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RESEARCH ARTICLE

Existence of asymptotically almost periodic solutions for some second-order hyperbolic integrodifferential equations

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Abstract

In this paper we study and obtain the existence of asymptotically almost periodic solutions to some classes of second-order hyperbolic integrodifferential equations of Gurtin–Pipkin type in a separable Hilbert space H. To illustrate our abstract results, the existence of asymptotically almost periodic mild solutions to the well-known Kirchoff plate equation is studied.

1 Introduction

Integrodifferential equations play an important role when it comes to describing various practical problems, see, e.g., [4–6, 13, 14, 17, 21–23]. In particular, these types of equations are utilized to study practical problems in which some memory effect is taken into account, such as the heat conduction in materials with memory or the sound propagation in viscoelastic media or in homogenization problems in perforated media (Darcy's Law), see, e.g., [1, 5, 15, 16, 18].

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Let $(H, \|\cdot\|_H, \langle\cdot,\cdot\rangle_H)$ be a separable Hilbert space. The main purpose of this paper consists of establishing the existence of asymptotically almost periodic mild solutions to a class of second-order hyperbolic integrodifferential equations of Gurtin–Pipkin type given by

$$\frac{d^2u}{dt^2} + A^2u - \int_{-\infty}^{t} g(t-s)A^2u(s)ds = f(t,u),\ t > 0 \tag{1}$$

with initial conditions

$$u(-t) = u_0(t),\ \ t \geq 0 \ \text{ and } \ u'(0) = u_1, \tag{2}$$

where $A : \mathcal{D}(A) \subset H \mapsto H$ is a positive self-adjoint operator which is bounded below, that is, there exists a constant $\omega > 0$ such that

$$\|Au\|_H \geq \omega\|u\|_H \ \text{ for all } \ u \in \mathcal{D}(A), \tag{3}$$

the function $f : [0,\infty) \times H \mapsto H$ is asymptotically almost periodic in the first variable uniformly in the second one, and the non-increasing differentiable relaxation (kernel) function $g : [0,\infty) \longrightarrow [0,\infty)$ satisfies the following assumptions,

(**A.1**) $g(0) > 0$ and $\beta := 1 - \int_0^\infty g(s)ds > 0$; and
(**A.2**) there exists a positive constant ξ such that $g'(t) \leq -\xi g(t)$ for all $t \geq 0$.

Our main task in this paper consists of showing that problem (1)–(2), under some suitable assumptions, has an asymptotically almost periodic mild solution. To achieve that, our strategy consists of transforming such a system into a first-order evolution Eq. (7) below. Under assumptions (**A.1**) and (**A.2**), it will be shown that the linear operator $\mathcal{A}$ appearing in Eq. (7) is the infinitesimal generator of a C_0-semigroup of contraction $(T(t))_{t\geq 0}$ which actually is exponentially stable (Theorems 2 and 3). Next, one makes use of the appropriate fixed-point tools to obtain the existence of an asymptotically almost periodic mild solution to Eq. (7) which, in turn, yields the existence of an asymptotically almost periodic mild solution to Eqs. (1)–(2).

Recall that the existence of almost periodic and asymptotically almost periodic solutions to integrodifferential equations is an important topic which has numerous applications, see, e.g., [8–10]. The novelty in this paper consists of using the semigroup approach to study the existence of solutions to some second-order integrodifferential equations in the case when the forcing term is asymptotically almost periodic. We show that if the forcing term f is asymptotically almost periodic, then, under some additional conditions on the parameters ((**A.1**) and (**A.2**)) of the system (1)–(2), there exists a mild solution which converges asymptotically to an almost periodic function. To the best of our knowledge, the existence of asymptotically almost periodic mid solutions to second-order integrodifferential equations of Gurtin–Pipkin type formulated in (1)–(2) is an untreated problem which constitutes the main motivation of this paper. For more information on the system (1)–(2) including its well-posedness or the asymptotic behavior of its solutions, we refer the reader to the work of Vlasov and Rautian [18–20].

Let us mention that assumptions (**A.1**)–(**A.2**) yield $g(t) \le g(0)e^{-\xi t}$ for all $t \ge 0$. Consequently, we have

$$\int_0^\infty s^k g(s)ds \le \frac{k!}{\xi^{k+1}} g(0) \qquad \text{for} \quad k = 0, 1, 2, \cdots .$$

Further, it can be shown that the relaxation function K defined by

$$K(t) = \sum_{k=1}^{\infty} a_k e^{-\gamma_k t}, \quad t \ge 0,$$

where $a_k > 0$, $\gamma_{k+1} > \gamma_k > 0$ for all $k \in \mathbb{N}$, $\gamma_k \to \infty$ as $k \to \infty$, and

$$\sum_{k=1}^{\infty} a_k < 1$$

satisfies assumptions (**A.1**)–(**A.2**) for any $\xi \in (0, \gamma_1]$.

2 Preliminaries

Fix once and for all a separable Hilbert space H whose inner product and norm are given respectively by $\langle \cdot, \cdot \rangle_H$ and $\| \cdot \|_H$. Let $(X, \| \cdot \|)$ and $(Y, \| \cdot \|)$ be two Banach spaces. If A is a linear operator, then the notations $\mathcal{D}(A)$ and $\rho(A)$ stand respectively for the domain and resolvent of A.

