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RESEARCH ARTICLE

Quantitative results for the limiting semigroup generated by the multidimensional Bernstein operators

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Abstract
A quantitative estimate for the Trotter's approximation theorem for the limiting semigroup of operators generated by the multidimensional Bernstein operators on a simplex is obtained. For this, an essential step consists in an explicit representation of the derivatives of higher order of multidimensional Bernstein operators.

Keywords Multidimensional Bernstein operators on a simplex · Trotter's approximation theorem · Limiting semigroup of operators

Mathematics Subject Classification 20Mxx · 41A36 · 41A10 · 41A25

1 Introduction

Let X be a Banach space, endowed with norm $\|\cdot\|$. Denote by $L(X)$ the space of bounded linear operators $T : X \to X$, endowed with norm $\|L\| = \sup\{\|Lx\|,\ x \in X, \|x\| = 1\}$. A C_0 semigroup of operators on the space X is a family of operators $\{T(t)\}_{t\geq 0}$, $T(t) \in L(X)$, with the properties

a) $T(t+s) = T(t)T(s)$, for $t, s \geq 0$;
b) $\lim_{t\to 0+} T(t)x = x$, for any $x \in X$, in the sense of norm of X.

As a general bibliography of the subject we mention [1–3, 5, 10, 17, 18]. A basic result concerning C_0 semigroups of operators is given by Trotter's approximation theorem.

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Theorem A [21] *Let* $(L_n)_{n\in\mathbb{N}}$ *be a sequence of bounded linear operators on a Banach space* X *and let* $(\rho_n)_{n\in\mathbb{N}}$ *be a decreasing sequence of positive real numbers tending to* 0. *Suppose that there exist* $M \geq 0$ *and* $\omega \in \mathbb{R}$ *such that*

$$\|L_n^k\| \leq M e^{\omega \rho_n k}, \ (k, n \in \mathbb{N}).$$

Moreover, assume that D *is a dense subspace of* X *and for every* $f \in D$ *the following Voronovskaja-type formula holds*

$$Af := \lim_{n\to\infty} \frac{L_n(f) - f}{\rho_n}$$

If $(\lambda I - A)(D)$ *is dense in* X *for some* $\lambda > \omega$, *then there exists a* C_0*-semigroup* $(T(t))_{t\geq 0}$ *such that for every* $f \in X$ *and every sequence* $(k(n))_n \in \mathbb{N}$ *of positive integers satisfying* $\lim_{n\to+\infty} k(n) \cdot \rho_n = t$, *we have*

$$T(t)f = \lim_{n\to\infty} L_n^{k(n)}(f).$$

A version of Trotter's approximation theorem is the following

Theorem B ([3], *a part of Corollary* 2.2.11) *Let* $(L_n)_{n\geq 1}$ *be a sequence of linear operators on the Banach space* E, *with* $\|L_n\| \leq 1$ *and let* $(\rho_n)_{n\geq 1}$ *be a sequence of positive real numbers such that* $\lim_{n\to\infty} \rho(n) = 0$. *Let* $A_0 : D_0 \to E$ *be a linear operator defined on a subspace* D_0 *of* E *and assume that* (i) *there is a family* $(E_i)_{i\in I}$ *of finite dimensional subspaces of* D_0 *which are invariant under each* L_n *and whose union* $\bigcup_{i\in I} E_i$ *is dense in* E; (ii) $\lim\limits_{n\to\infty} \frac{L_n(u)-u}{\rho(n)} = A(u)$ *for every* $u \in D_0$.

Then A_0 *is closable and its closure* $A : D(A) \to E$ *is the generator of a contraction* C_0*-semigroup* $(T(t))_{t\geq 0}$ *on* E *satisfying the following condition: if* $(k(n))_{n\geq 1}$ *is a sequence of positive integers with* $\lim\limits_{n\to\infty} k(n)/\rho(n) = t$, *then, for every* $f \in E$,

$$T(t)(f) = \lim_{n\to\infty} L_n^{k(n)}(f). \tag{1}$$

The iterates and the limiting semigroup generated by Bernstein operators were studied in [6–9, 12, 14, 16] among others. The semigroup generated by multidimensional Bernstein operators was considered in [6, 7, 15]. For the limiting groups generated by other positive linear operators we cite [4, 11, 13, 15, 19, 20].

2 Additional results for multidimensional Bernstein operators

We fix the following notation. Let $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $d \in \mathbb{N}$ be fixed. For a multi-index $\overline{k} \in \mathbb{N}_0^d$, $\overline{k} = (k_1, \dots, k_d)$, denote $|\overline{k}| = k_1 + \dots + k_d$ and $\overline{k}! = k_1! \dots k_d!$. For $n \in \mathbb{N}$, if $\overline{k} \in \mathbb{N}_0^d, |\overline{k}| \leq n$, define $\binom{n}{\overline{k}} = \frac{n!}{\overline{k}!(n-|\overline{k}|)!}$.

Define the d-simplex

$$\Delta_d := \{\overline{x} = (x_1, \dots, x_d) \mid x_i \geq 0,\ (1 \leq i \leq d),\ x_1 + \dots + x_d \leq 1\}. \tag{2}$$

Vectors $\overline{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$, $1 \leq i \leq d$, form the standard base of the space $\mathbb{R}^d$. If $\overline{x} = (x_1, \dots, x_d) \in \Delta_d$ put $|\overline{x}| = x_1 + \dots + x_d$. Hence $|\overline{x}| \leq 1$. If, in addition we take $\overline{k} = (k_1, \dots, k_d) \in \Lambda_d^n$, then define $\overline{x}^{\overline{k}} = x_1^{k_1} \dots x_d^{k_d}$. With this notation we define now

$$p_{n,\overline{k}}(\overline{x}) := \binom{n}{\overline{k}} \overline{x}^{\overline{k}} (1 - |\overline{x}|)^{n-|\overline{k}|}. \tag{3}$$

We extend the definition of $p_{n,\overline{k}}(\overline{x})$, for $\overline{k} \in \mathbb{Z}^d$, putting

$$p_{n,\overline{k}}(\overline{x}) := 0 \quad \text{if } \exists i \text{ such that } k_i < 0 \text{ or } |\overline{k}| > n. \tag{4}$$

In the case $d = 1$ and $\overline{k} = k$, $\overline{x} = x$ we write simply $p_{n,k}(x)$ instead of $p_{n,\overline{k}}(\overline{x})$.

