Semigroup Forum (2021) 102:134–159
https://doi.org/10.1007/s00233-020-10142-1

RESEARCH ARTICLE

Stability for coupled waves with locally disturbed Kelvin–Voigt damping

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Received: 15 January 2020 / Accepted: 17 August 2020 / Published online: 10 November 2020

Abstract

We consider a coupled wave system with partial Kelvin–Voigt damping in the interval $(-1, 1)$, where one wave is dissipative and the other is not. When the damping is effective in the whole domain $(-1, 1)$, it was proven in Portillo Oquendo and Sànez Pacheco (Appl Math Lett 67:16–20, 2017) that the energy is a non-increasing function of the time variable, with a rate equals to $t^{-\frac{1}{2}}$. In this paper, using the frequency domain method, we show the effect of the coupling and the non smoothness of the damping coefficient on the energy decay. Actually, as expected we show the lack of the exponential stability, that the semigroup loses speed and it decays polynomially with a slower rate than the one given in Portillo Oquendo and Sànez Pacheco (loc. cit.) [20].

Keywords Coupled system · Kelvin–Voigt damping · Frequency-domain approach

1 Introduction

When a vibrating source disturbs the first particle of a medium, a wave is created. This phenomenon begins to travel from particle to particle along the medium, which is typically modeled by a wave equation. In order to suppress those vibrations, the most common approach is by adding a damping. It is more likely to use one of two types:

Communicated by Markus Haase.

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1. The linear viscous damping or "external damping", it does mostly model an external frictional force, such that the auto-mobile shock absorber.
2. The Kelvin–Voigt damping, also called the "internal damping" or the "material damping", which is originated from the extension or compression of the vibrating particles.

In the recent years, many researchers showed interest in some problems involving this kind of damping. In control theory for instance, it was shown that when the Kelvin–Voigt damping coefficient does satisfy some geometrical control conditions, the semigroup corresponding to this system is exponentially stable (see [16, 21]). Nonetheless, when the damping is arbitrary localized with singular coefficient, it is not the case anymore (see [2, 15]). Actually, in one-dimensional case we can consider the following problem

$$\begin{cases} u_{tt} - [u_x(x,t) + b(x)u_{xt}]_x = 0 & -1 < x < 1,\ t \geq 0, \\ u(t,-1) = u(t,1) = 0 & t \geq 0, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x) & -1 \leq x \leq 1, \end{cases} \tag{1}$$

with $b \in L^\infty(-1,1)$ and

$$b(x) = \begin{cases} 0 & \text{for } x \in [0,1), \\ a(x) & \text{for } x \in (-1,0). \end{cases}$$

Under the assumption that the damping coefficient has a singularity at the interface of the damped and undamped regions, and behaves like x^α near the interface, it was proven by Liu and Zhang [17] that the semigroup corresponding to the system is polynomially or exponentially stable and that the decay rate depends on the parameter $\alpha \in (0,1]$. When $\alpha = 0$, Liu and Rao [15] showed that the system (1) is polynomially stable with an order equals to 2, and few years later, Liu and Liu [14] proved the lack of the exponential stability.

When we deal with systems involving quantities described by several components and pretend to control or observe all the state variables, it turns out that certain systems possess an internal structure that compensates the lack of control variables. Such a phenomenon is referred to as indirect stabilization or indirect control. For instance, Alabau et al. did study in [1] coupled waves with partial frictional damping

$$\begin{cases} u_{tt} - \Delta u + \alpha v = 0 & x \in \Omega,\ t \geq 0, \\ v_{tt} - \Delta v + \alpha u + \beta v_t = 0 & x \in \Omega,\ t \geq 0, \end{cases}$$

subjected to Dirichlet boundary conditions. It was proven that the semigroup corresponding to this system is not exponentially stable, but it is polynomially stable with a rate equal to $t^{-\frac{1}{2}}$. In 2016, Oquendo and Pacheco studied a wave equation with internal coupled terms where the Kelvin–Voigt damping is global in one equation. Although the damping is stronger than the frictional one, they had shown that the semigroup loses speed, with a slower rate, down to $t^{-\frac{1}{4}}$. For this kind of coupled visco-elastic models, we distinguish the case of a transmission problems, which have been intensively studied by the first author, Ammari and their collaborators in [2, 3, 8–11] (see also [4]). They did study the wave equation, the plate equation or a coupled wave-plate equations. With a non smooth and singular damping coefficient, it was shown an uniform and a non-uniform decay rates of the energy. In this work, we examine the behavior of a coupled waves system with a partial Kelvin–Voigt damping. We mainly consider the following system where the first wave is dissipation and the second is conservative:

$$\begin{cases} u_{tt}(x,t) - [u_x(x,t) + a(x)u_{xt}(x,t)]_x + v_t(x,t) = 0 & (x,t) \in (-1,1)\times(0,+\infty), \\ v_{tt}(x,t) - c\, v_{xx}(x,t) - u_t(x,t) = 0 & (x,t) \in (-1,1)\times(0,+\infty), \\ u(1,t) = v(1,t) = 0, u(-1,t) = v(-1,t) = 0 & t \in (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & x \in (-1,1), \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x) & x \in (-1,1), \end{cases} \tag{2}$$

where $c > 0$ and $a \in L^\infty(-1,1)$ is a non-negative function. In this paper we assume that the damping coefficient is a piece-wise function, in particular, we suppose that a has the following form $a = d.\mathbf{1}_{[0,1]}$, where d is a strictly positive constant. Since the damping is singular, this system can be seen as a coupling of a transmission wave equation with a conservative wave equation.

The natural energy of (u, v) solution of (2) at an instant t is given by

$$E(t) = \frac{1}{2}\int_{-1}^{1} \left(|u_t(x,t)|^2 + |v_t(x,t)|^2 + |u_x(x,t)|^2 + c\,|v_x(x,t)|^2\right) \mathrm{d}x, \quad \forall\, t > 0.$$

Multiplying the first equation of (2) by $\bar{u}_t$, the second one by $\bar{v}_t$, then by integrating by parts we end up with

$$E'(t) = -\int_{-1}^{1} a(x)\,|u_{xt}(x,t)|^2\,\mathrm{d}x, \quad \forall\, t > 0.$$

Therefore, the energy is a non-increasing function of the time variable t. We show the lack of the exponential stability and we prove that the semigroup corresponding to this system is polynomially stable for regular initial data, with a slower rate, down to $t^{-\frac{1}{12}}$.

This paper is organized as follows. In Sect. 2, we prove that system (2) is well-posed. In Sect. 3, we demonstrate that the energy of the system is strongly stable. In Sect. 4, we prove the lack of the exponential stability. In Sect. 5, we show the polynomial decay of the energy.

2 Well-posedness

In this section, using the semigroup theory, we discuss the well-posedness of the problem (2).

Let $\mathscr{H} = (H_0^1(-1,1))^2 \times (L^2(-1,1))^2$ be the Hilbert space endowed with the inner product defined, for $U_1 = (u^1, v^1, w^1, z^1) \in \mathscr{H}$ and $U_2 = (u^2, v^2, w^2, z^2) \in \mathscr{H}$, by

$$\langle U_1, U_2\rangle_{\mathcal{H}} = \left\langle u_x^1, u_x^2\right\rangle_{L^2(-1,1)} + \left\langle \sqrt{c}v_x^1, \sqrt{c}v_x^2\right\rangle_{L^2(-1,1)} + \left\langle w^1, w^2\right\rangle_{L^2(-1,1)} + \left\langle z^1, z^2\right\rangle_{L^2(-1,1)}.$$

By setting $y(t) = (u(t), v(t), u_t(t), v_t(t))$ and $y_0 = (u_0, v_0, u_1, v_1)$ we can rewrite the system (2) as a first order differential equation as follows

$$\dot{y}(t) = \mathscr{A}y(t), \quad y(0) = y_0, \tag{3}$$

where

$$\mathscr{A}(u^1, v^1, u^2, v^2) = \left(u^2, v^2, \left(u_x^1 + au_x^2\right)_x - v^2, c\, v_{xx}^1 + u^2\right),$$

with

$$(u^1, v^1, u^2, v^2) \in \mathscr{D}(\mathscr{A}) = \left\{(u^1, v^1, u^2, v^2) \in \mathscr{H}, (u^2, v^2) \in (H_0^1(-1,1))^2, v^1 \in H^2(-1,1) \cap H_0^1(-1,1), \left(u_x^1 + au_x^2\right)_x \in L^2(-1,1)\right\}.$$

For the well-posedness of the system (3), we have the following proposition:

Proposition 1 *For initial data* $y_0 = (u_0, v_0, u_1, v_1) \in \mathscr{H}$, *there exists a unique solution* $y = (u, v, u_t, v_t) \in C([0, +\infty), \mathscr{H})$ *to the problem* (3). *Moreover, if* $y_0 \in \mathscr{D}(\mathscr{A})$, *then*

$$y = (u, v, u_t, v_t) \in C([0, +\infty), \mathscr{D}(\mathscr{A})) \cap C^1([0, +\infty), \mathscr{H}).$$

Proof By Lumer–Phillips' theorem (see [18]), it suffices to show that $\mathscr{A}$ is dissipative and maximal.