Let $BC(\mathbb{R};X)$ (respectively, $BC(\mathbb{R} \times X;Y)$) stand for the Banach space of all bounded continuous functions from $\mathbb{R}$ into X (respectively, stand for the collection of all jointly continuous bounded functions from $\mathbb{R} \times X$ into Y) equipped with the sup-norm defined by $\|u\|_\infty := \sup_{t\in\mathbb{R}} \|u(t)\|$ for all $u \in BC(\mathbb{R};X)$. Finally, $C_0(\mathbb{R}_+;X)$ (respectively, $C_0(\mathbb{R}_+ \times X;Y)$) stands for the collection of all continuous functions $u : \mathbb{R}_+ \longrightarrow X$ such that $\lim_{t\to\infty} \|u(t)\| = 0$ (respectively, all jointly continuous functions $U : \mathbb{R}_+ \times X \longrightarrow Y$ such that $\lim_{t\to\infty} \|U(t,x)\| = 0$ for all $x \in X$).

Definition 1 A function $f \in BC(\mathbb{R};X)$ is almost periodic if for every $\varepsilon > 0$ there exists a relatively dense subset of $\mathbb{R}$, denoted by $\mathscr{T}(\varepsilon, f, X)$, such that

$$\|f(t+\tau) - f(t)\| < \varepsilon \qquad \forall\, t \in \mathbb{R}, \quad \forall\, \tau \in \mathscr{T}(\varepsilon, f, X).$$

Definition 2 A function $f \in BC(\mathbb{R}_+;X)$ is asymptotically almost periodic if there exist an almost periodic function g and $\phi \in C_0(\mathbb{R}_+;X)$ such that $f = g + \phi$.

Definition 3 A function $F \in BC(\mathbb{R} \times X;Y)$ is almost periodic in $t \in \mathbb{R}$ uniformly in $y \in Y$ if for each $\varepsilon > 0$ and any compact subset K of Y there exists a relatively dense subset of $\mathbb{R}$, denoted by $\mathscr{T}(\varepsilon, F, K, X)$, such that

$$\|F(t+\tau, y) - F(t, y)\| < \varepsilon \qquad \forall\, t \in \mathbb{R}, \quad \forall\, \tau \in \mathscr{T}(\varepsilon, F, K, Y), \quad \forall\, y \in K.$$

Definition 4 A function $F \in BC(\mathbb{R}_+ \times X;Y)$ is asymptotically almost periodic in $t \in \mathbb{R}_+$ uniformly $y \in Y$ if there exist an almost periodic function $G \in AP(\mathbb{R} \times X;Y)$ and $\Phi \in C_0(\mathbb{R}_+ \times X;Y)$ such that $F = G + \Phi$.

Lemma 1 [24] *A function $f \in BC(\mathbb{R}_+;X)$ is asymptotically almost periodic if and only if for every $\varepsilon > 0$ there exists $L(\varepsilon, f, X) > 0$ and a relatively dense subset of $\mathbb{R}_+$, denoted by $\mathscr{T}(\varepsilon, f, X)$, such that*

$$\|f(t+\tau) - f(t)\| < \varepsilon \qquad \forall\, t \geq L(\varepsilon, f, X), \quad \forall\, \tau \in \mathscr{T}(\varepsilon, f, X).$$

Lemma 2 [10] *A function $F \in BC(\mathbb{R} \times X;Y)$ is asymptotically almost periodic if for each $\varepsilon > 0$ and any compact subset K of Y there exists $L(\varepsilon, F, K, X) > 0$ and a relatively dense subset of $\mathbb{R}_+$, denoted by $\mathscr{T}(\varepsilon, F, K, X)$, such that*

$$\|F(t+\tau, y) - F(t, y)\| < \varepsilon \qquad \forall\, t \geq L(\varepsilon, F, K, X), \quad \forall\, \tau \in \mathscr{T}(\varepsilon, F, K, X), \quad \forall\, y \in K.$$

3 Preliminary settings

Through out this work we use $c > 1$ to represent a generic constant, independent of t and the initial data, and it may also vary from one line to another.

In order to study the system (1)–(2), we rewrite it as a first-order evolution equation which can be easily treated. Indeed, rewrite Eq. (1) as

$$\frac{d^2u}{dt^2} + A^2u - \int_0^\infty g(s)A^2u(t-s)ds = f(t, u), \; t > 0, \tag{4}$$

and introduce the following variable as in [2],

$$w(t, s) = u(t) - u(t-s), \qquad \forall\, s, t \geq 0,$$

which in turn satisfies the following system,

$$\begin{cases} \dfrac{\partial w}{\partial t} + \dfrac{\partial w}{\partial s} - \dfrac{du}{dt} = 0, & s, t > 0, \\[2ex] w(0, s) = u_0(0) - u_0(s), \; s > 0. & \end{cases}$$

In view of the above, Eqs. (1)–(2) become, under the assumption (**A.1**),

$$\begin{cases} \dfrac{d^2u}{dt^2} + \beta A^2u + \displaystyle\int_0^\infty g(s)A^2w(t, s)ds = f(t, u), & t > 0, \\[2ex] \dfrac{\partial w}{\partial t} + \dfrac{\partial w}{\partial s} - \dfrac{du}{dt} = 0, & s, t > 0, \\[2ex] u(0) = u_0, \quad u'(0) = u_1, \quad w(\cdot, 0) = 0, \quad w(0, \cdot) = w_0, & \end{cases} \tag{5}$$

where $u_0 = u_0(0)$, $w_0(s) = u_0(0) - u_0(s)$ and $\beta := 1 - \int_0^\infty g(s)ds > 0$.