With these preparations we can define the Bernstein operator on the simplex Δ_d:

$$B_n(f, \overline{x}) := \sum_{|\overline{k}| \in \Lambda_d^n} p_{n,\overline{k}}(\overline{x}) f\left(\frac{\overline{k}}{n}\right), \tag{5}$$

where $\frac{\overline{k}}{n} = \left(\frac{k_1}{n}, \dots \frac{k_d}{n}\right)$, $n \in \mathbb{N}$, $f : \Delta_d \to \mathbb{R}$, $\overline{x} \in \Delta_d$.

Let $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$. Suppose $|\alpha| \geq 1$, where $|\alpha| = \alpha_1 + \dots + \alpha_d$. If $f \in C^{|\alpha|}(\Delta_d)$ define

$$\frac{\partial^\alpha f}{\partial \overline{x}^\alpha} := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}. \tag{6}$$

If $|\alpha| = 0$, define $\frac{\partial^\alpha f}{\partial \overline{x}^\alpha} := f$.

For $\alpha \in \mathbb{N}_0^d$ denote by $C^\alpha(\Delta_d)$ the space of functions $f : \Delta_d \to \mathbb{R}$ which admits the partial derivative $\frac{\partial^\alpha f}{\partial \overline{x}^\alpha}$ continuous on Δ_d. For $1 \leq i \leq d$ consider functions $\pi_i : \Delta_d \to \mathbb{R}$, $\pi_i(\overline{x}) = x_i$.

The next lemma is easy to obtain and in great part well known, see for instance [2, Section 6.2].

Lemma 1 *For $\overline{x} = (x_1, \dots, x_d) \in \Delta_d$ we have*

i) $B_n(\pi_i - x_i, \overline{x}) = 0$, $(1 \leq i \leq d)$;
ii) $B_n((\pi_i - x_i)(\pi_j - x_j), \overline{x}) = -\frac{x_i x_j}{n}$, $(1 \leq i,j \leq d,\ i \neq j)$;
iii) $B_n((\pi_i - x_i)^2, \overline{x}) = \frac{x_i(1-x_i)}{n}$, $(1 \leq i \leq d)$;
iv) $B_n((\pi_i - x_i)^3, \overline{x}) = \frac{x_i(1-x_i)(1-2x_i)}{n^2}$, $(1 \leq i \leq d)$;
v) $B_n((\pi_i - x_i)^2(\pi_j - x_j), \overline{x}) = \frac{x_i x_j(2x_i - 1)}{n^2}$; $(1 \leq i,j \leq d,\ i \neq j)$;

vi) $B_n((\pi_i - x_i)(\pi_j - x_j)(\pi_m - x_m), \overline{x}) = \frac{2x_i x_j x_m}{n^2}$, $(1 \le i < j < m \le d)$;
vii) $B_n((\pi_i - x_i)^4, \overline{x}) = \frac{1}{n^2}\left(3 - \frac{6}{n}\right)x_i^2(1 - x_i)^2 + \frac{x_i(1-x_i)}{n^3}$, $(1 \le i \le d)$;
viii)

$$B_n((\pi_i - x_i)^2(\pi_j - x_j)^2, \overline{x}) = \frac{1}{n^2}\left(3 - \frac{6}{n}\right)x_i^2 x_j^3 + \left(-\frac{1}{n^2} + \frac{2}{n^3}\right)(x_i^2 x_j + x_i x_j^2) + \frac{n-1}{n^3}x_i x_j,$$
$(1 \le i,j \le d,\ i \ne j)$.

Theorem 1 *Let $\alpha \in \mathbb{N}_0^d$, $|\alpha| \ge 1$. Then for any $f \in C^{|\alpha|}(\Delta_d)$, $n \in \mathbb{N}$, $n \ge |\alpha|$ and $\overline{x} \in \Delta_d$ we have*

$$\frac{\partial^\alpha}{\partial \overline{x}^\alpha} B_n(f, \overline{x}) = \frac{n!}{(n - |\alpha|)!} \sum_{|\overline{k}| \le n - |\alpha|} p_{n-|\alpha|, \overline{k}}(\overline{x}) \times \\ \times \iint \cdots \int_{\left[0, \frac{1}{n}\right]^{|\alpha|}} \frac{\partial^\alpha}{\partial \overline{t}^\alpha} f\left(\frac{\overline{k}}{n} + \sum_{i \in I_\alpha} \left(\sum_{j=1}^{\alpha_i} t_{i,j}\right)\overline{e}_i\right) d\overline{t}_\alpha, \tag{7}$$

where $I_\alpha = \{i \in \{1, \ldots, d\} \mid \alpha_i \ge 1\}$ and

$$d\overline{t}_\alpha = \prod_{i \in I_\alpha} \prod_{j=1}^{\alpha_i} dt_{i,j}.$$

In the case $|\alpha| = 0$, the term $\iint \ldots \int_{\left[0, \frac{1}{n}\right]^{|\alpha|}} \frac{\partial^\alpha}{\partial \overline{t}^\alpha} f\left(\frac{\overline{k}}{n} + \sum_{i \in I_\alpha}\left(\sum_{j=1}^{\alpha_i} t_{i,j}\right)\overline{e}_i\right) d\overline{t}_\alpha$ is reduced to $f\left(\frac{\overline{k}}{n}\right)$.

Proof We consider only the case $d \ge 2$, since the proof the case $d = 1$ can be easily deduced from the case $d \ge 2$.

The following formula is well-known.

$$(p_{s,k}(x))' = s(p_{s-1,k-1}(x) - p_{s-1,k}(x)),\ s \in \mathbb{N},\ k \in \mathbb{Z},\ x \in [0, 1]. \tag{8}$$

We induct on $r := |\alpha|$. For $r = 0$ relation (7) is obvious. Suppose that relation (7) holds for any $d \ge 1$ and any α with $|\alpha| = r$ and let show that it is a true for a multi-index $\beta = (\beta_1, \ldots, \beta_d)$ with $|\beta| = r + 1$. Then there are a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $|\alpha| = r$ and an index $1 \le i \le d$ such that $\beta_i = \alpha_i + 1$ and $\beta_j = \alpha_j$, for $1 \le j \le d$, $j \ne i$. To simplify the notation, we can suppose that $i = d$. In other cases we make a renumbering of the variables.

