Semigroup Forum (2022) 105:551–569
https://doi.org/10.1007/s00233-022-10271-9

RESEARCH ARTICLE

Force recurrence of semigroup actions

Kesong Yan[1,2] · Fanping Zeng[1,2] · Rong Tian[3]

Received: 6 January 2022 / Accepted: 25 February 2022 / Published online: 5 April 2022

Abstract
We investigate the sets of countable discrete semigroups that force recurrence, that is, the recurrent properties of a point along a subset of a countable semigroup action. We show that a subset of a monoid forces recurrence (resp., forces minimality) if and only if it contains a broken IP-set (resp., broken syndetic set), and forces infinite recurrence implies it is contains a broken infinite IP-sets. As an example, we show that every subset with positive upper Banach density of infinite countable amenable groups forces infinite recurrence.

Keywords Recurrence · Semigroup actions · Minimality · Banach density

1 Introduction

By a *topological dynamical system* (or *dynamical system* for short) we mean a pair (X, G), where X is a compact metric space with a metric d and G is a topological group or semigroup acting continuously on X. Throughout the paper, the sets of integers, non-negative integers and positive integers are denoted by $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$, respectively. When $G = \mathbb{Z}$ (resp. $\mathbb{Z}_+$) the action is generated by a homeomorphism

Communicated by Jimmie D. Lawson.

✉ Kesong Yan
ksyan@mail.ustc.edu.cn

Fanping Zeng
fpzeng@gxu.edu.cn

Rong Tian
15078842780@163.com

1 School of Information and Statistics, Guangxi University of Finance and Economics, Nanning 530003, Guangxi, People's Republic of China

2 Guangxi Key Laboratory of Big Data in Finance and Economics, Guangxi University of Finance and Economics, Nanning 530003, Guangxi, People's Republic of China

3 College of Mathematics and Information Science, Guangxi University, Nanning 530004, Guangxi, People's Republic of China

(resp. a continuous map) $T : X \to X$, and we usually denote the dynamical system by (X, T), which is called a *cascade*.

Recurrence is a basic property of topological dynamical systems. Let (X, T) be a cascade. Recall that a point $x \in X$ is a *recurrent point* if there is some sequence of positive integers $n_i \to \infty$ such that $T^{n_i}x \to x$. Let $Rec(X, T)$ denote the set of all recurrent points of (X, T). The following result is the famous Birkhoff theorem (see [10, 19, 34] for example).

Theorem 1.1 (Birkhoff) *$Rec(X, T)$ is non-empty for every cascade (X, T).*

An important problem in topological dynamical systems is to investigate the recurrence of a point along some subset of $\mathbb{N}$. A subset A of $\mathbb{Z}_+$ is called a *topological recurrence set* if for every cascade (X, T), there is some sequence $\{n_i\}_{i=1}^{\infty}$ in A such that $n_i \to \infty$ and $T^{n_i}x \to x$ for some $x \in X$. Birkhoff's Theorem means that $\mathbb{Z}_+$ is a topological recurrence set. Amazingly, the notion of topological recurrence sets in topological dynamical systems are close related to the coloring problem in combinatorial mathematics, see e.g. [26, 34]. In [34], Weiss obtained an important characterization of a topological recurrence set, which clarified the relationship between recurrence sets and difference sets of syndetic sets.

Let (X, T) be a cascade. It is natural to ask that what conditions are satisfied to $S \subset \mathbb{N}$ such that there must be some recurrent point in the closure of $\{T^n x : n \in S\}$ for every point $x \in X$. This topic has been studied by some authors, see, e.g., [11, 17, 21, 29]. Such a set is said to force recurrence in [11]. A well-know result implies that if a subset of $\mathbb{N}$ has positive upper density, then it forces recurrence (see [11, Theorem 2]), where the upper density of $S \subset \mathbb{N}$ is defined as

$$\overline{d}(S) = \limsup_{N\to\infty} \frac{|S \cap [1, N]|}{N}.$$

A celebrated theorem of Furstenberg shows that the recurrence of topological dynamical systems is closely related to IP-sets [19, Theorem 2.17]. This terminology was derived from Furstenberg and Weiss in [20]. Following the idea of Furstenberg, Blokh and Fieldsteel [11] showed that a subset of $\mathbb{N}$ forces recurrence if and only if it contains a broken IP-set.

The theory of group or semigroup actions has attracted a lot of attention by many authors, for example, see works related to size and combinatorial properties [6], recurrence [1, 7, 8, 13, 15, 31], Lyapunov stability [9], topological entropy [25], sensitivity and chaos [28, 30, 33], transitivity and mixing [12, 27, 32, 35, 36], etc. Especially, Bergelson and McCutcheon [7] extended the notion of topological recurrence sets from the additive semigroup $\mathbb{N}$ to arbitrary countable semigroups, and explored their relationships with combinatorics.

In this paper, we focus on investigating the sets of countable discrete semigroups that force recurrence, which following the idea in [11]. We introduce the notion of force recurrence set for general semigroups. We prove that a subset of a monoid forces recurrence (resp., forces minimality) if and only if it contains a broken IP-set (resp., broken syndetic set), and forces infinite recurrence implies it is contains a broken infinite IP-sets. As an example, we show that every subset with positive upper Banach density of infinite countable amenable groups forces infinite recurrence.

2 Force recurrence

Throughout this paper, we let G be an infinite countable discrete semigroup. A semigroup G is a *monoid* if it has an identity e, and then we write $G^+ = G \setminus \{e\}$. By a *topological dynamical system* we mean that a triple (X, G, π) (simple for (X, G)), where X is a compact metric space with the metric d and $\pi : G \times X \to X$, $(g, x) \mapsto gx$ is a continuous mapping satisfying

(1) $\pi(e, x) = x$ for each $x \in X$ if G has an identity e;
(2) $\pi(s, \pi(t, x)) = \pi(st, x)$ for each $s, t \in G$ and $x \in X$.

If a non-empty compact subset $Y \subseteq X$ is G-invariant (i.e., $gy \in Y$ for any $g \in G$ and $y \in Y$), then (Y, G) is called a *subsystem* of (X, G).

For two dynamical systems (X, G) and (Y, G), their *product system* $(X \times Y, G)$ is defined by the diagonal action: $g(x, y) = (gx, gy)$ for all $x \in X$, $y \in Y$ and $g \in G$.

Let (X, G) be a dynamical system. A point $x \in X$ is called a *recurrent point* if $N_+(x, U)$ is non-empty for any neighborhood U of x, where

$$N_+(x, U) = \{g \in G^+ : gx \in U\}$$

is called the set of *return times* of the point x to U. Let $Rec(X, G)$ denote the set of recurrent points of (X, G).

Definition 2.1 We say that a set $S \subseteq G$ *forces recurrence* if whenever (X, G) is a dynamical system and $K \subseteq X$ is compact, and for some $x \in X$ and all $s \in S$, $sx \in K$, we have $K \cap Rec(X, G) \neq \emptyset$.

In this section, we mainly provide a characterization of subsets of the semigroup that force recurrence. Let G be a semigroup. For $g \in G$ and $S \subset G$, denote

$$g^{-1}S = \{h \in G : gh \in S\} \text{ and } Sg^{-1} = \{h \in G : hg \in S\}.$$

Theorem 2.2 *Let $\mathcal{P}$ be a non-empty family of infinite subsets of the semigroup G such that*

(1) for all $S \in \mathcal{P}$, there is some $g \in G^+$ such that $g^{-1}S \cap S \in \mathcal{P}$;
(2) $\mathcal{P}$ has the Ramsey property, that is, $S_1 \cup S_2 \in \mathcal{P}$ implies $S_1 \in \mathcal{P}$ or $S_2 \in \mathcal{P}$.