(1) We first prove that $\mathscr{A}$ is dissipative. Take $Z = (u, v, w, z) \in \mathscr{D}(\mathscr{A})$. Then

$$\langle \mathscr{A} Z, Z\rangle_{\mathscr{H}} = \langle w_x, u_x\rangle_{L^2(-1,1)} + c\,\langle z_x, v_x\rangle_{L^2(-1,1)} + \langle (u_x + aw_x)_x, w\rangle_{L^2(-1,1)} + \langle cv_{xx} + w, z\rangle_{L^2(-1,1)}.$$

By integration by parts and using the boundary conditions, it holds:

$$(\mathscr{A} Z, Z)_{\mathscr{H}} = -\langle aw_x, w_x\rangle_{L^2(-1,1)} = -\int_{-1}^{1} a|w_x|^2\, \mathrm{d}x \le 0. \tag{4}$$

This shows that $\mathscr{A}$ is dissipative.

(2) Let us now prove that $\mathscr{A}$ is maximal, i.e., that $\lambda I - \mathscr{A}$ is surjective for some $\lambda > 0$. So, for any given $(f, g, f_1, g_1) \in \mathscr{H}$, we solve the equation $\mathscr{A}(u, v, w, z) = (f, g, f_1, g_1)$, which is recast on the following way

$$\begin{cases} w = f \\ z = g \\ u_{xx} + (af_x)_x = f_1 + g \\ c\, v_{xx} = g_1 - f. \end{cases} \tag{5}$$

It is well known, by the Lax–Milgram theorem, that the system (5) admits a unique solution $(u, v) \in H_0^1(-1,1) \times H_0^1(-1,1)$. Moreover by multiplying the second and the third lines of (5) by $\overline{u}$ and $\overline{v}$ respectively, integrating over $(-1,1)$ then, using the Poincaré inequality and the Cauchy–Schwarz inequality, we find that there exists a constant $C > 0$ such that

$$\int_{-1}^{1} \left(|u_x(x)|^2 + |v_x(x)|^2\right) dx \le C \int_{-1}^{1} \left(|f_x(x)|^2 + |g_x(x)|^2 + |f_1(x)|^2 + |g_1(x)|^2\right) dx.$$

It follows that $(u, v, w, z) \in \mathscr{D}(\mathscr{A})$ and we have

$$\|(u, v, w, z)\|_{\mathscr{H}} \le C\|(f, g, f_1, g_1)\|_{\mathscr{H}}.$$

This implies that $0 \in \rho(\mathscr{A})$ and by contraction principle, we easily get $R(\lambda \mathrm{I} - \mathscr{A}) = \mathscr{H}$ for sufficient small $\lambda > 0$. The density of the domain of $\mathscr{A}$ follows from [18, Theorem 1.4.6]. Finally thanks to the Lumer–Phillips theorem (see [18, Theorem 1.4.3]), the operator $\mathscr{A}$ generates a C_0-semigroup of contractions on the Hilbert space $\mathscr{H}$ denoted by $(e^{t\mathscr{A}})_{t\ge 0}$. □

3 Strong stability

This section is devoted to prove that the energy of the system (2) is decreasing to zero as time goes to the infinity.

Theorem 1 *The semigroup* $(e^{t\mathscr{A}})_{t\ge 0}$ *is strongly stable in the energy space* $\mathscr{H}$, i.e.,

$$\lim_{t\to+\infty} \|e^{t\mathscr{A}} y_0\| = 0, \quad \forall\, y_0 \in \mathscr{H}.$$

Proof According to [5] the semigroup $(e^{t\mathscr{A}})_{t\ge 0}$ is strongly stable providing that $\mathscr{A}$ has no pure imaginary eigenvalues and the intersection of $\sigma(\mathscr{A})$ with $i\mathbb{R}$ is a countable set. Since the resolvent of the operator $\mathscr{A}$ is not compact, (see [16]), but $0 \in \rho(\mathscr{A})$, we only need to prove that $(i\mu I - \mathscr{A})$ is a one-to-one correspondence in the energy space $\mathscr{H}$ for all $\mu \in \mathbb{R}^*$. The proof will be done in two steps: in the first step we prove the injectivity of $(i\mu I - \mathscr{A})$ and in the second step we prove the subjectivity of the same operator.

Step 1 Let $(u, v, w, z) \in \mathscr{D}(\mathscr{A})$ be such that

$$\mathscr{A}(u, v, w, z) = i\mu(u, v, w, z), \tag{6}$$

or equivalently,

$$\begin{cases} w = i\mu u & \text{in } (-1,1), \\ z = i\mu v & \text{in } (-1,1), \\ (u_x + a w_x)_x - z = i\mu w & \text{in } (-1,1), \\ c\, v_{xx} + w = i\mu z & \text{in } (-1,1), \\ u(-1) = u(1) = 0, \ v(-1) = v(1) = 0. \end{cases} \tag{7}$$

Then, taking the real part of the scalar product of (6) with (u, v, w, z), we get

$$\mathrm{Re}(i\mu \|(u,v,w,z)\|^2_{\mathscr{H}}) = \mathrm{Re}\langle \mathscr{A}(u,v,w,z), (u,v,w,z)\rangle_{\mathscr{H}} = -d \int_0^1 |w_x|^2 dx = 0.$$

This leads to

$$w_x = 0 \quad \text{in } (0,1).$$

Using the first equation (7), we have

$$u_x = 0 \quad \text{in } (0,1),$$

which means that u is a constant in (0, 1), and since $u(1) = 0$, we obtain that

$$u = w = 0 \quad \text{in } (0,1).$$

Hence, from the third and the second equations of (7), one gets

$$u = w = v = z = 0 \quad \text{in } (0,1). \tag{8}$$

Using (8), (7) is reduced to the following problem

$$\begin{cases} w = i\mu u & \text{in } (-1,0), \\ z = i\mu v & \text{in } (-1,0), \\ \mu^2 u + u_{xx} - i\mu v = 0 & \text{in } (-1,0), \\ \mu^2 v + c\, v_{xx} + i\mu u = 0 & \text{in } (-1,0), \\ u(-1) = u(0) = 0, \ v(-1) = v(0) = 0. \end{cases} \tag{9}$$

Let $y = (u, v, u_x, v_x)$ and $y_x = (u_x, v_x, u_{xx}, v_{xx})$ then (9) is recast as follows

$$\begin{cases} y_x = A_\mu y \ \text{in}(-1,0), \\ y(0) = 0, \end{cases} \tag{10}$$

where

$$A_\mu = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\mu^2 & i\mu & 0 & 0 \\ -i\frac{\mu}{c} & -\frac{\mu^2}{c} & 0 & 0 \end{pmatrix}.$$

Since A_μ is a bounded operator, the unique solution of (10) is $y = 0$, therefore, $u = v = 0$ in $(-1, 0)$. Moreover, from the first and the second equation of

(9), we have $w = z = 0$ in $(-1, 1)$. Combining all these with (8), we deduce that $u = v = w = z = 0$ in $(-1, 1)$. This concludes the first part of this proof.

Step 2 Now given $(f, g) \in \mathscr{H}$, we solve the equation

$$(i\mu I - \mathscr{A})(u, v, w, z) = (f, g, f_1, g_1).$$

Or equivalently,

$$\begin{cases} w = i\mu u - f \\ z = i\mu v - g \\ \mu^2 u + u_{xx} + i\mu\,(au_x)_x - i\mu v = (af_x)_x - i\mu f - f_1 - g = F \\ \mu^2 v + c\,v_{xx} + i\mu u = -\mu g + f - g_1 = G. \end{cases} \tag{11}$$

Let us define the operator

$$\begin{aligned} A\ :\ (H_0^1(-1,1))^2 &\longrightarrow (H^{-1}(-1,1))^2 \\ (u, v) &\longmapsto (-u_{xx} - i\mu(au_x)_x + i\mu v, -cv_{xx} - i\mu u). \end{aligned}$$

First, we show that A is an isomorphism. For this purpose we consider the following two operators:

$$\begin{aligned} \tilde{A}\ :\ (H_0^1(-1,1))^2 &\longrightarrow (H^{-1}(-1,1))^2 \\ (u, v) &\longmapsto (-u_{xx} - i\mu(au_x)_x, -cv_{xx}), \end{aligned}$$

and C such that $A = C + \tilde{A}$.

