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RESEARCH ARTICLE

On a resolvent approach for perturbed semigroups and application to L^1-neutron transport theory

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Abstract We give new sufficient and practical conditions in terms of the generators ensuring the stability of the critical or the essential type of a perturbed C_0-semigroup in general Banach spaces. We apply our theoretical results in order to investigate the control and in particular the time asymptotic behavior of solutions to a broad class of transport equations in L^1-spaces and higher dimension. Our results improve, complete and enrich several earlier works.

Keywords Perturbations · Spectral analysis · Critical type stability · Essential type stability · Resolvent approach · Neutron transport theory

1 Introduction and main results

Let $\mathcal{X}$ be a complex Banach space and let $\mathcal{T} : D(\mathcal{T}) \subseteq \mathcal{X} \to \mathcal{X}$ be the infinitesimal generator of a C_0-semigroup $(\mathcal{U}(t))_{t\geq 0}$ acting on $\mathcal{X}$. We denote by $\mathcal{L}(\mathcal{X})$ the algebra of all bounded linear operators acting on $\mathcal{X}$. Let us consider the Cauchy problem

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$$\begin{cases} \dfrac{d\psi}{dt} = \mathcal{T}\psi + \mathcal{B}\psi \\ \\ \psi(0) = \psi_0 \end{cases} \tag{1.1}$$

where $\mathcal{B} \in \mathcal{L}(\mathcal{X})$ and $\psi_0 \in \mathcal{X}$. By the classical perturbation theory [7], the perturbed operator $\mathcal{A} := \mathcal{T} + \mathcal{B}$ generates also a strongly continuous semigroup $(\mathcal{V}(t))_{t\geq 0}$ given for every $t \geq 0$ by the *Dyson–Phillips expansion*

$$\mathcal{V}(t) = \sum_{j=0}^{m-1} \mathcal{U}_j(t) + \mathcal{R}_m(t), \tag{1.2}$$

where

$$\mathcal{U}_0(t) = \mathcal{U}(t), \;\; \mathcal{U}_j(t) = \int_0^t \mathcal{U}(t-s)\mathcal{B}\mathcal{U}_{j-1}(s)\, ds, \;\; (j \geq 1)$$

and

$$\mathcal{R}_m(t) = \sum_{j=m}^{+\infty} \mathcal{U}_j(t), \;\; (m \geq 1).$$

Consequently, for every $\psi_0 \in D(\mathcal{A})$, the Cauchy problem (1.1) admits the unique solution $\psi(t) = \mathcal{V}(t)\psi_0, t \geq 0$.

In order to investigate the asymptotic behavior of $\psi(t)$ (for large time), two fundamental approaches are at our disposal. The first approach, called the *resolvent approach*, is based on the *asymptotic spectrum* of the generator $\mathcal{A}$:

$$\sigma_{ass}(\mathcal{A}) := \{\lambda \in \mathbb{C} : \; \mathrm{Re}\lambda > \omega(\mathcal{U})\},$$

where $\omega(\mathcal{U})$ stands for the type of $(\mathcal{U}(t))_{t\geq 0}$.

This approach was initiated by Vidav [23] in a particular case and developed afterwords by Mokhtar-Kharroubi [16]. It is essentially based on the following hypothesis:

$$(\mathcal{H}(\mathcal{T}, \mathcal{B})) \begin{cases} \text{There exists an integer } j \text{ and } \omega > \omega(\mathcal{U}) \text{ such that} \\ \text{(i) } [\mathcal{B}(\lambda - \mathcal{T})^{-1}]^j \text{ is compact for every } \lambda \in \mathcal{R}_\omega; \\ \text{(ii) } \lim\limits_{|\mathrm{Im}\,\lambda| \to +\infty} \left\| [\mathcal{B}(\lambda - \mathcal{T})^{-1}]^j \right\| = 0 \text{ uniformly on } \mathcal{R}_\omega, \end{cases}$$

where $(\cdot - \mathcal{T})^{-1}$ denotes the resolvent operator of $\mathcal{T}$ and $\mathcal{R}_\omega = \{\lambda \in \mathbb{C} : \; \mathrm{Re}\lambda \geq \omega\}$. It has been shown in Ref. [16] that under the hypothesis $(\mathcal{H}(\mathcal{T}, \mathcal{B}))$ the part of the spectrum of the perturbed generator $\mathcal{A} = \mathcal{T} + \mathcal{B}$ lying in the half-plane $\mathcal{R}_\omega$ consists of at most of a finite number of isolated eigenvalues with finite algebraic multiplicity $\{\lambda_1, \ldots \lambda_n\}$. Let $\beta_1 = \sup\{\mathrm{Re}\lambda : \; \lambda \in \sigma(\mathcal{T} + \mathcal{K}) \text{ and } \mathrm{Re}\lambda < \omega\}$ and $\beta_2 = \min\{\mathrm{Re}\lambda_j, \; 1 \leq j \leq n\}$. Let P_j and D_j denote the spectral projection and the nilpotent operator associated with λ_j, $1 \le j \le n$, respectively. The solution of the Cauchy problem (1.1) fulfils

$$\Big\| \psi(t) - \sum_{j=1}^{n} e^{\lambda_j t} e^{D_j t} P_j \psi_0 \Big\| = o(e^{\beta^* t}) \text{ for every } \psi_0 \in D(\mathcal{A}^2),$$

where $\beta_1 < \beta^* < \beta_2$. The main disadvantage of this approach lies in the fact that the quantity $\psi(t) - \sum_{j=1}^{n} e^{\lambda_j t} e^{D_j t} P_j \psi_0$ can be evaluated only for initial data ψ_0 in $D(\mathcal{A}^2)$. This condition was weakened to $\psi_0 \in D(A)$ by D. Song [21] for special Banach spaces and by Song and Greenberg [22] in some special cases. Recently, Latrach and the first author [9] weakened the condition $\psi_0 \in D(\mathcal{A}^2)$ to $\psi_0 \in D(\mathcal{A})$ by considering instead of condition $(\mathcal{H}(\mathcal{T}, \mathcal{B}))$, the condition

$$(\tilde{\mathcal{H}}(\mathcal{T}, \mathcal{B})) \begin{cases} \text{There exist an integer } j \text{ and two reals } \omega > \omega(\mathcal{U}) \text{ and } C > 0 \text{ such that} \\ \text{(i) } [\mathcal{B}(\lambda - \mathcal{T})^{-1}]^j \text{ is compact for every } \lambda \in \mathcal{R}_\omega; \\ \text{(ii) } |\mathrm{Im}\,\lambda| \, \big\| [\mathcal{B}(\lambda - \mathcal{T})^{-1}]^j \big\| \le C \text{ uniformly on } \mathcal{R}_\omega. \end{cases}$$

Following [25], the *essential spectrum* of $\mathcal{O} \in \mathcal{L}(\mathcal{X})$ is defined by:

$$\sigma_{ess}(\mathcal{O}) := \{\lambda \in \sigma(\mathcal{O}) \text{ but } \lambda \text{ is not an eigenvalue of finite algebraic multiplicity}\},$$

and the *essential spectral radius* of $\mathcal{O}$ is

$$r_{ess}(\mathcal{O}) := \sup\{|\lambda| : \lambda \in \sigma_{ess}(\mathcal{O})\}.$$

Let $(\mathcal{W}(t))_{t\ge 0}$ be C_0-semigroup. According to [25, Lemma 2.1], there exists a real number $\omega_{ess}(\mathcal{W}) \in [-\infty, \omega(\mathcal{W})]$, called the *essential type* of $(\mathcal{W}(t))_{t\ge 0}$, such that

$$r_{ess}(\mathcal{W}(t)) = e^{\omega_{ess}(\mathcal{W})\, t} \text{ for every } t \ge 0.$$

The second approach, called the *semigroup approach*, is based on the *asymptotic spectrum* of $\mathcal{V}(t)$:

$$\sigma_{ass}(\mathcal{V}(t)) := \sigma(\mathcal{V}(t)) \cap \{\lambda \in \mathbb{C} : |\lambda| > e^{\omega(\mathcal{U})t}\}.$$

This approach makes use of the fact that if some remainder of the Dyson–Phillips expansion $\mathcal{R}_m(t)$ is compact (or strictly singular) for $t \ge 0$, then [16,26] $(\mathcal{U}(t))_{t\ge 0}$ and $(\mathcal{V}(t))_{t\ge 0}$ have the same essential type. Therefore, the part of the spectrum of the perturbed semigroup $\mathcal{V}(t)$ outside the circle $|\mu| = e^{t\omega(\mathcal{U})}$ consists only of at most isolated eigenvalues with finite algebraic multiplicities. If these eigenvalues exist, the semigroup $(\mathcal{V}(t))_{t\ge 0}$ may be decomposed in two parts: the first containing the time development of finitely many eigenmodes, the second being of faster decay. Applying now the spectral mapping theorem for the point spectrum [6, 3.7 Spectral Mapping Theorem for Point and Residual Spectrum], we deduce that for any fixed $\omega > \omega(\mathcal{U})$, the set $\sigma(\mathcal{T}+\mathcal{B})\cap\{\lambda\in\mathbb{C}:\ \mathrm{Re}\lambda\geq\omega\}$ consists of finitely many eigenvalues $\{\lambda_1,\ldots\lambda_n\}$. The solution of the Cauchy problem (1.1) satisfies

$$\left\|\psi(t)-\sum_{j=1}^{n}e^{\lambda_j t}e^{D_j t}P_j\psi_0\right\|=o(e^{\beta^* t})\text{ for every }\psi_0\in D(\mathcal{A}),$$

where P_j, D_j and β^* have the same meaning as above. For sake of completeness, we note that a similar result concerning the semigroup approach was given in [19] (see also [2, Chapter 19]).