Then S forces recurrence for all $S \in \mathcal{P}$.

Proof We shall follow the idea of the proof of [11, Theorem 3]. Let $S \in \mathcal{P}$, (X, G) be a dynamical system, $K \subseteq X$ is compact, and $x \in X$ satisfies $sx \in K$ for all $s \in S$. We will show that $K \cap Rec(X, G) \neq \emptyset$.

Set $K_1 = K$ and $S_1 = S$. Then we can find $p_1 \in G^+$ such that $p_1^{-1}S_1 \cap S_1 \in \mathcal{P}$. This implies $sx \in K_1 \cap p_1^{-1}K_1$ for every $s \in p_1^{-1}S_1 \cap S_1$. It follows that $K_1 \cap p_1^{-1}K_1$ is a non-empty compact subset of K_1. Let

$$K_1 \cap p_1^{-1}K_1 = \bigcup_{i=1}^{n_1} K_{1,i},$$

where each $K_{1,i}$ is a non-empty compact subset of X with $\operatorname{diam}(K_{1,i}) < 1/2$. For $i = 1, 2, \cdots, n_1$, let $S_{1,i} = \{s \in p_1^{-1}S_1 \cap S_1 : sx \in K_{1,i}\}$, then we have

$$p_1^{-1}S_1 \cap S_1 = \bigcup_{i=1}^{n_1} S_{1,i}.$$

Since $\mathcal{P}$ has the Ramsey property, one has $S_{1,i_1} \in \mathcal{P}$ for some i_1. Set

$$S_2 = S_{1,i_1} \text{ and } K_2 = K_{1,i_1}.$$

Then we have $S_2 \in \mathcal{P}$, $K_2 \subseteq K_1$, $\operatorname{diam}(K_2) < 1/2$ and $p_1(K_2) \subseteq K_1$.

We continue inductively. Assume that S_n, K_n and p_{n-1} have been found such that $S_n \in \mathcal{P}$, $K_n \subseteq K_{n-1}$, $\operatorname{diam}(K_n) < 1/n$, $sx \in K_n$ for all $s \in S_n$ and $p_{n-1}(K_n) \subseteq K_{n-1}$. Then we apply the above argument to S_n and K_n, there is $p_n \in G^+$ such that $p_n^{-1}S_n \cap S_n \in \mathcal{P}$. By the construction of S_n and K_n, we know that $sx \in K_n \cap p_n^{-1}K_n$ for any $s \in p_n^{-1}S_n \cap S_n$. Let

$$K_n \cap p_n^{-1}K_n = \bigcup_{i=1}^{m_n} K_{n,i},$$

where each $K_{n,i}$ is a non-empty compact subset of X with $\operatorname{diam}(K_{n,i}) < 1/(n+1)$. Let $S_{n,i} = \{s \in p_n^{-1}S_n \cap S_n : sx \in K_{n,i}\}$. Then

$$p_n^{-1}S_n \cap S_n = \bigcup_{i=1}^{m_n} S_{n,i},$$

which follows that $S_{n,i_n} \in \mathcal{P}$ for some i_n. Set

$$S_{n+1} = S_{n,i_n} \text{ and } K_{n+1} = K_{n,i_n}.$$

This completes the inductive process.

By induction, we obtain a sequence of non-empty compact sets $\{K_n\}_{n=1}^{\infty}$ and a sequence $\{p_n\}_{n=1}^{\infty}$ of G^+ such that

- $K_1 \supseteq K_2 \supseteq \cdots \supseteq K_n \supseteq \cdots$;
- $\operatorname{diam}(K_n) < 1/n$ for each $n \geq 2$;
- $p_n(K_{n+1}) \subseteq K_n$ for each $n \geq 1$.

Let y be the single point in $\bigcap_{n=1}^{\infty} K_n$. Then we have for all m, $p_m y \in K_m$. This shows that $y \in K \cap Rec(X, G)$. □

Next we give a characterization of the sets of semigroups that force recurrence. Before that, let us recall some notations. For a set A, denote by $\mathcal{P}_f(A)$ the set of all finite non-empty subsets of A.

Definition 2.3 Let G be a semigroup. Given a sequence $\{p_n\}_{n=1}^{\infty}$ in G, the *IP-set* generated by the sequence is defined by

$$FP(\{p_n\}_{n=1}^{\infty}) = \left\{ \prod_{n\in F} p_n : F \in \mathcal{P}_f(\mathbb{N}) \right\}, \text{ where } \prod_{n\in F} p_n = p_{n_1} \cdot p_{n_2} \cdot \ldots \cdot p_{n_k}$$

for $F = \{n_1, n_2, \ldots, n_k\} \in \mathcal{P}_f(\mathbb{N})$ with $n_1 < n_2 < \cdots < n_k$.

For each $L \in \mathbb{N}$, the *initial L-segment of $FP(\{p_n\}_{n=1}^{\infty})$* is defined as

$$FP(\{p_n\}_{n=1}^{L}) = \left\{ \prod_{n\in F} p_n : F \in \mathcal{P}_f(\{1, \ldots, L\}) \right\}.$$

A subset S of G *contains a broken IP-set* if there is a sequence $\{p_n\}_{n=1}^{\infty}$ in G such that for each $L \in \mathbb{N}$, there is $s_L \in G$ with $FP(\{p_n\}_{n=1}^{L}) \cdot s_L \subseteq S$.

Remark 2.4 The most interesting IP-sets are the infinite ones. However, If u is an idempotent, then $\{u\}$ is an IP-set. And even if G is a group, there may be many finite IP-sets. For example any finite subgroup of G is an IP-set.

Let (X, G) be a dynamical system. For $x \in X$ and $U \subseteq X$, let $N(x, U) = \{g \in G : gx \in U\}$. Following the idea of [19, Theorem 2.17], we can obtain

Lemma 2.5 *Let (X, G) be a dynamical system. If $x \in Rec(X, G)$, then $N(x, U)$ contains an IP-set for every neighborhood U of x.*

Proof Suppose x is a recurrence point for (X, G) and U is a neighborhood of U. Let $p_1 \in G^+$ satisfy

$$p_1 x \in U. \tag{2.1}$$

Now we find a neighborhood U_1 of x such that $U_1 \subset U$ and

$$z \in U_1 \Rightarrow p_1 z \in U. \tag{2.2}$$

For such U_1 we can find $p_2 \in G^+$ such that

$$p_2 x \in U_1. \tag{2.3}$$

Combining (2.1), (2.2) and (2.3), we have

$$gx \in U \text{ for } g = p_1, p_2 \text{ and } p_1 \cdot p_2. \tag{2.4}$$

We continue inductively. Assume that different elements $p_1, p_2, \cdots, p_n$ in G^+ have been found such that (2.4) is valid for all $g = p_{n_1} \cdot p_{n_2} \cdot \ldots \cdot p_{n_k}$ with $1 \le n_1 < n_2 < \cdots < n_k \le n$. Then we find a neighborhood U_{n+1} of x such that $U_{n+1} \subset U$ and

$$z \in U_{n+1} \Rightarrow gz \in U \tag{2.5}$$

for all of the above mentioned g. Thus, if $p_{n+1} \in G^+$ is defined such that

$$p_{n+1}x \in U_{n+1}, \tag{2.6}$$

then (2.4) will be valid whenever g replaced by $g \cdot p_{n+1}$ or by p_{n+1}. This completes the inductive process, and it is easy to check that $FP((p_n)_{n\in\mathbb{N}}) \subseteq N(x, U)$. □

The following theorem is the product version of the Hindman's theorem [22].

