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RESEARCH ARTICLE

Finite monogenic semigroups and saturated varieties of semigroups

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Abstract
J. M. Howie has shown that the finite monogenic semigroups are absolutely closed. We provide a new, simple and direct proof of Howie's result. We also show that all the varieties defined by the identities $ax = axa$, $axy = axay$ and $axy = axyay$ are saturated if and only if they are epimorphically closed.

Keywords Dominion · Zigzag equations · Absolutely closed semigroups · Epimorphically closed and saturated varieties

1 Introduction

In [6], Howie has shown that the finite monogenic semigroups are absolutely closed. In Theorem 4, we present a new and a simple alternative proof of the same result.

A semigroup S is said to be *permutative* if it satisfies a permutation identity

$$x_1x_2 \dots x_n = x_{i_1}x_{i_2} \dots x_{i_n} \ (n \geq 3) \tag{1}$$

for some non-trivial permutation i of the set $\{1,2,...,n\}$ Such semigroups are not saturated in general as all commutative semigroups are not saturated. The infinite monogenic semigroup $\langle a\rangle$ generated by a is not saturated since it is epimorphically embedded in the infinite cyclic group generated by a [7, Chapter VIII, Exercise 6(a)]. Thus the variety of all commutative semigroups is not saturated and, hence, all the varieties of permutative semigroups (the collection of all semigroups satisfying any, but fixed, non-trivial permutation identity only) are not saturated in general though these varieties are epimorphically closed (see [10, Theorem 4]). Therefore it is worthwhile to find epimorphically closed varieties of semigroups that are also

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saturated. In this paper we show that the varieties of semigroups defined by the identities $ax = axa$, $axy = axay$ and $axy = axyay$ are saturated if and only if they are epimorphically closed, thus, partially answer the open problem of whether the variety of semigroups defined by the identity $ax = axa$ is saturated or not posed by Higgins in his Ph.D. thesis [4].

2 Preliminaries

A morphism $\alpha : S \to T$ in the category of all semigroups is called an *epimorphism* (*epi* for short) if for all morphisms β, γ with $\alpha\beta = \alpha\gamma$ implies $\beta = \gamma$. Let U and S be semigroups with U as a subsemigroup of S. Following Isbell [8], we say that U *dominates* an element d of S if for every semigroup T and for all morphisms $\beta, \gamma : S \to T$, $u\beta = u\gamma$ for every $u \in U$ implies $d\beta = d\gamma$. The set of all elements of S dominated by U is called the *dominion* of U in S and we denote it by $Dom(U,S)$. It may be easily seen that $Dom(U,S)$ is a subsemigroup of S containing U.

Let U be a subsemigroup of S. Then U is said to be *closed* in S if $Dom(U,S) = U$. A semigroup U is said to be *absolutely closed* if it is closed in every containing semigroup S i.e., for any semigroup S such that U is a subsemigroup of S, $Dom(U, S) = U$. A semigroup U is said to be *saturated* if for every properly containing semigroup S, $Dom(U, S) \neq S$. It may be easily seen that $\alpha : S \to T$ is epi if and only if $i : S\alpha \to T$ is epi and the inclusion map $i : U \to S$ is epi if and only if $Dom(U, S) = S$.

A most useful characterization of semigroup dominions is provided by Isbell [8], called the Isbell zigzag theorem and is as follows:

Theorem 1 [8] *Let U be a subsemigroup of a semigroup S and $d \in S$. Then $d \in Dom(U, S)$ if and only if $d \in U$ or there is a series of factorizations of d as follows*:

$$d = u_0x_1 = y_1u_1x_1 = \cdots = y_mu_{2m-1}x_m = y_mu_{2m}, \tag{2}$$

where $m \geq 1, u_i \in U(i = 0, 1, 2, \ldots, 2m), x_i, y_i \in S(i = 1, 2, \ldots m)$ *and for* $i = 1, 2, \ldots, m - 1$

$$u_0 = y_1u_1, \qquad u_{2m-1}x_m = u_{2m},$$
$$u_{2i-1}x_i = u_{2i}x_{i+1}, \qquad y_iu_{2i} = y_{i+1}u_{2i+1}$$

The above series of factorizations of d is called a *zigzag* of length m in S over U with value d. In whatever follows, by zigzag equations, we shall mean equations of the type (2). Bracketed statements where ever used shall mean the dual to the other statements.