Let $\overline{x} \in \Delta_d$, $\overline{x} = (x_1, \ldots, x_d)$. Denote $|\overline{x}| = x_1 + \ldots + x_d$. Suppose $x_d > 0$. Define $\overline{z} = (x_1, \ldots, x_{d-1})$ and $|\overline{z}| = x_1 + \ldots x_{d-1}$. Then $|\overline{z}| < 1$. Denote also $y := \frac{x_d}{1-|\overline{z}|} \in [0, 1]$ and $m := n - |\alpha| = n - r$.

Let $\overline{k} \in \mathbb{N}_0^d$, with $|\overline{k}| = m$. Denote $\overline{\ell} := (k_1, \ldots k_{d-1})$. Then $|\overline{k}| = |\overline{\ell}| + k_d$. We can write

$$\begin{aligned} p_{m,\overline{k}}(\overline{x}) &= \frac{m!}{k_1! \dots k_d!(m-|\overline{k}|)!} x_1^{k_1} \dots x_d^{k_d} (1-|\overline{x}|)^{m-|\overline{k}|} \\ &= \frac{m!}{k_1! \dots k_{d-1}!(m-|\overline{\ell}|)!} x_1^{k_1} \dots x_{d-1}^{k_{d-1}} (1-|\overline{z}|)^{m-|\overline{\ell}|} \\ &\quad \times \frac{(m-|\overline{\ell}|)!}{k_d!(m-|\overline{k}|)!} \frac{x_d^{k_d}(1-|\overline{x}|)^{m-|\overline{k}|}}{(1-|\overline{z}|)^{m-|\overline{\ell}|}} \\ &= p_{m,\overline{\ell}}(\overline{z}) \cdot p_{m-|\overline{\ell}|,k_d}(y). \end{aligned} \tag{9}$$

For $\overline{k} \in \mathbb{N}_0^d, |\overline{k}| \le m$, denote

$$T_{\overline{k}} = \iint \dots \int_{\left[0,\frac{1}{n}\right]^{|\alpha|}} \frac{\partial^\alpha}{\partial \overline{t}^\alpha} f\left(\frac{\overline{k}}{n} + \sum_{i \in I_\alpha} \left(\sum_{j=1}^{\alpha_i} t_{i,j}\right) \overline{e}_i\right) d\overline{t}_\alpha.$$

By the hypothesis of induction we have

$$\frac{\partial^\alpha}{\partial \overline{x}^\alpha} B_n(f,\overline{x}) = \frac{n!}{m!} \sum_{|\overline{k}| \le m} p_{m,\overline{k}}(\overline{x}) T_{\overline{k}}.$$

Using relation (9) and the decomposition of the sum

$$\sum_{|\overline{k}| \le m} = \sum_{|\overline{\ell}| \le m} \sum_{k_d=0}^{m-|\overline{\ell}|}, \tag{10}$$

we can write

$$\frac{\partial^\alpha}{\partial \overline{x}^\alpha} B_n(f,\overline{x}) = \frac{n!}{m!} \sum_{|\overline{\ell}| \le m} p_{m,\overline{\ell}}(\overline{z}) \sum_{k_d=0}^{m-|\overline{\ell}|} p_{m-|\overline{\ell}|,k_d}(y) T_{\overline{k}}. \tag{11}$$

By relation (11) it follows

$$\begin{aligned} \frac{\partial^\beta}{\partial \overline{x}^\beta} B_n(f,\overline{x}) &= \frac{\partial}{\partial x_d} \frac{\partial^\alpha}{\partial \overline{x}^\alpha} B_n(f,\overline{x}) \\ &= \frac{n!}{m!} \sum_{|\overline{\ell}| \le m} p_{m,\overline{\ell}}(\overline{z}) \sum_{k_d=0}^{m-|\overline{\ell}|} \frac{\partial}{\partial x_d} p_{m-|\overline{\ell}|,k_d}(y) T_{\overline{k}}, \end{aligned}$$

where the first $d-1$ components of $\overline{k}$ are fixed and form vector $\overline{\ell}$. Then

$$\begin{aligned}
&\frac{\partial^\beta}{\partial \bar{x}^\beta} B_n(f,\bar{x})\\
&\quad = \frac{n!}{m!} \sum_{|\bar{\ell}|\le m} p_{m,\bar{\ell}}(\bar{z}) \sum_{k_d=0}^{m-|\bar{\ell}|} \frac{\mathrm{d}}{\mathrm{dy}} p_{m-|\bar{\ell}|,k_d}(y) \frac{1}{1-|\bar{z}|} \cdot T_{\bar{k}}\\
&\quad = \frac{n!}{m!} \sum_{|\bar{\ell}|\le m} p_{m,\bar{\ell}}(\bar{z}) \sum_{k_d=0}^{m-|\bar{\ell}|} \frac{m-|\bar{\ell}|}{1-|\bar{z}|} \Big(p_{m-|\bar{\ell}|-1,k_d-1}(y) - p_{m-|\bar{\ell}|-1,k_d}(y) \Big) \cdot T_{\bar{k}}\\
&\quad = \frac{n!}{m!} \sum_{|\bar{\ell}|\le m-1} \frac{m-|\bar{\ell}|}{1-|\bar{z}|} p_{m,\bar{\ell}}(\bar{z}) \sum_{k_d=0}^{m-|\bar{\ell}|} \Big(p_{m-|\bar{\ell}|-1,k_d-1}(y) - p_{m-|\bar{\ell}|-1,k_d}(y) \Big) \cdot T_{\bar{k}}\\
&\quad = \frac{n!}{(m-1)!} \sum_{|\bar{\ell}|\le m-1} p_{m-1,\bar{\ell}}(\bar{z}) \sum_{k_d=0}^{m-|\bar{\ell}|} \Big(p_{m-|\bar{\ell}|-1,k_d-1}(y) - p_{m-|\bar{\ell}|-1,k_d}(y) \Big) \cdot T_{\bar{k}}\\
&\quad = \frac{n!}{(m-1)!} \sum_{|\bar{\ell}|\le m-1} p_{m-1,\bar{\ell}}(\bar{z}) \sum_{k_d=0}^{m-|\bar{\ell}|-1} p_{m-|\bar{\ell}|-1,k_d}(y) \Big[T_{\bar{k}+\bar{e}_d} - T_{\bar{k}} \Big].
\end{aligned}$$

