)) = D(A(0)), \quad D(B(t)) = D(B(0)) \text{ for all } t \in \mathbb{R}_+, \tag{2.1}$$

and the embeddings

$$D(A(0)) \hookrightarrow D(B(0)) \hookrightarrow H$$

are dense and compact.

(A1) There exist two functions $a_1,\ b_1 : \mathbb{R}_+ \to \mathbb{R}_+^*$ of class C^1 and another continuous function $b_2 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+^*$ such that

$$b_1(t)\|w\|^2 \le \|B^{\frac{1}{2}}(t)w\|^2, \quad \forall w \in D(B^{\frac{1}{2}}(0)),\ \forall t \in \mathbb{R}_+, \tag{2.2}$$

$$\|B^{\frac{1}{2}}(t)w\|^2 \le a_1(t)\|A^{\frac{1}{2}}(t)w\|^2, \quad \forall w \in D(A^{\frac{1}{2}}(0)),\ \forall t \in \mathbb{R}_+ \tag{2.3}$$

and

$$\|B^{\frac{1}{2}}(t_1)w\|^2 \le b_2(t_1,t_2)\|B^{\frac{1}{2}}(t_2)w\|^2, \quad \forall w \in D(B^{\frac{1}{2}}(0)),\ \forall t_1, t_2 \in \mathbb{R}_+. \tag{2.4}$$

(A2) For any $t \in \mathbb{R}_+$, there exist two time-dependent linear operators on H

$$\tilde{A}(t) :\ D(A(0)) \to H \quad \text{and} \quad \tilde{B}(t) :\ D(B(0)) \to H \tag{2.5}$$

satisfying

$$\lim_{\tau\to t}\left[\left\|\left(\frac{A(\tau)-A(t)}{\tau-t} - \tilde{A}(t)\right)w_1\right\| + \left\|\left(\frac{B(\tau)-B(t)}{\tau-t} - \tilde{B}(t)\right)w_2\right\|\right] = 0 \tag{2.6}$$

for all $(w_1, w_2) \in D(A(0)) \times D(B(0))$.

(A3) The non-increasing class C^1 relaxation (kernel) function $g : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies

$$g_0 := \int_0^\infty g(s)ds < \frac{1}{a_1(t)}, \quad \forall t \in \mathbb{R}_+, \tag{2.7}$$

and there exists a positive constant θ_1 such that

$$-g'(s) \leq \theta_1 g(s), \quad \forall s \in \mathbb{R}_+. \tag{2.8}$$

Remark 1 Let us recall that the assumptions **(A0)–(A3)** hold for a wide range of linear operators $A(t)$ and $B(t)$, as well as the relaxation function g. Indeed, consider $\Omega \subset \mathbb{R}^N$ to be an open bounded domain with smooth boundary $\Gamma = \partial\Omega$ where $N \in \mathbb{N}^*$, and consider $H = L^2(\Omega)$ to be endowed with its standard inner product:

$$\langle f, h \rangle = \int_\Omega f(x)h(x)dx$$

for all $f, h \in L^2(\Omega)$.

Consider the case when $A(t)$ and $B(t)$ and g are given by

$$\begin{aligned} A(t) = -a(t)\Delta,\ B(t) = -b(t)\Delta,\ D(A(t)) = D(B(t)) \\ = H^2(\Omega) \cap H_0^1(\Omega) \text{ and } g(s) = \theta_0 e^{-\theta_1 s}, \end{aligned}$$

where $a,\ b : \mathbb{R}_+ \to \mathbb{R}_+^*$ are of class C^1 and $\theta_0,\ \theta_1 \in \mathbb{R}_+^*$ such that

$$\frac{\theta_0}{\theta_1} < \frac{a(t)}{b(t)}, \quad \forall t \in \mathbb{R}_+^*. \tag{2.9}$$

To make this work, we carefully choose $\tilde{A}(t)$ and $\tilde{B}(t)$ as follows:

$$\tilde{A}(t) = -a'(t)\Delta \text{ and } \tilde{B}(t) = -b'(t)\Delta,$$

where $a_1(t) = \frac{b(t)}{a(t)}$, $b_1(t) = c_0 b(t)$, $b_2(t_1, t_2) = \frac{b(t_1)}{b(t_2)}$, and c_0 is the Poincaré constant.

Under the previous assumptions and following a method derived from [10], we consider a new auxiliary variable η with its initial data η_0 defined by

$$\begin{cases} \eta(t, s) = u(t) - u(t - s), & \forall t,\ s \in \mathbb{R}_+, \\ \eta_0(s) = \eta(0, s) = u_0(0) - u_0(s), & \forall s \in \mathbb{R}_+, \end{cases} \tag{2.10}$$

and formulate (1.1)–(1.2) as a first-order nonautonomous evolution equation given by

$$\begin{cases} \mathcal{U}_t(t) = \mathcal{A}(t)\mathcal{U}(t), & \forall t \in \mathbb{R}_+^*, \\ \mathcal{U}(0) = \mathcal{U}_0, \end{cases} \tag{2.11}$$

where $\mathcal{U} = (u, u_t, \eta)^T$, $\mathcal{U}_0 = (u_0(0), u_1(0), \eta_0)^T \in \mathcal{H}(t)$,

$$\begin{aligned} &\mathcal{H}(t) = D(A^{\frac{1}{2}}(t)) \times H \times L_g(t), \\ &L_g(t) = \left\{ w : \mathbb{R}_+ \to D(B^{\frac{1}{2}}(t)), \quad \int_0^\infty g(s)\|B^{\frac{1}{2}}(t)w(s)\|^2 ds < \infty \right\}, \end{aligned}$$

and the time-dependent linear operators $\mathcal{A}(t)$ are given by

$$\mathcal{A}(t)\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} w_2 \\ \left(-A(t) + g_0 B(t)\right) w_1 - \displaystyle\int_0^{\infty} g(s) B(t) w_3(s) ds \\ -\dfrac{\partial w_3}{\partial s} + w_2 \end{pmatrix} \tag{2.12}$$

for all $\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathcal{D}(\mathcal{A}(t))$ where,

$$\mathcal{D}(\mathcal{A}(t)) = \left\{ \begin{array}{l} (w_1, w_2, w_3)^T \in D(A(t)) \times D(A^{\frac{1}{2}}(t)) \times L_g(t), \\ \frac{\partial w_3}{\partial s} \in L_g(t),\ w_3(0) = 0,\ w_3(s) \in D(B(t)),\ \forall s \in \mathbb{R}_+ \end{array} \right\}. \tag{2.13}$$

Based upon (2.1) and (2.4), the spaces $\mathcal{H}(t)$ and $L_g(t)$ do not depend on t, that is,

$$\mathcal{H}(t) = \mathcal{H}(0) \quad \text{and} \quad L_g(t) = L_g(0), \quad \forall t \in \mathbb{R}_+. \tag{2.14}$$

The space $L_g(t)$ is endowed with the classical inner product

$$\langle w_1, w_2 \rangle_{L_g(t)} = \int_0^{\infty} g(s) \left\langle B^{\frac{1}{2}}(t) w_1(s), B^{\frac{1}{2}}(t) w_2(s) \right\rangle ds.$$

Based upon (2.1) and (2.14), we have

$$\mathcal{D}(\mathcal{A}(t)) = \mathcal{D}(\mathcal{A}(0)), \quad \forall t \in \mathbb{R}_+. \tag{2.15}$$

On the other hand, keeping in mind the definition of η in (2.10), we have

$$\begin{cases} \eta_t(t,s) + \eta_s(t,s) = u_t(t), \ \forall t,\ s \in \mathbb{R}_+, \\ \eta_s(t,s) = u_t(t-s), \qquad\quad \forall t,\ s \in \mathbb{R}_+, \\ \eta(t,0) = 0, \qquad\qquad\qquad \forall t \in \mathbb{R}_+. \end{cases} \tag{2.16}$$

Therefore, we conclude from (2.12) and (2.16) that the systems (1.1)–(1.2) and (2.11) are equivalent.