The major drawback of this approach is that it is not applicable unless the unperturbed semigroup is explicit.

Even though these two approaches have a significant merit in their own right, it is of interest to find a balance between them to avoid the aforementioned drawbacks. On that account, the two approaches were linked firstly in the Hilbert spaces setting by Sbihi and subsequently in Banach spaces by Latrach and the authors. Specifically, if $\mathcal{X}$ is a Hilbert space, $\mathcal{T}$ is dissipative and there exists $\alpha>\omega(\mathcal{U})$ such that

$$(\alpha+i\beta-\mathcal{T})^{-1}\mathcal{B}(\alpha+i\beta-\mathcal{T})^{-1}\text{ is compact for all }\beta\in\mathbb{R},$$

and

$$\lim_{\beta\to\infty}\|\mathcal{B}^*(\alpha+i\beta-\mathcal{T})^{-1}\mathcal{B}\|+\|\mathcal{B}(\alpha+i\beta-\mathcal{T})^{-1}\mathcal{B}^*\|=0,\tag{1.3}$$

where $\mathcal{B}^*$ denotes the dual operator of $\mathcal{B}$, then [20, Theorem 2.3 and Lemma 2.8] $\mathcal{R}_1(t)$ is compact on $\mathcal{X}$ for all $t\geq 0$. This implies that, for each $t\geq 0$, $\mathcal{U}(t)$ and $\mathcal{V}(t)$ have the same essential spectrum. Further for a general Banach space $\mathcal{X}$, if the condition $(\tilde{\mathcal{H}}(\mathcal{T},\mathcal{B}))$ holds, then [8, Theorem 1.1] the remainder term $\mathcal{R}_{2j+3}$ is compact for each $t\geq 0$, and consequently, $(\mathcal{V}(t))_{t\geq 0}$ and $(\mathcal{U}(t))_{t\geq 0}$ have the same essential type.

In many applications, including multi-dimensional transport equations, the item (ii) in $(\tilde{\mathcal{H}}(\mathcal{T},\mathcal{B}))$, seems to be not always verified because 0 gives a singularity point in an integral representation of $KR(\lambda,T)K$, where T and K denotes respectively the streaming operator and the collision operator (see, Proposition 4.2 below and its proof). In this work, we present and describe an efficient method to avoid singularities. The idea of the method is to construct a sequence of operators $(\vartheta_0^\varepsilon(\mathcal{T},\mathcal{B})(\lambda))_{\varepsilon>0}$ which converges in $\mathcal{L}(\mathcal{X})$ to $(\lambda-\mathcal{T})^{-1}$ as ε goes to zero, uniformly on $\mathcal{R}_\omega$ (see, Lemma 3.4 below), and satisfies (see, Proposition 4.2 below): for every $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$|\mathrm{Im}\,\lambda|\ \left\|\mathcal{B}\vartheta_0^\varepsilon(\mathcal{T},\mathcal{B})(\lambda)\mathcal{B}\right\|\leq C_\varepsilon\text{ uniformly on }\mathcal{R}_\omega,$$

or more generally, we construct a sequence of operators $(\vartheta_j^\varepsilon(\mathcal{T},\mathcal{B})(\lambda))_{\varepsilon>0}$ which converges in $\mathcal{L}(\mathcal{X})$ to $\vartheta_j(\mathcal{T},\mathcal{B})(\lambda):=(\lambda-\mathcal{T})^{-1}[\mathcal{B}(\lambda-\mathcal{T})^{-1}]^j$ as ε goes to zero, uniformly on $\mathcal{R}_\omega$, and satisfies the hypothesis

$$(\mathcal{A}1(j))\begin{cases}\text{For every fixed } \varepsilon > 0, \text{ there exists } \nu \geq \max(\omega(\mathcal{U}), 0) \text{ satisfying}\\ \sup_{\xi > \nu} \int_{-\infty}^{+\infty} \|\vartheta_j^{\varepsilon}(\mathcal{T}, \mathcal{B})(\xi + i\tau)\| \, d\tau < \infty.\end{cases}$$

For further purposes, we recall the following:

Theorem 1.1 *(i)* [4] *Let $j \geq 1$, then, $\mathcal{U}_j(t)$ is compact for all $t \geq 0$ if and only if, the map $t \mapsto \mathcal{U}_j(t)$ is norm continuous for $t \geq 0$, and $(\lambda - \mathcal{T})^{-1}[\mathcal{B}(\lambda - \mathcal{T})^{-1}]^j$ is compact for all $s \in \mathbb{R}$.*
(ii) [5] *If the mapping $t \mapsto \mathcal{R}_j(t)$ is norm right continuous for every $t > 0$ and some $j \in \mathbb{N}$, then one has $\omega_{crit}(\mathcal{V}) = \omega_{crit}(\mathcal{U})$.*

(for the definition of the critical type $\omega_{crit}(\mathcal{W})$ of a given C_0-semigroup $(\mathcal{W}(t))_{t\geq 0}$, see Sect. 2). A natural relevant problem occurs here; namely, for $\mathcal{B} \in \mathcal{L}(\mathcal{X})$ and $j \in \mathbb{N}$ with $j \geq 1$, does exist necessary and sufficient conditions involving $\mathcal{T}$ and $\mathcal{B}$ to get norm continuity of the jth order term $\mathcal{U}_j(t)$ for $t > 0$, of the Dyson–Phillips expansion defining the C_0-semigroup $(\mathcal{V}(t))_{t\geq 0}$ generated by $\mathcal{T}+\mathcal{B}$. This problem has been addressed by several investigators in the Hilbert spaces setting (cf. [3,11,18,20] and [1]).

In [20], Sbihi gave a practical sufficient condition to get norm continuity of $\mathcal{R}_1(t)$ for $t \geq 0$, in the case where X is a Hilbert space. His result states as follows:

Lemma 1.1 [20, Corollary 2.9] *Assume that X is a Hilbert space, $\mathcal{T}$ is dissipative and for some $\alpha > 0$, the condition (1.3) holds true, then the map $t \mapsto \mathcal{U}_1(t)$ is norm continuous for $t \geq 0$.*

More recently, in [1], we showed the following:

Theorem 1.2 *If X is a Hilbert space then $\mathcal{V}(t) - \mathcal{U}(t)$ is norm continuous for $t \geq 0$ if and only if,*

$$\sup_{x \in H, \|x\|=1} \int_{|s|\geq a} \|R(\alpha + is, A)KR(\alpha + is, A)x\|^2 ds \to 0 \textit{ as } a \to +\infty$$

and

$$\sup_{x \in H, \|x\|=1} \int_{|s|\geq a} \|R(\alpha + is, A^*)K^*R(\alpha + is, A^*)x\|^2 ds \to 0 \textit{ as } a \to +\infty.$$