Theorem 2.6 (The finite product theorem, [4, 5, 37]) *Let G be a semigroup. If $S \subset G$ is an IP-set, $r \in \mathbb{N}$ and $S = \bigcup_{i=1}^{r} C_i$, then there is $i \in \{1, 2, \ldots, r\}$ such that C_i contains an IP-set.*

For a subset A of a topological space X, we denote $cl_X(A)$ the closure of A in X.

Lemma 2.7 *Let G be a monoid, let $S \subseteq G$, and let $\Sigma_2 = \{0, 1\}^G$ be the product space endowed with the product topology. Define the shift action of G on Σ_2 by $h\omega(g) = \omega(gh)$ for all $g, h \in G$ and all $\omega \in \Sigma_2$. Then (Σ_2, G) is a dynamical system. Define $1_S \in \Sigma_2$ by $1_S(s) = 1$ if and only if $s \in S$. Let $X = cl_{\Sigma_2}\{g1_S : g \in G\}$. Then X is an invariant closed subset of Σ_2 and (X, G) is a subsystem of (Σ_2, G). Let $K = \{x \in X : x(e) = 1\}$. Then K is a nonempty open and closed subset of X and $s1_S \in K$ for all $s \in S$.*

Proof Since G is countable, Σ_2 is a compact metric space. Sets of the form $\{\omega \in \Sigma_2 : \omega(g) = i\}$ for $g \in G$ and $i \in \{0, 1\}$ form a subbasis for the topology on Σ_2 so K is open and closed. It is routine to verify the rest of the assertions in the lemma. □

Theorem 2.8 *Suppose that G is a monoid. Then a subset S of G^+ forces recurrence if and only if it contains a broken IP-set.*

Proof Let $\mathcal{P}_{\text{bip}}$ denote the collection of all subsets of G that contains a broken IP-set. To prove S forces recurrence all $S \in \mathcal{P}_{\text{bip}}$, we only show that the family $\mathcal{P}_{\text{bip}}$ satisfies the conditions of Theorem 2.2.

Let $S \in \mathcal{P}_{\text{bip}}$ and suppose that S contains a broken $FP(\{p_n\}_{n=1}^{\infty})$. Fix $M \in \mathbb{N}$. Then for each $L \in \mathbb{N}$ with $L > M$, we can choose $s_L \in G$ such that

$$FP(\{p_n\}_{n=1}^{L}) \cdot s_L \subseteq S,$$

which implies

$$\begin{aligned} p_M^{-1}S \cap S &\supseteq \left(p_M^{-1} \cdot FP(\{p_n\}_{n=1}^{L}) \cdot s_L\right) \cap \left(FP(\{p_n\}_{n=1}^{L}) \cdot s_L\right) \\ &\supseteq \left[\left(p_M^{-1} \cdot FP(\{p_n\}_{n=1}^{L})\right) \cap \left(FP(\{p_n\}_{n=1}^{L})\right)\right] \cdot s_L \\ &\supseteq FP(\{p_n\}_{n=M+1}^{L}) \cdot s_L. \end{aligned}$$

This shows that $p_M^{-1}S \cap S$ contains a broken $FP(\{p_n\}_{n=M+1}^{\infty})$.

Next, we will show that $\mathcal{P}_{\text{bip}}$ has the Ramsey property. Suppose that $S \in \mathcal{P}_{\text{bip}}$ contains a broken $FP(\{p_n\}_{n=1}^{\infty})$ and $S = S_1 \cup S_2$. Without loss of generality, we may suppose that $S_1 \cap S_2 = \emptyset$. Let x be a point in $\{1,2\}^S$ that defined by

$$x(s) = i \ \text{ if and only if } \ s \in S_i. \tag{2.7}$$

For each $N \in \mathbb{N}$, there exists some $s_N \in G$ such that

$$FP(\{p_n\}_{n=1}^{N}) \cdot s_N \subset S. \tag{2.8}$$

Let $x_N \in \{1,2\}^{FP(\{p_n\}_{n=1}^{\infty})}$ be defined by

$$x_N(s) = \begin{cases} x(s \cdot s_N), \ s \in FP(\{p_n\}_{n=1}^{N}), \\ \\ \quad 1, \qquad \text{otherwise}. \end{cases}$$

Since $\{1,2\}^{FP(\{p_n\}_{n=1}^{\infty})}$ is a compact metric space, we can choose a subsequence $\{N_j\}_{j=1}^{\infty}$ such that x_{N_j} converges to some $y \in \{1,2\}^{FP(\{p_n\}_{n=1}^{\infty})}$. Write

$$C_i = \{s \in FP(\{p_n\}_{n=1}^{\infty}) : y(s) = i\} \ \text{ for } \ i = 1, 2.$$

By Theorem 2.6, there exists some $i \in \{1,2\}$ such that C_i is an IP-set, i.e., it contains $FP(\{q_n\}_{n=1}^{\infty})$ for some sequence $\{q_n\}_{n=1}^{\infty}$ in G. For each $L \in \mathbb{N}$, we can find some sufficiently large $j(L)$ such that $FP(\{q_n\}_{n=1}^{L}) \subseteq FP(\{p_n\}_{n=1}^{N_{j(L)}})$ and $x_{N_{j(L)}}(s) = y(s) = i$ for all $s \in FP(\{q_n\}_{n=1}^{L})$. This implies

$$x(s \cdot s_{N_{j(L)}}) = i \ \text{ for all } s \in FP(\{q_n\}_{n=1}^{L}).$$

Thus we have $FP(\{q_n\}_{n=1}^{L}) \cdot s_{N_{j(L)}} \subseteq S_i$. Therefore, S_i contains a broken $FP(\{q_n\}_{n=1}^{\infty})$.

Conversely, suppose that S is a set that forces recurrence. Let Σ_2, X, K and 1_S be as in Lemma 2.7. Then there is some point $y \in K \cap Rec(X, G)$. By Lemma 2.5, we know that $N(y, K)$ is an IP-set. So there is a sequence $\{p_n\}_{n=1}^{\infty}$ of G such that $FP(\{p_n\}_{n=1}^{\infty}) \subseteq N(y, K)$. This implies

$$y(s) = 1 \ \text{ for all } \ s \in FP(\{p_n\}_{n=1}^{\infty}). \tag{2.9}$$

For each $L \in \mathbb{N}$, there exists $s_L \in G$ such that $s_L 1_S \in V$, where

$$V = \{x \in X : x(s) = y(s) \text{ for all } s \in FP(\{p_n\}_{n=1}^{L})\}.$$

Thus, for every $s \in FP(\{p_n\}_{n=1}^{L})$, one has

$$1_S(s \cdot s_L) = s_L 1_S(s) = y(s) = 1.$$

This shows that

$$FP(\{p_n\}_{n=1}^{L}) \cdot s_L \subseteq S. \tag{2.10}$$

Therefore, S contains a broken $FP(\{p_n\}_{n=1}^{\infty})$. □

3 Force recurrence via Furstenberg family

In this section, we will consider more general forms of recurrence for semigroup actions via Furstenberg family. Let $\mathcal{P}$ be a non-empty collection of subsets of the semigroup G. We say that $\mathcal{P}$ is a *Furstenberg family* (or *family* for short) if it is hereditary upward, i.e., $S_1 \in \mathcal{P}$ and $S_1 \subseteq S_2$ implies $S_2 \in \mathcal{P}$.