Due to Lax–Milgram's theorem [13, Theorem 2.9.1] it is easy to show that $\tilde{A}$ is an isomorphism from $H_0^1(-1, 1)$ into $H^{-1}(-1, 1)$, then we could rewrite $A = \tilde{A}(Id - \tilde{A}^{-1}(-C))$.

To begin with, thanks to the compact embeddings

$$H_0^1(-1,1)^2 \hookrightarrow L^2(-1,1)^2 \text{ and } L^2(-1,1)^2 \hookrightarrow H^{-1}(-1,1)^2,$$

we notice that $\tilde{A}^{-1}$ is a compact operator. Secondly, it is clear that C is a bounded operator, therefore, thanks to the Fredholm alternative, we only have to prove that $(Id - \tilde{A}^{-1}(-C))$ is injective.

Let $(u, v) \in (H_0^1(-1, 1))^2$ such that $(Id - \tilde{A}^{-1}(-C))(u, v) = 0$, which implies that

$$(\tilde{A} - (-C))(u, v) = 0.$$

Or equivalently

$$\begin{cases} u_{xx} + i\mu\,(au_x)_x - i\mu v = 0 & \text{in } (-1, 1) \\ c\,v_{xx} + i\mu u = 0 & \text{in } (-1, 1) \\ u(-1) = u(1) = 0,\ v(-1) = v(1) = 0. \end{cases} \tag{12}$$

Multiplying the first equation of (12) by $\bar{u}$ and the conjugate of the second by v, after integration over $(-1, 1)$, it follows

$$-\int_{-1}^{1}|u_x|^2\mathrm{d}x+c\int_{-1}^{1}|v_x|^2\mathrm{d}x-i\mu\int_{-1}^{1}a|u_x|^2\mathrm{d}x=0.$$

Next, by taking the imaginary part, we can deduce that $u_x = 0$ in (0, 1) then, u is constant in (0, 1). Next, using the boundary condition $u(1) = 0$, we have $u = 0$ in (0,1). Moreover, using the second equation of (12), we obtain that $v = 0$ in (0, 1), which implies

$$\begin{cases} u_{xx} = i\mu v & \text{in } (-1,1) \\ v_{xx} = -i\frac{\mu}{c}u & \text{in } (-1,1) \\ u(0) = u(-1) = 0,\ v(0) = v(-1) = 0. \end{cases} \tag{13}$$

Let $y = (u, v, u_x, v_x)$ and $y_x = (u_x, v_x, u_{xx}, v_{xx})$, using the trace theorem we have:

$$\begin{cases} y_x = D_\mu y \ \text{ in } (-1,0) \\ y(0) = 0, \end{cases}$$

where

$$D_\mu = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & i\mu & 0 & 0 \\ -i\frac{\mu^2}{c} & 0 & 0 & 0 \end{pmatrix}.$$

Using the same approach used in the first step, we obtain the result that we are looking for (i.e., A is an isomorphism).

Now, rewriting the third and the fourth lines of (11), one gets

$$(u,v)-\mu^2A^{-1}(u,v)=A^{-1}(F,G).$$

Let $(u, v) \in \ker(Id - \mu^2A^{-1})$, i.e. $\mu^2(u, v) - A(u, v) = 0$, so we can notice that:

$$\begin{cases} \mu^2 u + u_{xx} + i\mu\,(au_x)_x - i\mu v = 0 & \text{in } (-1,1), \\ \mu^2 v + c\,v_{xx} + i\mu u = 0 & \text{in } (-1,1). \end{cases} \tag{14}$$

Furthermore, multiplying the first equation of (14) by $\bar{u}$, the conjugate of the second one by v, after integration over $(-1, 1)$ and taking the imaginary part, we deduce that

$$\int_{-1}^{1}a|u_x|^2\mathrm{d}x=d\int_{0}^{1}|u_x|^2\mathrm{d}x=0.$$

So, we get the same system as in the first step (see (7)). Thus, $\ker(I - \mu^2A^{-1}) = \{0_{(H^{-1}(-1,1))^2}\}$.

On the other hand, thanks to the compact embedding $H_0^1(-1,1)^2 \hookrightarrow L^2(-1,1)^2$ and $L^2(-1,1)^2 \hookrightarrow H^{-1}(-1,1)^2$, we see that A^{-1} is a compact operator. Now, following to Fredholm's alternative, the operator $(Id - \mu^2A^{-1})$ is bijective in $(H_0^1(-1,1))^2$. Finally, Eq. (11) has a unique solution in $H_0^1(-1,1)^2$. This completes the proof. □

4 Lack of exponential stability

In this section we prove under some assumptions on the speed wave propagation that system (2) is not exponentially stable.

Theorem 2 *The semigroup* $(e^{t\mathscr{A}})_{t\geq 0}$, *is not exponentially stable in the energy space provided that* $c > 1$ *and that*

$$\sin(2\sqrt{c}n\pi) \neq O\left(n^{-\frac{1}{2}}\right). \tag{15}$$

Noting that the assumption $c > 1$ is made just to make the calculation readable. The second assumption (15) can be fulfilled for instance by taking c such that $2\sqrt{c}$ is an integer number. To prove Theorem 2, we mainly use the following theorem.

Theorem 3 (see [12, 20]) *Let* $e^{t\mathscr{B}}$ *be a bounded* C_0*-semigroup on a Hilbert space* H *with generator* $\mathscr{B}$ *such that* $i\mathbb{R} \subset \rho(\mathscr{B})$. *Then* $e^{t\mathscr{B}}$ *is exponentially stable, i.e., there exist* $a > 0$ and $M > 0$ *such that*

$$\|e^{t\mathscr{B}}\|_{\mathscr{L}(H)} \leq Me^{-at}, \forall t \geq 0,$$

if and only if

$$\limsup_{\omega\in\mathbb{R},\,|\omega|\to\infty} \|(i\omega I - \mathscr{B})^{-1}\|_{\mathscr{L}(H)} < \infty.$$

Now, based on Theorem 3 we prove Theorem 2.

Proof (of Theorem 2) Our main objective is to show that:

$$\|(\lambda I - \mathscr{A})^{-1}\|_{\mathscr{L}(\mathscr{H})} \text{ is unbounded on the imaginary axis.} \tag{16}$$

For $n \in \mathbb{N}$ large enough, let $\lambda = \lambda_n = i\omega_n$, where

$$\omega_n = \sqrt{\frac{8c(c+1)n^2\pi^2 + 2c + \sqrt{\Delta}}{4c}} \tag{17}$$
$$\text{with } \Delta = (8c(c-1)\pi^2 n^2)^2 + 32(c+1)(c\pi n)^2 + 4c^2.$$

It is clear that $\omega_n \longrightarrow +\infty$ and in particular we have

$$\omega_n = \sqrt{c}\left(2n\pi + \frac{n^{-1}}{4\pi(c-1)} - \frac{cn^{-3}}{32\pi^3(c-1)^3} + o(n^{-4})\right) \tag{18}$$

and

$$\frac{1}{\omega_n} = \frac{1}{2n\pi\sqrt{c}} - \frac{1}{16\sqrt{c}(c-1)(\pi n)^3} + o(n^{-4}). \tag{19}$$

Define $(F_1, G_1, F_2, G_2) \in (H_0^1(0,1))^2 \times (L^2(0,1))^2$, such that

$$F_1 = F_1(x,n) = 0 \quad \forall x \in (-1,1),$$

$$G_1 = G_1(x,n) = \begin{cases} 0 & \text{in } (0,1) \\ g_1 = \dfrac{\sin(2n\pi x)}{2n\pi} & \text{in } (-1,0), \end{cases}$$

$$F_2 = F_2(x,n) = 0 \quad \forall x \in (-1,1),$$

$$G_2 = G_2(x,n) = \begin{cases} 0 & \text{in } (0,1) \\ g_2 = \dfrac{c\sin(2n\pi x)}{i\sqrt{\frac{2c}{c+1+\sqrt{(c-1)^2+\frac{4c}{\omega_n^2}}}}} & \text{in } (-1,0). \end{cases}$$

A straightforward calculation leads to

$$\|(F_1,G_1,F_2,G_2)\|^2_{\mathcal{H}} = \frac{1}{2} + \frac{1}{2\mu_-} \longrightarrow \frac{1}{2}\left(1 + \frac{1}{\sqrt{c}}\right) \quad \text{as} \quad n \nearrow +\infty. \tag{20}$$

Our goal is to prove that $\lim_{|\lambda|\to\infty} \|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} = \infty$. That is why, we want to solve the following resolvent equation

$$(\lambda I - \mathcal{A})(u^1, v^1, u^2, v^2) = (F_1, G_1, F_2, G_2). \tag{21}$$