Using (2.3) and (2.7), we conclude that $\mathcal{H}(t)$ endowed with the inner product

$$\begin{aligned} \left\langle (w_1, w_2, w_3)^T, (\tilde{w}_1, \tilde{w}_2, \tilde{w}_3)^T \right\rangle_{\mathcal{H}(t)} &= \left\langle A^{\frac{1}{2}}(t) w_1, A^{\frac{1}{2}}(t) \tilde{w}_1 \right\rangle - g_0 \left\langle B^{\frac{1}{2}}(t) w_1, B^{\frac{1}{2}}(t) \tilde{w}_1 \right\rangle \\ &\quad + \langle w_2, \tilde{w}_2 \rangle + \langle w_3, \tilde{w}_3 \rangle_{L_g(t)} \end{aligned}$$

is a Hilbert space with the following embedding $\mathcal{D}(\mathcal{A}(t)) \hookrightarrow \mathcal{H}(t)$ being dense (see, for example, [33]).

The following theorem ensures the well-posedness of (2.11):

Theorem 2.1 *Under assumptions* **(A0)-(A3)**, *for any* $\mathcal{U}_0 \in \mathcal{H}(0)$, *the system* (2.11) *has a unique (weak) solution*

$$\mathcal{U} \in C(\mathbb{R}_+, \mathcal{H}(0)). \tag{2.17}$$

Moreover, if $\mathcal{U}_0 \in \mathcal{D}(\mathcal{A}(0))$, *then the solution to* (2.11) *is a classical solution, that is,*

$$\mathcal{U} \in C^1(\mathbb{R}_+, \mathcal{H}(0)) \cap C(\mathbb{R}_+, \mathcal{D}(\mathcal{A}(0))). \tag{2.18}$$

Proof To prove Theorem 2.1, we make use of the semigroup theory approach. The proof is divided into three main steps.

Step 1. The first step consists of showing that the linear operators $\mathcal{A}(t)$ are dissipative for all $t \in \mathbb{R}_+$. Indeed, as in [22], letting $W = (w_1, w_2, w_3)^T \in \mathcal{D}(\mathcal{A}(t))$, we obtain,

$$\begin{aligned}\langle \mathcal{A}(t)W, W\rangle_{\mathcal{H}(t)} &= \left\langle A^{\frac{1}{2}}(t)w_2, A^{\frac{1}{2}}(t)w_1\right\rangle - g_0\left\langle B^{\frac{1}{2}}(t)w_2, B^{\frac{1}{2}}(t)w_1\right\rangle - \left\langle \frac{\partial w_3}{\partial s}, w_3\right\rangle_{L_g(t)}\\ &+ \langle w_2, w_3\rangle_{L_g(t)}\\ &+\left\langle \left(-A(t) + g_0 B(t)\right) w_1 - \int_0^\infty g(s)B(t)w_3(s)ds, w_2\right\rangle.\end{aligned} \tag{2.19}$$

It is clear from the definitions of $A^{\frac{1}{2}}(t)$ and $B^{\frac{1}{2}}(t)$, and the fact that H is a real Hilbert space, that

$$\langle (-A(t) + g_0 B(t))\, w_1, w_2\rangle = -\left\langle A^{\frac{1}{2}}(t)w_2, A^{\frac{1}{2}}(t)w_1\right\rangle + g_0\left\langle B^{\frac{1}{2}}(t)w_2, B^{\frac{1}{2}}(t)w_1\right\rangle$$

and

$$\begin{aligned}\left\langle -\int_0^\infty g(s)B(t)w_3(s)ds, w_2\right\rangle &= -\int_0^\infty g(s)\left\langle B^{\frac{1}{2}}(t)w_3(s), B^{\frac{1}{2}}(t)w_2\right\rangle ds\\ &= -\langle w_2, w_3\rangle_{L_g(t)}.\end{aligned}$$

On the other hand, using (**A3**), we see that

$$\lim_{s\to\infty} g(s)B^{\frac{1}{2}}(t)w_3(s) = 0.$$

Next, integrating by parts with respect to s and using the property $w_3(0) = 0$ (definition of $\mathcal{D}(\mathcal{A}(t))$), we deduce that

$$-\left\langle \frac{\partial w_3}{\partial s}, w_3\right\rangle_{L_g(t)} = \frac{1}{2}\int_0^\infty g'(s)\|B^{\frac{1}{2}}(t)w_3(s)\|^2 ds.$$

Consequently, inserting these three formulas in the previous identity (2.19), we get

$$\langle \mathcal{A}(t)W, W\rangle_{\mathcal{H}(t)} = \frac{1}{2}\int_0^\infty g'(s)\|B^{\frac{1}{2}}(t)w_3(s)\|^2 ds \le 0, \tag{2.20}$$

as g is non increasing, which yields $\mathcal{A}(t)$ is dissipative.

Using (2.8) and the fact that g is non-increasing and $w_3 \in L_g(t)$, we have

$$\begin{aligned}\left|\int_0^\infty g'(s)\|B^{\frac{1}{2}}(t)w_3(s)\|^2 ds\right| &= -\int_0^\infty g'(s)\|B^{\frac{1}{2}}(t)w_3(s)\|^2 ds\\ &\leq \theta_1 \int_0^\infty g(s)\|B^{\frac{1}{2}}(t)w_3(s)\|^2 ds\\ &< \infty\end{aligned}$$

and so the integral in the right hand side of (2.20) is well defined.

Step 2. In this step, we prove that $I - \mathcal{A}(t)$ is surjective for all $t \in \mathbb{R}_+$, where I stands for the identity operator. Indeed, let $F = (f_1, f_2, f_3)^T \in \mathcal{H}(t)$, we show that there exists

$$W = (w_1, w_2, w_3)^T \in \mathcal{D}(\mathcal{A}(t))$$

satisfying

$$(I - \mathcal{A}(t))W = F, \tag{2.21}$$

which is equivalent to

$$\begin{cases} w_2 = w_1 - f_1, \\ (A(t) - g_0 B(t) + I)\, w_1 + \displaystyle\int_0^\infty g(s)B(t)w_3(s)ds = f_1 + f_2, \\ w_3 + \dfrac{\partial w_3}{\partial s} = w_1 + f_3 - f_1. \end{cases} \tag{2.22}$$

We note that the third Eq. in (2.22) with $w_3(0) = 0$ has the unique solution

$$w_3(s) = \left(1 - e^{-s}\right) w_1 + e^{-s}\int_0^s e^y(f_3(y) - f_1)dy. \tag{2.23}$$

Next, plugging (2.23) into the second Eq. in (2.22), we get

$$(A(t) - g_1 B(t) + I)\, w_1 = \tilde{f}(t), \tag{2.24}$$

where

$$g_1 = \int_0^\infty g(s)e^{-s}ds$$

and

$$\tilde{f}(t) = f_1 + f_2 - \int_0^\infty g(s)e^{-s}\left(\int_0^s e^y B(t)(f_3(y) - f_1)dy\right)ds.$$