The following theorems are from Khan [11].

Theorem 2 [11] *Let U be a subsemigroup of a semigroup S and let $d \in Dom(U, S)\backslash U$. If (2) is a zigzag of minimal length m in S over U with value d, then $x_i, y_i \in S\backslash U$ for all $i = 1, 2, \ldots m$.*

Theorem 3 [11] *Let U be a subsemigroup of a semigroup S such that $Dom(U, S) = S$. Then, for any $d \in S \setminus U$ and any positive integer k, if* (2) *is a zigzag of minimal length m in S over U with value d, then there exist $b_1, b_2, \dots, b_k \in U$ and $d_k \in S \setminus U$ such that $d = b_1 b_2 \dots b_k d_k$.*

3 Finite monogenic semigroups are absolutely closed

In [6], Howie has shown that finite monogenic semigroups are absolutely closed by showing that such semigroups are totally division-ordered. In the next theorem, we provide a new, simple and direct proof of Howie's result.

Theorem 4 *Finite monogenic semigroups are absolutely closed.*

Proof Let $U = \langle a \rangle$ be a finite monogenic semigroup of *index n* and *period r*. Then

$$a^n = a^{n+qr} \text{ for every positive integer } q. \tag{3}$$

Let S be any semigroup containing U as a subsemigroup. Take $d \in Dom(U, S) \setminus U$. By Theorem 1, d has a zigzag of the following type:

$$d = a^{n_0} x_1 = y_1 a^{n_1} x_1 = \cdots = y_m a^{n_{2m-1}} x_m = y_m a^{n_{2m}}, \tag{4}$$

where $m \geq 1, n_i$ is an integer $(0 \leq i \leq 2m)$;$x_j, y_j \in S \, (1 \leq j \leq m)$ and for $i = 1, 2, \dots, m-1$

$$a^{n_0} = y_1 a^{n_1}, \qquad a^{n_{2m-1}} x_m = a^{n_{2m}},$$
$$a^{n_{2i-1}} x_i = a^{n_{2i}} x_{i+1}, \qquad y_i a^{n_{2i}} = y_{i+1} a^{n_{2i+1}}.$$

To complete the proof of the theorem, we first prove the following lemmas:

Lemma 1 *If $d = y_1^p a^{n+s} x_1$, where p and s are non-negative integers, then $d \in U$.*

Proof It follows from the repeated application of the Eq. (3) and the suitable choice of integer q. ☐

Lemma 2 *If $n_0 < n_1$, then $d \in U$.*

Proof We have

$$\begin{aligned} d &= y_1 a^{n_1} x_1 && \text{(by Eq. (4))} \\ &= y_1 a^{n_0} a^{n_1 - n_0} x_1 && (\text{as } n_0 < n_1) \\ &= y_1^2 a^{n_1} a^{n_1 - n_0} x_1 && \text{(by Eq. (4))} \\ &= y_1^2 a^{n_0} a^{2(n_1 - n_0)} x_1 && (\text{as } n_0 < n_1). \end{aligned}$$

Continuing in this way, we obtain $d = y_1^n a^{n_0} a^{n(n_1 - n_0)} x_1 \in U$ by Lemma 1, as required. ☐

Lemma 3 *If $n_0 \geq n_1 \geq \cdots \geq n_{2m}$, then $d \in U$.*

Proof We have

$$\begin{aligned}
d &= a^{n_0}x_1 && \text{(by Eq. (4))}\\
&= a^{n_0-n_1}a^{n_1}x_1 && \text{(where } a^{n_0-n_1} \in U^1)\\
&= a^{n_0-n_1}a^{n_2}x_2 && \text{(by Eq. (4))}\\
&= a^{n_0-n_1}a^{n_2-n_3}a^{n_3}x_2 && \text{(where } a^{n_0-n_1}a^{n_2-n_3} \in U^1)\\
&\vdots\\
&= a^{n_0-n_1}a^{n_2-n_3}\ldots a^{n_{2m-2}-n_{2m-1}}a^{n_{2m-1}}x_m\\
&\quad \text{(where } a^{n_0-n_1}a^{n_2-n_3}\ldots a^{n_{2m-2}-n_{2m-1}} \in U^1)\\
&= a^{n_0-n_1}a^{n_2-n_3}\ldots a^{n_{2m-2}-n_{2m-1}}a^{n_{2m}} \in U && \text{(by Eq. (4)).}
\end{aligned}$$