Step 1 For all $x \in (0,1)$, we have

$$\begin{cases} \lambda u^1 - u^2 = 0 \\ \lambda v^1 - v^2 = 0 \\ \lambda u^2 - (1+\lambda d)u^1_{xx} + v^2 = 0 \\ \lambda v^2 - cv^1_{xx} - u^2 = 0 \\ v^1(1) = u^1(1) = 0. \end{cases} \tag{22}$$

Let

$$\eta_+ = \frac{-\lambda(1+\lambda d - c) + \omega_n\sqrt{r}e^{i\frac{\phi}{2}}}{2(1+\lambda d)} \quad \text{and} \quad \eta_- = \frac{-\lambda(1+\lambda d - c) - \omega_n\sqrt{r}e^{i\frac{\phi}{2}}}{2(1+\lambda d)}$$

where

$$r = \sqrt{a^2+b^2}, \quad \cos(\phi) = \frac{a}{r} \quad \text{and} \quad \sin(\phi) = \frac{b}{r},$$

with

$$a = -(1-c)^2 + d^2\omega^2 - \frac{4c}{\omega_n^2}$$
$$b = -2d\left((1-c)\omega_n + \frac{2c}{\omega_n}\right).$$

It is important to note that

$$\sqrt{a} = d\omega - \frac{(c-1)^2}{2d}\omega^{-1} - \frac{(c-1)^4 + 16cd^2}{8d^3}\omega^{-3} + o(\omega^{-3}),$$
$$\frac{b}{a} = \frac{2(c-1)}{d} + \frac{2(c-1) - 4cd}{d}\omega^{-3} + o(\omega^{-4})$$

and

$$i\frac{\sqrt{r}}{d}e^{i\frac{\phi}{2}} = \lambda - \frac{c-1}{d} - \frac{(c-1)^3 - d^2(c-1) + 2cd^2}{d^3}\lambda^{-2} + \frac{d^2(c-1)^2 - (c-1)^4 - 2cd^3(c-1) - 2cd^2}{d^4}\lambda^{-3} + o(\omega^{-3}).$$

Then, we have

$$\eta_+ = -\lambda + \frac{c}{d} - \frac{c}{d^2}\lambda^{-1} + \frac{(c-1)^3 + d^2(c+1) + 2c}{2d^3}\lambda^{-2} + \frac{(c-1)^4 - (c-1)^3 - d^2(c-1)(c-2) - 2c}{2d^4} + o(\omega^{-3}), \tag{23}$$

and

$$\eta_- = -\frac{(c-1)^3 + d^2(c+1)}{2d^3}\lambda^{-2} + \frac{(c-1)^3(2-c) + d^2(c-1)(c-2-2cd)}{2d^4}\lambda^{-3} + o(\omega^{-3}). \tag{24}$$

A straightforward calculation leads to

$$(u^1 + \eta_+ v^1)_{xx} = (\beta_+)^2(u^1 + \eta_+ v^1) \tag{25}$$

$$(u^1 + \eta_- v^1)_{xx} = (\beta_-)^2(u^1 + \eta_- v^1), \tag{26}$$

where

$$(\beta_\pm)^2 = \frac{c\lambda^2 - \lambda\eta_\pm(1+\lambda d)}{c(1+\lambda d)}.$$

So, for n large enough we get

$$\beta_{\pm} = \frac{\omega_n}{\sqrt{2c(1+(d\omega_n)^2)}}\sqrt{r_{\pm}}e^{i\frac{\phi_{\pm}}{2}},$$

where

$$r_{\pm} = \sqrt{a_{\pm}^2 + b_{\pm}^2}, \quad \cos(\phi_{\pm}) = \frac{a_{\pm}}{r_{\pm}} \quad \text{and} \quad \sin(\phi_{\pm}) = \frac{b_{\pm}}{r_{\pm}},$$

with

$$a_{\pm} = -(1+c) - (d\omega_n)^2 \pm \sqrt{r}\left(-d\omega_n \cos\left(\frac{\phi}{2}\right) + \sin\left(\frac{\phi}{2}\right)\right)$$
$$b_{\pm} = cd\omega_n \pm \sqrt{r}\left(-\cos\left(\frac{\phi}{2}\right) - d\omega_n \sin\left(\frac{\phi}{2}\right)\right).$$

Noting that

$$|a_+| = 2(d\omega)^2 + \frac{c^2-3c+6}{2} + o(\omega^{-1}),$$
$$\sqrt{|a_+|} = \sqrt{2}d\omega + \frac{c^2-3c+6}{4\sqrt{2}d}\omega^{-1} + o(\omega^{-1}),$$
$$b_+ = \left(d(2cd+1-c) + \frac{(c-1)^3}{d}\right)\omega^{-1} + o(\omega^{-1}),$$

and

$$\frac{b_+}{a_+} = o(\omega^{-2}).$$

Then, one gets

$$\beta_+ = \frac{\lambda}{\sqrt{c}} - \frac{(c-1)(c-2)}{8\sqrt{2}d^2} + o(\omega^{-1}), \tag{27}$$

and

$$\beta_+^2 = \frac{\lambda^2}{c} + o(1). \tag{28}$$

Similarly we have

$$b_- = 2cd\omega + \left(d(c-1-2cd) - \frac{(c-1)^3}{d}\right)\omega^{-1} + o(\omega^{-1}),$$
$$\sqrt{b_-} = \sqrt{2cd\omega}\left(1 + \left(\frac{c-1-2cd}{4c} - \frac{(c-1)^3}{4cd^2}\right)\omega^{-2}\right) + o(\omega^{-2}),$$
$$a_- = -2c + o(\omega^{-1}),$$

and

$$\frac{a_-}{b_-} = -\frac{\omega^{-1}}{d} + o(\omega^{-2}),$$

then, consequently we obtain

$$\beta_- = \sqrt{\frac{\omega}{d}} \mathrm{e}^{\frac{i\pi}{4}} - \frac{\omega^{-\frac{1}{2}}}{2d^{\frac{3}{2}}} \mathrm{e}^{-i\frac{\pi}{4}} + o(\omega^{-1}), \tag{29}$$

and

$$\begin{aligned} \beta_-^2 = \frac{\lambda}{d} - \frac{1}{d^2} + \frac{(c-1)^3 + d(c-1) + 2c}{2cd^3} \lambda^{-1} \\ - \frac{(c-1)^3(2-c) + d(c-1)(c-2-2cd) + 2c}{2cd^4} \lambda^{-2} + o(\omega^{-2}). \end{aligned} \tag{30}$$

Next, from (25), we get

$$(u^1 + \eta_+ v^1) = c_1 e^{x\beta_+} + c_2 e^{-x\beta_+},$$

and

$$(u^1 + \eta_- v^1) = c_3 e^{x\beta_-} + c_4 e^{-x\beta_-}.$$

Recalling that $u^1(1) = v^1(1) = 0$, we can rewrite the last two equations as follows:

$$(u^1 + \eta_+ v^1) = c_1 (e^{x\beta_+} - e^{(2-x)\beta_+}), \tag{31}$$

$$(u^1 + \eta_- v^1) = c_3 (e^{x\beta_-} - e^{(2-x)\beta_-}). \tag{32}$$

Hence, by combining (31) and (32), we obtain

$$u^1(x) = -\frac{c_1 \eta_-}{\eta_+ - \eta_-} \left(\mathrm{e}^{\beta_+ x} - \mathrm{e}^{\beta_+(2-x)} \right) + \frac{c_3 \eta_+}{\eta_+ - \eta_-} \left(\mathrm{e}^{\beta_- x} - \mathrm{e}^{\beta_-(2-x)} \right), \tag{33}$$

and

$$v^1(x) = \frac{c_1}{\eta_+ - \eta_-} \left(\mathrm{e}^{\beta_+ x} - \mathrm{e}^{\beta_+(2-x)} \right) - \frac{c_3}{\eta_+ - \eta_-} \left(\mathrm{e}^{\beta_- x} - \mathrm{e}^{\beta_-(2-x)} \right). \tag{34}$$

Step 2 For all $x \in (-1, 0)$ we have

$$\begin{cases} \lambda u^1 - u^2 = 0 \\ \lambda v^1 - v^2 = g_1 \\ \lambda u^2 - u^1_{xx} + v^2 = 0 \\ \lambda v^2 - c v^1_{xx} - u^2 = g_2 \\ v^1(-1) = u^1(-1) = 0. \end{cases} \tag{35}$$