This problem becomes distinctly more complicated when going from Hilbert spaces to Banach spaces. Notice that according to Theorem 1.1, the norm continuity of one of the terms $(\mathcal{U}_j(t))_{t\geq 0}$, $j \geq 1$, implies that the critical type $\omega_{crit}(\mathcal{V})$ of the perturbed semigroup $(\mathcal{U}(t))_{t\geq 0}$ equals to the critical type $\omega_{crit}(\mathcal{U})$ of the unperturbed semigroup $(\mathcal{V}(t))_{t\geq 0}$. This enables us to study the control for $t \geq 0$ or in particular the time asymptotic behavior of solutions to several evolution equations. Indeed, according to the partial spectral mapping theorem (see, Theorem 2.1 below), for $\beta > \omega(\mathcal{U})$, if $\sigma(\mathcal{T}+\mathcal{B}) \cap \{\lambda \in \mathbb{C} : Re\lambda = \beta\} = \emptyset$ and P is the spectral projection associated to the closed part of the spectrum $\sigma(\mathcal{T}+\mathcal{B}) \cap \{\lambda \in \mathbb{C} : Re\lambda > \beta\}$, then there exists $M > 0$ satisfying

$$\|\mathcal{V}(t)(I-P)\| \leq Me^{\beta t} \text{ for all } t \geq 0.$$

In this paper we provide sufficient conditions in terms of the resolvent operator $(\cdot - \mathcal{T})^{-1}$ of the generator $\mathcal{T}$ of the semigroup $(\mathcal{U}(t))_{t\geq 0}$ and the perturbation operator $\mathcal{B}$, ensuring the norm continuity of the mapping $t \mapsto \mathcal{R}_j(t)$ for $t > 0$ (see Theorem 3.1 and Corollary 3.4 below). Furthermore, we show that, if condition $(\mathcal{A}1(j))$ and assertion (i) of hypothesis $(\mathcal{H}(\mathcal{T},\mathcal{B}))$ hold true simultaneously, then the j^{th}-order term $\mathcal{R}_j(t)$ of the Dyson–Phillips expansion (4.4) is compact on $\mathcal{X}$ for every $t \geq 0$ (see Theorem 3.2 and Corollary 3.6 below). These theoretical results apply directly to discuss the control for every $t \geq 0$ and consequently the time asymptotic behavior (for large times) of solutions to a broad class of multi-dimensional neutron transport equations on L^1-spaces.

2 Critical spectrum

Let us recall the concept of critical spectrum introduced by R. Nagel and J. Poland [17]. Let $(\mathcal{W}(t))_{t\geq 0}$ be a C_0-semigroup on a given Banach space $\mathcal{Y}$. We consider the Banach space $\tilde{\mathcal{Y}} := \ell^\infty(\mathcal{Y})$ of all bounded sequences in $\mathcal{Y}$ endowed with the norm

$$\|\tilde{y}\| := \sup_{n\in\mathbb{N}} \|y_n\| \text{ for } \tilde{y} = (y_n)_{n\in\mathbb{N}} \in \tilde{\mathcal{Y}}.$$

We extend the semigroup $(\mathcal{W}(t))_{t\geq 0}$ to $\tilde{\mathcal{Y}}$ and obtain a new semigroup $(\tilde{\mathcal{W}}(t))_{t\geq 0}$ defined by

$$\tilde{\mathcal{W}}(t)\tilde{y} := (\mathcal{W}(t)y_n)_{n\in\mathbb{N}} \text{ for } \tilde{y} = (y_n)_{n\in\mathbb{N}} \in \tilde{\mathcal{Y}}.$$

Let $\tilde{Y}_{\mathcal{W}}$ be the subspace of strong continuity of $(\tilde{\mathcal{W}}(t))_{t\geq 0}$

$$\tilde{Y}_{\mathcal{W}} := \left\{\tilde{y} \in \tilde{Y} : \lim_{t\downarrow 0} \left\|\tilde{\mathcal{W}}(t)\tilde{y} - \tilde{y}\right\| = 0\right\}.$$

This subspace is closed and $(\tilde{\mathcal{W}}(t))_{t\geq 0}$-invariant. On the quotient space $\hat{Y} := \tilde{Y}/\tilde{Y}_{\mathcal{W}}$, the semigroup $(\tilde{\mathcal{W}}(t))_{t\geq 0}$ induces the quotient semigroup $(\hat{\mathcal{W}}(t))_{t\geq 0}$ given by

$$\hat{\mathcal{W}}(t)\hat{y} := \tilde{\mathcal{W}}(t)\tilde{y} + \tilde{Y}_{\mathcal{W}} \text{ for } \hat{y} = \tilde{y} + \tilde{Y}_{\mathcal{W}} \text{ and } t \geq 0.$$

The *critical spectrum* of $\mathcal{W}(t)$ is then defined as

$$\sigma_{crit}(\mathcal{W}(t)) := \sigma(\hat{\mathcal{W}}(t))$$

and the critical spectral radius is defined as

$$r_{crit}(\mathcal{W}(t)) := r(\hat{\mathcal{W}}(t))$$

while the *critical type* or the *critical growth bound* of $(\mathcal{W}(t))_{t\geq 0}$ is defined as

$$\omega_{crit}(\mathcal{W}) := \omega(\hat{\mathcal{W}}).$$

The critical spectrum enjoys nice properties (see, [5,17] for general theory); in particular, we have

Theorem 2.1 [17, Proposition 4.3] *Let $(\mathcal{W}(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space $\mathcal{Y}$ with infinitesimal generator Ξ. Then*

$$\omega(\mathcal{W}) = \max\{s(\Xi), \omega_{crit}(\mathcal{W})\}.$$

Moreover, the following partial spectral mapping theorem holds

$$\sigma(\mathcal{W}(t)) \cap \mathcal{Q}_{crit}(\mathcal{W}(t)) = e^{t\sigma(\Xi)} \cap \mathcal{Q}_{crit}(\mathcal{W}(t)) \text{ for each } t \geq 0,$$

where $\mathcal{Q}_{crit}(\mathcal{W}(t)) := \{\lambda \in \mathbb{C} : |\lambda| > r_{crit}(\mathcal{W}(t))\}$.

3 Resolvent approach for perturbed semigroups

3.1 Norm continuity of $\mathcal{U}_j(\cdot)$

For every $\varepsilon > 0$ small enough, set

$$\mathcal{U}_0^{\varepsilon}(t) := \mathcal{U}(t)\, \chi_{(\varepsilon,+\infty)}(t), \;\; t \geq 0. \tag{3.1}$$

Remark 3.1 (1) For every $\varepsilon > 0$ and $x \in \mathcal{X}$, $\mathcal{U}_0^{\varepsilon}(\cdot)x = 0$ on $(0, \varepsilon)$ and since $(\mathcal{U}(t))_{t\geq 0}$ is a strongly continuous semigroup on $\mathcal{X}$, the map $\mathcal{U}_0^{\varepsilon}(\cdot)x$ is continuous on $(\varepsilon, +\infty)$ (and therefore, the map $\mathcal{U}_0^{\varepsilon}(\cdot)x$ is measurable on $(0, +\infty)$) and for every $\omega > \omega(\mathcal{U})$, there exists $M \geq 1$ satisfying

$$\|\mathcal{U}(t)\| \leq Me^{\omega t} \text{ for every } t > 0,$$

and therefore,

$$\|\mathcal{U}_0^{\varepsilon}(t)\| \leq Me^{\omega t} \text{ for every } t > 0. \tag{3.2}$$

(2) It is well known that (see, [7] for example) for every $\omega > \omega(\mathcal{U})$, there exists $M \geq 1$ satisfying

$$\|\mathcal{U}_j(t)\| \leq M_j\, t^j\, e^{\omega t} \text{ for every } t > 0 \text{ and } j \geq 0 \tag{3.3}$$

where $M_j := (M^{j+1}\, \|\mathcal{B}\|^j)/j!$, $j \geq 0$.

Let $\varepsilon > 0$. Formally, we set

$$\mathcal{U}_j^\varepsilon(t)x = \int_0^t \mathcal{U}_{j-1}^\varepsilon(t-s)\,\mathcal{B}\mathcal{U}_0^\varepsilon(s)x\,ds,\ t \geq 0,\ x \in \mathcal{X} \text{ and } j \geq 1.$$

In the following lemma, we prove that the operators $\mathcal{U}_j^\varepsilon(t)$ are well defined for almost all $t \geq 0$ and all $j \in \mathbb{N}$.

Lemma 3.1 *For every $j \geq 0$, $\varepsilon > 0$ and $x \in \mathcal{X}$, $\mathcal{U}_j^\varepsilon(\cdot)x = 0$ on $(0, \varepsilon)$, the map $\mathcal{U}_j^\varepsilon(\cdot)x$ is continuous on $(\varepsilon, +\infty)$, and therefore, it is measurable on $(0, +\infty)$. Furthermore, for every $\omega > \omega(\mathcal{U})$, there exists $M \geq 1$ satisfying*

$$\|\mathcal{U}_j^\varepsilon(t)\| \leq M_j\, t^j\, e^{\omega t} \text{ for every } t > 0,\ \varepsilon > 0 \text{ and } j \geq 0 \tag{3.4}$$

where $M_j := (M^{j+1}\,\|\mathcal{B}\|^j)/j!$, $j \geq 0$.