It remains to rewrite Eq. (5) as a first-order evolution evolution. For that, let $V := \mathcal{D}(A)$ be equipped with the inner product defined by,

$$\langle v, \tilde{v}\rangle_V := \langle Av, A\tilde{v}\rangle_H, \qquad \forall\, v, \tilde{v} \in V.$$

It follows from the continuous embedding $\mathcal{D}(A) \hookrightarrow H$ (see Eq. (3)) that $(V, \|\cdot\|_V)$ is a Hilbert space.

Let

$$W = L^2_g((0,\infty);V) := \left\{ w : (0,\infty) \longrightarrow V : \int_0^\infty g(s)\|w(s)\|^2_V ds < \infty \right\}$$

be equipped with the inner product

$$\langle w, \tilde{w}\rangle_W := \int_0^\infty g(s)\langle w(s), \tilde{w}(s)\rangle_V ds.$$

Clearly, the completeness of $(V, \|\cdot\|_V)$ yields $(W, \|\cdot\|_W)$ is a Hilbert space.

The state space of our problem is given by $\mathcal{H} := V \times H \times W$ which is equipped with the inner product defined by

$$\left\langle U, \tilde{U}\right\rangle_{\mathcal{H}} := \beta\langle u, \tilde{u}\rangle_V + \langle v, \tilde{v}\rangle_H + \langle w, \tilde{w}\rangle_W,$$

for any $U = (u, v, w)^T$ and $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{w})^T$ in $\mathcal{H}$.

Setting $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \longrightarrow \mathcal{H}$

$$\mathcal{A}U := \begin{pmatrix} v \\ -\beta A^2 u - \displaystyle\int_0^\infty g(s)A^2 w(s)ds \\ v - \dfrac{dw}{ds} \end{pmatrix} \tag{6}$$

for all $U = (u, v, w)^T \in \mathcal{D}(\mathcal{A})$ where

$$\mathcal{D}(\mathcal{A}) = \left\{ U \in \mathcal{H} : \beta u + \int_0^\infty g(s)w(s)ds \in \mathcal{D}(A^2), \quad v \in \mathcal{D}(A), \quad \frac{dw}{ds} \in W, \quad w(0) = 0 \right\}$$

it follows that Eq. (5) can be rewritten as a first-order evolution equation in the form

$$\begin{cases} \dfrac{dU}{dt} = \mathcal{A}U(t) + F(t, U),\ t > 0, \\ U(0) = U_0, \end{cases} \tag{7}$$

where $U = (u, u', w)^T$, $U_0 = (u_0, u_1, w_0)^T \in \mathcal{H}$ and $F : \mathbb{R}_+ \times \mathcal{H} \longrightarrow \mathcal{H}$ is defined by

$$F(t,U) := \begin{pmatrix} 0 \\ f(t,u) \\ 0 \end{pmatrix} \quad \text{for all} \quad U \in \mathcal{H}.$$

4 Exponential stability of the C_0-semigroup associated with $\mathcal{A}$

To establish our existence results, we will be using Theorem 1 given below.

Theorem 1 [11] *Let $B : \mathcal{D}(B) \subset H \mapsto H$ be a densely defined linear operator. If B is dissipative and $0 \in \rho(B)$, then B is the infinitesimal generator of a C_0-semigroup of contractions on H.*

Theorem 2 *Under assumptions* (**A.1**) *and* (**A.2**), *the linear operator $\mathcal{A}$ defined in Eq.* (6) *is the infinitesimal generator of a C_0-semigroup of contraction $(T(t))_{t\geq 0}$.*

Proof We start by establishing the dissipativeness of the linear operator $\mathcal{A}$. Indeed, for any $U = (u, v, w)^T \in \mathcal{D}(\mathcal{A})$, we have

$$\begin{aligned}
\langle \mathcal{A}U, U\rangle_{\mathcal{H}} =& \beta\langle v, u\rangle_V - \left\langle \beta A^2 u + \int_0^\infty g(s)A^2 w(s)ds, v\right\rangle_H + \langle v - \frac{dw}{ds}, w\rangle_W \\
=& \beta\langle v, u\rangle_V - \left\langle \beta Au + \int_0^\infty g(s)Aw(s)ds, Av\right\rangle_H \\
&+ \int_0^\infty g(s)\langle v - \frac{dw}{ds}, w(s)\rangle_V ds \\
=& \beta\langle v, u\rangle_V - \beta\langle Au, Av\rangle_H - \int_0^\infty g(s)\langle Aw(s), Av\rangle_H ds \\
&+ \int_0^\infty g(s)\langle v, w(s)\rangle_V ds - \int_0^\infty g(s)\langle \frac{dw}{ds}, w(s)\rangle_V ds \\
=& -\frac{1}{2}\int_0^\infty g(s)\frac{d}{ds}\|w(s)\|_V^2 ds \\
=& -\frac{1}{2}\left(\lim_{T\to\infty}\left[g(s)\|w(s)\|_V^2\right]_{s=0}^{s=T} - \int_0^\infty g'(s)\|w(s)\|_V^2 ds\right) \\
=& \frac{1}{2}\int_0^\infty g'(s)\|w(s)\|_V^2 ds \leq 0
\end{aligned}$$

and hence $\mathcal{A}$ is dissipative.