Hence the lemma is proved. □

Lemma 4 *Let i be the least positive integer such that $n_{2i-1} < n_{2i}$ $[n_{2i} < n_{2i+1}]$ and $m \geq 2$. Then the length m of the zigzag (4) can be reduced to $m-1$.*

Proof We prove the lemma for the case $n_{2i-1} < n_{2i}$, the case $n_{2i} < n_{2i+1}$ follows dually. If $i = 1$, then $n_0 \geq n_1$ and $n_1 < n_2$. Now

$$\begin{aligned}
d &= a^{n_0}x_1 && \text{(by Eq. (4))}\\
&= a^{n_0-n_1}a^{n_1}x_1 && (\text{as } n_0 \geq n_1)\\
&= a^{n_0-n_1}a^{n_2}x_2 && \text{(by Eq. (4)) and}\\
a^{n_0-n_1}a^{n_2} &= a^{n_0}a^{n_2-n_1} && (\text{as } n_2 > n_1)\\
&= y_1a^{n_1}a^{n_2-n_1} && \text{(by Eq. (4))}\\
&= y_1a^{n_2}\\
&= y_2a^{n_3} && \text{(by Eq. (4)).}
\end{aligned}$$

Thus the length of the zigzag can be reduced to $m-1$ in this case. Now suppose $i \geq 2$. Since i is the least positive integer such that $n_{2i-1} < n_{2i}$ so that $n_{2i-2} \geq n_{2i-1}$. Now, we have

$$\begin{aligned}
a^{n_{2i-3}}x_{i-1} &= a^{n_{2i-2}}x_i && \text{(by Eq. (4))}\\
&= a^{n_{2i-2}-n_{2i-1}}a^{n_{2i-1}}x_i && (\text{as } n_{2i-2} \geq n_{2i-1})\\
&= a^{n_{2i-2}-n_{2i-1}}a^{n_{2i}}x_{i+1} && \text{(by Eq. (4)) and}\\
y_{i-1}a^{n_{2i-2}-n_{2i-1}}a^{n_{2i}} &= y_{i-1}a^{n_{2i-2}}a^{n_{2i}-n_{2i-1}} && (\text{as } n_{2i-1} > n_{2i})\\
&= y_ia^{n_{2i-1}}a^{n_{2i}-n_{2i-1}} && \text{(by Eq. (4))}\\
&= y_ia^{n_{2i}}\\
&= y_{i+1}a^{n_{2i+1}} && \text{(by Eq. (4)).}
\end{aligned}$$

Therefore the length of the zigzag is reduced by one in this case as well. Thus the proof of the lemma is completed. □

By Lemmas 1–4, it is sufficient to prove the theorem for $m = 1$.

To complete the proof of the theorem, consider the following cases:

Case 1 $n_0 \geq n_1$. Then, we have

$$d = a^{n_0}x_1 = a^{n_0-n_1}a^{n_1}x_1 = a^{n_0-n_1}a^{n_2} \in U.$$

Case 2 $n_0 < n_1$. Then by Lemma 2, $d \in U$, as required. Hence the proof of the theorem is completed. □

4 Saturated varieties of semigroups

A variety $\mathcal{V}$ of semigroups is said to be *saturated* if each member of $\mathcal{V}$ is saturated and *epimorphically closed* if for any $S \in \mathcal{V}$ and $\alpha : S \to T$ is epi implies $T \in \mathcal{V}$. Alternatively for any subsemigroup U of a semigroup S with $U \in \mathcal{V}$ and $Dom(U, S) = S$ implies $S \in \mathcal{V}$. An identity $u = v$ is said to be *preserved* under epis if for any semigroups U and S with $Dom(U,S) = S$ and U satisfies $u = v$ implies S also satisfies $u = v$.