Proof We proceed by mathematical induction on $j \in \mathbb{N}$. For $j = 0$, the result follows from the first item of Remark 3.1. Let $j \geq 0$ and assume that the result holds for j. Then, for $\varepsilon > 0$ and $x \in \mathcal{X}$, we have

$$\mathcal{U}_{j+1}^\varepsilon(t)x = \int_0^t \mathcal{U}_j^\varepsilon(t-s)\,\mathcal{B}\mathcal{U}_0^\varepsilon(s)x\,ds,\ t \geq 0. \tag{3.5}$$

Since $\mathcal{U}_0^\varepsilon(\cdot)x = 0$ on $(0, \varepsilon)$, then, $\mathcal{U}_{j+1}^\varepsilon(\cdot)x = 0$ on $(0, \varepsilon)$. Linking (3.2), (3.4) and (3.5), we get

$$\|\mathcal{U}_{j+1}^\varepsilon(t)\| \leq M_{j+1}\, t^{j+1}\, e^{\omega t},\ t > 0.$$

To achieve the proof, it remains to show that the map $\mathcal{U}_{j+1}^\varepsilon(\cdot)x$ is continuous on $(\varepsilon, +\infty)$. For $\varepsilon < t_0 < t$, we have

$$\begin{aligned}
&\mathcal{U}_{j+1}^\varepsilon(t)x - \mathcal{U}_{j+1}^\varepsilon(t_0)x \\
&\quad= \int_0^t \mathcal{U}_j^\varepsilon(t-s)\,\mathcal{B}\mathcal{U}_0^\varepsilon(s)x\,ds - \int_0^{t_0} \mathcal{U}_j^\varepsilon(t_0-s)\,\mathcal{B}\mathcal{U}_0^\varepsilon(s)x\,ds \\
&\quad= \int_0^{t_0} [\mathcal{U}_j^\varepsilon(t-s) - \mathcal{U}_j^\varepsilon(t_0-s)]\,\mathcal{B}\mathcal{U}_0^\varepsilon(s)x\,ds + \int_{t_0}^t \mathcal{U}_j^\varepsilon(t-s)\,\mathcal{B}\mathcal{U}_0^\varepsilon(s)x\,ds
\end{aligned}$$

and hence,

$$\begin{aligned}
&\|\mathcal{U}_{j+1}^\varepsilon(t)x - \mathcal{U}_{j+1}^\varepsilon(t_0)x\| \\
&\quad\leq \left\| \int_0^{t_0} [\mathcal{U}_j^\varepsilon(t-s) - \mathcal{U}_j^\varepsilon(t_0-s)]\,\mathcal{B}\mathcal{U}_0^\varepsilon(s)x\,ds \right\| + \int_{t_0}^t \|\mathcal{U}_j^\varepsilon(t-s)\,\mathcal{B}\mathcal{U}_0^\varepsilon(s)x\|\,ds
\end{aligned}$$

and similarly for $\varepsilon < t < t_0$, we have

$$\|\mathcal{U}^{\varepsilon}_{j+1}(t_0)x - \mathcal{U}^{\varepsilon}_{j+1}(t)x\| \\ \leq \left\| \int_0^t [\mathcal{U}^{\varepsilon}_j(t_0 - s) - \mathcal{U}^{\varepsilon}_j(t - s)]\, \mathcal{B}\mathcal{U}^{\varepsilon}_0(s)x\, ds \right\| + \int_t^{t_0} \|\mathcal{U}^{\varepsilon}_j(t_0 - s)\, \mathcal{B}\mathcal{U}^{\varepsilon}_0(s)x\|\, ds.$$

Now the continuity of $\mathcal{U}^{\varepsilon}_{j+1}(\cdot)x$ at $t_0 \in (\varepsilon, +\infty)$ follows from the fact that $\|\mathcal{U}^{\varepsilon}_j(t')\, \mathcal{B}\mathcal{U}^{\varepsilon}_0(s)x\|$ is uniformly bounded for t' and s belonging to bounded subsets of $(0, +\infty)$ [see (3.4)] together with the boundedness of $\mathcal{B}$ and the Lebesgue dominated convergence theorem. This ends the proof. □

Lemma 3.2 *For every $j \in \mathbb{N}$, the sequence $(\mathcal{U}^{\varepsilon}_j(t))_{\varepsilon>0}$ converges in $\mathcal{L}(\mathcal{X})$ to $\mathcal{U}_j(t)$ as ε goes to 0, uniformly on bounded and closed intervals of* $(0, +\infty)$.

Proof We proceed by induction on $j \in \mathbb{N}$. For $[a, b] \subset (0, +\infty)$, we have

$$\|\mathcal{U}^{\varepsilon}_0(t) - \mathcal{U}_0(t)\| = \|\mathcal{U}(t)\chi_{(0,\varepsilon)}(t)\| = 0 \text{ for every } t \in [a, b] \text{ and } 0 < \varepsilon < a.$$

This gives the result for $j = 0$. Let $j \in \mathbb{N}$ and assume that the sequence $(\mathcal{U}^{\varepsilon}_j(t))_{\varepsilon>0}$ converges in $\mathcal{L}(X)$ to $\mathcal{U}_j(t)$ as ε goes to 0, uniformly on bounded and closed intervals of $(0, +\infty)$. We will show the result for $j + 1$. Notice, for every $\varepsilon > 0$ and $t \in (0, +\infty)$, we have

$$\begin{aligned}
\|\mathcal{U}^{\varepsilon}_{j+1}(t) - \mathcal{U}_{j+1}(t)\| \leq & \sup_{\substack{x \in \mathcal{X} \\ \|x\| = 1}} \Big\| \int_0^t \mathcal{U}^{\varepsilon}_j(t-s)\mathcal{B}\mathcal{U}^{\varepsilon}_0(s)x\, ds, \\
& \qquad - \int_0^t \mathcal{U}_j(t-s)\mathcal{B}\mathcal{U}_0(s)x\, ds \Big\| \\
\leq & \sup_{\substack{x \in \mathcal{X} \\ \|x\| = 1}} \Big\| \int_0^t [\mathcal{U}^{\varepsilon}_j(t-s) - \mathcal{U}_j(t-s)]\mathcal{B}\mathcal{U}^{\varepsilon}_0(s)x\, ds \Big\| \\
& + \sup_{\substack{x \in \mathcal{X} \\ \|x\| = 1}} \Big\| \int_0^t \mathcal{U}_j(t-s)\mathcal{B}[\mathcal{U}^{\varepsilon}_0(s) - \mathcal{U}_0(s)]x\, ds \Big\| \\
\leq & \int_0^t \|\mathcal{U}^{\varepsilon}_j(t-s) - \mathcal{U}_j(t-s)\|\, \|\mathcal{B}\|\, \|\mathcal{U}^{\varepsilon}_0(s)\|\, ds \\
& + \int_0^t \|\mathcal{U}_j(t-s)\|\, \|\mathcal{B}\|\, \|\mathcal{U}^{\varepsilon}_0(s) - \mathcal{U}_0(s)\|\, ds.
\end{aligned}$$

Now, taking into account (3.3), (3.4) and the boundedness of $\mathcal{B}$, the result follows from the Lebesgue dominated convergence theorem. □

Set

$$\vartheta_j(\mathcal{T}, \mathcal{B})(\lambda) = (\lambda - T)^{-1}[\mathcal{B}(\lambda - T)^{-1}]^j \text{ for } \mathrm{Re}\lambda > \omega(\mathcal{U}) \text{ and } j \geq 1.$$

For $\varepsilon > 0$, set by induction,

$$\mathcal{L}(\mathcal{X}) \ni \vartheta_0^{\varepsilon}(\mathcal{T}, \mathcal{B})(\lambda) := \int_0^{+\infty} e^{-\lambda t} \mathcal{U}_0^{\varepsilon}(t)\, dt \text{ for } \mathrm{Re}\lambda > \omega(\mathcal{U}) \tag{3.6}$$

in the strong sense, and

$$\mathcal{L}(\mathcal{X}) \ni \vartheta_j^{\varepsilon}(\mathcal{T}, \mathcal{B})(\lambda) := \vartheta_{j-1}^{\varepsilon}(\mathcal{T}, \mathcal{B})(\lambda)\, \mathcal{B}\, \vartheta_0^{\varepsilon}(\mathcal{T}, \mathcal{B})(\lambda), \text{ for } \mathrm{Re}\lambda > \omega(\mathcal{U}) \text{ and } j \geq 1.$$

For $\omega > \omega(\mathcal{U})$, set

$$\mathcal{R}_{\omega} := \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \geq \omega\}.$$

Remark 3.2 In the strong sense, we have [7]

$$\vartheta_j(\mathcal{T}, \mathcal{B})(\lambda) = \int_0^{+\infty} e^{-\lambda t}\, \mathcal{U}_j(t)\, dt \text{ for } \mathrm{Re}\lambda > \omega(\mathcal{U}) \text{ and } j \geq 0. \tag{3.7}$$