The next step consists of showing that $0 \in \rho(\mathcal{A})$. Indeed, for any $F = (f_1, f_2, f_3)^T \in \mathcal{H}$, consider the solvability of the equation

$$\mathcal{A}U = F.$$

Equivalently,

$$v = f_1 \qquad \text{in} \qquad V, \tag{8a}$$

$$-\beta A^2 u - \int_0^\infty g(s) A^2 w(s) ds = f_2 \qquad \text{in} \qquad H, \tag{8b}$$

$$v - \frac{dw}{ds} = f_3 \qquad \text{in} \qquad W. \tag{8c}$$

From Eq. (8a) and Eq. (8c), we have

$$w(s) = sf_1 - \int_0^s f_3(\tau) d\tau, \quad \forall\, s \geq 0. \tag{9}$$

Clearly, $w(0) = 0$ and $\dfrac{dw}{ds} = f_1 - f_3 \in W$. Using assumption (**A.2**) and the remarks on the relaxation function g (Introduction) we obtain

$$\begin{aligned}
\int_0^\infty g(s)\|w(s)\|_V^2 ds \leq & 2\int_0^\infty g(s)\|sf_1\|_V^2 ds + 2\int_0^\infty \int_0^s g(s)\|f_3(\tau)\|_V^2 d\tau ds \\
\leq & \frac{2}{\xi^3} g(0)\|f_1\|_V^2 + 2\int_0^\infty \int_\tau^\infty g(s)\|f_3(\tau)\|_V^2 ds d\tau \\
\leq & \frac{2}{\xi^3} g(0)\|f_1\|_V^2 - \frac{2}{\xi}\int_0^\infty \int_\tau^\infty g'(s)\|f_3(\tau)\|_V^2 ds d\tau \\
= & \frac{2}{\xi^3} g(0)\|f_1\|_V^2 + \frac{2}{\xi}\int_0^\infty g(\tau)\|f_3(\tau)\|_V^2 d\tau \\
= & \frac{2}{\xi^3} g(0)\|f_1\|_V^2 + \frac{2}{\xi}\|f_3\|_W^2 < \infty
\end{aligned}$$

and hence $w \in W$.

A combination of Eq. (8b) and Eq. (9) yields

$$\beta A^2 u = -\left(\int_0^\infty s g(s) ds\right) A^2 f_1 + \int_0^\infty \int_0^s g(s) A^2 f_3(\tau) d\tau ds - f_2. \tag{10}$$

Consider both the bilinear form $B : V \times V \longrightarrow \mathbb{R}$ and the linear form $L : V \longrightarrow \mathbb{R}$ defined by

$$B(u, \varphi) := \beta\langle u, \varphi\rangle_V \quad \text{for all} \quad (u, \varphi) \in V \times V$$

and

$$L(\varphi) := -\left(\int_0^\infty g(s) ds\right)\langle f_1, \varphi\rangle)_V + \int_0^\infty \int_0^s g(s)\langle f_3(\tau), \varphi\rangle_V d\tau ds - \langle f_2, \varphi\rangle_H \quad \text{for all} \quad \varphi \in V.$$

Clearly, the bilinear form B is bounded and coercive.

Now

$$
\begin{aligned}
|L(\varphi)| \leq & \int_0^\infty sg(s)ds\|f_1\|_V\|\varphi\|_V + \int_0^\infty\int_\tau^\infty g(s)\|f_3(\tau)\|_V\|\varphi\|_V ds d\tau + \|f_2\|_H\|\varphi\|_H \\
\leq & \left(\frac{1}{\xi^2}g(0)\|f_1\|_V + \frac{\sqrt{1-\beta}}{\xi}\|f_3\|_W + \frac{1}{\omega}\|f_2\|_H\right)\|\varphi\|_V, \qquad \forall\,\varphi \in V
\end{aligned}
$$

and which yields the linear form L is bounded.

Consequently, the well-known Lax–Milgram Lemma does guarantee the existence of a unique $u \in V$ satisfying

$$
B(u, \varphi) = L(\varphi), \qquad \forall\,\varphi \in V. \tag{11}
$$

The classical regularity argument entails that $u \in \mathcal{D}(A^2)$ and satisfies Eq. (10). Combining this with both Eq. (8a) and Eq. (9) we deduce that $0 \in \rho(\mathcal{A})$.