From the third and the fourth equations of (22) and of (35), we can deduce, thanks to the regularity of the states, that

$$(1+\lambda d)u_x^1(0^+) = u_x^1(0^-), \tag{36}$$

$$v_x^1(0^+) = v_x^1(0^-). \tag{37}$$

and

$$(1+\lambda d)u_{xx}^1(0^+) = u_{xx}^1(0^-), \tag{38}$$

$$v_{xx}^1(0^+) = v_{xx}^1(0^-). \tag{39}$$

We denote by

$$\alpha_+ = \frac{\lambda}{2}\left(c-1+\sqrt{(1-c)^2+\frac{4c}{\omega_n^2}}\right) = (c-1)\lambda - \frac{c}{c-1}\lambda^{-1} - \frac{c^2}{(c-1)^3} + o(\omega^{-3}), \tag{40}$$

and

$$\alpha_- = \frac{\lambda}{2}\left(c-1-\sqrt{(1-c)^2+\frac{4c}{\omega_n^2}}\right) = \frac{c}{c-1}\lambda^{-1} + o(\omega^{-1}) \tag{41}$$

and we define, for n large enough, $\mu_\pm$ as follows

$$\mu_\pm = \frac{\sqrt{2c}}{\sqrt{c+1-\left(\pm\sqrt{(c-1)^2+\frac{4c}{\omega_n^2}}\right)}},$$

in particular with the choice of ω_n in (17), one gets

$$\mu_\pm^2 = \frac{\lambda}{\lambda - \frac{\alpha_\pm}{c}}.$$

Besides, we have

$$\mu_+ = \sqrt{c}\left(1 - \frac{c}{2(c-1)}\lambda^{-2} + o(\omega^{-2})\right), \tag{42}$$

$$\mu_- = 1 + \frac{\lambda^{-2}}{2(c-1)} + o(\omega^{-2}), \tag{43}$$

and

$$\frac{\mu_+}{\mu_-} = \sqrt{c}\left(1 - \frac{c+1}{2(c-1)}\lambda^{-2} + o(\omega^{-2})\right). \tag{44}$$

We set

$$\omega_1^+(x) = (u^2 + \alpha_+ v^2 + \mu_+(u_x^1 + \alpha_+ v_x^1), \tag{45}$$

$$\omega_1^-(x) = (u^2 + \alpha_+ v^2 - \mu_+(u_x^1 + \alpha_+ v_x^1)), \tag{46}$$

$$\omega_2^+(x) = (u^2 + \alpha_- v^2 + \mu_-(u_x^1 + \alpha_- v_x^1)), \tag{47}$$

$$\omega_2^-(x) = (u^2 + \alpha_- v^2 - \mu_-(u_x^1 + \alpha_- v_x^1)). \tag{48}$$

Now, define $Y = (\omega_1^+, \omega_1^-, \omega_2^+, \omega_2^-)^t$ and $Z = (g_{1x}, g_2)^t$. Then, we have

$$Y_x = AY + BZ, \tag{49}$$

where

$$A = \begin{pmatrix} \mu_+(\lambda - \frac{\alpha_+}{c}) & 0 & 0 & 0 \\ 0 & \mu_+(-\lambda + \frac{\alpha_+}{c}) & 0 & 0 \\ 0 & 0 & \mu_-(\lambda - \frac{\alpha_-}{c}) & 0 \\ 0 & 0 & 0 & \mu_-(-\lambda + \frac{\alpha_-}{c}) \end{pmatrix},$$

and

$$B = \begin{pmatrix} -\alpha_+ & -\mu_+\frac{\alpha_+}{c} \\ -\alpha_+ & \mu_+\frac{\alpha_+}{c} \\ -\alpha_- & -\mu_-\frac{\alpha_-}{c} \\ -\alpha_- & \mu_-\frac{\alpha_-}{c} \end{pmatrix}.$$

Then, a straightforward calculation leads to:

$$\mu_+\left(\lambda - \frac{\alpha_+}{c}\right) = 2in\pi.$$

Using the boundary condition at -1, we get

$$\omega_1^+(-1) = -\omega_1^-(-1) \quad \text{and} \quad \omega_2^+(-1) = -\omega_2^-(-1). \tag{50}$$

Taking (50) into account, the solution of (49), is written as follows

$$\begin{aligned}\omega_1^+(x) &= \omega_1^+(-1)\mathrm{e}^{2in\pi x} \\ &\quad - \frac{\alpha_+}{2}\left[\left(1 - \frac{\mu^+}{\mu^-}\right)(x+1)\mathrm{e}^{2in\pi x} + \frac{1}{2n\pi}\left(1 + \frac{\mu^+}{\mu^-}\right)\sin(2n\pi x)\right],\end{aligned} \tag{51}$$

$$\omega_1^-(x) = -\omega_1^+(-1)\mathrm{e}^{-2in\pi x} - \frac{\alpha_+}{2}\left[\left(1-\frac{\mu^+}{\mu^-}\right)(x+1)\mathrm{e}^{-2in\pi x} + \frac{1}{2n\pi}\left(1+\frac{\mu^+}{\mu^-}\right)\sin(2n\pi x)\right], \tag{52}$$

$$\omega_2^+(x) = \omega_2^+(-1)\mathrm{e}^{\mu_-\left(\lambda-\frac{\alpha_-}{c}\right)(x+1)} + \frac{\alpha_-}{2in\pi + \mu_-\left(\lambda-\frac{\alpha_-}{c}\right)}\left[\mathrm{e}^{-2in\pi x} + \mathrm{e}^{\mu_-\left(\lambda-\frac{\alpha_-}{c}\right)(x+1)}\right], \tag{53}$$

$$\omega_2^-(x) = -\omega_2^+(-1)\mathrm{e}^{-\mu_-\left(\lambda-\frac{\alpha_-}{c}\right)(x+1)} - \frac{\alpha_-}{2in\pi + \mu_-\left(\lambda-\frac{\alpha_-}{c}\right)}\left[\mathrm{e}^{2in\pi x} + \mathrm{e}^{-\mu_-\left(\lambda-\frac{\alpha_-}{c}\right)(x+1)}\right]. \tag{54}$$

Taking the trace of ω_1^+ and of ω_1^-, respectively in (51)–(52) and in (45)–(46), on the boundary 0, on top of that, by using the continuity of the states u_2 and v_2 we obtain

$$\begin{aligned}
(\omega_1^+ + \omega_1^-)(0^-) &= \alpha_+\left(\frac{\mu_+}{\mu_-} - 1\right) = 2u^2(0^-) + 2\alpha_+ v^2(0^-)\\
&= 2\lambda(u^1(0^-) + \alpha_+ v^1(0^-)) = 2\lambda(u^1(0^+) + \alpha_+ v^1(0^+))\\
&= \frac{2\lambda}{\eta_+ - \eta_-}\big(c_1(1-e^{2\beta_+})(\alpha_+ - \eta_-) + c_3(1-e^{2\beta_-})(\eta_+ - \alpha_+)\big),
\end{aligned}$$

where we have used the expressions of u^1 and v^1 in (33) and in (34).

This implies that

$$c_3 = \frac{1-e^{2\beta_+}}{1-e^{2\beta_-}}A_n c_1 + \frac{B_n}{1-e^{2\beta_-}}, \tag{55}$$

where

$$\begin{aligned}
A_n = \frac{\eta_- - \alpha_+}{\eta_+ - \alpha_+} &= \frac{c-1}{c}\left(1 + \frac{\lambda^{-1}}{d} - \frac{\lambda^{-2}}{c-1} + o(\omega^{-2})\right)\\
&= \frac{c-1}{c}\left(1 + \frac{n^{-1}}{2i\pi d\sqrt{c}} - \frac{n^{-2}}{4\pi^2 c(c-1)} + o(\omega^{-2})\right),
\end{aligned} \tag{56}$$

and

$$\begin{aligned}
B_n &= \frac{\alpha_+(\eta_+ - \eta_-)\left(\frac{\mu_+}{\mu_-} - 1\right)}{2\lambda(\eta_+ - \alpha_+)} \\
&= \frac{(c-1)(\sqrt{c}-1)}{2c}\left(1 - \frac{c-1}{d}\lambda^{-1} - \left(\frac{1}{(c-1)^2} + \frac{\sqrt{c}(c+1)}{2(\sqrt{c}-1)(c-1)}\right)\lambda^{-2}\right. \\
&\quad \left. + o(\omega^{-2})\right) \\
&= \frac{(c-1)(\sqrt{c}-1)}{2c}\left(1 - \frac{c-1}{2i\pi d\sqrt{c}}n^{-1} - \left(\frac{1}{(c-1)^2} + \frac{\sqrt{c}(c+1)}{2(\sqrt{c}-1)(c-1)}\right)\right. \\
&\quad \left. \times \frac{n^{-2}}{4\pi^2 c} + o(n^{-2})\right),
\end{aligned} \tag{57}$$

where we used here (23), (24), (40), (41), (44) and (17).