For a family $\mathcal{P}$, the *block family* of $\mathcal{P}$, denote by $b\mathcal{P}$, is the family consisting of sets $S \subset G$ for which there exists some $P \in \mathcal{P}$ such that for every finite subset F of P one has $F \cdot s_F \subseteq S$ for some $s_F \in G$. It is easy to check that

$$b\mathcal{P} = \{S \subseteq G : (\exists P \in \mathcal{P})(\forall F \in \mathcal{P}_f(G))(\exists s_F \in G) \text{ such that } (P \cap F) \cdot s_F \subseteq S\}.$$

3.1 Force family recurrence

Let $\mathcal{P}$ be a family of the semigroup G and (X, G) be a topological dynamical system. A point $x \in X$ is called a *$\mathcal{P}$-recurrent point* if $N(x, U) \in \mathcal{P}$ for any neighborhood U of x. Denote the set of all $\mathcal{P}$-recurrent points of (X, G) by $Rec_{\mathcal{P}}(X, G)$. We note that the recurrence in Sect. 2 can be regard as $\mathcal{P}_+$-recurrence, where $\mathcal{P}_+$ denote the family of all non-empty subsets of G that have non-identity elements of G.

Definition 3.1 Let $\mathcal{P}$ be a non-empty family of the semigroup G. We say that a set $S \subseteq G$ *forces $\mathcal{P}$-recurrence* if whenever (X, G) is a dynamical system and $K \subseteq X$ is compact, and for some $x \in X$ and all $s \in S$, $sx \in K$, we have $K \cap Rec_{\mathcal{P}}(X, G) \neq \emptyset$.

Following the idea of Theorem 2.8, we have the following general result.

Theorem 3.2 *Let $\mathcal{P}$ be a non-empty family of the monoid G. If S is a subset of G that forces $\mathcal{P}$-recurrence, then $S \in b\mathcal{P}$.*

Proof Let Σ_2, X, K and 1_S be as in Lemma 2.7. Clearly, $s1_S \in K$ for all $s \in S$. Thus there exists a $\mathcal{P}$-recurrent point $y \in K$. Notice that K is also a non-empty open subset of X. Let $P = N(y, K)$. Then $P \in \mathcal{P}$. For each non-empty finite subset F of P, there exists $s_F \in G$ such that $s_F 1_S \in V$, where

$$V = \{x \in X : x(s) = y(s) \text{ for all } s \in F\}.$$

Thus, for every $s \in F$, one has

$$1_S(s \cdot s_L) = s_L 1_S(s) = y(s) = 1.$$

This shows that $F \cdot s_L \subseteq S$. Therefore, $S \in b\mathcal{P}$. □

Let $\mathcal{P}$ be a family of the semigroup G. Denote by $\mathcal{P}_{\text{force}}$ the collection of all subsets of G that force $\mathcal{P}$-recurrence. It is easy to see that $\mathcal{P}_{\text{force}}$ is a family, and it is not empty if and only if $Rec_{\mathcal{P}}(X, G)$ is non-empty for every topological dynamical system (X, G). In addition, a subset S of G forces $\mathcal{P}$-recurrence if and only if whenever (X, G) is a dynamical system and $x \in X$, $cl_X\{gx : g \in S\} \cap Rec_{\mathcal{P}}(X, G) \neq \emptyset$.

Theorem 3.3 *Let $\mathcal{P}$ be a family of the monoid G. If $\mathcal{P}_{\text{force}}$ is not empty, then we have*

(1) $\mathcal{P}_{\text{force}}$ has the Ramsey property, that is, $S_1 \cup S_2 \in \mathcal{P}_{\text{force}}$ implies $S_1 \in \mathcal{P}_{\text{force}}$ or $S_2 \in \mathcal{P}_{\text{force}}$;
(2) $\mathcal{P}_{\text{force}} = b\mathcal{P}_{\text{force}}$.

Proof (1) Let $S \in \mathcal{P}_{\text{force}}$ and $S = S_1 \cup S_2$. If neither S_1 nor S_2 forces $\mathcal{P}$-recurrence, then there exist topological dynamical systems (X, G), (Y, G) and $x \in X$, $y \in Y$ such that neither $K_1 = cl_X\{gx : g \in S_1\}$ nor $K_2 = cl_Y\{gy : g \in S_2\}$ contains $\mathcal{P}$-recurrence points. Consider the product system $(X \times Y, G)$ and $K = cl_{X\times Y}\{(gx, gy) : g \in S\}$. Since S forces $\mathcal{P}$-recurrence, there is some $\mathcal{P}$-recurrence point $(z_1, z_2) \in K$. Without loss of generality, we may assume that $(z_1, z_2) \in cl_{X\times Y}\{(gx, gy) : g \in S_1\}$. Then $z_1 \in K_1$ is a $\mathcal{P}$-recurrence point of (X, G), which is a contradiction. Thus, $\mathcal{P}_{\text{force}}$ has the Ramsey property.

(2) It is obvious that $\mathcal{P}_{\text{force}} \subseteq b\mathcal{P}_{\text{force}}$. Let $S \in b\mathcal{P}_{\text{force}}$. Then there exists some $\widetilde{S} \in \mathcal{P}_{\text{force}}$ such that for every non-empty finite subset F of G, there exists $s_F \in G$ such that $(\widetilde{S} \cap F) \cdot s_F \subseteq S$.

Let (X, G) be a topological dynamical system, K a compact subset of X and $x \in X$ such that $sx \in K$ for all $s \in S$. Since G is countable, we can find an increasing sequence $\{F_n\}_{n=1}^{\infty}$ of non-empty finite subsets of G such that

$$F_1 \subset F_2 \subset \cdots \subset F_n \subset \cdots \text{ and } \bigcup_{n=1}^{\infty} F_n = G.$$

Let $z_n = s_{F_n}x$ for all $n \in \mathbb{N}$. Since X is a compact metric space, we can find $z \in X$ and a subsequence $\{n_i\}_{i=1}^{\infty}$ such that z_{n_i} convergence to z. Given $g \in \widetilde{S}$, then $gs_{F_{n_i}} \in S$, and thus $gz_{n_i} \in K$, for all sufficiently large i. By the continuity of g, we have $gz_{n_i} \to gz \in K$. This shows that $gz \in K$ for all $g \in \widetilde{S}$. Since $\widetilde{S}$ forces $\mathcal{P}$-recurrence, there is some $\mathcal{P}$-recurrence point $y \in K$. Thus, $S \in \mathcal{P}_{\text{force}}$. □

Denote by $\mathcal{P}_{\text{ip}}$ the family of all sets that contains some IP-set. It is obvious that $b\mathcal{P}_{\text{ip}} = \mathcal{P}_{\text{bip}}$. Thus, by Lemma 2.5, Theorems 2.8, 3.2 and 3.3, we have

Corollary 3.4 *Suppose that G is a monoid and S is a subset of G^+. Then the following conditions are equivalent:*

(1) S forces recurrence;
(2) S forces $\mathcal{P}_{\text{ip}}$-recurrence;
(3) $S \in b\mathcal{P}_{\text{ip}}$.