To complete this step, we need to prove that (2.24) has a solution $w_1 \in D(A^{\frac{1}{2}}(t))$. Then, substituting w_1 in (2.23) and the first Eq. in (2.22), we obtain $W \in \mathcal{D}(\mathcal{A}(t))$ satisfying (2.21). Since $g_1 < g_0$, then $A(t) - g_1 B(t)$ is a positive definite operator thanks to (2.3) and (2.7). Therefore, $A(t) - g_1 B(t) + I$ is a self-adjoint linear positive definite operator. Applying the Lax-Milgram Theorem and classical regularity arguments, we conclude that (2.24) has a unique solution $w_1 \in D(A^{\frac{1}{2}}(t))$ satisfying, using (2.23),

$$(A(t) - g_0 B(t))w_1 + \int_0^{\infty} g(s)B(t)w_3(s)ds \in H.$$

This proves that $I - \mathcal{A}(t)$ is surjective. We note that (2.20) and (2.21) mean that, for any $t \in \mathbb{R}_+$, $-\mathcal{A}(t)$ is a maximal monotone operator. Hence, using Lummer-Phillips Theorem (see [35]), we deduce that $\mathcal{A}(t)$ is an infinitesimal generator of a C_0-semigroup of contraction on $\mathcal{H}(t)$.

Step 3. Condition (2.6) yields the applications $h_1,\ h_2 : \mathbb{R}_+ \to H$ given by

$$h_1(t) = A(t)w_1 \quad \text{and} \quad h_2(t) = B(t)w_2$$

are differentiable and their derivatives are, respectively,

$$\tilde{h}_1(t) = \tilde{A}(t)w_1 \quad \text{and} \quad \tilde{h}_2(t) = \tilde{B}(t)w_2.$$

Now, let

$$W = (w_1, w_2, w_3)^T \in D(\mathcal{A}(0))$$

and $h : \mathbb{R}_+ \to \mathcal{H}(0)$ defined by $h(t) = \mathcal{A}(t)W$.

We prove in this step that h is differentiable and that its derivative is the function

$$\tilde{h}(t) = \begin{pmatrix} 0 \\ \left(-\tilde{A}(t) + g_0\tilde{B}(t)\right) w_1 - \displaystyle\int_0^{\infty} g(s)\tilde{B}(t)w_3(s)ds \\ 0 \end{pmatrix}.$$

Notice that, given (2.1), (2.5) and (2.13), we have $\tilde{h}(t) \in \mathcal{H}(0)$, for any $t \in \mathbb{R}_+$. On the other hand, we have, for any $\tau, t \in \mathbb{R}_+$ with $\tau \neq t$,

$$\frac{h(\tau) - h(t)}{\tau - t} = \frac{1}{\tau - t}\begin{pmatrix} 0 \\ w \\ 0 \end{pmatrix},$$

where

$$w = -(A(\tau) - A(t))w_1 + g_0(B(\tau) - B(t))w_1 - \int_0^{\infty} g(s)(B(\tau) - B(t))w_3(s)ds.$$

Then

$$\frac{h(\tau) - h(t)}{\tau - t} - \tilde{h}(t) = \begin{pmatrix} 0 \\ \tilde{w} \\ 0 \end{pmatrix},$$

where

$$\tilde{w} = -\left(\frac{A(\tau) - A(t)}{\tau - t} - \tilde{A}(t)\right) w_1 + g_0 \left(\frac{B(\tau) - B(t)}{\tau - t} - \tilde{B}(t)\right) w_1 \\ - \int_0^\infty g(s) \left(\frac{B(\tau) - B(t)}{\tau - t} - \tilde{B}(t)\right) w_3(s) ds,$$

which yields

$$\begin{aligned} \left\| \frac{h(\tau) - h(t)}{\tau - t} - \tilde{h}(t) \right\|_{\mathcal{H}(t)} &= \|\tilde{w}\| \\ &\leq \left\| \left(\frac{A(\tau) - A(t)}{\tau - t} - \tilde{A}(t)\right) w_1 \right\| \\ &\quad + g_0 \left\| \left(\frac{B(\tau) - B(t)}{\tau - t} - \tilde{B}(t)\right) w_1 \right\| \\ &\quad + \int_0^\infty g(s) \left\| \left(\frac{B(\tau) - B(t)}{\tau - t} - \tilde{B}(t)\right) w_3(s) \right\| ds, \end{aligned}$$

so we get from (2.6) that

$$\lim_{\tau \to t} \left\| \frac{h(\tau) - h(t)}{\tau - t} - \tilde{h}(t) \right\|_{\mathcal{H}(t)} = 0.$$

Based upon the properties shown in the previous steps, we conclude that $\mathcal{A}(\cdot)$ generates a unique evolution family on $\mathcal{H}(0)$ (see [35]). Consequently, (2.11) is well-posed in the sense of Theorem 2.1. □

3 Asymptotic stability

In this section, we look at the asymptotic behavior of solutions to (2.11). For that, we assume the following additional conditions are met:

(A4) There exist three continuous functions, a_2, $\tilde{a}$, $\tilde{b} : \mathbb{R}_+ \to \mathbb{R}_+$ that satisfy the following conditions,

$$\|A^{\frac{1}{2}}(t)w\|^2 \leq a_2(t)\|B^{\frac{1}{2}}(t)w\|^2, \quad \forall w \in D(A^{\frac{1}{2}}(0)), \ \forall t \in \mathbb{R}_+, \tag{3.1}$$

$$\|\tilde{A}^{\frac{1}{2}}(t)w\|^2 \leq \tilde{a}(t)\|A^{\frac{1}{2}}(t)w\|^2, \quad \forall w \in D(A^{\frac{1}{2}}(0)), \ \forall t \in \mathbb{R}_+, \tag{3.2}$$

and

$$\|\tilde{B}^{\frac{1}{2}}(t)w\|^2 \leq \tilde{b}(t)\|B^{\frac{1}{2}}(t)w\|^2, \quad \forall w \in D(B^{\frac{1}{2}}(0)), \ \forall t \in \mathbb{R}_+. \tag{3.3}$$

(A5) The kernel g satisfies $g_0 > 0$ and there exists a positive constant θ_2 such that

$$g'(s) \leq -\theta_2 g(s), \quad \forall s \in \mathbb{R}_+, \tag{3.4}$$

$$\sqrt{\tilde{b}(t)} < \frac{\theta_2}{2} \quad \text{and} \quad \left\| g_0 a_1 \sqrt{\tilde{b}} + \sqrt{\tilde{a}} \right\|_{L^\infty(\mathbb{R}_+)} \quad \text{is small enough.} \tag{3.5}$$

Remark 2 Consider the example given in Remark 1. Observe that, for

$$a_2(t) = \frac{a(t)}{b(t)}, \quad \tilde{a}(t) = \frac{|a'(t)|}{a(t)} \quad \text{and} \quad \tilde{b}(t) = \frac{|b'(t)|}{b(t)}, \quad \forall t \in \mathbb{R}_+^*, \tag{3.6}$$

such that (3.5) holds, the assumptions **(A4)** and **(A5)** are also fulfilled with $\theta_2 = \theta_1$. In the autonomous case, we have $\tilde{a} = \tilde{b} = 0$, and then (3.5) is trivial.