In [5] Higgins gave a condition for a variety to be saturated. In fact he proved the following.

Theorem 5 [5, *Theorem* 6] *Suppose $\mathcal{V}$ is a variety not equal to the variety of all semigroups. Then $\mathcal{V}$ is saturated only if each set of identities which define $\mathcal{V}$ contains a non-trivial identity, not a permutation identity, for which at least one side contains no repeated variable.*

However, Khan [11], has proved that for permutative variety the above condition is both necessary and sufficient. In fact we have the following.

Theorem 6 [11, *Theorem* 5.4] *A permutative variety is saturated if and only if it admits an identity I such that I is not a permutation identity, and at least one side of I has no repeated variable.*

Further, Khan in [9] gave the sufficient condition for heterotypical variety to be saturated by proving the following.

Theorem 7 [9, *Corollary* 3.3] *If a semigroup variety $\mathcal{V}$ admits a heterotypical identity of which at least one side has no repeated variable, then $\mathcal{V}$ is saturated.*

The sufficient condition for homotypical variety to be saturated is given by Higgins [5] and is as follows.

Theorem 8 [5, *Theorem* 16] *A sufficient condition for a homotypical variety $\mathcal{V}$ to be saturated is that $\mathcal{V}$ admits an identity $x_1x_2 \dots x_n = f(x_1, x_2, \dots, x_n)$ for which $|x_i|_f > 1$ for some $1 \le i \le n$ and such that f neither begins with x_1 nor ends with x_n.*

Moreover in [1], the authors have shown that the variety of left [right] seminormal bands is saturated in the category of all bands and is as follows.

Theorem 9 [1, *Theorem* 3.4] *The variety of left [right] seminormal bands is saturated in the category of all bands.*

The next theorems are from [3, 5].

Theorem 10 [3, *Theorem* 4] *Generalised inverse semigroups are saturated. In particular the variety of normal bands is saturated.*

Theorem 11 [5, *Corollary* 11] *A semigroup S is saturated if* S^n *is saturated for some integer* $n > 1$.

However, the determination of all saturated varieties of semigroups is still an open problem and Higgins in [4], posed whether the variety of semigroups defined by the identity $ax = axa\,[xa = axa]$ is saturated or even epimorphically closed. The answer to this question is very important for determining the saturated varieties of semigroups. If the answer is no, then it immediately determines the saturated varieties of bands. For if a semigroup S satisfying the identity $ax = axa$ is not saturated, then S^2 is a band and is not saturated by Theorem 11. As each non-normal variety of bands contains S^2 or its dual (see Figure 4 of [2]) and normal bands are saturated by Theorem 10, then these are the only saturated varieties of bands. Moreover if the variety of *left* [*right*] regular bands is saturated, then, again by Theorem 11, the variety of semigroups defined by the identity $ax = axa\,[xa = axa]$ is saturated.

In this section, we show that the varieties defined by the identities $ax = axa$, $axy = axay$ and $axy = axyay$ are saturated if and only if they are epimorphically closed and, therefore, partially answer the Higgins problem. Throughout this section, $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$ will denote the varieties of semigroups defined by the identities $ax = axa$, $axy = axay$ and $axy = axyay$ respectively.

It is well known that saturated varieties of semigroups are epimorphically closed, but converse is not true in general. In the following theorem, we show that the variety $\mathcal{V}_2$ of semigroups is saturated if it is epimorphically closed.

Theorem 12 *The variety* $\mathcal{V}_2$ *is saturated if it is epimorphically closed.*

Proof Suppose $\mathcal{V}_2$ is epimorphically closed. Assume on contrary that $\mathcal{V}_2$ is not saturated. Then there exist $U \in \mathcal{V}_2$ and a semigroup S containing U properly as a subsemigroup such that $Dom(U, S) = S$. Therefore

$$axy = axay \text{ for all } a, x, y \in S. \tag{5}$$

Take any $d \in S \backslash U$ and let (2) be the zigzag equations in S over U with value d of minimal length m. By Theorem 2, $x_i, y_i \in S \backslash U$ for all $i = 1, 2, \ldots, m$. Now, by Theorem 3, there exist $u'_{2i-1} \in U$ and $x'_i \in S \backslash U$ such that

$$x_i = u'_{2i-1} x'_i \text{ for all } i = 1, 2, \ldots, m. \tag{6}$$

To prove the theorem, we first prove the following lemma.