Lemma 3.3 *For every $j \geq 0$ and $\varepsilon > 0$, we have, in the strong sense,*

$$\int_0^{+\infty} e^{-\lambda t}\, \mathcal{U}_j^{\varepsilon}(t)\, dt = \vartheta_j^{\varepsilon}(\mathcal{T}, \mathcal{B})(\lambda) \text{ for } Re\lambda > \omega(\mathcal{U}) \tag{3.8}$$

and consequently for $\omega > \omega(\mathcal{U})$, there exists $M_j \geq 1$ satisfying

$$\|\vartheta_j^{\varepsilon}(\mathcal{T}, \mathcal{B})(\lambda)\| \leq M_j \Big(\int_0^{+\infty} t^j e^{(\omega-\omega')\, t}\, dt \Big) \text{ for } Re\lambda > \omega' > \omega. \tag{3.9}$$

Accordingly, by applying the Lebesgue dominated convergence theorem, we obtain

Corollary 3.1 *For every $j \geq 1$, $\varepsilon > 0$, $\omega > \omega(\mathcal{U})$ and $x \in X$, the function*

$$R_{\omega} \ni \lambda \longmapsto \vartheta_j^{\varepsilon}(\mathcal{T}, \mathcal{B})(\cdot)x \in \mathcal{X}$$

is holomorphic on the interior of R_{ω}.

Proof of Lemma 3.3 We proceed by induction. For $j = 0$, the result follows from (3.6). Let $j \geq 0$ and assume that (3.8) holds for j. For $\mathrm{Re}\lambda > \omega(\mathcal{U})$ and $x \in X$, we have by Fubini's theorem

$$\begin{aligned} \int_0^{+\infty} e^{-\lambda t}\, \mathcal{U}_{j+1}^{\varepsilon}(t)x\, dt &= \int_0^{+\infty} dt\, e^{-\lambda t} \int_0^{+\infty} ds\, \mathcal{U}_j^{\varepsilon}(t-s)\, \mathcal{B}\mathcal{U}_0^{\varepsilon}(s)x \\ &= \int_0^{+\infty} ds \int_0^{+\infty} dt\ e^{-\lambda t}\, \mathcal{U}_j^{\varepsilon}(t-s)\, \mathcal{B}\mathcal{U}_0^{\varepsilon}(s)x \end{aligned}$$

where $\mathcal{U}_j^\varepsilon(.)$ is identified to its natural extension ($\mathcal{U}_j^\varepsilon(u) = 0$ for $u < 0$) on $\mathbb{R}$. Now using the change of unknowns $t \mapsto u := t - s$, we get

$$\begin{aligned}\int_0^{+\infty} e^{-\lambda t}\,\mathcal{U}_{j+1}^\varepsilon(t)x\,dt &= \int_0^{+\infty} ds\; e^{-\lambda s}\int_0^{+\infty} du\; e^{-\lambda u}\,\mathcal{U}_j^\varepsilon(u)\,\mathcal{B}\mathcal{U}_0^\varepsilon(s)x\\ &= \int_0^{+\infty} ds\; e^{-\lambda s}\,\vartheta_j^\varepsilon(\mathcal{T},\mathcal{B})(\lambda)\,\mathcal{B}\mathcal{U}_0^\varepsilon(s)x.\end{aligned}$$

And, by linearity and continuity,

$$\begin{aligned}\int_0^{+\infty} e^{-\lambda t}\,\mathcal{U}_{j+1}^\varepsilon(t)x\,dt &= \vartheta_j^\varepsilon(\mathcal{T},\mathcal{B})(\lambda)\,\mathcal{B}\,\vartheta_0^\varepsilon(\mathcal{T},\mathcal{B})(\lambda)x\\ &= \vartheta_{j+1}^\varepsilon(\mathcal{T},\mathcal{B})(\lambda)x.\end{aligned}$$

To achieve the proof, we use formula (3.4). □

Lemma 3.4 *Let $j \in \mathbb{N}$ and $\omega > \omega(\mathcal{U})$, then uniformly on $\mathcal{R}_\omega$, the sequence $(\vartheta_j^\varepsilon(\mathcal{T},\mathcal{B})(\lambda))_{\varepsilon>0}$ converges in $\mathcal{L}(\mathcal{X})$ to $\vartheta_j(\mathcal{T},\mathcal{B})(\lambda)$ as ε goes to zero.*

Proof For every $\lambda \in \mathcal{R}_\omega$, from (3.8) and (3.7), we have

$$\begin{aligned}\|\vartheta_j^\varepsilon(\mathcal{T},\mathcal{B})(\lambda) - \vartheta_j(\mathcal{T},\mathcal{B})(\lambda)\| &= \sup_{x\in\mathcal{X}\ \|x\|=1}\left\|\int_0^{+\infty} e^{-\lambda t}\,[\mathcal{U}_j^\varepsilon(t)x - \mathcal{U}_j(t)x]\,dt\right\|\\ &\le \int_0^{+\infty} e^{-\omega t}\,\|\mathcal{U}_j^\varepsilon(t) - \mathcal{U}_j(t)\|\,dt.\end{aligned}$$

Let $\omega' \in (\omega(\mathcal{U}),\omega)$. From (3.3) and (3.4), there exists $M' \ge 0$ satisfying

$$\max(\|\mathcal{U}_j^\varepsilon(t)\|, \|\mathcal{U}_j(t)\|) \le M_j'\,t^j\,e^{\omega' t} \text{ for every } t \ge 0$$

where $M_j' := ((M')^{j+1}\,\|\mathcal{B}\|^j)/j!$. We have

$$0 \le e^{-\omega t}\,\|\mathcal{U}_j^\varepsilon(t) - \mathcal{U}_j(t)\| \le 2M_j'\,t^j\,e^{(\omega'-\omega)t} \text{ for every } t \ge 0$$

and the map: $(0,+\infty) \ni t \longmapsto 2M_j'\,t^j\,e^{(\omega'-\omega)t} \in \mathbb{R}_+$, belongs to $L^1(0,+\infty)$. Now we achieve the proof by using Lemma 3.2 and applying the Lebesgue dominated convergence theorem. □

The following lemma will be crucial in the proof of the two results below.

Lemma 3.5 *For every $j \ge 1$, $\varepsilon > 0$ and $\omega > \omega(\mathcal{U})$, the function*

$$R_\omega \ni \lambda \longmapsto \vartheta_j^\varepsilon(\mathcal{T},\mathcal{B})(\lambda) \in \mathcal{L}(\mathcal{X})$$

is holomorphic on the interior of R_ω, and,

$$\frac{d\vartheta_j^\varepsilon(\mathcal{T},\mathcal{B})}{d\lambda}(\lambda)x = \Upsilon(\lambda)x := -\int_0^{+\infty} te^{-\lambda t}\,\mathcal{U}_j^\varepsilon(t)x\,dt \text{ for every } \lambda \in int(\mathcal{R}_\omega) \text{ and } x \in \mathcal{X},$$

where int($\mathcal{R}_\omega$) denotes the interior of $\mathcal{R}_\omega$.

Proof For each $t > 0$ and $z \in \mathcal{R}_\omega$, we put $\phi_t(z) := e^{-zt}$. Let $\lambda \in \text{int}(\mathcal{R}_{\omega'})$ where $\omega' > \omega$. Note that $\phi_t'(\lambda) = -te^{-\lambda t}$. For each $x \in \mathcal{X}$ and $h \in \mathbb{C}$ such that

$$\lambda + h \in \text{ int}(\mathcal{R}_{\omega'}) \text{ and } |h^{-1}(\phi_t(h) - 1) + t| < 1, \tag{3.10}$$

we have according to Lemma 3.3,

$$\begin{aligned}
&\left\| h^{-1}[\vartheta_j^\varepsilon(\mathcal{T},\mathcal{B})(\lambda+h)x - \vartheta_j^\varepsilon(\mathcal{T},\mathcal{B})(\lambda)x] + \int_0^{+\infty} te^{-\lambda t}\mathcal{U}_j^\varepsilon(t)x\,dt \right\| \\
&= \left\| \int_0^{+\infty} [h^{-1}(\phi_t(\lambda+h) - \phi_t(\lambda)) + te^{-\lambda t}]\,\mathcal{U}_j^\varepsilon(t)x\,dt \right\| \\
&\le \int_0^{+\infty} |h^{-1}(\phi_t(h) - 1) + t|\,e^{-Re\lambda\,t}\,\|\mathcal{U}_j^\varepsilon(t)x\|\,dt.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\|h^{-1}[\vartheta_j^\varepsilon(\mathcal{T},\mathcal{B})(\lambda+h) - \vartheta_j^\varepsilon(\mathcal{T},\mathcal{B})(\lambda)] - \Upsilon(\lambda)\| \\
&= \sup_{x\in\mathcal{X},\ \|x\|\le 1} \left\| h^{-1}[\vartheta_j^\varepsilon(\mathcal{T},\mathcal{B})(\lambda+h)x - \vartheta_j^\varepsilon(\mathcal{T},\mathcal{B})(\lambda)x] + \int_0^{+\infty} te^{-\lambda t}\mathcal{U}_j^\varepsilon(t)x\,dt \right\| \\
&\le \int_0^{+\infty} |h^{-1}(\phi_t(h) - 1) + t|\,e^{-Re\lambda\,t}\,\|\mathcal{U}_j^\varepsilon(t)\|\,dt.
\end{aligned}$$