Remark 3 In the sequel, we will make extensive use of Young's inequality, which is stated as follows: let $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+^*$ and $1 < p, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\alpha\beta \leq \varepsilon(t)\alpha^p + (p\varepsilon(t))^{-\frac{p}{q}} q^{-1} \beta^q, \quad \forall t, \alpha, \beta \in \mathbb{R}_+. \tag{3.7}$$

When $p = q = 2$, we get the special case

$$\alpha\beta \leq \varepsilon(t)\alpha^2 + \frac{1}{4\varepsilon(t)}\beta^2, \quad \forall t, \alpha, \beta \in \mathbb{R}_+. \tag{3.8}$$

Theorem 3.1 *Assume that* **(A0)-(A5)** *hold. Then, for any* $\mathcal{U}_0 \in \mathcal{H}(0)$*, there exists a positive constant* λ *such that the solution to* (2.11) *satisfies*

$$\|\mathcal{U}(t)\|_{\mathcal{H}(0)}^2 \leq \frac{\lambda e^{\tilde{\xi}(t)}}{M(t) - M_2(t)}, \quad \forall t \in \mathbb{R}_+, \tag{3.9}$$

where the functions $M(\cdot)$, $M_2(\cdot)$ *and* $\tilde{\xi}(\cdot)$ *are defined in the proof (see* (3.43), (3.44), (3.45), (3.48) *and* (3.50) *below).*

Proof Let us assume that **(A0)-(A5)** hold and let $\mathcal{U}_0 \in \mathcal{H}(0)$. The energy functional E associated with the solution of (2.11) corresponding to $\mathcal{U}_0$ is given by

$$\begin{aligned} E(t) &= \frac{1}{2} \|\mathcal{U}(t)\|_{\mathcal{H}(0)}^2 \\ &= \frac{1}{2} \left(\left\| A^{\frac{1}{2}}(t)u(t) \right\|^2 + \|u_t(t)\|^2 - g_0 \left\| B^{\frac{1}{2}}(t)u(t) \right\|^2 \right. \\ &\quad \left. + \int_0^\infty g(s) \left\| B^{\frac{1}{2}}(t)\eta(t,s) \right\|^2 ds \right). \end{aligned} \tag{3.10}$$

In order to complete the proof of Theorem 3.1, we need the next lemmas, where throughout the proofs, c, c_1, c_2, $\cdots$, stand for some positive generic constants which do not depend upon t, and c can be different from a given line to another.

Lemma 3.2 *The energy functional $E(\cdot)$ satisfies the estimate*

$$E'(t) = \frac{1}{2}\int_0^\infty g'(s)\left\|B^{\frac{1}{2}}(t)\eta(t,s)\right\|^2 ds - g_0\left\langle \tilde{B}^{\frac{1}{2}}(t)u(t), B^{\frac{1}{2}}(t)u(t)\right\rangle + \left\langle \tilde{A}^{\frac{1}{2}}(t)u(t), A^{\frac{1}{2}}(t)u(t)\right\rangle + \int_0^\infty g(s)\left\langle \tilde{B}^{\frac{1}{2}}(t)\eta(t,s), B^{\frac{1}{2}}(t)\eta(t,s)\right\rangle ds. \tag{3.11}$$

Proof Multiplying (1.1) by u_t and integrating by parts, one gets,

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2 + \left\langle A^{\frac{1}{2}}(t)u(t), A^{\frac{1}{2}}(t)u_t(t)\right\rangle - \left\langle \int_0^\infty g(s)B^{\frac{1}{2}}(t)u(t-s)ds, B^{\frac{1}{2}}(t)u_t(t)\right\rangle = 0. \tag{3.12}$$

Now

$$\frac{1}{2}\frac{d}{dt}\left\|A^{\frac{1}{2}}(t)u(t)\right\|^2 = \left\langle \tilde{A}^{\frac{1}{2}}(t)u(t), A^{\frac{1}{2}}(t)u(t)\right\rangle + \left\langle A^{\frac{1}{2}}(t)u(t), A^{\frac{1}{2}}(t)u_t(t)\right\rangle. \tag{3.13}$$

A similar result can be obtained for $B(\cdot)$, that is, using the first Eq. in (2.10), one obtains,

$$\begin{aligned}
&\left\langle \int_0^\infty g(s)B^{\frac{1}{2}}(t)u(t-s)ds, B^{\frac{1}{2}}(t)u_t(t)\right\rangle \\
&= \left\langle \int_0^\infty g(s)B^{\frac{1}{2}}(t)\left[u(t)-\eta(t,s)\right]ds, B^{\frac{1}{2}}(t)u_t(t)\right\rangle \\
&= -\left\langle \int_0^\infty g(s)B^{\frac{1}{2}}(t)\eta(t,s)ds, B^{\frac{1}{2}}(t)u_t(t)\right\rangle + g_0\left\langle B^{\frac{1}{2}}(t)u(t), B^{\frac{1}{2}}(t)u_t(t)\right\rangle \\
&= -\left\langle \int_0^\infty g(s)B^{\frac{1}{2}}(t)\eta(t,s)ds, B^{\frac{1}{2}}(t)u_t(t)\right\rangle + \frac{g_0}{2}\frac{d}{dt}\left\|B^{\frac{1}{2}}(t)u(t)\right\|^2 \\
&\quad -g_0\left\langle \tilde{B}^{\frac{1}{2}}(t)u(t), B^{\frac{1}{2}}(t)u(t)\right\rangle.
\end{aligned} \tag{3.14}$$

Using the first Eq. in (2.16), we obtain

$$\begin{aligned}
&\frac{1}{2}\int_0^\infty g(s)\frac{\partial}{\partial s}\left\|B^{\frac{1}{2}}(t)\eta(t,s)\right\|^2 ds + \frac{1}{2}\int_0^\infty g(s)\frac{\partial}{\partial t}\left\|B^{\frac{1}{2}}(t)\eta(t,s)\right\|^2 ds \\
&= \int_0^\infty g(s)\left\langle B^{\frac{1}{2}}(t)\frac{\partial}{\partial s}\eta(t,s), B^{\frac{1}{2}}(t)\eta(t,s)\right\rangle ds \\
&\quad + \int_0^\infty g(s)\left\langle \left[\tilde{B}^{\frac{1}{2}}(t)\eta(t,s) + B^{\frac{1}{2}}(t)\frac{\partial}{\partial t}\eta(t,s)\right], B^{\frac{1}{2}}(t)\eta(t,s)\right\rangle ds \\
&= \int_0^\infty g(s)\left\langle B^{\frac{1}{2}}(t)u_t(t), B^{\frac{1}{2}}(t)\eta(t,s)\right\rangle ds \\
&\quad + \int_0^\infty g(s)\left\langle \tilde{B}^{\frac{1}{2}}(t)\eta(t,s), B^{\frac{1}{2}}(t)\eta(t,s)\right\rangle ds.
\end{aligned} \tag{3.15}$$