Arguing as above, it is not hard to show that $I - \mathcal{A}$ is surjective, which yields $\mathcal{A}$ is densely defined in $\mathcal{H}$. Therefore, using Theorem 1, we deduce that the operator $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions which we denote $(T(t))_{t\geq 0}$. □

Definition 5 [12] If $G : \mathbb{R}_+ \longrightarrow \mathcal{H}$ is a continuous function, then the function $U : \mathbb{R}_+ \longrightarrow \mathcal{H}$ is said to be a mild solution to the first-order evolution equation

$$
\frac{d\boldsymbol{\Phi}}{dt} = \mathcal{A}\boldsymbol{\Phi} + G(t), \;\; t > 0; \;\; \boldsymbol{\Phi}(0) = \boldsymbol{\Phi}_0 \tag{12}
$$

if it satisfies

$$
U(t) = T(t)\boldsymbol{\Phi}_0 + \int_0^t T(t-s)G(s)ds, \qquad \forall\,t \geq 0.
$$

From Definition 5, we have

Definition 6 If $F : \mathbb{R}_+ \times \mathcal{H} \longrightarrow \mathcal{H}$ is jointly continuous, then a function $U : \mathbb{R}_+ \longrightarrow \mathcal{H}$ is said to be a mild solution to Eq. (7) if it satisfies

$$
U(t) = T(t)U_0 + \int_0^t T(t-s)F(s, U(s))ds, \qquad \forall\,t \geq 0.
$$

Theorem 3 *Under assumptions* (**A.1**)–(**A.2**), *the* C_0*-semigroup* $(T(t))_{t\geq 0}$ *is exponentially stable, that is, there exist two positive constants* $M > 0$ *and* $\delta > 0$ *such that*

$$
\|T(t)\| \leq Me^{-\delta t}, \qquad \forall\,t \geq 0.
$$

Proof We make extensive use of the multiplier method. Indeed, setting $F \equiv 0$ in Eq. (7) (i.e., $f \equiv 0$), then the energy functional associated with the resulting homogeneous equation can be formulated as follows,

$$
E(t) := \frac{1}{2}\|U(t)\|^2_{\mathcal{H}}, \qquad \forall\,t \geq 0,
$$

where U is the mild solution to the corresponding homogeneous equation to Eq. (7), which exists and is given by

$$U(t) = T(t)U_0, \qquad \forall\, t \geq 0.$$

Simple computations show that, the energy functional satisfies

$$E'(t) = \frac{1}{2}\int_0^\infty g'(s)\|w(t,s)\|_V^2 ds \leq 0, \quad \forall\, t \geq 0.$$

Indeed, taking the inner product of both sides of Eq. (7) with $U(t)$ in $\mathcal{H}$ it follows that

$$\langle U'(t), U(t)\rangle_{\mathcal{H}} = \langle \mathcal{A}U(t), U(t)\rangle_{\mathcal{H}} = \frac{1}{2}\int_0^\infty g'(s)\|w(t,s)\|_V^2 ds, \quad \forall\, t \geq 0,$$

which proves our claim.

Define the functionals I_1 and I_2 by setting

$$I_1(t) := \langle u'(t), u(t)\rangle_H \qquad \text{and} \qquad I_2(t) := -\left\langle u'(t), \int_0^\infty g(s)w(t,s)ds \right\rangle_H, \qquad \forall\, t \geq 0.$$

Differentiating I_1 and using Eq. (5), we obtain

$$\begin{aligned} I_1'(t) =& \|u'(t)\|_H^2 + \left\langle -\beta A^2 u(t) - \int_0^\infty g(s)A^2 w(t,s)ds, u(t) \right\rangle_H \\ =& \|u'(t)\|_H^2 - \beta\|Au(t)\|_H^2 - \int_0^\infty g(s)\langle Aw(t,s), Au(t)\rangle_H ds. \end{aligned} \tag{13}$$

Next, we estimate the last term in the above inequality. Indeed, using the Cauchy–Schwarz, Young, and Hölder inequalities, we obtain

$$\begin{aligned} -\int_0^\infty g(s)\langle Aw(t,s), Au(t)\rangle_H ds \leq& \|Au(t)\|_H \int_0^\infty g(s)\|Aw(t,s)\|_H ds \\ \leq& \frac{\beta}{2}\|Au(t)\|_H^2 + \frac{1}{2\beta}\left(\int_0^\infty g(s)\|Aw(t,s)\|_H ds\right)^2 \\ \leq& \frac{\beta}{2}\|Au(t)\|_H^2 + \frac{1-\beta}{2\beta}\int_0^\infty g(s)\|Aw(t,s)\|_H^2 ds. \end{aligned}$$

Inserting this estimate in Eq. (13), we get

$$I_1'(t) \leq \|u'(t)\|_H^2 - \frac{\beta}{2}\|u(t)\|_V^2 + c\int_0^\infty g(s)\|w(t,s)\|_V^2 ds, \qquad \forall\, t \geq 0. \tag{14}$$

Differentiating I_2 and exploiting Eq. (5), we get

$$
\begin{aligned}
I_2'(t) = & \left\langle \beta A^2 u(t) + \int_0^\infty g(s)A^2 w(t,s)ds, \int_0^\infty g(s)w(t,s)ds \right\rangle_H \\
& - \left\langle u'(t), \int_0^\infty g(s)w_t(t,s)ds \right\rangle_H \\
= & \left\langle \beta A u(t), \int_0^\infty g(s)Aw(t,s)ds \right\rangle_H + \left\langle \int_0^\infty g(s)Aw(t,s)ds, \int_0^\infty g(s)Aw(t,s)ds \right\rangle_H \\
& - \left\langle u'(t), \int_0^\infty g(s)(u'(t) - w_s(t,s))ds \right\rangle_H \\
= & \beta \left\langle Au(t), \int_0^\infty g(s)Aw(t,s)ds \right\rangle_H + \left\| \int_0^\infty g(s)Aw(t,s)ds \right\|_H^2 \\
& - (1-\beta)\|u'(t)\|_H^2 + \left\langle u'(t), \int_0^\infty g(s)w_s(t,s)ds \right\rangle_H .
\end{aligned} \tag{15}
$$