Note that, by using (3.4), one has

$$0 \le |h^{-1}(\phi_t(h) - 1) + t|\,e^{-Re\lambda\,t}\,\|\mathcal{U}_j^\varepsilon(t)\| \le M_j\,t^j\,e^{(\omega - Re\lambda)\,t} \le M_j\,t^j\,e^{(\omega-\omega')\,t}$$

for each h satisfying (3.10) and

$$\int_0^{+\infty} M_j\,t^j\,e^{(\omega-\omega')\,t}\,dt < \infty.$$

Therefore, by applying the Lebesgue dominated convergence theorem, we get

$$
\begin{aligned}
&\lim_{h\to 0} \|h^{-1}[\vartheta_j^{\varepsilon}(\mathcal{T},\mathcal{B})(\lambda+h)-\vartheta_j^{\varepsilon}(\mathcal{T},\mathcal{B})(\lambda)]-\Upsilon(\lambda)\| \\
&\quad = \lim_{h\to 0} \left\| h^{-1}[\vartheta_j^{\varepsilon}(\mathcal{T},\mathcal{B})(\lambda+h)-\vartheta_j^{\varepsilon}(\mathcal{T},\mathcal{B})(\lambda)] + \int_0^{+\infty} te^{-\lambda t}\mathcal{U}_j^{\varepsilon}(t)\,dt \right\| \\
&\quad \le \lim_{h\to 0} \int_0^{+\infty} |h^{-1}(\phi_t(h)-1)+t|\, e^{-Re\lambda\, t}\, \|\mathcal{U}_j^{\varepsilon}(t)\|\,dt \\
&\quad = \int_0^{+\infty} \lim_{h\to 0} |h^{-1}(\phi_t(h)-1)+t|\, e^{-Re\lambda\, t}\, \|\mathcal{U}_j^{\varepsilon}(t)\|\,dt = 0.
\end{aligned}
$$

This ends the proof. □

For each fixed $j \ge 0$ and $\varepsilon > 0$, let us consider the following hypotheses

$$
(\mathcal{A}1(j,\varepsilon)) \begin{cases} \text{There exists } \nu \ge \max(\omega(\mathcal{U}),0) \text{ satisfying} \\ \sup\limits_{\xi>\nu} \displaystyle\int_{-\infty}^{+\infty} \|\vartheta_j^{\varepsilon}(\mathcal{T},\mathcal{B})(\xi+i\tau)\|\,d\tau < \infty \end{cases}
$$

and

$(\mathcal{A}1(j))$ The hypothesis $(\mathcal{A}1(j,\varepsilon))$ hols for every (small) $\varepsilon > 0$.

Now, we are in a position to state the following

Theorem 3.1 *Let $j \ge 0$ and assume that $(\mathcal{A}1(j))$ holds true. Then, $\mathcal{U}_j(\cdot)$ is norm continuous on $(0,+\infty)$.*

Proposition 3.1 *Let $j \ge 0$ and $\varepsilon > 0$. If the hypothesis $(\mathcal{A}1(j,\varepsilon))$ holds then, $\mathcal{U}_j^{\varepsilon}(\cdot)$ is norm continuous on $(\varepsilon,+\infty)$. Furthermore, in the strong sense and in $\mathcal{L}(\mathcal{X})$, we have for $\xi > \nu$,*

$$
\mathcal{U}_j^{\varepsilon}(t) = \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} e^{\lambda t}\,\vartheta_j^{\varepsilon}(\mathcal{T},\mathcal{B})(\lambda)\,d\lambda, \quad t > \varepsilon. \tag{3.11}
$$

To prove this result, we need to recall some relevant definitions and auxiliary results.

Definition 3.1 (See [7, Definition 6.4.1].) Let $\mathcal{Y}$ be a Banach space and let $\vartheta(\cdot)$ denote an $\mathcal{Y}$-valued function defined on the half-plane $\{\lambda \in \mathbb{C}: \ \mathrm{Re}\lambda > \alpha\}$ where $\alpha \in \mathbb{R}$. We say that $\vartheta(\cdot)$ belongs to the class $H_p(\alpha,\mathcal{Y})$ if the following conditions are satisfied:

(a) $\vartheta(\cdot)$ is a function on complex numbers to $\mathcal{Y}$, which is holomorphic for $\mathrm{Re}\lambda > \alpha$;

(b) $\sup_{\gamma>\alpha}\{\int_{-\infty}^{+\infty} \|\vartheta(\gamma+i\tau)\|^p\,d\tau\}^{\frac{1}{p}} < +\infty$;

(*c*) $\lim_{\gamma\to\alpha}\vartheta(\gamma+i\ell)=\vartheta(\alpha+i\ell)$ exists for almost all values of ℓ and

$$\int_{-\infty}^{+\infty}\|\vartheta(\alpha+i\tau)\|^p\,d\tau<+\infty.$$

The following result shows that every function in $H_p(\alpha,\mathcal{Y})$ may be represented by a generalized Laplace integral.

Proposition 3.2 (See [7, p. 230].) *Let* $\vartheta(\cdot)\in H_p(\alpha,\mathcal{Y})$ *where* $\alpha\geq 0$. *Let* $\gamma>\alpha$ *and* $\beta q>1$ *where* $\frac{1}{p}+\frac{1}{q}=1$. *Then*

$$\theta_\beta(t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{\lambda t}\lambda^{-\beta}\vartheta(\lambda)\,d\lambda$$

defines a continuous function on $(0,+\infty)$ *to* $\mathcal{Y}$ *and*

$$\vartheta(\lambda)=\lambda^\beta\int_0^{+\infty}e^{-\lambda t}\theta_\beta(t)\,dt$$

the integral being absolutely convergent for Re$\lambda>\alpha$.
For $p=1$ *(see* [7, p. 230]*), we may take* $\beta=0$*, obtaining*

$$\vartheta(\lambda)=\int_0^{+\infty}e^{-\lambda t}\theta_0(t)\,dt.$$

Let $\vartheta(\cdot)$ be an $\mathcal{Y}$-valued function defined on a half-plane $P_\alpha:=\{\lambda\in\mathbb{C}\colon \mathrm{Re}\lambda>\alpha\}$ where $\mathcal{Y}$ is a complex Banach space and $\alpha\in\mathbb{R}$. Assume that the function $\vartheta(\cdot)$ satisfies Conditions (a) and (b) (except may be Condition (c)) of Definition 3.1, say

$$\vartheta(\cdot)\in\Xi_p(\alpha,\mathcal{Y}).\tag{3.12}$$

Then, naturally, for each $\alpha'>\alpha$, this function satisfies

(a') $\vartheta(\cdot)$ is a function on complex numbers to $\mathcal{Y}$, which is holomorphic for Re$\lambda>\alpha'$;
(b') $\sup_{\gamma>\alpha'}\{\int_{-\infty}^{+\infty}\|\vartheta(\gamma+i\tau)\|^p\,d\tau\}^{\frac{1}{p}}<\sup_{\gamma>\alpha}\{\int_{-\infty}^{+\infty}\|\vartheta(\gamma+i\tau)\|^p\,d\tau\}^{\frac{1}{p}}<+\infty$;
(c') The holomorphy of $\vartheta(\cdot)$ on P_α implies its continuity on P_α; thus,

$$\lim_{\gamma\to\alpha'}\vartheta(\gamma+i\ell)=\vartheta(\alpha'+i\ell)\text{ exists for all values of }\ell\in\mathbb{R}.$$