Substituting (3.13)-(3.15) into (3.12) yields

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2 + \frac{1}{2}\frac{d}{dt}\left\|A^{\frac{1}{2}}(t)u(t)\right\|^2 - \left\langle \tilde{A}^{\frac{1}{2}}(t)u(t), A^{\frac{1}{2}}(t)u(t)\right\rangle - \frac{g_0}{2}\frac{d}{dt}\left\|B^{\frac{1}{2}}(t)u(t)\right\|^2$$
$$+\frac{1}{2}\int_0^\infty g(s)\frac{\partial}{\partial s}\left\|B^{\frac{1}{2}}(t)\eta(t,s)\right\|^2 ds + \frac{1}{2}\int_0^\infty g(s)\frac{\partial}{\partial t}\left\|B^{\frac{1}{2}}(t)\eta(t,s)\right\|^2 ds$$
$$-\int_0^\infty g(s)\left\langle \tilde{B}^{\frac{1}{2}}(t)\eta(t,s), B^{\frac{1}{2}}(t)\eta(t,s)\right\rangle ds + g_0\left\langle \tilde{B}^{\frac{1}{2}}(t)u(t), B^{\frac{1}{2}}(t)u(t)\right\rangle = 0. \tag{3.16}$$

Integrating by parts with respect to s and using the properties

$$\eta(t,0) = 0 \quad \text{and} \quad \lim_{s\to\infty} g(s) = 0,$$

the formula in (3.16) becomes,

$$\frac{1}{2}\frac{d}{dt}\left[\|u_t\|^2 + \left\|A^{\frac{1}{2}}(t)u(t)\right\|^2 + \int_0^\infty g(s)\left\|B^{\frac{1}{2}}(t)\eta(t,s)\right\|^2 ds - g_0\left\|B^{\frac{1}{2}}(t)u(t)\right\|^2\right]$$
$$-\frac{1}{2}\int_0^\infty g'(s)\left\|B^{\frac{1}{2}}(t)\eta(t,s)\right\|^2 ds - \int_0^\infty g(s)\left\langle \tilde{B}^{\frac{1}{2}}(t)\eta(t,s), B^{\frac{1}{2}}(t)\eta(t,s)\right\rangle ds$$
$$+g_0\left\langle \tilde{B}^{\frac{1}{2}}(t)u(t), B^{\frac{1}{2}}(t)u(t)\right\rangle - \left\langle \tilde{A}^{\frac{1}{2}}(t)u(t), A^{\frac{1}{2}}(t)u(t)\right\rangle = 0, \tag{3.17}$$

and the result follows. □

Lemma 3.3 *There exists a positive constant c_1 such that the functional*

$$I_1(t) = \langle u(t), u_t(t)\rangle \tag{3.18}$$

satisfies, for any continuous function $\varepsilon_1 : \mathbb{R}_+ \to \mathbb{R}_+^*$,

$$I_1'(t) \le \|u_t(t)\|^2 - (1 - g_0 a_1(t) - \varepsilon_1(t)a_1(t))\left\|A^{\frac{1}{2}}(t)u(t)\right\|^2$$
$$+\frac{c_1}{\varepsilon_1(t)}\int_0^\infty g(s)\left\|B^{\frac{1}{2}}(t)\eta(t,s)\right\|^2 ds. \tag{3.19}$$

Proof Differentiating I_1 with respect to t and using (1.1), we get

$$I_1'(t) = \|u_t(t)\|^2 - \left\|A^{\frac{1}{2}}(t)u(t)\right\|^2 + \left\langle \int_0^\infty g(s)B^{\frac{1}{2}}(t)u(t-s)ds, B^{\frac{1}{2}}(t)u(t)\right\rangle. \tag{3.20}$$

Using the same computations as those in (3.14) and then (2.3), one gets,

$$
\begin{aligned}
I_1'(t) &= \|u_t(t)\|^2 - \left\| A^{\frac{1}{2}}(t)u(t) \right\|^2 + g_0 \left\| B^{\frac{1}{2}}(t)u(t) \right\|^2 \\
&\quad - \left\langle \int_0^\infty g(s) B^{\frac{1}{2}}(t)\eta(t,s)ds,\, B^{\frac{1}{2}}(t)u(t) \right\rangle \\
&\leq \|u_t(t)\|^2 - (1 - g_0 a_1(t)) \left\| A^{\frac{1}{2}}(t)u(t) \right\|^2 \\
&\quad - \left\langle \int_0^\infty g(s) B^{\frac{1}{2}}(t)\eta(t,s)ds,\, B^{\frac{1}{2}}(t)u(t) \right\rangle. \qquad (3.21)
\end{aligned}
$$

Applying the Cauchy–Schwarz inequality, Young's inequality on the last term of this inequality and (2.3) yields (3.19). □

Lemma 3.4 *There exists a positive constant c_2 such that the functional*

$$
I_2(t) = \left\langle -u_t(t), \int_0^\infty g(s)\eta(t,s)ds \right\rangle \qquad (3.22)
$$

satisfies, for any continuous functions $\varepsilon_2,\ \varepsilon_3 : \mathbb{R}_+ \to \mathbb{R}_+^*$,

$$
\begin{aligned}
I_2'(t) &\leq -(g_0 - \varepsilon_2(t)) \|u_t(t)\|^2 + \varepsilon_3(t)(1 + a_1(t)) \left\| A^{\frac{1}{2}}(t)u(t) \right\|^2 \\
&\quad + \left[g_0 + c_2 \left(\frac{1 + a_2(t)}{\varepsilon_3(t)} + \frac{1}{\varepsilon_2(t) b_1(t)} \right) \right] \int_0^\infty g(s) \left\| B^{\frac{1}{2}}(t)\eta(t,s) \right\|^2 ds.
\end{aligned} \qquad (3.23)
$$

Proof Differentiating with respect to t and exploiting Eq. (1.1) gives

$$
\begin{aligned}
I_2'(t) &= \left\langle A^{\frac{1}{2}}(t)u(t), \int_0^\infty g(s) A^{\frac{1}{2}}(t)\eta(t,s)ds \right\rangle - \left\langle u_t(t), \int_0^\infty g(s)\eta_t(t,s)ds \right\rangle \\
&\quad - \left\langle \int_0^\infty g(s) B^{\frac{1}{2}}(t)u(t-s)ds, \int_0^{+\infty} g(s) B^{\frac{1}{2}}(t)\eta(t,s)ds \right\rangle. \qquad (3.24)
\end{aligned}
$$

Again from the first Eq. in (2.10) and in (2.16) we have, as for in (3.17),

$$
-\left\langle u_t(t), \int_0^\infty g(s)\eta_t(t,s)ds \right\rangle = -\left\langle u_t(t), \int_0^\infty g'(s)\eta(t,s)ds \right\rangle - g_0\|u_t(t)\|^2 \qquad (3.25)
$$

and

$$-\left\langle\int_0^\infty g(s)B^{\frac{1}{2}}(t)u(t-s)ds, \int_0^\infty g(s)B^{\frac{1}{2}}(t)\eta(t,s)ds\right\rangle$$
$$=\left\langle\int_0^\infty g(s)B^{\frac{1}{2}}(t)\eta(t,s)ds, \int_0^\infty g(s)B^{\frac{1}{2}}(t)\eta(t,s)ds\right\rangle$$
$$-g_0\left\langle B^{\frac{1}{2}}(t)u(t), \int_0^\infty g(s)B^{\frac{1}{2}}(t)\eta(t,s)ds\right\rangle. \tag{3.26}$$