Next, we estimate the terms in the above inequality. Using the Cauchy–Schwarz, Young and Hölder inequalities, it follows that, for any $\eta > 0$,

$$
\begin{aligned}
\beta \left\langle Au(t), \int_0^\infty g(s)Aw(t,s)ds \right\rangle_H \leq & \eta |Au(t)\|_H^2 + \frac{\beta^2}{4\eta}\left(\int_0^\infty g(s)\|Aw(t,s)\|_H ds \right)^2 \\
\leq & \eta \|u(t)\|_V^2 + \frac{c}{\eta} \int_0^\infty g(s)\|w(t,s)\|_V^2 ds.
\end{aligned}
$$

Using Hölder's inequality, we have

$$
\begin{aligned}
\left\| \int_0^\infty g(s)Aw(t,s)ds \right\|_H^2 \leq & \left(\int_0^\infty g(s)ds \right) \int_0^\infty g(s)\|Aw(t,s)\|_H^2 ds \\
= & (1-\beta) \int_0^\infty g(s)\|w(t,s)\|_V^2 ds.
\end{aligned}
$$

Lastly, using integration by parts together with both Young's and Hölder's inequalities, it follows that, for any $\eta > 0$,

$$
\begin{aligned}
\left\langle u'(t), \int_0^\infty g(s)w_s(t,s)ds \right\rangle_H = & \left\langle u'(t), g(s)w(t,s)\Big|_{s=0}^\infty - \int_0^\infty g'(s)w(t,s)ds \right\rangle_H \\
= & -\left\langle u'(t), \int_0^\infty g'(s)w(t,s)ds \right\rangle_H \\
\leq & \eta \|u'(t)\|_H^2 + \frac{1}{4\eta}\left(\int_0^\infty g'(s)\|w(t,s)\|_H ds \right)^2 \\
\leq & \eta \|u'(t)\|_H^2 - \frac{c}{\eta} \int_0^\infty g'(s)\|w(t,s)\|_V ds.
\end{aligned}
$$

Plugging the above estimates in Eq. (15), we obtain

$$\begin{aligned} I_2'(t) \leq & -((1-\beta)-\eta)\|u'(t)\|_H^2 + \eta\|u(t)\|_V^2 \\ & + \left((1-\beta)+\frac{c}{\eta}\right)\int_0^\infty g(s)\|w(t,s)\|_V^2 ds \\ & - \frac{c}{\eta}\int_0^\infty g'(s)\|w(t,s)\|_V^2 ds. \end{aligned} \tag{16}$$

Define the functional $\mathscr{L}$ by

$$\mathscr{L}(t) := NE(t) + \varepsilon_1 I_1(t) + \varepsilon_2 I_2(t), \qquad \forall\, t \geq 0,$$

where N, ε_1, ε_2 are positive constants to be specified later.

From Eq. (14) and Eq. (16) we have

$$\begin{aligned} \mathscr{L}'(t) \leq & -\left[((1-\beta)-\eta)\varepsilon_2 - \varepsilon_1\right]\|u'(t)\|_H^2 - \left(\frac{1}{2}\beta\varepsilon_1 - \eta\varepsilon_2\right)\|u(t)\|_V^2 \\ & + \left[c\varepsilon_1 + \left((1-\beta)+\frac{c}{\eta}\right)\varepsilon_2\right]\int_0^\infty g(s)\|w(t,s)\|_V^2 ds \\ & + \left(\frac{N}{2} - \frac{c}{\eta}\varepsilon_2\right)\int_0^\infty g'(s)\|w(t,s)\|_V^2 ds. \end{aligned} \tag{17}$$

Now choose $\eta > 0$ small enough so that

$$\eta < \min\left\{\frac{1}{2}(1-\beta),\ \frac{1}{8}\beta(1-\beta)\right\}.$$

Consequently, for any fixed $\varepsilon_2 > 0$, we pick $\varepsilon_1 > 0$ satisfying

$$\frac{1}{4}(1-\beta)\varepsilon_2 < \varepsilon_1 < \frac{1}{2}(1-\beta)\varepsilon_2.$$

Then,

$$c_1 := ((1-\beta)-\eta)\varepsilon_2 - \varepsilon_1 > \frac{1}{2}(1-\beta)\varepsilon_2 - \varepsilon_1 > 0$$

and

$$c_2 := \frac{1}{2}\beta\varepsilon_1 - \eta\varepsilon_2 > \frac{1}{2}\beta\varepsilon_1 - \frac{1}{8}\beta(1-\beta) = \frac{1}{2}\beta\left(\varepsilon_1 - \frac{1}{4}(1-\beta)\varepsilon_2\right) > 0.$$