Now, using similar relations to (9) and (10), but with $m-1$ instead of m we obtain

$$\frac{\partial^\beta}{\partial \bar{x}^\beta} B_n(f,\bar{x}) = \frac{n!}{(m-1)!} \sum_{|\bar{k}|\le m-1} p_{m-1,\bar{k}}(\bar{x}) \Big[T_{\bar{k}+\bar{e}_d} - T_{\bar{k}} \Big]. \tag{12}$$

Finally, we have

$$\begin{aligned}
T_{\bar{k}+\bar{e}_d} - T_{\bar{k}} &= \iint \dots \int_{\left[0,\frac{1}{n}\right]^{|\alpha|}} \Big\{ \frac{\partial^\alpha}{\partial \bar{t}^\alpha} f\Big(\frac{\bar{k}}{n} + \frac{1}{n}\bar{e}_d + \sum_{i\in I_\alpha} \Big(\sum_{j=1}^{\alpha_i} t_{i,j} \Big) \bar{e}_i \Big)\\
&\qquad - \frac{\partial^\alpha}{\partial \bar{t}^\alpha} f\Big(\frac{\bar{k}}{n} + \sum_{i\in I_\alpha} \Big(\sum_{j=1}^{\alpha_i} t_{i,j} \Big) \bar{e}_i \Big) \Big\} d\bar{t}_\alpha\\
&= \iint \dots \int_{\left[0,\frac{1}{n}\right]^{|\alpha|}} \Big[\int_0^{\frac{1}{n}} \frac{\partial}{\partial x_d} \frac{\partial^\alpha}{\partial \bar{t}^\alpha} f\Big(\frac{\bar{k}}{n} + s\bar{e}_d + \sum_{i\in I_\alpha} \Big(\sum_{j=1}^{\alpha_i} t_{i,j} \Big) \bar{e}_i \Big) ds \Big] d\bar{t}_\alpha.
\end{aligned}$$

Because $\beta_d = \alpha_d + 1 \ge 1$ it follows that $d \in I_\beta$. Then we can denote s by t_{d,β_d}. Let us use the notation $d\bar{t}_\beta = \prod_{i\in I_\beta} \prod_{j=1}^{\beta_i} dt_{i,j}$. Then

$$s\bar{e}_d + \sum_{i\in I_\alpha} \Big(\sum_{j=1}^{\alpha_i} t_{i,j} \Big) \bar{e}_i = \sum_{i\in I_\beta} \Big(\sum_{j=1}^{\beta_i} t_{i,j} \Big) \bar{e}_i.$$

Also, $\left[0,\frac{1}{n}\right]^{|\alpha|}\times\left[0,\frac{1}{n}\right]=\left[0,\frac{1}{n}\right]^{|\beta|}$, $dt_{d,\beta_d}d\bar{t}_\alpha=d\bar{t}_\beta$ and $\frac{\partial}{\partial x_d}\frac{\partial^\alpha}{\partial\bar{t}^\alpha}=\frac{\partial^\beta}{\partial\bar{t}^\beta}$. Thus,

$$T_{\bar{k}+\bar{e}_d}-T_{\bar{k}}=\iint\cdots\int_{\left[0,\frac{1}{n}\right]^{|\beta|}}\frac{\partial^\beta}{\partial\bar{t}^\beta}f\left(\frac{\bar{k}}{n}+\sum_{i\in I_\beta}\left(\sum_{j=1}^{\beta_i}t_{i,j}\right)\bar{e}_i\right)d\bar{t}_\beta. \tag{13}$$

From relations (12) and (13) and since $m-1=n-|\beta|$ one obtains

$$\begin{aligned}\frac{\partial^\beta}{\partial\bar{x}^\beta}B_n(f,\bar{x})=&\frac{n!}{(n-|\beta|)!}\sum_{|\bar{k}|\le m-1}p_{n-|\beta|,\bar{k}}(\bar{x})\\&\times\iint\cdots\int_{\left[0,\frac{1}{n}\right]^{|\beta|}}\frac{\partial^\beta}{\partial\bar{t}^\beta}f\left(\frac{\bar{k}}{n}+\sum_{i\in I_\beta}\left(\sum_{j=1}^{\beta_i}t_{i,j}\right)\bar{e}_i\right)d\bar{t}_\beta.\end{aligned} \tag{14}$$

The relation above can be also extended by continuity at a point $\bar{x}$ with $x_d=0$. The induction step is proved. □

Let $\alpha\in\mathbb{N}_0^d$. Denote

$$K^\alpha(\Delta_d)=\left\{f\in C^\alpha(\Delta_d)\mid\frac{\partial^\alpha f}{\partial\bar{x}^\alpha}(\bar{x})\ge 0,\ (\bar{x}\in\Delta_d)\right\}. \tag{15}$$

The following corollaries are immediate.

Corollary 1 *For any $n\in\mathbb{N}$ we have*

$$B_n(K^\alpha(\Delta))\subset K^\alpha(\Delta_d). \tag{16}$$

Let $\alpha\in\mathbb{N}_0^d$. If $f\in C^\alpha(\Delta_d)$ denote $\left\|\frac{\partial^\alpha f}{\partial\bar{x}^\alpha}\right\|=\max_{\bar{x}\in\Delta_d}\left|\frac{\partial^\alpha f}{\partial\bar{x}^\alpha}(\bar{x})\right|$.

Corollary 2 *For any $n\in\mathbb{N}$, any $\alpha\in\mathbb{N}_0^d$ and any $f\in C^\alpha(\Delta_d)$ we have*

$$\left\|\frac{\partial^\alpha}{\partial\bar{x}^\alpha}B_n(f)\right\|\le\frac{n!}{(n-|\alpha|)!\,n^{|\alpha|}}\left\|\frac{\partial^\alpha f}{\partial\bar{x}^\alpha}\right\|. \tag{17}$$

By induction one obtains

Corollary 3 *For any $n\in\mathbb{N}$, any $\alpha\in\mathbb{N}_0^d$, any $j\in\mathbb{N}_0$ and any $f\in C^\alpha(\Delta_d)$ we have*

$$\left\|\frac{\partial^\alpha}{\partial\bar{x}^\alpha}(B_n)^j(f)\right\|\le\left(\frac{n!}{(n-|\alpha|)!\,n^{|\alpha|}}\right)^j\left\|\frac{\partial^\alpha f}{\partial\bar{x}^\alpha}\right\|. \tag{18}$$

Remark 1 For $|\alpha| \geq 2$ it follows

$$\frac{n!}{(n-|\alpha|)!\,n^{|\alpha|}} \leq \frac{n!}{(n-2)!\,n^2} = \frac{n-1}{n}.$$