Now, from (3.25) and (3.26), Eq. (3.24) becomes,

$$\begin{aligned} I_2'(t) = -g_0\|u_t(t)\|^2 &- \left\langle u_t(t), \int_0^\infty g'(s)\eta(t,s)\,ds\right\rangle \\ &+\left\langle A^{\frac{1}{2}}(t)u(t), \int_0^\infty g(s)A^{\frac{1}{2}}(t)\eta(t,s)\,ds\right\rangle \\ &-g_0\left\langle B^{\frac{1}{2}}(t)u(t), \int_0^\infty g(s)B^{\frac{1}{2}}(t)\eta(t,s)\,ds\right\rangle \\ &+\left\|\int_0^\infty g(s)B^{\frac{1}{2}}(t)\eta(t,s)\,ds\right\|^2. \end{aligned} \tag{3.27}$$

Using Cauchy–Schwarz inequality, Young's inequality, **(A1)**, **(A3)** and **(A4)** on the last four terms yields, for the second term,

$$-\left\langle u_t(t), \int_0^\infty g'(s)\eta(t,s)\,ds\right\rangle$$
$$\le \varepsilon_2(t)\|u_t(t)\|^2 + \frac{c}{\varepsilon_2(t)b_1(t)}\int_0^\infty g(s)\left\|B^{\frac{1}{2}}(t)\eta(t,s)\right\|^2 ds, \tag{3.28}$$

the third term

$$\left\langle A^{\frac{1}{2}}(t)u(t), \int_0^\infty g(s)A^{\frac{1}{2}}(t)\eta(t,s)\,ds\right\rangle \le \varepsilon_3(t)\|A^{\frac{1}{2}}(t)u(t)\|^2$$
$$+\frac{ca_2(t)}{\varepsilon_3(t)}\int_0^\infty g(s)\|B^{\frac{1}{2}}(t)\eta(t,s)\|^2\,ds, \tag{3.29}$$

the fourth term

$$-g_0\left\langle B^{\frac{1}{2}}(t)u(t), \int_0^\infty g(s)B^{\frac{1}{2}}(t)\eta(t,s)\,ds\right\rangle \le \varepsilon_3(t)a_1(t)\|A^{\frac{1}{2}}(t)u(t)\|^2$$
$$+\frac{c}{\varepsilon_3(t)}\int_0^\infty g(s)\|B^{\frac{1}{2}}(t)\eta(t,s)\|^2\,ds, \tag{3.30}$$

and the fifth term

$$\left\| \int_0^\infty g(s) B^{\frac{1}{2}}(t)\eta(t,s)\, ds \right\|^2 \leq \left(\int_0^\infty g(s) \left\| B^{\frac{1}{2}}(t)\eta(t,s) \right\| ds \right)^2$$
$$\leq \left(\int_0^\infty \sqrt{g(s)}\sqrt{g(s)} \left\| B^{\frac{1}{2}}(t)\eta(t,s) \right\| ds \right)^2$$
$$\leq \left(\int_0^\infty g(s)\, ds \right) \left(\int_0^\infty g(s) \left\| B^{\frac{1}{2}}(t)\eta(t,s) \right\|^2 ds \right)$$
$$\leq g_0 \int_0^\infty g(s) \left\| B^{\frac{1}{2}}(t)\eta(t,s) \right\|^2 ds. \tag{3.31}$$

Combining all the above estimates yields (3.23). □

Lemma 3.5 *Let* $M_1 \in \mathbb{R}_+^*$, $M : \mathbb{R}_+ \to \mathbb{R}_+^*$ *be a differentiable function and let* $\varepsilon_1, \varepsilon_2 : \mathbb{R}_+ \to \mathbb{R}_+^*$ *be given continuous functions. Then the functional*

$$F(t) = M_1 I_1(t) + I_2(t) + M(t)E(t), \tag{3.32}$$

satisfies

$$F'(t) \leq M'(t)E(t)$$
$$- \min\{A_1(t), A_2(t), A_3(t)\} \left(\|u_t(t)\|^2 + \left\| A^{\frac{1}{2}}(t)u(t) \right\|^2 \right.$$
$$\left. + \int_0^\infty g(s) \left\| B^{\frac{1}{2}}(t)\eta(t,s) \right\|^2 ds \right), \tag{3.33}$$

where

$$A_1(t) = g_0 - \varepsilon_2(t) - M_1,$$
$$A_2(t) = (1 - g_0 a_1(t) - \varepsilon_1(t)a_1(t))\, M_1 - \varepsilon_3(t)(1 + a_1(t))$$
$$- \left(g_0 \sqrt{\tilde{b}(t)} a_1(t) + \sqrt{\tilde{a}(t)} \right) M(t)$$

and

$$A_3(t) = \left(\frac{\theta_2}{2} - \sqrt{\tilde{b}(t)} \right) M(t) - \frac{c_1 M_1}{\varepsilon_1(t)} - \left[g_0 + c_2 \left(\frac{1 + a_2(t)}{\varepsilon_3(t)} + \frac{1}{\varepsilon_2(t) b_1(t)} \right) \right].$$

Proof Direct differentiation gives

$$F'(t) = M_1 I_1'(t) + I_2'(t) + M'(t)E(t) + M(t)E'(t). \tag{3.34}$$

We can also estimate every term of $E'(t)$ given in (3.11), using Cauchy–Schwarz inequality and Young's inequality with the help of (2.3) and (3.2)-(3.4) to get, for the first term of $E'(t)$,

$$\frac{1}{2}\int_0^\infty g'(s)\left\|B^{\frac{1}{2}}(t)\eta(t,s)\right\|^2 ds \le -\frac{\theta_2}{2}\int_0^\infty g(s)\left\|B^{\frac{1}{2}}(t)\eta(t,s)\right\|^2 ds,$$

the second term of $E'(t)$

$$\left|g_0\left\langle \tilde{B}^{\frac{1}{2}}(t)u(t), B^{\frac{1}{2}}(t)u(t)\right\rangle\right| \le \ g_0\sqrt{\tilde{b}(t)}\left\|B^{\frac{1}{2}}(t)u(t)\right\|^2 \le g_0\sqrt{\tilde{b}(t)}a_1(t)\left\|A^{\frac{1}{2}}(t)u(t)\right\|^2,$$

the third term of $E'(t)$

$$\left\langle \tilde{A}^{\frac{1}{2}}(t)u(t), A^{\frac{1}{2}}(t)u(t)\right\rangle \le \sqrt{\tilde{a}(t)}\left\|A^{\frac{1}{2}}(t)u(t)\right\|^2,$$

and the fourth term of $E'(t)$

$$\left\langle\int_0^\infty g(s)\tilde{B}^{\frac{1}{2}}(t)\eta(t,s)ds, B^{\frac{1}{2}}(t)\eta(t,s)\right\rangle \le \sqrt{\tilde{b}(t)}\int_0^\infty g(s)\left\|B^{\frac{1}{2}}(t)\eta(t,s)\right\|^2 ds.$$

Now, $E'(t)$ can be estimated as follows:

$$\begin{aligned} E'(t) \le & -\frac{\theta_2}{2}\int_0^\infty g(s)\left\|B^{\frac{1}{2}}(t)\eta(t,s)\right\|^2 ds && (3.35)\\ & + \left(g_0\sqrt{\tilde{b}(t)}a_1(t) + \sqrt{\tilde{a}(t)}\right)\left\|A^{\frac{1}{2}}(t)u(t)\right\|^2 \\ & + \sqrt{\tilde{b}(t)}\int_0^\infty g(s)\left\|B^{\frac{1}{2}}(t)\eta(t,s)\right\|^2 ds && (3.36)\end{aligned}$$