Finally, we choose N large enough so that $N > \dfrac{2c}{\eta}\varepsilon_2$ and $\mathscr{L} \sim E$. Thus Eq. (17) becomes, for some fixed $\alpha > 0$,

$$\begin{aligned}\mathscr{L}'(t) \leq & - c_1\|u'(t)\|_H^2 - c_2\|u(t)\|_V^2 + c\int_0^\infty g(s)\|w(t,s)\|_V^2 ds \\ \leq & - \alpha E(t) + c\int_0^\infty g(s)\|w(t,s)\|_V^2 ds \\ \leq & - \alpha E(t) + \frac{c}{\xi}\int_0^\infty \xi g(s)\|w(t,s)\|_V^2 ds \\ \leq & - \alpha E(t) - \frac{c}{\xi}\int_0^\infty g'(s)\|w(t,s)\|_V^2 ds \\ \leq & - \alpha E(t) - cE'(t), \qquad \forall\, t \geq 0.\end{aligned}$$

Set $\mathcal{F} := \mathscr{L} + cE$, then $\mathcal{F} \sim E$ and thus we deduce from the above estimate that

$$\mathcal{F}'(t) \leq -\lambda \mathcal{F}(t), \qquad \forall\, t \geq 0,$$

for some fixed $\lambda > 0$. A simple integration over $(0, t)$ yields

$$\mathcal{F}(t) \leq \mathcal{F}(0)e^{-\lambda t}, \qquad \forall\, t \geq 0$$

which in turn yields

$$E(t) \leq CE(0)e^{-\lambda t}, \qquad \forall\, t \geq 0.$$

Thus

$$\|T(t)U_0\| = \|U(t)\| \leq \sqrt{C}e^{-\frac{\lambda}{2}t}\|U_0\|, \qquad \forall\, t \geq 0.$$

Consequently,

$$\|T(t)\| \leq Me^{-\delta t} \ \text{ for all } \ t \geq 0,$$

where $M = \sqrt{C}$ and $\delta = \frac{\lambda}{2}$. □

5 Existence of asymptotically almost periodic mild solutions

The next lemma can be proved using similar arguments as in [10]. However for the sake of completeness, we provide the proof.

Lemma 3 *Let $g \in AAP(X)$. Then the function v defined by*

$$v(t) = \int_0^t T(t-s)g(s)ds$$

is asymptotically almost periodic.

Proof Let $\varepsilon > 0$, since $g \in AAP(X)$, there exists $L = L(\frac{\delta\varepsilon}{2M}, g, X)$ and a relatively dense subset $\mathscr{T}(\frac{\delta\varepsilon}{2M}, g, X)$ of $\mathbb{R}_+$ such that

$$\|g(t+\tau)-g(t)\| < \frac{\delta\varepsilon}{2M}, \quad \forall\, t \ge L, \quad \forall\, \tau \in \mathscr{T}\Big(\frac{\delta\varepsilon}{2M}, g, X\Big).$$

Let L_1 be a positive constant such that $L_1 > \frac{1}{\delta}\ln(\frac{6M^2\|g\|_\infty}{\delta\varepsilon})$, then any $t \ge L + L_1$ and $\tau \in \mathscr{T}\Big(\frac{\delta\varepsilon}{2M}, g, X\Big)$ we have

$$\begin{aligned}\|v(t+\tau)-v(t)\| =& \left\|\int_0^{t+\tau} T(t+\tau-s)g(s)ds - \int_0^t T(t-s)g(s)ds\right\| \\ \le& \underbrace{\int_0^{\tau} \|T(t+\tau-s)g(s)\|ds}_{:=I_3} \\ &+ \underbrace{\int_0^t \|T(t-s)(g(s+\tau)-g(s))\|ds}_{:=I_4}.\end{aligned}$$

Next, we estimate each term on the right-hand side of the above inequality.

$$\begin{aligned}I_3 \le& \int_0^{\tau} \|T(t)\|\|T(\tau-s)g(s)\|ds \\ \le& Me^{-\delta t}\int_0^{\tau} Me^{-\delta(\tau-s)}\|g(s)\|ds \\ \le& M^2\|g\|_\infty e^{-\delta t}\int_0^{\tau} e^{-\delta s}ds = \frac{1}{\delta}M^2\|g\|_\infty e^{-\delta t}. \\ I_4 =& \left(\int_0^L + \int_L^t\right)\|T(t-s)(g(s+\tau)-g(s))\|ds \\ =& \int_0^L \|T(t-L)T(L-s)(g(s+\tau)-g(s))\|ds \\ &+ \int_L^t \|T(t-s)(g(s+\tau)-g(s))\|ds \\ \le& M^2e^{-\delta(t-L)}\int_0^L e^{-\delta(L-s)}\|g(s+\tau)-g(s)\|ds \\ &+ M\int_L^t e^{-\delta(t-s)}\|g(s+\tau)-g(s)\|ds \\ \le& 2M^2\|g\|_\infty e^{-\delta(t-L)}\int_0^L e^{-\delta(L-s)}ds + M\int_L^t e^{-\delta(t-s)}\frac{\delta\varepsilon}{2M}ds \\ \le& \frac{2}{\delta}M^2\|g\|_\infty e^{-\delta(t-L)} + \frac{\varepsilon}{2}.\end{aligned}$$