Furthermore,

$$\begin{aligned}\int_{-\infty}^{+\infty}\|\vartheta(\alpha'+i\tau)\|^p\,d\tau&\leq\sup_{\gamma>\alpha}\left\{\int_{-\infty}^{+\infty}\|\vartheta(\gamma+i\tau)\|^p\,d\tau\right.\\&=\left(\sup_{\gamma>\alpha}\left\{\int_{-\infty}^{+\infty}\|\vartheta(\gamma+i\tau)\|^p\,d\tau\right\}^{1/p}\right)^p<+\infty\,.\end{aligned}$$

This means that $\vartheta(\cdot)$ belongs to $H_p(\alpha', \mathcal{Y})$ for every $\alpha' > \alpha$, and consequently, we deduce the following

Corollary 3.2 *Let* $\vartheta(\cdot) \in \Xi_p(\alpha, \mathcal{Y})$ *where* $\alpha \geq 0$. *Let* $\gamma > \alpha$ *and* $\beta q > 1$ *where* $\frac{1}{p} + \frac{1}{q} = 1$. *Then, the conclusion of Proposition 3.2 holds true.*

Proof of Proposition 3.1 By using the hypothesis $(\mathcal{A}1(j, \varepsilon))$ and Corollary 3.1 (resp. Lemma 3.5), we infer from (3.12) that $\vartheta_j^\varepsilon(\mathcal{T}, \mathcal{B})(\cdot)x$ (resp $\vartheta_j^\varepsilon(\mathcal{T}, \mathcal{B})(\cdot)$) belongs to $\Xi_1(\nu, \mathcal{X})$ for each $x \in H$ (resp. $\Xi_1(\nu, \mathcal{L}(\mathcal{X}))$). This implies, according to Corollary 3.2, that the continuous function

$$\theta_j^\varepsilon(t) := \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t}\, \vartheta_j^\varepsilon(\mathcal{T}, \mathcal{B})(\lambda)\, d\lambda,\ \gamma > \nu,\ t > 0,$$

satisfies,

$$\int_0^{+\infty} e^{-\lambda t} \theta_j^\varepsilon(t)\, dt = \vartheta_j^\varepsilon(\mathcal{T}, \mathcal{B})(\lambda),$$

where the integrals are considered in the strong sense (resp. $\mathcal{L}(\mathcal{X})$). Keeping in mind that (Lemma 3.3) in the strong sense,

$$\int_0^{+\infty} e^{-\lambda t}\, \mathcal{U}_j^\varepsilon(t)\, dt = \vartheta_j^\varepsilon(\mathcal{T}, \mathcal{B})(\lambda) \text{ for } \mathrm{Re}\lambda > \omega(\mathcal{U}),$$

the uniqueness of the Laplace transform yields by continuity in the strong sense (see Lemma 3.1),

$$\mathcal{U}_j^\varepsilon(t) = \theta_j^\varepsilon(t) \text{ for every } t > \varepsilon.$$

Therefore, (3.11) follows in the strong sense and in $\mathcal{L}(\mathcal{X})$; moreover, $\mathcal{U}_j^\varepsilon(\cdot)$ is norm continuous on $(\varepsilon, +\infty)$. This ends the proof. □

Proof of Theorem 3.1 The result follows from Proposition 3.1 on the basis of Lemma 3.2. □

As a consequence of Theorem 3.1 we obtain the following

Corollary 3.3 *Assume that there exists* $j \geq 0$ *such that the hypothesis* $(\mathcal{A}1(j))$ *holds. Then, the semigroups* $(\mathcal{U}(t))_{t\geq 0}$ *and* $(\mathcal{V}(t))_{t \geq 0}$ *have the same critical type* ω_{crit}.

Remark 3.3 For each $j \geq 0$ and $\varepsilon > 0$, let consider the following conditions

$$(\mathcal{A}1'(j, \varepsilon)) \begin{cases} \text{There exists } \nu \geq \max(\omega(\mathcal{U}), 0) \text{ satisfying} \\ \lim_{a \to +\infty} \sup_{\xi > \nu} \int_{|\tau| > a} \|\vartheta_j^\varepsilon(\mathcal{T}, \mathcal{B})(\xi + i\tau)\|\, d\tau = 0 \end{cases}$$

and

$(\mathcal{A}1'(j))$ The hypothesis $(\mathcal{A}1'(j,\varepsilon))$ hols for every (small) $\varepsilon > 0$.

Note that for every $j \geq 0$ and $\varepsilon > 0$, Condition $(\mathcal{A}1'(j,\varepsilon))$ implies Condition $(\mathcal{A}1(j,\varepsilon))$. Therefore, for every $j \geq 0$, Condition $(\mathcal{A}1'(j))$ implies condition $(\mathcal{A}1(j))$.

Let us consider the following hypothesis.

$$(\mathcal{A}2)\begin{cases}\text{For every } \varepsilon > 0, \text{ there exist } \nu > \max(\omega(\mathcal{U}),0) \text{ and } C_\varepsilon > 0 \text{ satisfying}\\ |Im\lambda|\,\|\mathcal{B}\vartheta_0^\varepsilon(\mathcal{T},\mathcal{B})(\lambda)\mathcal{B}\| \leq C_\varepsilon \text{ for every } \lambda \in \mathcal{R}_\nu.\end{cases}$$

Now we state another consequence of Theorem 3.1. This result is very useful in applications.

Corollary 3.4 *Assume that* $(\mathcal{A}2)$ *holds true. Then for each* $j \geq 4$, *the operator* $\mathcal{U}_j(t)$ *is norm continuous for each* $t > 0$ *and therefore, the semigroups* $(\mathcal{U}(t))_{t\geq 0}$ *and* $(\mathcal{V}(t))_{t\geq 0}$ *have the same critical type* ω_{crit}. *Furthermore* (3.11) *holds in the strong sense and in* $\mathcal{L}(\mathcal{X})$, *for each* $\varepsilon > 0$, $\xi > \nu$ *and* $j \geq 4$.

Proof From $(\mathcal{A}2)$, we have for $\varepsilon > 0$, $j \geq 4$, $\mathrm{Re}\lambda \geq \nu$ and $\mathrm{Im}\lambda \neq 0$,

$$\begin{aligned}\|\vartheta_j^\varepsilon(\mathcal{T},\mathcal{B})(\lambda)\| &\leq \|\vartheta_0^\varepsilon(\mathcal{T},\mathcal{B})(\lambda)[\mathcal{B}\vartheta_0^\varepsilon(\mathcal{T},\mathcal{B})(\lambda)]^4\|\,\|[\mathcal{B}\vartheta_0^\varepsilon(\mathcal{T},\mathcal{B})(\lambda)]^{j-4}\|\\ &\leq \|\mathcal{B}\vartheta_0^\varepsilon(\mathcal{T},\mathcal{B})(\lambda)\mathcal{B}\|^2\,\|\vartheta_0^\varepsilon(\mathcal{T},\mathcal{B})(\lambda)\|^3\,\|[\mathcal{B}\vartheta_0^\varepsilon(\mathcal{T},\mathcal{B})(\lambda)]^{j-4}\|\\ &\leq \frac{C_\varepsilon^2}{Im^2\lambda}\,\|\mathcal{B}\|^{j-4}\,\|\vartheta_0^\varepsilon(\mathcal{T},\mathcal{B})(\lambda)\|^{j-1}.\end{aligned}$$

Since the operator $\mathcal{B}$ is assumed to be bounded, then by the fact that $\|\vartheta_0^\varepsilon(\mathcal{T},\mathcal{B})(\lambda)\|$ is uniformly bounded on $\mathcal{R}_\nu$ (apply (3.9) with $\omega' = \nu$ and $\omega = \frac{\omega(\mathcal{U})+\nu}{2}$), there exists $\tilde{C}_\varepsilon > 0$ verifying

$$\lim_{a\to+\infty}\sup_{\xi>\nu}\int_{|\tau|>a}\|\vartheta_j^\varepsilon(\mathcal{T},\mathcal{B})(\xi+i\tau)\|\,d\tau \leq \tilde{C}_\varepsilon\lim_{a\to+\infty}\int_{|\tau|>a}\frac{d\tau}{\tau^2} = 0.$$

Now, the result follows from the remark above and Theorem 3.1. □

3.2 Compactness of $\mathcal{U}_j(\cdot)$

For $j \geq 0$, let us consider the following hypothesis

$$(\mathcal{A}3(j))\begin{cases}\text{There exists } \nu > \max(\omega(\mathcal{U}),0) \text{ such that}\\ \text{for every (small) } \varepsilon > 0,\ \vartheta_j^\varepsilon(\mathcal{T},\mathcal{B})(\lambda) \text{ is compact for all } \lambda \in \mathcal{R}_\nu.\end{cases}$$

The second main result of this section is the following

Theorem 3.2 *Let $j \geq 0$ and assume that $(\mathcal{A}1(j))$ and $(\mathcal{A}3(j))$ hold true. Then, $\mathcal{U}_j(t)$ is compact for every $t > 0$.*

Proof From the hypothesis $(\mathcal{A}1(j))$ and according to Proposition 3.1 we get, in the strong sense and in $\mathcal{L}(\mathcal{X})$, for each $\varepsilon > 0$ and $\xi > \nu$,

$$\mathcal{U}_j^\varepsilon(t) = \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} \exp(\lambda t)\, \vartheta_j^\varepsilon(\mathcal{T}, \mathcal{B})(\lambda)\, d\lambda, \quad t > 0.$$

Now the use of the convex compactness property [24] gives that $\mathcal{U}_j^\varepsilon(t)$ is compact for every $t > 0$ and $\varepsilon > 0$. Now, the result follows from Lemma 3.2. □

Consequently, we get

Corollary 3.5 *Assume that there exists $j \geq 0$ such that the hypotheses $(\mathcal{A}1(j))$ and $(\mathcal{A}3(j))$ hold. Then, the semigroups $(\mathcal{U}(t))_{t\geq 0}$ and $(\mathcal{V}(t))_{t\geq 0}$ have the same essential type ω_{ess}.*

Now we state another consequence of Theorem 3.2. This result is very useful in applications.