For $k \in \mathbb{N}$, $f \in C^k(\varDelta_d)$ define

$$\mu_k(f) := \sup_{\alpha \in \mathbb{N}_0^d,\ |\alpha|=k} \left\| \frac{\partial^\alpha f}{\partial \overline{x}^\alpha} \right\|. \tag{19}$$

Corollary 4 *For any $n \in \mathbb{N}$, any $j \in \mathbb{N}_0$, any $k \in \mathbb{N}$, $k \geq 2$, and any $f \in C^k(\varDelta)$ we have*

$$\mu_k((B_n)^j(f)) \leq \left(\frac{n-1}{n}\right)^j \mu_k(f). \tag{20}$$

Proof Let $j \geq 0$. There exists $\alpha_0 \in \mathbb{N}_0^d$ with $|\alpha_0| = k$ such that $\mu_k((B_n)^{j+1}(f)) = \left\| \frac{\partial^{\alpha_0}}{\partial \overline{x}^{\alpha_0}} (B_n)^{j+1}(f) \right\|$. Then using relation (17) and Remark 1 we obtain

$$\begin{aligned}\mu_k((B_n)^{j+1}(f)) &= \left\| \frac{\partial^{\alpha_0}}{\partial \overline{x}^{\alpha_0}} B_n((B_n)^{j+1}(f)) \right\| \leq \frac{n-1}{n} \left\| \frac{\partial^{\alpha_0}}{\partial \overline{x}^{\alpha_0}} (B_n)^j(f) \right\| \\ &\leq \frac{n-1}{n} \mu_k((B_n)^j(f)).\end{aligned}$$

So that we can apply the induction. □

Corollary 5 *We have $B_n(\varPi_m) \subset \varPi_m$, $m \geq 0$, where $\varPi_m$ is the set of polynomials with d variables with total degree at most m.*

Proof Take a monomial function $f(\overline{x}) = \overline{x}^\gamma$, $\gamma = (\gamma_1, \ldots, \gamma_d)$, with $|\gamma| \leq m$. Then $\frac{\partial^{\gamma_j+1}}{\partial x_j^{\gamma_j+1}} f = 0$ on $\mathbb{R}^d$, for $1 \leq j \leq d$. From Theorem 1 we deduce that $\frac{\partial^{\gamma_j+1}}{\partial x_j^{\gamma_j+1}} B_n(f) = 0$ on $\varDelta_d$, for every $1 \leq j \leq d$. It is easy to see that $B_n(f)$ is a polynomial of the form $\sum_{s\in I} a_s \overline{x}^{\beta_s}$, where I is finite, $\beta_s = (\beta_{s,1}, \ldots, \beta_{s,d}) \in \mathbb{N}^d$, $\beta_{s,j} \leq \gamma_j$, for $1 \leq j \leq d$, $s \in I$ and $a_s \in \mathbb{R}$, for $s \in I$. Therefore $B_n(f) \in \varPi_m$. It follows $B_n(\varPi_m) \subset \varPi_m$. □

3 A quantitative estimate for Trotter's theorem

Consider operator

$$Af(\overline{x}) = \frac{1}{2} \sum_{i=1}^d \frac{\partial^2 f(\overline{x})}{\partial x_i^2} x_i(1-x_i) - \sum_{1\leq i<j\leq d} \frac{\partial^2 f(\overline{x})}{\partial x_i \partial x_j} x_i x_j,\ f \in C^2(\varDelta_d). \tag{21}$$

In the following lemma we give a Voronovskaja type theorem for operators B_n.

Lemma 2

$$\lim_{n\to\infty} n(B_n(f,\overline{x})-f(\overline{x})) = Af(\overline{x}),\ f\in C^2(\Delta_d).$$

Remark 2 There exists a semigroup of bounded linear operators $\{T(t)\}_{t\geq 0}$, $T(t) : C(\Delta_d)\to C(\Delta_d)$, such that

$$\lim_{n\to\infty} B_n^{m_n}(f) = T(t),\ t\geq 0,$$

for any sequences of integers $(m_n)_n$ such that $\frac{m_n}{n}=t$. This fact follows, for instance, from Theorem B, with the choices: $L_n=B_n$, $E=C(\Delta_d)$, $D_0=C^2(\Delta_d)$, $A_0=A$ and $E_i=\Pi_i$, $i\geq 0$, where Π_i is the space of polynomials with total degree i, see Corollary 5.

Lemma 3 *For $g\in C^4(\Delta_d)$ we have*

$$\left\| B_n(g) - g - \frac{1}{n}Ag\right\| \leq \frac{C_d^1}{n^2}\mu_3(g), \tag{22}$$

where

$$C_d^1 = \frac{1}{3}d^3 - \frac{1}{2}d^2 + \frac{1}{3}d. \tag{23}$$

and $\mu_3(g)$ is defined in (19).

Proof For $g\in C^4(\Delta_d)$, $\overline{x},\overline{t}\in\Delta_d$ we get

$$\begin{aligned} g(\overline{t}) &= g(\overline{x}) + \sum_{i=1}^{d}\frac{\partial g_i(\overline{x})}{\partial x_i}(t_i-x_i)\\ &+\frac{1}{2}\Big[\sum_{i=1}^{d}\frac{\partial^2 g_i(\overline{x})}{\partial x_i^2}(t_i-x_i)^2 + 2\sum_{1\leq i<j\leq d}\frac{\partial^2 g_i(\overline{x})}{\partial x_i\partial x_j}(t_i-x_i)(t_j-x_j)\Big]\\ &+\frac{1}{6}\Big[\sum_{i=1}^{d}\frac{\partial^3 g_i(\xi)}{\partial x_i^3}(t_i-x_i)^3 + 3\sum_{1\leq i,j\leq d,\ i\neq j}\frac{\partial^3 g_i(\xi)}{\partial x_i^2\partial x_j}(t_i-x_i)^2(t_j-x_j)\\ &+6\sum_{1\leq i<j<k\leq d}\frac{\partial^3 g_i(\xi)}{\partial x_i\partial x_j\partial x_k}(t_i-x_i)(t_j-x_j)(t_k-x_k)\Big], \end{aligned}$$

where ξ belongs to the interval $[\overline{x},\overline{t}]\subset\Delta_d$. Then, using relation (21) and Lemma 1 we obtain $B_n(g,\overline{x}) = g(\overline{x}) + \frac{1}{n}A(g,\overline{x}) + R_3(\overline{x})$, and

$$\begin{aligned}|R_3(\overline{x})| &\le \frac{\mu_3(g)}{6}\Big|\sum_{i=1}^{d}\frac{x_i(1-x_i)(1-2x_i)}{n^2}+3\sum_{1\le i,j\le d,\ i\ne j}\frac{x_ix_j(2x_i-1)}{n^2}\\ &\quad+6\sum_{1\le i<j<k\le d}\frac{2x_ix_jx_k}{n^2}\Big|\\ &=\frac{\mu_3(g)}{6n^2}[d+3d(d-1)+2d(d-1)(d-2)]\\ &=\frac{\mu_3(g)}{n^2}\left[\frac{1}{3}d^3-\frac{1}{2}d^2+\frac{1}{3}d\right].\end{aligned}$$