Combining the above estimates, we obtain,

$$\begin{aligned}\|v(t+\tau)-v(t)\| &\leq \frac{1}{\delta}M^2\|g\|_\infty e^{-\delta t}+\frac{2}{\delta}M^2\|g\|_\infty e^{-\delta(t-L)}+\frac{\varepsilon}{2}\\ &\leq \frac{3}{\delta}M^2\|g\|_\infty e^{-\delta(t-L)}+\frac{\varepsilon}{2}\leq \frac{3}{\delta}M^2\|g\|_\infty e^{-\delta L_1}+\frac{\varepsilon}{2}\\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon \qquad \forall\, t\geq L \quad \text{and} \quad \forall\, \tau\in \mathscr{T}\Big(\frac{\delta\varepsilon}{2M},g,X\Big),\end{aligned}$$

which yields $v \in AAP(X)$. □

Theorem 4 *Under assumptions* (**A.1**)–(**A.2**), *suppose that the initial data* $U_0 \in \mathcal{H}$ *and that the function* f *belongs to* $AAP(\mathbb{R}_+ \times H, H)$. *Further, suppose that there exists* $K > 0$ *such that*

$$\|f(t,u)-f(t,v)\|_H \leq K\|u-v\|_H \qquad \forall\, u,v\in H \quad \text{and} \quad \forall\, t\in \mathbb{R}_+. \tag{18}$$

Then, Eq. (7) *has a unique asymptotically almost periodic mild solution whenever K is small enough.*

Proof First of all, note that using the assumptions on f and the composition of asymptotically almost periodic functions (see [3]) it follows that $F \in AAP(\mathbb{R}_+ \times \mathcal{H}, \mathcal{H})$. Furthermore, F is Lipschitzian in the second variable uniformly in the first one with K as a Lipschitz constant.

Consider the nonlinear integral operator Γ given on $AAP(\mathcal{H})$ by

$$\Gamma U(t) = T(t)U_0 + \int_0^t T(t-s)F(s,U(s))ds,$$

for any $t \in \mathbb{R}_+$.

Using Lemmas 1 and 3 it follows that Γ is well-defined and maps $AAP(\mathcal{H})$ into itself. Now, for any $U, \tilde{U} \in AAP(\mathcal{H})$ we have

$$\begin{aligned}\|\Gamma U(t)-\Gamma\tilde{U}(t)\|_{\mathcal{H}} &\leq \int_0^t \|T(t-s)\big(F(s,U(s))-F(s,\tilde{U}(s))\big)\|_{\mathcal{H}}ds\\ &\leq \int_0^t Me^{-\delta(t-s)}\|F(s,U(s))-F(s,\tilde{U}(s))\|_{\mathcal{H}}ds\\ &\leq MK\|U-V\|_\infty \int_0^\infty e^{-\delta s}ds\\ &= \delta^{-1}MK\|U-V\|_\infty \qquad \forall\, t\geq 0.\end{aligned}$$

Thus Γ is a strict contraction whenever K is small enough, that is, $\delta^{-1}MK < 1$. Therefore, using the Banach contraction mapping theorem, we deduce that Γ has a unique fixed point which obviously is the only asymptotically mild solution to Eq. (7). □

Corollary 1 *Under assumptions* (**A.1**)–(**A.2**), *suppose that the initial data* $u_0 \in \mathcal{D}(A)$ *and that the function f belongs to* $AAP(\mathbb{R}_+ \times H, H)$. *Further, suppose that f satisties*

Eq. (18). *Then, the system* (1)–(2) *has a unique asymptotically almost periodic mild solution whenever* $K < \delta M^{-1}$.

6 Example

In order to illustrate our previous abstract results, we consider the so-called Kirchhoff plate equation with infinite memory. For that, let $\Omega \subset \mathbb{R}^n$ be an open bounded subset with smooth boundary $\partial\Omega$ and let $H = L^2(\Omega)$ be equipped with its usual L^2 -norm given, for all $u \in L^2(\Omega)$, by

$$\|u\|_{L^2(\Omega)} = \left(\int_\Omega |u(x)|^2 dx \right)^{\frac{1}{2}}.$$

Consider the so-called Kirchhoff plate with infinite memory given by

$$\begin{cases} u_{tt}(x,t) + \Delta^2 u(x,t) - \displaystyle\int_{-\infty}^{t} g(t-s)\Delta^2 u(x,s)ds = f(t, u(x,t)), & \text{in} \quad \Omega \times (0,\infty), \\ u(x,t) = \Delta u(x,t) = 0, & \text{on} \quad \partial\Omega \times [0,\infty), \\ u(x,-t) = u_0(x,t), \qquad u_t(x,0) = u_1(x) & \text{in} \quad \Omega \times (0,\infty), \end{cases} \tag{19}$$

where u_0 u_1 are given initial data and f belongs to $AAP(\mathbb{R}_+ \times L^2(\Omega), L^2(\Omega))$ and satisfies Eq. (18).

Consequently, letting $A = -\Delta$ with domain $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$, one can easily see that Eq. (19) is a particular case of our system (1)–(2). Therefore, under assumptions of Corollary 1, we conclude that Eq. (19) has a unique asymptotically almost periodic solution if the Lipschitz constant K is small enough. That is, Eq. (19) has a unique mild solution which converges asymptotically to an almost periodic function.

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