Combining (3.11), (3.19), (3.23) and (3.35) leads to

$$\begin{aligned} F'(t) \le & -A_1(t)\|u_t(t)\|^2 - A_2(t)\left\|A^{\frac{1}{2}}(t)u(t)\right\|^2 \\ & -A_3(t)\int_0^\infty g(s)\left\|B^{\frac{1}{2}}(t)\eta(t,s)\right\|^2 ds + M'(t)E(t), && (3.37)\end{aligned}$$

so, (3.33) follows. □

Lemma 3.6 *Let $\varepsilon_4, \varepsilon_5 : \mathbb{R}_+ \to \mathbb{R}_+^*$ be continuous functions. Then there exists a positive constant c_3 such that the functional F satisfies*

$$(M(t) - M_2(t))E(t) \le F(t) \le (M(t) + M_2(t))E(t), \qquad (3.38)$$

where

$$M_2(t) = \frac{c_3}{1 - g_0 a_1(t)}\max\left\{\varepsilon_5(t)M_1 + \varepsilon_4(t), \frac{a_1(t)M_1}{\varepsilon_5(t)b_1(t)}, \frac{1}{\varepsilon_4(t)b_1(t)}\right\}.$$

Proof We see that

$$E(t) \leq \frac{1}{2}\left[\|u_t(t)\|^2 + \left\|A^{\frac{1}{2}}(t)u(t)\right\|^2 + \int_0^\infty g(s)\left\|B^{\frac{1}{2}}(t)\eta(t,s)\right\|^2 ds\right] \tag{3.39}$$

and, using (2.3),

$$E(t) \geq \frac{1-g_0a_1(t)}{2}\left[\|u_t(t)\|^2 + \left\|A^{\frac{1}{2}}(t)u(t)\right\|^2 + \int_0^\infty g(s)\left\|B^{\frac{1}{2}}(t)\eta(t,s)\right\|^2 ds\right]. \tag{3.40}$$

On the other hand, using Young's inequality and assumption **(A1)**, we have, for any continuous functions $\varepsilon_4, \varepsilon_5 : \mathbb{R}_+ \to \mathbb{R}_+^*$,

$$|I_1(t)| \leq \varepsilon_5(t)\|u_t(t)\|^2 + \frac{ca_1(t)}{\varepsilon_5(t)b_1(t)}\left\|A^{\frac{1}{2}}(t)u(t)\right\|^2 \tag{3.41}$$

and

$$|I_2(t)| \leq \varepsilon_4(t)\|u_t(t)\|^2 + \frac{c}{\varepsilon_4(t)b_1(t)}\int_0^\infty g(s)\left\|B^{\frac{1}{2}}(t)\eta(t,s)\right\|^2 ds. \tag{3.42}$$

Therefore, by combining (3.40)-(3.42), we get

$$|F(t) - M(t)E(t)| \leq M_2(t)E(t),$$

which gives (3.38). □

We choose the functions M_j and ε_j carefully. Thanks to the properties of $g_0,\ a_1,\ b_1,\ \tilde{a}$ and $\tilde{b}$ assumed in **(A0)-(A5)**, one can choose

$$M_1 = \frac{g_0}{2}, \quad \varepsilon_1(t) = \frac{1-g_0a_1(t)}{2a_1(t)}, \qquad \varepsilon_2(t) = \frac{g_0^2}{2}a_1(t), \quad \varepsilon_3(t) = \frac{g_0(1-g_0a_1(t))}{8(1+a_1(t))},$$

$$\varepsilon_4(t) = \frac{2}{\sqrt{b_1(t)\left(4+g_0^2a_1(t)\right)}}, \quad \varepsilon_5(t) = \frac{g_0a_1(t)}{\sqrt{b_1(t)\left(4+g_0^2a_1(t)\right)}}$$

and

$$M(t) > \max\left\{\frac{c_3}{2(1-g_0a_1(t))}\sqrt{\frac{4+g_0^2a_1(t)}{b_1(t)}}, \frac{M_3(t)}{\frac{\theta_2}{2}-\sqrt{\tilde{b}(t)}}\right\}, \tag{3.43}$$

where

$$M_3(t) = g_0 + \frac{g_0}{8}(1-g_0a_1(t)) + \frac{c_1g_0a_1(t)}{1-g_0a_1(t)} + c_2\left(\frac{8(1+a_1(t))(1+a_2(t))}{g_0(1-g_0a_1(t))} + \frac{2}{g_0^2a_1(t)b_1(t)}\right). \tag{3.44}$$

Then

$$A_1(t) = \frac{g_0}{2}(1 - g_0 a_1(t)),$$
$$A_2(t) = \frac{g_0}{4}(1 - g_0 a_1(t)) - \left(g_0 a_1(t)\sqrt{\tilde{b}(t)} + \sqrt{\tilde{a}(t)}\right) M(t),$$
$$A_3(t) \geq \frac{g_0}{8}(1 - g_0 a_1(t)), \quad M(t) > \max\left\{M_2(t), \frac{M_3(t)}{\frac{\theta_2}{2} - \sqrt{\tilde{b}(t)}}\right\}$$

and

$$M_2(t) = \frac{c_3}{2(1 - g_0 a_1(t))}\sqrt{\frac{4 + g_0^2 a_1(t)}{b_1(t)}}. \tag{3.45}$$

On the other hand, we assume that the second assumption in (3.5) holds such that

$$\left(g_0 a_1(t)\sqrt{\tilde{b}(t)} + \sqrt{\tilde{a}(t)}\right) M(t) \leq \frac{g_0}{8}(1 - g_0 a_1(t)) \tag{3.46}$$

(notice that (3.46) is possible as $M_2(t)$ and $M_3(t)$ depend neither on $\tilde{a}$ nor on $\tilde{b}$), so we get

$$A_2(t) \geq \frac{g_0}{8}(1 - g_0 a_1(t)),$$

and then, combining (3.33) and (3.39), we find

$$F'(t) \leq \left[M'(t) - \frac{g_0}{4}(1 - g_0 a_1(t))\right] E(t),$$

therefore, according to (3.38),

$$F'(t) \leq \xi(t) F(t), \tag{3.47}$$

where

$$\xi(t) = \max\left\{\frac{M'(t) - \frac{g_0}{4}(1 - g_0 a_1(t))}{M(t) - M_2(t)}, \frac{M'(t) - \frac{g_0}{4}(1 - g_0 a_1(t))}{M(t) + M_2(t)}\right\}. \tag{3.48}$$

By integrating (3.47), we arrive to

$$F(t) \leq F(0) e^{\tilde{\xi}(t)}, \tag{3.49}$$

where

$$\tilde{\xi}(t) = \int_0^t \xi(s) ds. \tag{3.50}$$

Consequently, exploiting again (3.38), we conclude (3.9). □

Remark 4 If $\frac{M_3}{\frac{\theta_2}{2}-\sqrt{\tilde{b}}}$ and M_2 are bounded, then one can choose M as a constant satisfying

$$M > \max\left\{ \|M_2\|_{L^\infty(\mathbb{R}_+)}, \left\| \frac{M_3}{\frac{\theta_2}{2}-\sqrt{\tilde{b}}} \right\|_{L^\infty(\mathbb{R}_+)} \right\},$$

therefore

$$\xi(t) = \frac{-\frac{g_0}{4}(1-g_0a_1(t))}{M+M_2(t)} \le \frac{-\frac{g_0}{4}(1-g_0a_1(t))}{M+\|M_2\|_{L^\infty(\mathbb{R}_+)}} \quad \text{and} \quad \frac{1}{M-M_2(t)} \le \frac{1}{M-\|M_2\|_{L^\infty(\mathbb{R}_+)}},$$

hence (3.9) implies that there exist positive constants λ_0 and λ_1 such that

$$\|\mathcal{U}(t)\|^2_{\mathcal{H}(0)} \le \lambda_0 e^{-\lambda_1 \int_0^t (1-g_0a_1(s))ds}. \tag{3.51}$$