Using (51)–(52) and (38)–(39), one gets

$$\begin{aligned}
(\omega_1^+ - \omega_1^-)'(0^-) &= 2in\pi\alpha_+\left(\frac{\mu_+}{\mu_-} - 1\right) \\
&= 2\mu_+(u^1 + \alpha_+ v^1)_{xx}(0^-) = 2\mu_+((1+\lambda d)u^1 + \alpha_+ v^1)_{xx}(0^+) \\
&= \frac{2\mu_+\left[c_1\beta_+^2(1 - \mathrm{e}^{2\beta_+})(\alpha_+ - (1+\lambda d)\eta_-) + c_3\beta_-^2(1 - \mathrm{e}^{2\beta_-})((1+\lambda d)\eta_+ - \alpha_+)\right]}{\eta_+ - \eta_-}.
\end{aligned}$$

Then we obtain

$$c_1 = \frac{1 - e^{2\beta_-}}{1 - e^{2\beta_+}} A_n' c_3 + \frac{B_n'}{1 - e^{2\beta_+}}, \tag{58}$$

where

$$\begin{aligned}
A_n' &= \frac{\beta_-^2(\alpha_+ - (1+\lambda d)\eta_+)}{\beta_+^2(\alpha_+ - (1+\lambda d)\eta_-)} \\
&= \frac{c}{c-1}\left(1 - \frac{\lambda^{-1}}{d} + \left(\frac{(c-1)^3}{2cd^2} + \frac{c-1}{2cd} + \frac{3-c}{2d^2} + \frac{1}{2}\right)\lambda^{-2} + o(\omega^{-2})\right) \\
&= \frac{c}{c-1}\left(1 - \frac{n^{-1}}{2i\pi d\sqrt{c}} + \left(\frac{(c-1)^3}{2cd^2} + \frac{c-1}{2cd} + \frac{3-c}{2d^2} + \frac{1}{2}\right)\frac{n^{-2}}{4\pi^2 c} + o(n^{-2})\right),
\end{aligned} \tag{59}$$

and

$$
\begin{aligned}
B_n' &= \frac{in\pi\alpha_+(\eta_+ - \eta_-)\left(\frac{\mu_+}{\mu_-} - 1\right)}{\mu_+\beta_+^2(\alpha_+ - (1+\lambda d)\eta_-)} \\
&= \frac{n\pi(c-\sqrt{c})}{2\omega}\left(-1 + \frac{c}{d}\lambda^{-1} + \left(\frac{c+\sqrt{c}+3}{2(c-1)^2} - \frac{c+1+d^2}{2d^2}\right)\lambda^{-2}\right) + o(\omega^{-2}) \\
&= \frac{\sqrt{c}-1}{2}\left(-1 + \frac{\sqrt{c}}{2i\pi d}n^{-1} + \left(\frac{\sqrt{c}+4}{c-1} - \frac{c+1+d}{d^2}\right)\frac{n^{-2}}{8c\pi^2} + o(n^{-2})\right).
\end{aligned} \tag{60}
$$

Here we have used (23), (24), (28), (30), (40), (41), (42), (44) and (17).

Combining (55) and (58), we find that

$$
c_1 = \frac{1}{1-e^{2\beta_+}} \times \frac{A_n' B_n + B_n'}{1 - A_n A_n'} = \frac{c_1'}{1-e^{2\beta_+}}, \tag{61}
$$

and

$$
c_3 = \frac{1}{1-e^{2\beta_-}} \times \frac{A_n B_n' + B_n}{1 - A_n A_n'} = \frac{c_3'}{1-e^{2\beta_-}}, \tag{62}
$$

where, thanks to (56), (57), (59) and (60), we have

$$
c_1' = O(1) \quad \text{and} \quad c_3' = O(1). \tag{63}
$$

On the other hand, by donating $\theta = -i\mu_-\left(\lambda - \frac{\alpha_-}{c}\right)$ and by using the same argument as previously, one gets

$$
\begin{aligned}
(\omega_2^+ + \omega_2^-)(0^-) &= 2i\sin(\theta)\omega_2^+(-1) + \frac{2\alpha_-}{2n\pi - \theta}\sin(\theta) \\
&= 2\lambda(u^1 + \alpha_- v^1)(0^-) = 2\lambda(u^1 + \alpha_- v^1)(0^+) \\
&= \frac{2\lambda}{\eta_+ - \eta_-}\left(c_1'(\alpha_- - \eta_-) + c_3'(\eta_+ - \alpha_-)\right).
\end{aligned}
$$

It is clear that $\theta \neq 0[\pi]$, whence we can see that

$$
\omega_2^+(-1) = \frac{\lambda}{i\sin(\theta)(\eta_+ - \eta_-)}\left[c_1'(\alpha_- - \eta_-) + c_3'(\eta_+ - \alpha_-)\right] - \frac{\alpha_-}{2in\pi + i\theta}. \tag{64}
$$

Noting that from (17), (18), (41) and (43), we have

$$
\theta = \omega\left(1 - \frac{3}{2(c-1)}\omega^{-2} + o(\omega^{-2})\right) = \sqrt{c}\left(2n\pi + \frac{c\pi - 12}{4c\pi^2(c-1)}n^{-1} + o(n^{-1})\right). \tag{65}
$$

Then, from (18), (23), (24), (40), (42) and (65) we deduce that

$$\omega_2^+(-1) \sim \frac{2\pi n\sqrt{c}\,c_3'}{\sin(\theta)}. \tag{66}$$

Using (36)–(37), (45)–(46) and (51)–(52), we get

$$\begin{aligned}\omega_1^+(-1) &= \frac{(\omega_1^+ - \omega_1^-)(0^-)}{2} = \mu_+(u^1 + \alpha_+ v^1)_x(0^-)\\ &= \mu_+((1+\lambda d)u^1 + \alpha_+ v^1)_x(0^+)\\ &= \frac{\mu_+}{\eta_+ - \eta_-}\left[c_1\beta_+(1+e^{2\beta_+})(\alpha_+ - (1+\lambda d)\eta_-)\right.\\ &\quad \left. + c_3\beta_-(1+e^{2\beta_-})((1+\lambda d)\eta_+ - \alpha_+)\right].\end{aligned} \tag{67}$$

Then, from (18), (23), (24), (29), (29), (41) and (42) we deduce that

$$\omega_1^+(-1) \sim c_3'\sqrt{\frac{c}{d}}\mathrm{e}^{-i\frac{\pi}{4}}(2\pi\sqrt{c}n)^{\frac{3}{2}}. \tag{68}$$

Next, for all $x \in (-1,0)$ we have

$$\begin{aligned}v_x^1(x) &= \frac{1}{2\mu_-\mu_+(\alpha_+ - \alpha_-)}\left[\alpha_-(\omega_1^+(x) - \omega_1^-(x)) - \alpha_+(\omega_2^+(x) - \omega_2^-(x))\right]\\ &= \frac{1}{2\mu_-\mu_+(\alpha_+ - \alpha_-)}\left[\mu_-\left(2\omega_1^+(-1)\cos(2n\pi x)\right.\right.\\ &\quad \left. - i\alpha_+\left(1 - \frac{\mu_+}{\mu_-}\right)(x+1)\sin(2n\pi x)\right)\\ &\quad - \mu_+\left(2\omega_2^+(-1)\cos\left(\theta(x+1)\right) + \frac{2\alpha_-}{2in\pi + i\theta}(\cos(2n\pi x)\right.\\ &\quad \left.\left. + \cos\left(\theta(x+1)\right))\right)\right],\end{aligned} \tag{69}$$

where we have used (45)–(48) and (51)–(54). This further leads to

$$\begin{aligned}\|v_x^1\|^2_{L^2(-1,0)} \geq \max\left\{\frac{|\omega_1^+(-1)|^2}{2\mu_+^2|\alpha_+ - \alpha_-|^2}, \frac{|\omega_2^+(-1)|^2}{\mu_-^2|\alpha_+ - \alpha_-|^2}\right\} - \frac{|\alpha_+|^2(\mu_+ - \mu_-)^2}{4\mu_-^2\mu_+^2|\alpha_+ - \alpha_-|^2}\\ - \min\left\{\frac{|\omega_1^+(-1)|^2}{2\mu_+^2|\alpha_+ - \alpha_-|^2}, \frac{|\omega_2^+(-1)|^2}{\mu_-^2|\alpha_+ - \alpha_-|^2}\right\} - \frac{2|\alpha_-|^2}{\mu_-^2\mu_+^2(2n\pi + \theta)^2|\alpha_+ - \alpha_-|^2}.\end{aligned} \tag{70}$$

Since from assumption (15) we have

$$\sin(\theta) \neq O(n^{-\frac{1}{2}}),$$

then by using (17), (42), (43), (40), (41), (66) and (68), we can show that the second and the fourth terms of the right hand side of (70) are bounded, while the sum of the first and the third terms tends to the infinity as n goes to $+\infty$, whence we obtain

$$\|v_x^1\|^2_{L^2(-1,0)} \longrightarrow +\infty \quad \text{as} \quad n \nearrow +\infty. \tag{71}$$

Last but not least, we have

$$\|(i\omega_n I - \mathscr{A})^{-1}(F_1, G_1, F_2, G_2)\|_{\mathscr{H}} = \|(u^1, v^1, u^2, v^2)\|^2_{\mathscr{H}} \geq \int_{-1}^{0} |v_x^1(x)|^2 \mathrm{d}x \longrightarrow +\infty, \tag{72}$$

as $n \nearrow +\infty$. Finally, we conclude, using (72) and (20) that

$$\limsup_{\omega \in \mathbb{R}, |\omega| \to \infty} \|(i\omega I - \mathscr{A})^{-1}\|_{\mathscr{L}(\mathscr{H})} = +\infty.$$

So, $e^{t\mathscr{A}}$ is not exponentially stable in the energy space. This completes the proof. □

5 Polynomial stabilization

This section aims to prove the polynomial stability given by the following theorem.