Furthermore, we have

$$\mathcal{P}_{+,\text{force}} = \mathcal{P}_{\text{ip,force}} = b\mathcal{P}_{+,\text{force}} = b\mathcal{P}_{\text{ip,force}} = b\mathcal{P}_{\text{ip}}.$$

3.2 Force infinite recurrence

Let (X, G) be a dynamical system. A point $x \in X$ is called a *infinite recurrent point* if $N(x, U)$ is infinite for any neighborhood U of x. Denote by $\mathcal{P}_{\inf}$ the family of all infinite subsets of G. Then x is an infinite recurrence point if and only if it is a $\mathcal{P}_{\inf}$-recurrent point.

The following lemma can be found in [35, Lemma 3.18].

Lemma 3.5 *Let (X, G) be a dynamical system. If x is an infinite recurrence point, then $N(x, U)$ contains an infinite IP-set for every neighborhood U of x.*

Similar to the proof of Theorem 2.2, we have the following result.

Theorem 3.6 *Let $\mathcal{P}$ be a non-empty family of infinite subsets of the semigroup G such that*

(1) for all $S \in \mathcal{P}$, there exist infinitely many $g \in G$ such that $g^{-1}S \cap S \in \mathcal{P}$;
(2) $\mathcal{P}$ has the Ramsey property, that is, $S_1 \cup S_2 \in \mathcal{P}$ implies $S_1 \in \mathcal{P}$ or $S_2 \in \mathcal{P}$.

Then S forces $\mathcal{P}_{\inf}$-recurrence for all $S \in \mathcal{P}$.

Proof Let $S \in \mathcal{P}$, (X, G) be a dynamical system, $K \subseteq X$ is compact, and $x \in X$ satisfies $sx \in K$ for all $s \in S$. We will show that $K \cap Rec_{\mathcal{P}_{\inf}}(X, G) \neq \emptyset$.

Set $K_1 = K$ and $S_1 = S$. Then we can find $p_1 \in G$ such that $p_1^{-1}S_1 \cap S_1 \in \mathcal{P}$. This implies $sx \in K_1 \cap p_1^{-1}K_1$ for every $s \in p_1^{-1}S_1 \cap S_1$. It follows that $K_1 \cap p_1^{-1}K_1$ is a non-empty compact subset of K_1. Let

$$K_1 \cap p_1^{-1}K_1 = \bigcup_{i=1}^{n_1} K_{1,i},$$

where each $K_{1,i}$ is a non-empty compact subset of X with $\mathrm{diam}(K_{1,i}) < 1/2$. For $i = 1, 2, \cdots, n_1$, let $S_{1,i} = \{s \in p_1^{-1}S_1 \cap S_1 : sx \in K_{1,i}\}$, then we have

$$p_1^{-1}S_1 \cap S_1 = \bigcup_{i=1}^{n_1} S_{1,i}.$$

Since $\mathcal{P}$ has the Ramsey property, one has $S_{1,i_1} \in \mathcal{P}$ for some i_1. Set

$$S_2 = S_{1,i_1} \text{ and } K_2 = K_{1,i_1}.$$

Then we have $S_2 \in \mathcal{P}$, $K_2 \subseteq K_1$, $\mathrm{diam}(K_2) < 1/2$ and $p_1(K_2) \subseteq K_1$.

We continue inductively. Assume that S_n, K_n and p_{n-1} have been found such that $S_n \in \mathcal{P}$, $K_n \subseteq K_{n-1}$, $\mathrm{diam}(K_n) < 1/n$, $sx \in K_n$ for all $s \in S_n$ and $p_{n-1}(K_n) \subseteq K_{n-1}$. Then we apply the above argument to S_n and K_n, by Condition (1), there is $p_n \neq p_i$ for $i = 1, 2, \cdots, n-1$, such that $p_n^{-1}S_n \cap S_n \in \mathcal{P}$. By the construction of S_n and K_n, we know that $sx \in K_n \cap p_n^{-1}K_n$ for any $s \in p_n^{-1}S_n \cap S_n$. Let

$$K_n \cap p_n^{-1} K_n = \bigcup_{i=1}^{m_n} K_{n,i},$$

where each $K_{n,i}$ is a non-empty compact subset of X with $\mathrm{diam}(K_{n,i}) < 1/(n+1)$. Let $S_{n,i} = \{s \in p_n^{-1} S_n \cap S_n : sx \in K_{n,i}\}$. Then

$$p_n^{-1} S_n \cap S_n = \bigcup_{i=1}^{m_n} S_{n,i},$$

which follows that $S_{n,i_n} \in \mathcal{P}$ for some i_n. Set

$$S_{n+1} = S_{n,i_n} \quad \text{and} \quad K_{n+1} = K_{n,i_n}.$$

This completes the inductive process.

By induction, we obtain a sequence of non-empty compact sets $\{K_n\}_{n=1}^{\infty}$ and a sequence $\{p_n\}_{n=1}^{\infty}$ of G such that

- $K_1 \supseteq K_2 \supseteq \cdots \supseteq K_n \supseteq \cdots$;
- $\mathrm{diam}(K_n) < 1/n$ for each $n \geq 2$;
- $p_n(K_{n+1}) \subseteq K_n$ for each $n \geq 1$;
- $p_i \neq p_j$ for each $i \neq j$.

Let y be the single point in $\bigcap_{n=1}^{\infty} K_n$. Then we have for all m, $p_m y \in K_m$. This shows that $y \in K \cap Rec_{\mathcal{P}_{\inf}}(X, G)$. □

Next, we provide a characterization of subsets of the semigroup that force infinite recurrence via infinite IP-sets. Let $\mathcal{P}_{\mathrm{inf,ip}}$ denote the family of all subsets of the semigroup G that contains some infinite IP-set. We have the following lemma:

Lemma 3.7 *Let G be a semigroup which is either right or left cancellative. Then $b\mathcal{P}_{\mathrm{inf,ip}}$ has the Ramsey property.*

Proof This is established in Corollary 5.4 in the Appendix. □

The idea is that the proof involves results about the Stone–Čech compactification of G which are not needed for the rest of the results of this paper, so we leave it to an Appendix.

Theorem 3.8 *Suppose that G is a monoid and $S \subseteq G$. Statements (1) and (2) are equivalent and imply statement (3). If G is either right or left cancellative, then all three statement are equivalent.*

(1) S forces $\mathcal{P}_{\inf}$-recurrence;
(2) S forces $\mathcal{P}_{\mathrm{inf,ip}}$-recurrence;
(3) $S \in b\mathcal{P}_{\mathrm{inf,ip}}$,

Proof It follows directly from Lemma 3.5 and Theorem 3.2 that (1) ⇔ (2) ⇒ (3). Now we only show that (3) ⇒ (1) if G is either right or left cancellative.