□

Lemma 4 *For any $g\in C^4(\Delta_d)$ and $t\ge 0$ we have*

$$\|T(t)g-g-tAg\|\le\frac{t^2}{2}\sum_{k=2}^{4}C_d^k\mu_k(g),$$

where

$$C_d^2=\frac{1}{2}d^2,\ C_d^3=d^3-d^2+\frac{1}{2}d,\ C_d^4=\frac{1}{4}d^4 \tag{24}$$

and $\mu_k(g)$, $k=2,3,4$ are defined in (19).

Proof First we use the known inequality

$$\|T(t)g-g-tAg\|\le\frac{t^2}{2}\|A^2g\|.$$

In the sequel we use abbreviated notations for sums of the form $\sum_i a_i$, $\sum_{i,j}a_{i,j}$, $\sum_{i,j,k}a_{i,j,k}$, $\sum_{i,j,k,\ell}a_{i,j,k,\ell}$. We suppose that all the indices are in the set $\{1,2,\dots,d\}$ and are different from each other in the case of these sums. The terms are unique taken as indicated in the generic form described by the sum. For instance, $\sum_{i,j}x_ix_j$ is the abbreviation of $\sum_{1\le i<j\le d}x_ix_j$ and $\sum_{i,j}x_i^2x_j$ is the abbreviation of $\sum_{1\le i,j\le d,\ i\ne j}x_i^2x_j$. We also use the convention that if the number of indices is strictly greater than d, then the corresponding sum is null.

From (21) one obtains, after certain calculations, for $g\in C^4(\Delta_d)$ and $\overline{x}\in\Delta_d$:

$$
\begin{aligned}
A^2(g,\bar{x}) = {} & \frac{1}{4}\sum_i \frac{\partial^4 g(\bar{x})}{\partial x_i^4} x_i^2(1-x_i)^2 \\
& + \frac{1}{2}\sum_i \frac{\partial^3 g(\bar{x})}{\partial x_i^3} x_i(1-x_i)(1-2x_i) \\
& - \frac{1}{2}\sum_i \frac{\partial^2 g(\bar{x})}{\partial x_i^2} x_i(1-x_i) \\
& + \frac{1}{2}\sum_{i,j} \frac{\partial^4 g(\bar{x})}{\partial x_i^2 \partial x_j^2} x_i(1-x_i)x_j(1-x_j) \\
& - \frac{1}{2}\sum_{i,j} \frac{\partial^4 g(\bar{x})}{\partial x_i^3 \partial x_j} x_i^2(1-x_i)x_j \\
& - \frac{1}{2}\sum_{i,j,k} \frac{\partial^4 g(\bar{x})}{\partial x_i \partial x_j \partial x_k^2} x_i x_j x_k(1-x_k) \\
& - \frac{1}{2}\sum_{i,j,k} \frac{\partial^4 g(\bar{x})}{\partial x_i^2 \partial x_j \partial x_k} x_i(1-x_i)x_j x_k - \frac{1}{2}\sum_{i,j} \frac{\partial^3 g(\bar{x})}{\partial x_i^2 \partial x_j} x_i(1-x_i)x_j \\
& - \frac{1}{2}\sum_{i,j} \frac{\partial^4 g(\bar{x})}{\partial x_i^3 \partial x_j} x_i^2(1-x_i)x_j - \frac{1}{2}\sum_{i,j} \frac{\partial^3 g(\bar{x})}{\partial x_i^2 \partial x_j} x_i(1-2x_i)x_j \\
& + \sum_{i,j} \frac{\partial^4 g(\bar{x})}{\partial x_i^2 \partial x_j^2} x_i^2 x_j^2 + \sum_{i,j} \frac{\partial^3 g(\bar{x})}{\partial x_i^2 \partial x_j} x_i^2 x_j \\
& + \sum_{i,j} \frac{\partial^2 g(\bar{x})}{\partial x_i \partial x_j} x_i x_j + 6\sum_{i,j,k,\ell} \frac{\partial^4 g(\bar{x})}{\partial x_i \partial x_j \partial x_k \partial x_\ell} x_i x_j x_k x_\ell \\
& + 2\sum_{i,j,k} \frac{\partial^4 g(\bar{x})}{\partial x_i^2 \partial x_j \partial x_k} x_i^2 x_j x_k + 6\sum_{i,j,k} \frac{\partial^3 g(\bar{x})}{\partial x_i \partial x_j \partial x_k} x_i x_j x_k.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|A^2 g\| \le {} & \frac{1}{4} d\mu_4(g) + \frac{1}{2} d\mu_3(g) \\
& + \frac{1}{2} d\mu_2(g) + \frac{1}{2}\cdot\frac{d(d-1)}{2}\mu_4(g) + \frac{1}{2} d(d-1)\mu_4(g) \\
& + \frac{1}{2} d(d-1)\mu_3(g) + \frac{1}{2}\frac{d(d-1)(d-2)}{2}\mu_4(g) \\
& + \frac{1}{2}\frac{d(d-1)(d-2)}{2}\mu_4(g) \\
& + \frac{1}{2} d(d-1)\mu_4(g) + \frac{1}{2} d(d-1)\mu_3(g) \\
& + \frac{d(d-1)}{2}\mu_4(g) + d(d-1)\mu_3(g) \\
& + \frac{d(d-1)}{2}\mu_2(g) \\
& + 6\frac{d(d-1)(d-2)(d-3)}{24}\mu_4(d) \\
& + 2\frac{d(d-1)(d-2)}{2}\mu_4(g) \\
& + 6\frac{d(d-1)(d-2)}{6}\mu_3(g) \\
= {} & \frac{1}{4} d^4\mu_4(g) + \left(d^3 - d^2 + \frac{1}{2}d\right)\mu_3(g) \\
& + \frac{1}{2} d^2\mu_2(g).
\end{aligned}
$$

□

The main result is the following.