Theorem 4 *The semigroup of contractions* $(e^{t\mathscr{A}})_{t\geq 0}$ *is polynomially stable of order* $\frac{1}{12}$, i.e., *there exists* $C > 0$ *such that*

$$\|e^{t\mathscr{A}}(u_0, v_0, u_1, v_1)\|_{\mathscr{H}} \leq \frac{C}{(1+t)^{\frac{1}{12}}} \|(u_0, v_0, u_1, v_1)\|_{\mathscr{D}(\mathscr{A})}$$

for all $t > 0$ *and for all* $(u_0, v_0, u_1, v_1) \in \mathscr{D}(\mathscr{A})$.

We recall here the Batty–Duychaerts [6] and the Borichev–Tomilov [7] results given by the following

Theorem 5 ([6, 7]) *Let* $\mathscr{B}$ *be a generator of a* C_0*-semigroup of contractions in a Hilbert space* $\mathscr{X}$ *with a domain* $\mathscr{D}(\mathscr{B})$ *such that* $i\mathbb{R} \subset \sigma(\mathscr{B})$. *Then* $e^{t\mathscr{B}}$ *is polynomially stable of order* $\frac{1}{\gamma}, \gamma > 0$, i.e., *there exists* $C > 0$ *such that*

$$\|e^{t\mathscr{B}} U_0\|_{\mathscr{X}} \leq \frac{C}{(1+t)^{\frac{1}{\gamma}}} \|U_0\|_{\mathscr{D}(\mathscr{B})}, \quad \forall\, t \geq 0,\ \forall\, U_0 \in \mathscr{D}(\mathscr{B}),$$

if and only if

$$\limsup_{\beta \in \mathbb{R}, |\beta| \to \infty} \|\beta^{-\gamma}(i\beta - \mathscr{B})^{-1}\|_{\mathscr{L}(\mathscr{X})} < +\infty.$$

Based on Theorem 5, we are able now to prove our main result given in Theorem 4. For that, let us consider the following

Proposition 2 *The operator $\mathscr{A}$ defined in* (3) *satisfies:*

$$\limsup_{\beta\in\mathbb{R},\,|\beta|\to\infty} \|\beta^{-12}(i\beta-\mathscr{A})^{-1}\|_{\mathscr{L}(\mathscr{H})} < +\infty. \tag{73}$$

Proof To prove (73) we use an argument of contradiction. In fact, if (73) is false, then there exist $\beta_n \in \mathbb{R}_+$ and $Y_n = (u_n^1, v_n^1, u_n^2, v_n^2) \in \mathscr{D}(\mathscr{A})$ such that

$$\|Y_n\|_{\mathscr{H}} = 1,\ \beta_n \nearrow +\infty \text{ and } \beta^\gamma(i\beta_n \mathrm{I} - \mathscr{A})Y_n := (f_n^1, g_n^1, f_n^2, g_n^2) \longrightarrow 0 \text{ in } \mathscr{H} \text{ as } n \nearrow +\infty. \tag{74}$$

Equivalently, we have

$$\beta_n^\gamma\left(i\beta_n u_n^1 - u_n^2\right) = f_n^1 \longrightarrow 0 \quad \text{in} H_0^1(-1,1), \tag{75}$$

$$\beta_n^\gamma\left(i\beta_n v_n^1 - v_n^2\right) = g_n^1 \longrightarrow 0 \quad \text{in} \quad H_0^1(-1,1), \tag{76}$$

$$\beta_n^\gamma\left(i\beta_n u_n^2 - \left(u_{nx}^1 + a u_{nx}^2\right)_x + v_n^2\right) = f_n^2 \longrightarrow 0 \quad \text{in} \quad L^2(-1,1), \tag{77}$$

$$\beta_n^\gamma\left(i\beta_n v_n^2 - c v_{nxx}^1 - u_n^2\right) = g_n^2 \longrightarrow 0 \quad \text{in} \quad L^2(-1,1). \tag{78}$$

Denote

$$T_n = u_{nx}^1 + a u_{nx}^2.$$

Taking the real part of $\langle \beta^\gamma (i\beta_n \mathrm{I} - \mathscr{A})Y_n, Y_n\rangle_{\mathscr{H}}$, we get by the dissipation property of the semigroup of the operator $\mathscr{A}$:

$$\beta_n^\gamma \int_0^1 d.|u_{nx}^2|^2\, \mathrm{d}x \longrightarrow 0,$$

which leads to

$$\beta_n^{\frac{\gamma}{2}} \|u_{nx}^2\|_{L^2(0,1)} \longrightarrow 0. \tag{79}$$

Now thanks to (75) and (79), we obtain

$$\beta_n^{\frac{\gamma}{2}+1} \|u_{nx}^1\|_{L^2(0,1)} \longrightarrow 0. \tag{80}$$

From (79) and (80), it follows

$$\beta_n^{\frac{\gamma}{2}} \|T_n\|_{L^2(0,1)} \longrightarrow 0. \tag{81}$$

Taking the inner product of (77) with u_n^2 in $L^2(0,1)$, we get

$$\beta_n^{\frac{3\gamma}{4}}\left(i\beta_n\|u_n^2\|^2_{L^2(0,1)}+\langle T_n,u^2_{nx}\rangle_{L^2(0,1)}+T_n(0^+)\overline{u_n^2}(0^+)+\langle v_n^2,u_n^2\rangle_{L^2(0,1)}\right)=o(1). \tag{82}$$

Thanks to (74), (79) and (81), it is clear that the second and the last terms converge to zero. Furthermore, we have

$$\beta_n^{\frac{3\gamma}{4}}T_n(0^+)\overline{u_n^2}(0^+)\leq C\beta_n^{\frac{\gamma}{2}}\left(\|T_n\|^{\frac12}_{L^2(0,1)}.\|u^2_{nx}\|^{\frac12}_{L^2(0,1)}.\|T_n'\|^{\frac12}_{L^2(0,1)}.\|u^2_n\|^{\frac12}_{L^2(0,1)}\right).$$

From (77), we can see that $\|\beta_n u_n^2+v_n^2\|_{L^2(0,1)}\sim\|T_n'\|_{L^2(0,1)}$ which implies that

$$\begin{aligned}\beta_n^{\frac{3\gamma}{4}}|T_n(0^+)|.|\overline{u_n^2}(0^+)|&\leq C\beta_n^{\frac{3\gamma}{4}}\|T_n\|^{\frac12}_{L^2(0,1)}.\|u^2_{nx}\|^{\frac12}_{L^2(0,1)}\\&\quad\times\left(\|\beta_nu_n^2\|^{\frac12}_{L^2(0,1)}+\|v_n^2\|^{\frac12}_{L^2(0,1)}+o(1)\right).\|u_n^2\|^{\frac12}_{L^2(0,1)}\\&\leq C\|\beta_n^{\frac{\gamma}{2}}T_n\|^{\frac12}_{L^2(0,1)}.\|\beta_n^{\frac{\gamma}{2}}u^2_{nx}\|^{\frac12}_{L^2(0,1)}\\&\quad\times\left(\|\beta_nu_n^2\|^{\frac12}_{L^2(0,1)}+\|v_n^2\|^{\frac12}_{L^2(0,1)}\right)\|\beta_n^{\frac{\gamma}{2}}u_n^2\|^{\frac12}_{L^2(0,1)}+o(1)\\&\leq\left(1+\beta_n^{\frac12+\frac{\gamma}{4}}.\|u_n^2\|_{L^2(0,1)}\right)o(1).\end{aligned} \tag{83}$$

Combining (82) and (83), one gets

$$\beta_n^{\frac12+\frac{3\gamma}{8}}\|u_n^2\|_{L^2(0,1)}\longrightarrow 0. \tag{84}$$