By Theorem 3.6 and Lemma 3.7, it suffices to prove for all $S \in \mathcal{P}_{\mathrm{inf,ip}}$, there are infinitely many $g \in G$ such that $g^{-1}S \cap S \in b\mathcal{P}_{\mathrm{inf,ip}}$. Let $S \in b\mathcal{P}_{\mathrm{inf,ip}}$ and suppose that S contains a broken infinite IP-set $FP(\{p_n\}_{n=1}^{\infty})$. Fix $M \in \mathbb{N}$. Then for each $L \in \mathbb{N}$ with $L > M$, we can choose $s_L \in G$ such that

$$FP(\{p_n\}_{n=1}^{L}) \cdot s_L \subseteq S,$$

which implies

$$\begin{aligned} g^{-1}S \cap S &\supseteq \left(g^{-1} \cdot FP(\{p_n\}_{n=1}^{L}) \cdot s_L\right) \cap \left(FP(\{p_n\}_{n=1}^{L}) \cdot s_L\right) \\ &\supseteq \left[\left(g^{-1} \cdot FP(\{p_n\}_{n=1}^{L})\right) \cap \left(FP(\{p_n\}_{n=1}^{L})\right)\right] \cdot s_L \\ &\supseteq FP(\{p_n\}_{n=M+1}^{L}) \cdot s_L. \end{aligned}$$

for all $g \in FP(\{p_n\}_{n=1}^{M})$, and thus $g^{-1}S \cap S$ contains a broken infinite IP-set $FP(\{p_n\}_{n=M+1}^{\infty})$. Therefore, $g^{-1}S \cap S \in b\mathcal{P}_{\mathrm{inf,ip}}$ for all $g \in FP(\{p_n\}_{n=1}^{\infty})$. □

3.3 Force minimality

Recall that a dynamical system (X, G) is called *minimal* if it contains no proper subsystem, i.e., the orbit $\mathrm{orb}(x, G) = \{gx : g \in G\}$ of x is dense in X for all $x \in X$. A point x is called *a minimal point* if it belonging to some minimal subsystem of (X, G). Note that x is a minimal point of (X, G) if and only if $cl_X\{gx : g \in G\}$ is minimal.

Let G be a semigroup. A subset $S \subseteq G$ is called *syndetic* if there exists a finite subset F of G such that $F^{-1}S = \bigcup_{g \in F} g^{-1}S = G$. Denote by $\mathcal{P}_s$ the family of all syndetic sets in G.

It is a routine Zorn's Lemma argument to show that any dynamical system contains a minimal dynamical system. The proof of the following lemma can be found in [15, Proposition 5.21] with the caution that they use the left-right switches of both the definition of syndetic and the action of G on X.

Lemma 3.9 *Let (X, G) be a dynamical system and $x \in X$. Then x is a minimal point if and only if it is an $\mathcal{P}_s$-recurrent point.*

Definition 3.10 We say that a set $S \subset G$ *forces minimality* if whenever (X, G) is a dynamical system and $K \subseteq X$ is compact, and for some $x \in X$ and all $s \in S$, $sx \in K$, there exists a minimal subset non-disjoint from K.

Now we prove the following theorem.

Theorem 3.11 *Let G be a monoid and S a subset of G. Then the following conditions are equivalent:*

(1) S forces $\mathcal{P}_s$-recurrence;
(2) S forces minimality;
(3) $S \in b\mathcal{P}_s$;

Proof (1) $\Rightarrow$ (2) Let S be a set that forces $\mathcal{P}_s$-recurrence. Suppose that (X, G) is a dynamical system, K is a compact subset of X, and $x \in X$ is a point such that $sx \in K$ for all $s \in S$. Then there exists a $\mathcal{P}_s$-recurrence point $z \in K$. By Lemma 3.9, one has z is a minimal point so $z \in cl_X\{gx : g \in G\} \cap K$. Therefore, S forces minimality.

(2) $\Rightarrow$ (3) Let S be a set that forces minimality. Let Σ_2, X, K and 1_S be as in Lemma 2.7. Clearly, $s1_S \in K$ for all $s \in S$. By the force minimality, there exists a minimal point $y \in K$. Notice that K is also a non-empty open subset of X. By Lemma 3.9, one has $N(y, K) \in \mathcal{P}_\mathrm{s}$. For each $F \in \mathcal{P}_f(G)$, there exists $s_F \in G$ such that $s_F 1_S \in V$, where

$$V = \{x \in X : x(s) = y(s) \text{ for all } s \in F\}.$$

Thus, for every $s \in N(y, K) \cap F$, one has

$$1_S(s \cdot s_L) = s_L 1_S(s) = y(s) = sy(e) = 1.$$

This shows that

$$(N(x, K) \cap F) \cdot s_L \subseteq S. \tag{3.1}$$

Therefore, $S \in b\mathcal{P}_\mathrm{s}$.

(3) $\Rightarrow$ (1) Suppose that $S \in b\mathcal{P}_\mathrm{s}$. Then there exists some $\widetilde{S} \in \mathcal{P}_\mathrm{s}$ such that for every $F \in \mathcal{P}_f(G)$, there exists $s_F \in G$ such that

$$(\widetilde{S} \cap F) \cdot s_F \subseteq S.$$

Now let (X, G) be a dynamical system, K a compact subset of X and $x \in X$ such that $sx \in K$ for all $s \in S$. Since G is countable, we can find an increasing sequence $\{F_n\}_{n=1}^{\infty}$ of finite subsets of G such that

$$F_1 \subset F_2 \subset \cdots \subset F_n \subset \cdots \quad \text{and} \quad \bigcup_{n=1}^{\infty} F_n = G.$$

Without loss of generality, we may assume that $F_1 \cap \widetilde{S} \neq \emptyset$. Pick some $r \in F_1 \cap \widetilde{S}$, and let $z_n = s_{F_n} x \in r^{-1}K$ for all $n \in \mathbb{N}$. By the compactness of K and the continuity of r, we can find a subsequence $\{n_i\}_{i=1}^{\infty}$ such that z_{n_i} convergence to $z \in r^{-1}K$. Given $g \in r^{-1}\widetilde{S}$. Then we have $rg \in \widetilde{S}$, which implies for all sufficiently large i,

$$rgz_{n_i} = (rg \cdot s_{F_{n_i}})x \in K.$$

By the continuity of rg we have $rgz \in K$. This shows that $gz \in r^{-1}K$ for all $g \in r^{-1}\widetilde{S}$.

Let F be a finite subset of G such that $F^{-1}\widetilde{S} = G$. Choose a finite subset H of G such that $F = rH$. Then we can obtain

$$cl_X\{gz : g \in G\} \subseteq \bigcup_{h \in H} (rh)^{-1} K. \tag{3.2}$$

Indeed, for each $g \in G$, there is $h \in H$ such that $rhg \in \widetilde{S}$, which implies $hg \in r^{-1}\widetilde{S}$, and thus $gz \in (rh)^{-1}K$. For the closed invariant subset $cl_X\{gz : g \in G\}$ we can find a non-empty minimal subset $Y \subseteq cl_X\{gz : g \in G\}$. Furthermore, we know that every point in Y is $\mathcal{P}_s$-recurrent by Lemma 3.9. Last, we only show that $Y \cap K \neq \emptyset$. Pick $y \in Y$, choose a sequence $\{g_n\}_{n=1}^{\infty}$ of G such that $g_n z \to y$. By (3.2), we know that for each n, there exists some $h_n \in H$ such that $rh_n g_n z \in K$. Since H is finite, we may assume that h_n is constantly equal to h. It follows that

$$rhg_n z \to rhy \in K.$$

Thus, $rhy \in Y \cap K$. This completes the proof. □

3.4 Force non-wandering

In this subsection, we study the non-wandering for semigroup actions. Let (X, G) be a dynamical system. For a non-empty family $\mathcal{P}$ of non-empty sets of the semigroup G, we say that a point $x \in X$ is *$\mathcal{P}$-non-wandering* if $N(U, U) \in \mathcal{P}$ for every neighborhood U of x, where

$$N(U, U) = \{g \in G : U \cap g^{-1}U \neq \emptyset\}.$$

Denote the set of all $\mathcal{P}$-non-wandering points of (X, G) by $\Omega_{\mathcal{P}}(X, G)$.

Definition 3.12 We say that a subset $S \subseteq G$ *forces $\mathcal{P}$-non-wandering* if whenever (X, G) is a dynamical system and $K \subset X$ is compact, and for some $x \in X$ and all $s \in S$, $sx \in K$, then we have $K \cap \Omega_{\mathcal{P}}(X, G) \neq \emptyset$.