Lemma 5 *If for* $k = 0, 1, \ldots, m-1$, $s_k = u_0u_1'u_2\ u_3' \ldots u_{2k-1}'u_{2k}$, *then the following holds*:

$$s_k = s_{k-1}u_{2k-1}'y_{k+1}u_{2k+1} \text{ for all } k = 1, 2, \ldots, m-1. \tag{7}$$

Proof To prove the result, we use induction on k. For $k = 1$, we have

$$\begin{aligned} s_1 = u_0u_1'u_2 &= y_1u_1u_1'u_2 && \text{(by zigzag equations (2))} \\ &= y_1u_1u_1'y_1u_2 && \text{(by Eq. (5))} \\ &= u_0u_1'y_2u_3 && \text{(by zigzag equations (2)).} \end{aligned}$$

Therefore the result holds for $k = 1$. Assume inductively that the result holds for $k = j$. Thus

$$s_j = s_{j-1}u_{2j-1}'y_{j+1}u_{2j+1}. \tag{8}$$

We show that the result also holds for $k = j + 1$. Now, we have

$$\begin{aligned} s_{j+1} &= s_ju_{2j+1}'u_{2j+2} && \text{(by definition of } s_k) \\ &= s_{j-1}u_{2j-1}'y_{j+1}u_{2j+1}u_{2j+1}'u_{2j+2} && \text{(by Eq. (8))} \\ &= s_{j-1}u_{2j-1}'y_{j+1}u_{2j+1}u_{2j+1}'y_{j+1}u_{2j+2} && \text{(by Eq. (5))} \\ &= s_{j-1}u_{2j-1}'y_{j+1}u_{2j+1}u_{2j+1}'y_{j+2}u_{2j+3} && \text{(by zigzag equations (2))} \\ &= s_ju_{2j+1}'y_{j+2}u_{2j+3} && \text{(by Eq. (8)).} \end{aligned}$$

Therefore the result holds for $k = j + 1$ and, hence, the lemma follows. □

Now to complete the proof of the theorem, we have

$$\begin{aligned} d &= y_1u_1x_1 && \text{(by zigzag equations (2))} \\ &= y_1u_1u_1'x_1' && \text{(by Eq. (6))} \\ &= y_1u_1u_1'u_1x_1' && \text{(by Eq. (5))} \\ &= y_1u_1u_1'u_1u_1'x_1' && \text{(by Eq. (5))} \\ &= y_1u_1u_1'u_1x_1 && \text{(by Eq. (6))} \\ &= u_0u_1'u_2x_2 && \text{(by zigzag equations (2))} \\ &= u_0u_1'y_2u_3x_2 && \text{(by Lemma 5)} \\ &= u_0u_1'y_2u_3u_3'x_2' && \text{(by Eq. (6))} \\ &= u_0u_1'y_2u_3u_3'u_3u_3'x_2' && \text{(by Eq. (5))} \\ &= u_0u_1'y_2u_3u_3'u_3x_2 && \text{(by Eq. (6))} \\ &= s_1u_3'u_3x_2 && \text{(by Lemma 5)} \\ &= s_1u_3'u_4x_3 && \text{(by zigzag equations (2))} \\ &= s_2x_3 && \text{(by definition of } s_k). \end{aligned}$$