From (3.44) and (3.45), we observe that $\frac{M_3}{\frac{\theta_2}{2}-\sqrt{\tilde{b}}}$ and M_2 are bounded if and only if

$$\begin{cases} \|a_1\|_{L^\infty(\mathbb{R}_+)} < \infty, \quad g_0 < \frac{1}{\|a_1\|_{L^\infty(\mathbb{R}_+)}}, \quad \inf_{t\in\mathbb{R}_+} a_1(t) > 0, \quad \inf_{t\in\mathbb{R}_+} b_1(t) > 0, \\ \|a_2\|_{L^\infty(\mathbb{R}_+)} < \infty \quad \text{and} \quad \|\tilde{b}\|_{L^\infty(\mathbb{R}_+)} < \frac{\theta_2^2}{4}, \end{cases} \tag{3.52}$$

so (3.51) is reduced to the exponential stability estimate, for $\tilde{\lambda}_1 = \lambda_1(1 - g_0\|a_1\|_{L^\infty(\mathbb{R}_+)})$,

$$\|\mathcal{U}(t)\|^2_{\mathcal{H}(0)} \le \lambda_0 e^{-\tilde{\lambda}_1 t}. \tag{3.53}$$

Remark 5 Let us construct a solution to (2.11) which converges to 0 as $t \to \infty$. For that, it is enough to construct a C_0-semigroup $(\mathcal{U}(t))_{t\ge 0}$ that is exponentially stable. Indeed, let $\Omega \subset \mathbb{R}^N$, for $N \in \mathbb{N}^*$, be an open bounded domain with smooth boundary $\Gamma = \partial\Omega$ and let $H = L^2(\Omega)$ equipped with its standard L^2-topology. Consider, for $m \in \mathbb{N}^*$, $L = \Delta^m$ with $D(L) = H^{2m}(\Omega) \cap H_0^m(\Omega)$. Obviously, $-L$ is a positive selfadjoint linear operator on $L^2(\Omega)$ with compact resolvent. Further, $D((-L)^{\frac{1}{2}}) = H_0^m(\Omega)$.

Consider the case when $A(t) = -a(t)\Delta^m$, $B(t) = -b(t)\Delta^m$, $D(A(t)) = D(B(t)) = H^{2m}(\Omega) \cap H_0^m(\Omega)$ with

$$a(t) = \alpha + r(t), \quad b(t) = \beta + k(t) \quad \text{for all} \quad t \in \mathbb{R}_+,$$

where $\alpha \ge \beta > 0$, $\theta_1 = \theta_2 = 1$ (yielding $g_0 = 1$), $r, k : \mathbb{R}_+ \to \mathbb{R}_+^*$ are class C^1 bounded functions such that $\|r'\|_{L^\infty(\mathbb{R}_+)} < \infty$ and $\|k'\|_{L^\infty(\mathbb{R}_+)} < \infty$ and that,

i) $\inf\limits_{t\in\mathbb{R}_+} r(t) > \beta + \|k\|_{L^\infty(\mathbb{R}_+)}$;

ii) $\sqrt{\frac{\|k'\|_{L^\infty(\mathbb{R}_+)}}{\beta}} < \frac{1}{2}$; and

iii) $\frac{\beta + \|k\|_{L^\infty(\mathbb{R}^+)}}{2\alpha} + \sqrt{\frac{\|r'\|_{L^\infty(\mathbb{R}_+)}}{\beta}}$ is small enough (to guarantee (3.46)).

In view of the above, it is not hard to see that (3.52) holds. Therefore, the solution to (2.11) converges to 0 as $t \to \infty$.

4 Applications

In this section, we present two examples that fit into our abstract model, namely (1.1)–(1.2). Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with smooth boundary Γ, where $N \in \mathbb{N}^*$. In both cases, we will assume that $H = L^2(\Omega)$ is equipped its standard L^2-topology.

4.1 Wave equations

The abstract model (1.1)–(1.2) includes the following nonautonomous wave equation,

$$\begin{cases} u_{tt}(x,t) - a(t)\Delta u(x,t) + b(t)\displaystyle\int_0^\infty g(s)\Delta u(x,t-s)ds = 0, & \forall (x,t) \in \Omega \times \mathbb{R}_+^*, \\ u(x,t) = 0, & \forall (x,t) \in \Gamma \times \mathbb{R}_+^*, \\ u(x,-t) = u_0(x,t), \quad u_t(x,0) = u_1(x), & \forall (x,t) \in \Omega \times \mathbb{R}_+, \end{cases} \tag{4.1}$$

where $A(t) = -a(t)\Delta$, $B(t) = -b(t)\Delta$, $D(A(t)) = D(B(t)) = H^2(\Omega) \cap H_0^1(\Omega)$. Theorems 2.1 and 3.1 hold true under the assumptions given in Remarks 1 and 2.

4.2 Petrovsky type systems

The following nonautonomous Petrovsky type system fits into our abstract model (1.1)–(1.2),

$$\begin{cases} u_{tt}(x,t) + a(t)\Delta^2 u(x,t) - b(t)\displaystyle\int_0^\infty g(s)\Delta^2 u(x,t-s)ds = 0, & \forall (x,t) \in \Omega \times \mathbb{R}_+^*, \\ u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = 0, & \forall (x,t) \in \Gamma \times \mathbb{R}_+^*, \\ u(x,-t) = u_0(x,t), \quad u_t(x,0) = u_1(x), & \forall (x,t) \in \Omega \times \mathbb{R}_+. \end{cases} \tag{4.2}$$

where $A(t) = a(t)\Delta^2$, $B(t) = b(t)\Delta^2$, $D(A(t)) = D(B(t)) = H^4(\Omega) \cap H_0^2(\Omega)$, and assumptions of Remarks 1 and 2 yield both Theorems 2.1 and 3.1.

5 General comments and issues

Under some appropriate assumptions on the time-dependent operators $A(t)$ and $B(t)$, as well as the relaxation (kernel) function g, we established the well-posedness and asymptotic stability of the solutions to the system (1.1)–(1.2) as time t goes to infinity. In light of our findings, we would like to propose the following questions, which, to the best of our knowledge, remain unanswered:

(1) Will we be in the presence of a discrete or distributed delay by adding

$$\int_0^\infty f(s)C(t)u_t(t-s)ds \quad \text{or} \quad f(t)C(t)u_t(t-\tau),$$

to (1.1), where $C(t)$ is an operator, $f : \mathbb{R}_+ \to \mathbb{R}$ is a function, and τ is a fixed positive real number?

(2) Can we apply the previous theory to a larger class of relaxation functions g, that is,

$$g'(s) \leq -\theta_2(s)g(s), \quad \forall s \in \mathbb{R}_+$$

instead of (3.4), where $\theta_2 : \mathbb{R}_+ \to \mathbb{R}_+$ is a function?

(3) Can we establish similar results when $D(A(t))$ and $D(B(t))$ are no longer constant in time t?

(4) What about the damping case?

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