Moreover, multiplying (77) by $\beta_n^{-\frac{\gamma}{2}}(1-x)T_n$, integrating over the interval (0, 1) and then, taking into account (81), an integration by parts leads to

$$\begin{aligned}&\mathrm{Re}\langle i\beta_n^{\frac12+\frac{3\gamma}{8}}u_n^2,(1-x)\beta_n^{\frac12-\frac{3\gamma}{8}+\frac{\gamma}{2}}T_n\rangle_{L^2(0,1)}+\frac{\beta_n^{\frac{\gamma}{2}}}{2}\left(|T_n(0^+)|^2-\|T_n\|^2_{L^2(0,1)}\right)\\&\quad+\beta_n^{\frac{\gamma}{2}}\mathrm{Re}\langle v_n^2,(1-x)T_n\rangle_{L^2(0,1)}=o(1).\end{aligned} \tag{85}$$

We suppose that $\gamma\geq\dfrac43$. It is clear from (74), (81) and (84) that the first, the third and the last terms of (85) converge to zero, whence one gets

$$\beta_n^{\frac{\gamma}{4}}.|T_n(0^+)|\longrightarrow 0. \tag{86}$$

Taking into account (80), the trace formula gives

$$\beta_n^{\frac{\gamma}{2}+1}.|u_n^1(0^+)|\longrightarrow 0. \tag{87}$$

Substituting (76) into (77), taking the inner product with $\beta_n^{3-\gamma}v_n^1$ in $L^2(0,1)$ and then, by integrating by parts, we end up with

$$i\beta_n^4\left\langle u_n^2, v_n^1\right\rangle_{L^2(0,1)} + \beta_n^3\left\langle T_n, v_n^1\right\rangle_{L^2(0,1)} + i\beta_n^4\|v_n^1\|_{L^2(0,1)}^2 + \beta_n^3 T_n(0^+)\overline{v_n^1(0^+)} = o(1). \tag{88}$$

Now, taking $\gamma \geq 12$ and using (74), (81), (84) and (86), we can see that the first, the second and the fourth terms of (88) converge to zero, therefore

$$\beta_n^2.\|v_n^1\|_{L^2(0,1)} \longrightarrow 0. \tag{89}$$

From (76) and (89), it follows

$$\beta_n\|v_n^2\|_{L^2(0,1)} \longrightarrow 0. \tag{90}$$

Multiplying (78) with $\beta_n^{-\gamma}(1-x)\overline{v_{nx}^1}$, integrating over (0, 1) and then, by taking the real part we get

$$\frac{c}{2}\left(|v_{nx}^1(0^+)|^2 - \|v_{nx}^1\|_{L^2(0,1)}^2\right) = \mathrm{Re}\left\langle u_n^2, (1-x)v_{nx}^1\right\rangle_{L^2(0,1)}$$
$$- \mathrm{Re}\langle i\beta_n v_n^2, (1-x)v_{nx}^1\rangle_{L^2(0,1)} + o(1).$$

Using (74), (84) and (90) leads to

$$|v_{nx}^1(0^+)|^2 - \|v_{nx}^1\|_{L^2(0,1)}^2 \longrightarrow 0. \tag{91}$$

We take the inner product of (78) with $\beta_n^{-\gamma}xv_n^1$ in $L^2(0,1)$ in order to have

$$c\left(\int_0^1 x|v_{nx}^1(x)|^2\,\mathrm{d}x + \left\langle v_{nx}^1, v_n^1\right\rangle_{L^2(0,1)}\right) = \left\langle u_n^2, xv_n^1\right\rangle_{L^2(0,1)} - i\beta_n\langle v_n^2, xv_n^1\rangle_{L^2(0,1)} + o(1).$$

Using (74), (84) and (90), we deduce that

$$\int_0^1 x|v_{nx}^1(x)|^2\,\mathrm{d}x \longrightarrow 0.$$

This implies in particular, that for every ε in (0, 1), we have

$$\|v_{nx}^1\|_{L^2(\varepsilon,1)} \longrightarrow 0 \ \text{ as } n \nearrow +\infty. \tag{92}$$

Multiplying (78) with $\beta_n^{-\gamma}(1-x)\overline{v_{nx}^1}$, integrating over $(0,\varepsilon)$ and then, by taking the real part we find

$$\frac{c}{2}\left(|v_{nx}^1(\varepsilon)|^2 - \|v_{nx}^1\|_{L^2(\varepsilon,1)}^2\right) = \mathrm{Re}\left\langle u_n^2, (1-x)v_{nx}^1\right\rangle_{L^2(\varepsilon,1)} - \mathrm{Re}\langle i\beta_n v_n^2, (1-x)v_{nx}^1\rangle_{L^2(\varepsilon,1)} + o(1).$$

Besides, from (74), (84), (90) and (92), it follows

$$|v_{nx}^1(\varepsilon)| \longrightarrow 0 \ \text{ as } n \nearrow +\infty.$$

Then, we deduce that

$$v_{nx}^1(x) \longrightarrow 0 \ \text{ a.e. in } [0,1] \text{ as } n \nearrow +\infty. \tag{93}$$

Now, (74) and (93) allow the use of the dominated convergence theorem, which leads to

$$\|v^1_{nx}\|_{L^2(0,1)} \longrightarrow 0. \tag{94}$$

Therefore, we obtain

$$|v^1_n(0^+)| \longrightarrow 0. \tag{95}$$

By combining (91) and (94), one gets

$$|v^1_{nx}(0^+)| \longrightarrow 0. \tag{96}$$

Furthermore, taking the inner product of (76) with $\beta_n^{1-\gamma}(1-x)v^1_{nx}$ and then, by considering the imaginary part, one gets

$$\begin{aligned}
&\beta_n^2 Re(v^1_{nx}, (1-x)v^1_n) - Im\beta_n(v^2_n, (1-x)v^1_{nx}) = o(1)\\
&= \frac{1}{2}(\beta_n^2|v^1_n(0^+)|^2 - \beta_n^2\|v^1_n\|^2) - \beta_n Im\langle v^2_n, (1-x)v^1_{nx}\rangle.
\end{aligned}$$

Adding to this (95), (89) and (90) we can deduce that

$$\beta_n|v^1_n(0^+)| \longrightarrow 0. \tag{97}$$

Thanks to (86), (87), (95) and (96), one gets

$$\beta_n^{\frac{\gamma}{2}+1}.u^1_n(0^-) \longrightarrow 0, \tag{98}$$

$$\beta^{\frac{\gamma}{4}}.u^1_{nx}(0^-) \longrightarrow 0, \tag{99}$$

$$\beta_n v^1_n(0^-) \longrightarrow 0, \tag{100}$$

$$v^1_{nx}(0^-) \longrightarrow 0. \tag{101}$$

Next, inserting (75) into (77), inserting (76) into (78) and consider both equations in the interval (0, 1), leads to

$$-\beta_n^2 u^1_n - u^1_{nxx} + v^2_n = \beta_n^{-\gamma} f^2_n + i\beta_n^{1-\gamma} f^1_n, \tag{102}$$

and

$$-\beta_n^2 v^1_n - cv^1_{nxx} - u^2_n = \beta_n^{-\gamma} g^2_n + i\beta_n^{1-\gamma} g^1_n. \tag{103}$$

A straightforward calculation shows that the real part of the inner product of (102) with $(x+1).u^1_{nx}$ and that the real part of the inner product of (103) with $(x+1).v^1_{nx}$, leads to

$$\frac{1}{2}\int_{-1}^{0}\left(|\beta_n u_n^1|^2+|u_{nx}^1|^2\right)\mathrm{d}x=\frac{1}{2}\left(|u_{nx}^1(0^-)|^2+\beta_n^2|u_n^1(0^-)|^2\right)\\ -\operatorname{Re}\langle v_n^2,(x+1)u_{nx}^1\rangle_{L^2(-1,0)}+o(1), \tag{104}$$

and

$$\frac{1}{2}\int_{-1}^{0}\left(|\beta_n v_n^1|^2+c|v_{nx}^1|^2\right)\mathrm{d}x=\frac{1}{2}\left(c|v_{nx}^1(0^-)|^2+\beta_n^2|v_n^1(0^-)|^2\right)\\ +\operatorname{Re}\langle u_n^2,(x+1)v_{nx}^1\rangle_{L^2(-1,0)}+o(1), \tag{105}$$

where we have used (74)–(78).

On the other hand, from (74), (84), (90) and (98)–(101), we get

$$\int_{-1}^{0}\left(|\beta_n u_n^1|^2+|u_{nx}^1|^2\right)\mathrm{d}x\longrightarrow 0, \tag{106}$$

and

$$\int_{-1}^{0}\left(|\beta_n v_n^1|^2+c|v_{nx}^1|^2\right)\mathrm{d}x\longrightarrow 0. \tag{107}$$

Now by summing (80) (84), (89), (90), (106) and (107), we can see that

$$\|Y_n\|_{\mathscr{H}}\longrightarrow 0. \tag{108}$$

This contradicts (74). And so, (73) holds true with $\gamma\geq 12$. This completes the proof. □

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