Continuing in this way, we get

$$\begin{aligned}
d &= s_{m-1}x_m \\
&= s_{m-2}u'_{2m-3}y_m u_{2m-1}x_m && \text{(by Lemma 5)} \\
&= s_{m-2}u'_{2m-3}y_m u_{2m-1}u'_{2m-1}x'_m && \text{(by Eq. (6))} \\
&= s_{m-2}u'_{2m-3}y_m u_{2m-1}u'_{2m-1}u_{2m-1}u'_{2m-1}x'_m && \text{(by Eq. (5))} \\
&= s_{m-2}u'_{2m-3}y_m u_{2m-1}u'_{2m-1}u_{2m-1}x_m && \text{(by Eq. (6))} \\
&= s_{m-2}u'_{2m-3}y_m u_{2m-1}u'_{2m-1}u_{2m} && \text{(by zigzag equations (2))} \\
&= s_{m-1}u'_{2m-1}u_{2m} && \text{(by Lemma (5))} \\
&= u_0u'_1u_2u'_3u_4\ldots u'_{2m-1}u_{2m} \in U,
\end{aligned}$$

a contradiction, as required. This completes the proof of the theorem. □

Corollary 1 *The variety $\mathcal{V}_1$ of semigroups is saturated if it is epimorphically closed.*

In the next theorem we show that the variety $\mathcal{V}_3$ of semigroups is saturated if it is epimorphically closed.

Theorem 13 *The variety $\mathcal{V}_3$ is saturated if it is epimorphically closed.*

Proof Suppose the variety $\mathcal{V}_3$ is epimorphically closed. Assume on contrary that $\mathcal{V}_3$ is not saturated. Then there exist $U \in \mathcal{V}_3$ and a semigroup S containing U properly as a subsemigroup such that $Dom(U,S) = S$. Since $\mathcal{V}_3$ is epimorphically closed, therefore, we have

$$axy = axyay \text{ for all } a, x, y \in S. \tag{9}$$

Take any $d \in S\backslash U$ and let (2) be the zigzag equations in S over U with value d of minimal length m. By Theorem 2, $x_i, y_i \in S\backslash U$ for all $i = 1, 2, \ldots, m$. Now, by Theorem 3, there exist $u'_i, v'_i \in U$ and $x'_i, y'_i \in S\backslash U$ such that

$$y_i = y'_iu'_i \text{ and } x_i = v'_ix'_i \text{ for all } i = 1, 2, \ldots, m. \tag{10}$$

Lemma 6 *If for $k = 1, 2, \ldots, m-1$, $s_k = u_0v'_1u'_1v'_1u_2v'_2\,u'_2v'_2u_4\ldots v'_ku'_kv'_ku_{2k}$. Then following holds:*

$$s_k = s_ky_{k+1}u_{2k+1} \text{ for all} k = 1, 2, \ldots, m-1. \tag{11}$$

Proof We prove the lemma by induction on k. For $k = 1$, we have

$$\begin{aligned}
s_1 = u_0v'_1u'_1v'_1u_2 &= y_1(u_1v'_1u'_1v'_1)u_2 && \text{(by zigzag equations (2))} \\
&= y_1u_1v'_1u'_1v'_1u_2y_1u_2 && \text{(by Eq. (9))} \\
&= u_0v'_1u'_1v'_1u_2y_2u_3 && \text{(by zigzag equations (2))} \\
&= s_1y_2u_3.
\end{aligned}$$

Therefore the result holds for $k = 1$. Assume inductively that result holds for $k = j$. Thus

$$s_j = s_jy_{j+1}u_{2j+1}. \tag{12}$$

We show that the result also holds for $k = j + 1$. Now, we have

$$
\begin{aligned}
s_{j+1} &= s_j v'_{j+1} u'_{j+1} v'_{j+1} u_{2j+2} && \text{(by definition of} s_k) \\
&= s_j y_{j+1} (u_{2j+1} v'_{j+1} u'_{j+1} v'_{j+1}) u_{2j+2} && \text{(by Eq. (12))} \\
&= s_j y_{j+1} u_{2j+1} v'_{j+1} u'_{j+1} v'_{j+1} u_{2j+2} y_{j+1} u_{2j+2} && \text{(by Eq. (9))} \\
&= s_j v'_{j+1} u'_{j+1} v'_{j+1} u_{2j+2} y_{j+1} u_{2j+2} && \text{(by Eq. (12))} \\
&= s_{j+1} y_{j+1} u_{2j+2} && \text{(by definition of} s_k) \\
&= s_{j+1} y_{j+2} u_{2j+3}.
\end{aligned}
$$

The the last equality follows from zigzag equations (2). Thus the result also holds for $k = j + 1$ and, hence, the proof of the lemma follows. □