Theorem 3.13 *Let $\mathcal{P}$ be a non-empty family of non-empty sets of the semigroup G such that*

(1) right shift invariant: $S \in \mathcal{P}$ implies $Sg^{-1} \in \mathcal{P}$ for all $g \in G$;
(2) $\mathcal{P}$ has the Ramsey property.

Then S forces $\mathcal{P}$-non-wandering for all $S \in \mathcal{P}$.

Proof Let $S \in \mathcal{P}$ and K be a compact set in a dynamical system (X, G) such that for some point $x \in X$ and all $s \in S$, $sx \in K$. Write $K = \bigcup_{i=1}^{n_1} K_{1,i}$, where each $K_{1,i}$ is a non-empty compact subset with $\mathrm{diam}(K_{1,i}) < 1$. For $i = 1, 2, \ldots, n_1$, let $S_{1,i} = \{s \in S : sx \in K_{1,i}\}$. Then we have $S = \bigcup_{i=1}^{n_1} S_{1,i}$. Since $\mathcal{P}$ has the Ramsey property, one has $S_{1,i_1} \in \mathcal{P}$ for some $i_1 \in \{1, 2, \ldots, n_1\}$. Set

$$S_1 = S_{1,i_1} \text{ and } K_1 = K_{1,i_1}.$$

By induction, we obtain a sequence of non-empty compact sets $\{K_n\}_{n=1}^{\infty}$ and a sequence $\{S_n\}_{n=1}^{\infty} \subset \mathcal{P}$ such that

- $K \supseteq K_1 \supseteq K_2 \supseteq \cdots \supseteq K_n \supseteq \cdots$;
- $S \supseteq S_1 \supseteq S_2 \supseteq \cdots \supseteq S_n \supseteq \cdots$;

- $\mathrm{diam}(K_n) < 1/n$ for each $n \geq 1$;
- $sx \in K_n$ for all $s \in S_n$.

Let y be the single point in $\bigcap_{n=1}^{\infty} K_n$. Then for every neighborhood U of y, there exists some n_U such that $K_{n_U} \subset U$. Pick some $s \in S_{n_U}$, then we have $sx \in K_{n_U} \subset U$ and $h(sx) = (hs)x \in K_{n_U} \subseteq U$ for all $h \in S_{n_U}s^{-1} \in \mathcal{P}$. This implies $S_{n_U}s^{-1} \subset N(U, U)$, and hence $N(U, U) \in \mathcal{P}$. □

4 Density of group and force recurrence

The notions of upper Banach density of group have been studied from several points of view (see, for example, [2, 14]). Let G be a countable discrete infinite semigroup. For a subset A in G and a finite set $F \subset G$, define

$$\overline{D}_F(A) = \sup_{g \in G} \frac{|A \cap Fg|}{|F|}.$$

The *upper Banach density* of A is defined by

$$BD^*(A) = \inf_{F \in \mathcal{P}_f(G)} \overline{D}_F(A) \tag{4.1}$$

Recall that an infinite countable discrete group G is called *amenable* if there exists a sequence of finite subsets $F_n \subset G$ such that for every $g \in G$,

$$\lim_{n \to +\infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0, \tag{4.2}$$

where $|\cdot|$ denotes the cardinality of a set and $\triangle$ stands for the symmetric difference of sets. A sequence satisfying condition (4.2) is called a *Følner sequence* (see [18]). The basic example of an amenable group is the group $G = \mathbb{Z}^d$ for some $d \in \mathbb{N}$, and $\{F_n = [0, n-1]^d : n \in \mathbb{N}\}$ is a Følner sequence of G.

Lemma 4.1 *Let G be a countably infinite discrete amenable group, let $A \subseteq G$ such that $BD^*(A) > 0$, and let F be a finite subset of G. There exists $g \in G \setminus F$ such that $BD^*(A \cap g^{-1}A) > 0$.*

Proof This follows immediately from Proposition 2.2 (ii) of [3]. □

The proof of Lemma 4.2 is adapted from the proof of [23, Theorem 11.11].

Lemma 4.2 *Let G be a countably infinite discrete amenable group and let $S \subseteq G$ such that $BD^*(S) > 0$. Then $S \in b\mathcal{P}_{\mathrm{inf,ip}}$.*

Proof Let e be the identity of G and let $D_1 = S$. By Lemma 4.1, pick $g_1 \in G \setminus \{e\}$ such that $BD^*(D_1 \cap g_1^{-1}D_1) > 0$.

Let $n \in \mathbb{N}$ and assume we have chosen $(D_k)_{k=1}^n$ and $(g_k)_{k=1}^n$ such that for $k \in \{1, 2, \cdots, n\}$,

(1) $g_k \in G$ and $D_k \subset G$;
(2) $BD^*(D_k \cap g_k^{-1}D_k) > 0$;
(3) if $k < n$, then $D_{k+1} = D_k \cap g_k^{-1}D_k$; and
(4) if $k < n$, then $g_{k+1} \notin FP(\{g_t\}_{t=1}^k)$.

Let $D_{n+1} = D_n \cap g_n^{-1}D_n$ and let $F = FP(\{g_t\}_{t=1}^n) \cup \{e\}$. Pick by Lemma 4.1 some $g_{n+1} \in G \setminus F$ such that $BD^*(D_{n+1} \cap g_{n+1}^{-1}D_{n+1}) > 0$. One easily shows by induction that for each n,

$$D_{n+1} = S \cap \left(\bigcap_{g \in FP(\{g_t\}_{t=1}^n)} g^{-1}S \right).$$

Let $P = FP(\{g_t\}_{t=1}^\infty)$. By hypothesis (4), P is an infinite IP-set. Given finite nonempty $F \subset P$ pick $n \in \mathbb{N}$ such that $F \subseteq FP(\{g_t\}_{t=1}^n)$ and pick $s_F \in D_{n+1}$. Then $F \cdot s_F \subseteq S$ so $S \in b\mathcal{F}_{\text{inf,ip}}$. □

Theorem 4.3 *Let G be an infinite countable discrete amenable group. If $S \subseteq G$ has positive upper Banach density, then S forces $\mathcal{P}_{\text{inf}}$-recurrence.*

Proof Since G is a group, this is an immediate consequence of Lemma 4.2 and Theorem 3.8. □

Question 4.4 *Let G be an arbitrary countable discrete group or semigroup and let S be a subset of G with positive upper Banach density. Must S force $\mathcal{P}_{\text{inf}}$-recurrence?*

Acknowledgements The authors would like to thank the referee for many valuable and constructive comments and suggestions, especially providing Lemmas 4.1, 4.2 and the Appendix, which help us to improve the paper. We also thank Hongzhi Hu for a very careful reading.

Funding The authors are supported by NNSF of China (11861010, 11761012) and NSF of Guangxi Province (2018GXNSFFA281008). The first author is also supported by the Cultivation Plan of Thousands of Young Backbone Teachers in Higher Education Institutions of Guangxi Province, Program for Innovative Team of Guangxi University of Finance and Economics and Project of Guangxi Key Laboratory of Quantity Economics.

Appendix: When $b\mathcal{P}_{\text{inf,ip}}$ has the Ramesey property

In this section we utilize the algebraic structure of the Stone–Čech compactification βG of a discrete semigroup $(G, \cdot)$. We shall assume that the reader is familiar with the basic facts about this structure. For an elementary introduction see [24, Part I].