Theorem 2 *For* $f \in C^4(\Delta_d)$, $m \in \mathbb{N}$, $n \in \mathbb{N}$, $t \geq 0$ *we have*

$$\|(B_n)^m f - T(t)f\| \leq \left|\frac{m}{n} - t\right| \frac{d^2}{2}\mu_2(f) + \frac{1}{n}\left[C_d^1\mu_3(f) + \frac{1}{2}\sum_{k=2}^{4} C_d^k \mu_k(f)\right] \tag{25}$$

where C_d^k, $k = 1, 2, 3, 4$ *are given in* (23) *and* (24).

Proof The method of proof is a modification of the method used in [12] and consists in a modification of a telescopic sum argument.

Since $\|(B_n)^m\| = 1$, for $m \geq 1$ it follows $\|T(t)\| = 1$, $t > 0$.

Consider the decomposition

$$\|(B_n)^m f - T(t)f\| \leq \left\|T\left(\frac{m}{n}\right)f - T(t)f\right\| + \left\|(B_n)^m f - T\left(\frac{m}{n}\right)f\right\|. \tag{26}$$

From relation (21) we deduce $\|Af\| \leq \left(\frac{1}{2}d + \frac{d(d-1)}{2}\right)\mu_2(f) = \frac{d^2}{2}\mu_2(f)$. We obtain successively:

$$\begin{aligned}\left\|T\left(\frac{m}{n}\right)f - T(t)f\right\| &= \left\|\int_t^{\frac{m}{n}} T(u)Afdu\right\| \\ &\leq \left|\frac{m}{n} - t\right| \sup_{u\in\left[\frac{m}{n},t\right]} \|T(u)Af\| \\ &\leq \left|\frac{m}{n} - t\right| \cdot \|Af\| \\ &\leq \left|\frac{m}{n} - t\right| \cdot \frac{d^2}{2}\mu_2(f).\end{aligned} \tag{27}$$

For the second term one can use a telescopic sum:

$$\begin{aligned}\left\|(B_n)^m f - T\left(\frac{m}{n}\right)f\right\| &= \left\|\sum_{j=0}^{m-1} T\left(\frac{m-1-j}{n}\right)\left(B_n - T\left(\frac{1}{n}\right)\right)(B_n)^j f\right\| \\ &\leq \sum_{j=0}^{m-1}\left\|\left(B_n - T\left(\frac{1}{n}\right)\right)(B_n)^j f\right\|.\end{aligned} \tag{28}$$

We can write

$$\begin{aligned}&\left\|\left(B_n - T\left(\frac{1}{n}\right)\right)(B_n)^j f\right\| \\ &\quad\leq \left\|\left(B_n - I - \frac{1}{n}A\right)(B_n)^j f\right\| + \left\|\left(T\left(\frac{1}{n}\right) - I - \frac{1}{n}A\right)(B_n)^j f\right\|\end{aligned} \tag{29}$$

From Lemmas 3 and 4 it results for any j

$$\left\| \left(B_n - I - \frac{1}{n} A \right) (B_n)^j f \right\| \leq \frac{C_d^1}{n^2} \mu_3((B_n)^j f) \tag{30}$$

$$\left\| \left(T\left(\frac{1}{n}\right) - I - \frac{1}{n} A \right) (B_n)^j f \right\| \leq \frac{1}{2n^2} \sum_{k=2}^{4} C_d^k \mu_k((B_n)^j f). \tag{31}$$

But using Corollary 4 we have $\mu_k((B_n)^j f) \leq \left(\frac{n-1}{n}\right)^j \mu_k(f)$, for $j \geq 0$, $k = 2, 3, 4$. Then by using (28), (29), (30) and (31) one obtains

$$\begin{aligned} \left\| (B_n)^m f - T\left(\frac{m}{n}\right) f \right\| &\leq \sum_{j=0}^{m-1} \left(\frac{n-1}{n}\right)^j \frac{1}{n^2} \left[C_d^1 \mu_3(f) + \frac{1}{2} \sum_{k=2}^{4} C_d^k \mu_k(f) \right] \\ &= \frac{1}{n} \left[C_d^1 \mu_3(f) + \frac{1}{2} \sum_{k=2}^{4} C_d^k \mu_k(f) \right]. \end{aligned} \tag{32}$$

From (27) and (32) it results (25). □

Finally, we compare our result with others, obtained previously.

Remark 3 A quantitative version of Trotter's theorem for the semigroup generated by Bernstein operators defined on the simplex Δ_d was obtained by Campiti and Tacelli [6, 7] for functions belonging to the space $C^{2,\alpha}(\Delta_d)$, with $0 < \alpha < 1$. The space $C^{2,\alpha}(\Delta_d)$ consists of real functions f defined on Δ_d, which admit second derivatives on Δ_d and for which the following condition

$$\sup_{\substack{x,y \in \Delta_d \\ x \neq y}} \frac{1}{\|x - y\|^\alpha} \sum_{i,j=1}^{d} \left| \frac{\partial^2 f}{\partial x_i \partial x_j}(x) - \frac{\partial^2 f}{\partial x_i \partial x_j}(y) \right| < \infty$$

is satisfied. In [6, Theorem 2.3], completed in [7], the following estimate is obtained:

$$\begin{aligned} \|T(t)f - (B_n)^{k(n)} f\| \leq \frac{t\psi(f)}{n^{\alpha/(4+\alpha)}} + \Bigg(\left| \frac{k(n)}{n} - t \right| \\ + \frac{\sqrt{k(n)}}{n} \Bigg) \left(\|Af\| + \frac{\psi(f)}{n^{\alpha/(4+\alpha)}} \right), \end{aligned} \tag{33}$$

for every $t \geq 0$, $f \in C^{2,\alpha}(\Delta_d)$ and sequence $(k(n))_{n \geq 1}$ of positive integers, where $\psi(f)$ depends only on f. On other hand, relation (25) with m replaced by $k(n)$ is of the form

$$\begin{aligned} \|T(t)f - (B_n)^{k(n)} f\| \leq C_1(f) \left| \frac{k(n)}{n} - t \right| \\ + C_2(f) \frac{1}{n}, \ t \geq 0, \ f \in C^4(\Delta_d). \end{aligned} \tag{34}$$

The first remark is that in the hypothesis $k(n)/n \to t$, $(n \to \infty)$, relation (33) is generally stronger, because it is valid for the greater space $C^{2,\alpha}(\Delta_d)$, instead of space $C^4(\Delta_d)$.