Now to complete the proof of the theorem, we have

$$
\begin{aligned}
d &= y_1 u_1 x_1 && \text{(by zigzag equations (2))} \\
&= y'_1 (u'_1 u_1 v'_1) x'_1 && \text{(by Eq. (10))} \\
&= y'_1 u'_1 u_1 v'_1 u'_1 v'_1 x'_1 && \text{(by Eq. (9))} \\
&= y_1 (u_1 v'_1 u'_1 v'_1) x'_1 && \text{(by Eq. (10))} \\
&= y_1 u_1 v'_1 u'_1 v'_1 u_1 v'_1 x'_1 && \text{(by Eq. (9))} \\
&= u_0 v'_1 u'_1 v'_1 u_1 x_1 && \text{(by Eq. (10) \textit{and zigzag equations} (2))} \\
&= u_0 v'_1 u'_1 v'_1 u_2 x_2 && \text{(by zigzag equations (2))} \\
&= s_1 x_2 && \text{(by definition of } s_k) \\
&= s_1 y_2 u_3 x_2 && \text{(by Lemma 6)} \\
&= s_1 y'_2 (u'_2 u_3 v'_2) x'_2 && \text{(by Eq. (10))} \\
&= s_1 y'_2 u'_2 u_3 v'_2 u'_2 v'_2 x'_2 && \text{(by Eq. (9))} \\
&= s_1 y_2 (u_3 v'_2 u'_2 v'_2) x'_2 && \text{(by Eq. (10))} \\
&= s_1 y_2 u_3 v'_2 u'_2 v'_2 u_3 v'_2 x'_2 && \text{(by Eq. (9))} \\
&= s_1 y_2 u_3 v'_2 u'_2 v'_2 u_3 x_2 && \text{(by Eq. (10))} \\
&= s_1 y_2 u_3 v'_2 u'_2 v'_2 u_4 x_3 && \text{(by zigzag equations (2))} \\
&= s_1 v'_2 u'_2 v'_2 u_4 x_3 && \text{(by Lemma 6)} \\
&= s_2 x_3 && \text{(by definition of } s_k).
\end{aligned}
$$

Proceeding in this way, we obtain the following

$$
\begin{aligned}
d &= s_{m-1} x_m \\
&= s_{m-1} y_m u_{2m-1} x_m && \text{(by Lemma 6)} \\
&= s_{m-1} y'_m (u'_m u_{2m-1} v'_m) x'_m && \text{(by Eq. (10))} \\
&= s_{m-1} y'_m u'_m u_{2m-1} v'_m u'_m v'_m x'_m && \text{(by Eq. (9))} \\
&= s_{m-1} y_m (u_{2m-1} v'_m u'_m v'_m) x'_m && \text{(by Eq. (10))} \\
&= s_{m-1} y_m u_{2m-1} v'_m u'_m v'_m u_{2m-1} v'_m x'_m && \text{(by Eq. (9))} \\
&= s_{m-1} y_m u_{2m-1} v'_m u'_m v'_m u_{2m-1} x_m && \text{(by Eq. (10))} \\
&= s_{m-1} y_m u_{2m-1} v'_m u'_m v'_m u_{2m} && \text{(by zigzag equations (2))} \\
&= s_{m-1} v'_m u'_m v'_m u_{2m} && \text{(by Lemma 6)} \\
&= u_0 v'_1 u'_1 v'_1 u_2 \ldots v'_m u'_m v'_m u_{2m} \in U,
\end{aligned}
$$

4a contradiction, as required. This completes the proof of the theorem. □

From the above theorems, one can immediately obtain the following corollary.

Corollary 2 *Let U be a semigroup satisfying the identity $ax = axa\,(axy = axay,\ axy = axyay)$ and S be any semigroup such that $Dom(U, S) = S$. If S either satisfies $a^2 = a$ or $ax = axa\,(axy = axay,\ axy = axyay)$, then $U = S$.*

Now we pose the following problem:

Problem 1 Let U be a semigroup satisfying the identities $a^2 = a$ and $ax = axa$ and let S be any semigroup with U as a subsemigroup and $Dom(U, S) = S$. Does S satisfy the identity $a^2 = a$?

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