We shall show that if $\beta G \setminus G$ is a subsemigroup of βG, in particular if G is either right or left cancellative, then both $\mathcal{P}_{\text{inf,ip}}$ and $b\mathcal{P}_{\text{inf,ip}}$ have the Ramsey property. An exact characterization of when $\beta G \setminus G$ is a subsemigroup of βG is given in [24, Theorem 4.28].

Lemma 5.1 *Assume that G is a semigroup, p is an idempotent in $\beta G \setminus G$, and $A \in p$. Then A contains an infinite IP-set. In fact there exists an injective sequence $(x_n)_{n=1}^\infty$ in G such that $FP(\{x_n\}_{n=1}^\infty) \subseteq A$.*

Proof Let $A^\star = \{x \in A : x^{-1}A \in p\}$. By [24, Lemma 4.14] if $x \in A^\star$, then $x^{-1}A^\star \in p$. Choose $x_1 \in A^\star$. Inductively let $n \in \mathbb{N}$ and assume we have chosen injective $(x_t)_{t=1}^n$ in G such that $E = FP(\{x_t\}_{t=1}^n) \subseteq A^\star$. Since $p \in \beta G \setminus G$, $G \setminus \{x_1, x_2, \ldots, x_n\} \in p$ so

$$\left(A^\star \cap \bigcap_{y \in E} y^{-1}A^\star\right) \setminus \{x_1, x_2, \ldots, x_n\} \in p.$$

Pick $x_{n+1} \in \left(A^\star \cap \bigcap_{y\in E} y^{-1}A^\star\right) \setminus \{x_1, x_2, \ldots, x_n\}$. □

Lemma 5.2 *Assume that G is a semigroup, $\beta G \setminus G$ is a subsemigroup of βG, and $A \subseteq G$. If A contains an infinite IP-set $FP(\{x_n\}_{n=1}^\infty)$, then there is an idempotent $p \in \beta G \setminus G$ such that for every $m \in \mathbb{N}$, $FP(\{x_n\}_{n=m}^\infty) \in p$.*

Proof We claim that for each $m \in \mathbb{N}$, $FP(\{x_n\}_{n=m}^\infty)$ is infinite. To see this let $m > 1$ and let $E = FP(\{x_n\}_{n=1}^{m-1})$. Then

$$FP(\{x_n\}_{n=1}^\infty) = E \cup FP(\{x_n\}_{n=m}^\infty) \cup \bigcup_{y\in E} y \cdot FP(\{x_n\}_{n=m}^\infty)$$

so one of the listed sets is infinite and thus $FP(\{x_n\}_{n=m}^\infty)$ is infinite.

Let $\mathcal{A} = \{FP(\{x_n\}_{n=m}^\infty) : m \in \mathbb{N}\}$. Then $\mathcal{A}$ is a nested family of infinite sets so by [24, Corollary 3.14], there is some $q \in \beta G \setminus G$ such that $\mathcal{A} \subseteq q$. That is, $(\beta G \setminus G) \cap \bigcap_{m=1}^\infty \overline{FP(\{x_n\}_{n=m}^\infty)} \neq \emptyset$. By [24, Lemma 5.11], $\bigcap_{m=1}^\infty \overline{FP(\{x_n\}_{n=m}^\infty)}$ is a semigroup so $(\beta G \setminus G) \cap \bigcap_{m=1}^\infty \overline{FP(\{x_n\}_{n=m}^\infty)}$ is a compact right topological semigroup so by [16, Lemma 1], there is an idempotent $p \in (\beta G \setminus G) \cap \bigcap_{m=1}^\infty \overline{FP(\{x_n\}_{n=m}^\infty)}$. □

Theorem 5.3 *Assume that G is a semigroup, $\beta G \setminus G$ is a subsemigroup of βG, and $A \subseteq G$.*

(1) $A \in \mathcal{P}_{\text{inf,ip}}$ if and only if there is an idempotent $p \in \beta G \setminus G$ such that $A \in p$;
(2) $A \in b\mathcal{P}_{\text{inf,ip}}$ if and only if there exists an idempotent $p \in \beta G \setminus G$ and $q \in \beta G$ such that $A \in p \cdot q$.

Proof (1) The necessity follows from Lemma 5.2 and the sufficiency follows from Lemma 5.1.

(2) To establish the necessity, pick a sequence $(x_n)_{n=1}^\infty$ in G such that $FP(\{x_n\}_{n=1}^\infty)$ is infinite and for each $m \in \mathbb{N}$ there exists $s_m \in G$ such that $FP(\{x_n\}_{n=1}^m) \cdot s_m \subseteq A$. Pick by Lemma 5.2 an idempotent $p \in \beta G \setminus G$ such that for every $m \in \mathbb{N}$, $FP(\{x_n\}_{n=m}^\infty) \in p$. Pick $q \in \beta G$ such that $\{\{s_m : m > n\} : n \in \mathbb{N}\} \subseteq q$. We claim that $A \in p \cdot q$. To see this, it suffices to show that $FP(\{x_n\}_{n=1}^\infty) \subseteq \{y \in G : y^{-1}A \in q\}$ by [24, Lemma 4.12] so let $z \in FP(\{x_n\}_{n=1}^\infty)$ and pick $F \in \mathcal{P}_f(\mathbb{N})$ such that $z = \prod_{t\in F} x_t$. Let $n = \max F$. Then $\{s_m : m > n\} \in q$ and $\{s_m : m > n\} \subseteq z^{-1}A$.

For the sufficiency, pick an idempotent $p \in \beta G \setminus G$ and $q \in \beta G$ such that $A \in p \cdot q$. Let $B = \{y \in G : y^{-1}A \in q\}$. Then $B \in p$ by [24, Lemma 4.12] so pick by Lemma 5.1 an injective sequence $(x_n)_{n=1}^\infty$ in G such that $FP(\{x_n\}_{n=1}^\infty) \subseteq B$. Now let

$m \in \mathbb{N}$. It suffices to show that there exists $s_m \in G$ such that $FP(\{x_n\}_{n=1}^m)\cdot s_m \subseteq A$. Let $E = FP(\{x_n\}_{n=1}^m)$. Then E is a finite subset of B so we may pick $s_m \in \bigcap_{y\in E} y^{-1}A$. □

Corollary 5.4 *Assume that G is a semigroup and $\beta G \setminus G$ is a subsemigroup of βG. Then $\mathcal{P}_{\mathrm{inf,ip}}$ and $b\mathcal{P}_{\mathrm{inf,ip}}$ have the Ramsey property. In particular if G is infinite and is either right cancellative or left cancellative, then $\mathcal{P}_{\mathrm{inf,ip}}$ and $b\mathcal{P}_{\mathrm{inf,ip}}$ have the Ramsey property.*

Proof Let S_1 and S_2 be subsets of G. If $S_1 \cup S_2 \in \mathcal{P}_{\mathrm{inf,ip}}$, pick an idempotent $p \in \beta G \setminus G$ such that $S_1 \cup S_2 \in p$. Since p is an ultrafilter, either $S_1 \in p$ or $S_2 \in p$. If $S_1 \cup S_2 \in b\mathcal{P}_{\mathrm{inf,ip}}$, pick an idempotent $p \in \beta G \setminus G$ and $q \in \beta G$ such that $S_1 \cup S_2 \in p \cdot q$. Since $p \cdot q$ is an ultrafilter, either $S_1 \in p \cdot q$ or $S_2 \in p \cdot q$. The "in particular" conclusions follow from [24, Corollary 4.29]. □

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