In the case when $f \in C^4(\Delta_d)$, in order to make an asymptotic comparison, let fix f and t and denote $\beta = \frac{\alpha}{4(\alpha+1)} \in (0, 1/8)$. We can make this comparison in two cases.

If $\liminf_{n\to\infty} \left|\frac{k(n)}{n} - t\right| n^{\beta} \in (0,\infty) \cup \{\infty\}$, then the two estimates have the same order of convergence to 0, namely $\mathrm{O}\left(\left|\frac{k(n)}{n} - t\right|\right)$,

In the case when $\left|\frac{k(n)}{n} - t\right| = \mathrm{o}\left(n^{-\beta}\right)$ $(n \to \infty)$ relation (34), i.e., (25) is stronger than relation (33).

Remark 4 Another estimate for approximation of the semigroup generated by the Bernstein operators on a simplex was given by Mangino and Raşa [15] in the form:

$$\begin{aligned} &\|(B_n)^{k_n} f - T(t)f\|_\infty \\ &\le \left(tC_n + \left(\left| t - \frac{k(n)}{n} \right| + \frac{\sqrt{k(n)}}{n} \right)(C_n + 1) \right) \|f\|_3, \end{aligned} \tag{35}$$

where $C_n = \frac{1}{n} + \frac{1}{6} d^3 \sqrt{n} \sup_{\overline{x}\in\Delta_d} \sqrt{B_n(\|\overline{x} - \bullet\|^4, \overline{x})}$, $f \in C^3(\Delta_d)$, $\|f\|_3 = \sum_{|\alpha|\le 3} \|D^\alpha f\|$. This estimate has also a larger domain of applicability: $C^3(\Delta)$. It remains to compare (35) with (25) for $f \in C^4(\Delta)$. Fix d and $t > 0$. Consider a sequence $(k(n))_n$, such that $\lim_{n\to\infty} \frac{k(n)}{n} = t$. We can make the comparison in two cases.

If $\liminf_{n\to\infty} \left|\frac{k(n)}{n} - t\right| \sqrt{n} \in (0,\infty) \cup \{\infty\}$, then, by taking into account that $C_n = \mathrm{O}\left(\frac{1}{\sqrt{n}}\right)$, it follows that the two estimates have the same order, namely $\mathrm{O}\left(\left|\frac{k(n)}{n} - t\right|\right)$.

In the case when $\left|\frac{k_n}{n} - t\right| = \mathrm{o}\left(n^{-\frac{1}{2}}\right)$, estimate (25) is stronger than relation (35).

References

1. Altomare, F.: Iterates of Markov operators and constructive approximation of semigroups. Construct. Math. Anal. **2**, 22–39 (2019)
2. Altomare, F., Campiti, M.: Korovkin-type Approximation Theory and Its Applications. De Gruyter Studies in Mathematics, vol. 17. De Gruyter, Berlin (1994)
3. Altomare, F., Cappelletti Montano, M., Leonessa, V., Raşa, I.: Markov Operators, Positive Semigroups and Approximation Processes. De Gruyter Studies in Mathematics, vol. 61. De Gruyter, Berlin (2014)
4. Attalienti, A.: Generalized Bernstein-Durrmeyer operators and the associated limit semigroup. J. Approx. Theory **99**, 289–309 (1999)
5. Butzer, P.L., Berens, H.: Semi-groups of Operators and Approximation Theory. Grundlehren der mathematischen Wissenschaften, vol. 145. Springer, Berlin (1967)
6. Campiti, M., Tacelli, C.: Rate of convergence in Trotter's approximation theorem. Constr. Approx. **28**, 333–341 (2008)

7. Campiti, M., Tacelli, C.: Erratum to: Rate of convergence in Trotter's approximation theorem. Constr. Approx. **31**, 459–462 (2010)
8. Cheney, E.W., Sharma, A.: Bernstein power series. Can. J. Math. **16**, 241–252 (1964)
9. da Silva, M.R.: The Limiting of the Bernstein Iterates: Properties and Applications. Ph.D Thesis, Imperial College, University of London (1978)
10. Engel, K.-J., Nagel, R.: One-Parameter Semigroups for Linear Evolution Equations. Graduate Texts in Mathematics, vol. 194. Springer, New York (2000)
11. Gonska, H., Heilmann, M., Raşa, I.: Convergence of iterates of genuine and ultraspherical Durrmeyer operators to the limiting semigroup: C^2-estimates. J. Approx. Theory **160**, 243–255 (2009)
12. Gonska, H., Raşa, I.: The limiting semigroup of the Bernstein iterates: degree of convergence. Acta Math. Hung. **111**, 119–130 (2006)
13. Karlin, S., Zieger, Z.: Iteration of positive approximation operators. J. Approx. Theory **3**, 310–339 (1970)
14. Kelinsky, R.P., Rivlin, T.J.: Iterates of Bernstein polynomials. Pac. J. Math. **21**, 511–520 (1967)
15. Mangino, E., Raşa, I.: A quantitative version of Trotter's theorem. J. Approx. Theory **146**, 149–156 (2007)
16. Minea, B.: On quantitative estimation for the limiting semigroup of linear positive operators. Bull. Transl. Univ. Braşov **6**(55), 31–36 (2013)
17. Nagel, R. (ed.): One-parameter Semigroups of Positive Operators. Lecture Notes in Mathematics, vol. 1184. Springer, Berlin (1986)
18. Schnabl, R.: Zum globalen Saturationsproblem der Folge der Bernstein–Operatoren. Acta Sci. Math. (Szeged) **31**, 351–358 (1970)
19. Sikkema, P.C.: Über Potenzen von verallgemeinerten Bernstein–Operatoren. Mathematica (Cluj) **8**(31), 173–180 (1966)
20. Smuc, M.: On quantitative estimation for the limiting semigroup of positive operators. Bull. Transl. Univ. Braşov **11**(2), 235–262 (2018)
21. Trotter, H.F.: Approximation of semi-groups of operators. Pac. J. Math. **8**, 887–919 (1958)