Corollary 3.6 *Assume that the hypothesis $(\mathcal{A}2)$ holds and there exists $j \geq 4$ such that the hypothesis $(\mathcal{A}3(j))$ holds. Then the operator $\mathcal{U}_j(t)$ is compact for each $t > 0$ and therefore, the semigroups $(\mathcal{U}(t))_{t\geq 0}$ and $(\mathcal{V}(t))_{t\geq 0}$ have the same essential type ω_{ess}.*

4 Application to L^1-neutron transport theory

In this section, we consider the following concrete initial value problem governing the time evolution of the distribution of neutrons in nuclear reactor, namely the neutron transport equation (see [13,14] and the references therein)

$$\begin{cases} \dfrac{\partial \psi}{\partial t}(x, v, t) = -v\dfrac{\partial \psi}{\partial x}(x, v, t) - \sigma(v)\,\psi(x, v, t) + \displaystyle\int_V \kappa(x, v, v')\,\psi(x, v', t)\, d\mu(v') \\ \qquad\qquad := T\psi(x, v, t) + K\psi(x, v, t) := A\psi(x, v, t) \\ \psi_{|_{\Gamma^-}} = 0 \ \text{ and } \psi(x, v, 0) = \psi_0(x, v) \end{cases} \tag{4.1}$$

where $(x, v) \in \Omega \times V$ with Ω and V are smooth open subsets of $\mathbb{R}^N$ $(N \geq 1)$, Ω and V denote respectively the space of positions and the space of velocities, dx and $d\mu$ denote respectively the Lebesgue measure on $\mathbb{R}^N$ and a positive Radon measure on $\mathbb{R}^N$ with support V. The function $\psi(x, v)$ represents the number (or probability) density of particles having the position x and the velocity v. The functions $\sigma(\cdot)$ and $\kappa(\cdot, \cdot, \cdot)$ are called, respectively, the *collision frequency* and *the scattering kernel*, and,

$$\Gamma^- = \{(x, v) \in \partial\Omega \times V : \ v \text{ is incomming at } x \in \partial\Omega\}.$$

The operators A, T and K the integral part of A, are called respectively the *transport operator*, the *streaming operator* and the *collision operator*.

Let us consider the Banach space

$$X = L^1(\Omega \times V, dx \otimes d\mu(v)).$$

We assume that

$$\sigma(\cdot) \in L^\infty(V),\ \ K \in \mathcal{L}(X) \text{ and } \Omega \text{ is bounded and convex.}$$

From [25], the unbounded operator

$$\begin{cases} T\psi(x, v) = -v\dfrac{\partial \psi}{\partial x}(x, v) - \sigma(v)\,\psi(x, v), \quad \psi \in D(T) \\ D(T) = \{\psi \in X : \dfrac{\partial \psi}{\partial x} \in X \text{ and } \psi_{|_{\Gamma^-}} = 0\} \end{cases} \tag{4.2}$$

generates the following explicit positive C_0-semigroup

$$U(t)\psi(x, v) = \begin{cases} e^{-\sigma(v)t}\psi(x - vt, v) & \text{if } t < \tau(x, v) \\ 0 & \text{otherwise,} \end{cases}$$

where $\tau(x, v) = \inf\{t > 0 : \ x - tv \notin \Omega\}$. The type of $(U(t))_{t\geq 0}$ is given [16, Formulae (2.4), p. 10] by

$$\omega(T) = \begin{cases} -\infty & \text{if } 0 \notin \overline{V} \\ -\liminf\limits_{V \ni v \to 0} \sigma(v) & \text{if } 0 \in \overline{V}. \end{cases}$$

Let us write the evolution problem (4.1) as an abstract Cauchy problem:

$$\frac{\partial \varphi}{\partial t} = (T + K)\varphi, \quad \varphi(0) = \varphi_0 \in X,$$

where the streaming operator T is the closed unbounded operator with dense domain defined by (4.2), and the collision operator K is the bounded integral operator having the form

$$X \ni \psi \mapsto K\varphi := \int_{\mathbb{V}} \kappa(x, v, v')\ \psi(x, v')\, d\mu(v') \in X. \tag{4.3}$$

Note that from the classical perturbation theory, the perturbed (transport) operator $A := T + K$ generates also a strongly continuous semigroup $(V(t))_{t\geq 0}$ in X given by the Dyson–Phillips expansion

$$V(t) = \sum_{j=0}^{m-1} U_j(t) + R_m(t), \tag{4.4}$$

where

$$U_0(t) = U(t), \quad U_j(t) = \int_0^t U(t-s) K U_{j-1}(s)\, ds, \quad (j \geq 1)$$

and

$$R_m(t) = \sum_{j=m}^{+\infty} U_j(t), \quad (m \geq 1).$$

Following [13,14], we adopt the following definition

Definition 4.1 A collision operator $K \in \mathcal{L}(X)$ is said to be *regular* if it belongs to the closure in the operator norm topology of the class of collision operators with kernels in the form

$$\kappa(x, v, v') = \sum_{i \in I} \alpha_i(x) f_i(v) g_i(v'),$$

with $\alpha_i(.) \in L^\infty(\Omega)$, $f_i(.) \in L^1(V, d\mu)$, $g_i(.) \in L^\infty(V, d\mu)$, $i \in I$, where I is finite.

For every $(x, s) \in \Omega \times (0, +\infty)$, we define the map

$$\begin{aligned} f_{x,s} : V &\longrightarrow \Omega \\ v &\longmapsto (x - sv)\, \chi_{[0, \tau(x,v)]}(s). \end{aligned}$$

Let us assume as in [10] that the Radon measure $d\mu$ on $\mathbb{R}^N$ satisfies the following geometrical properties,

$$\int_{\alpha_1 \leq |v| \leq \alpha_2} d\mu(v) \int_0^{\alpha_3} \chi_A(tv)\, dt \to 0 \text{ as } |A| \to 0 \tag{4.5}$$

for every $\alpha_1 < \alpha_2 < +\infty$ and $\alpha_3 < +\infty$, where $|A|$ is the Lebesgue measure of A and χ_A denotes the characteristic function of A.

Let us assume that there exists a positive measure $d\nu$ on $\mathbb{R}^N$ satisfying

$$\begin{cases} f_{x,s}(d\mu) = (1/s^N)\, d\nu \ \text{ for every } x \in \Omega \text{ and } s \in (0, +\infty), \ \text{ and} \\ \text{the canonical injection } j : L^1(\Omega, dx) \longrightarrow L^1(\Omega, d\nu) \text{ is continuous.} \end{cases} \tag{4.6}$$

Here, $f_{x,s}(d\mu)$ denotes the image of the measure μ on $\mathbb{R}^N$ by the map $f_{x,s}$.

Remark 4.1 The set of Radon measures μ on $\mathbb{R}^N$ satisfying condition (4.6) is non-empty, in fact, it contains the usual Lebesgue measure dv on $\mathbb{R}^N$.

For any Borelian subset $V_0 \subset V$, we consider the following geometrical property.

Definition 4.2 Let $n_0 \in \mathbb{N}^*$. A Borelian subset V_0 of V is said to be $(\Omega - \Omega)$-finite of order n_0, if it has a finite $\mu-$measure and for each $x \in (\Omega - \Omega) := \{x - y : x, y \in \Omega\}$, the set

$$\Delta_x(V_0) := \{t \in (0, +\infty) \text{ such that } (x/t) \in V_0\}$$

is a union of at most n_0 intervals.

Let us assume the following geometrical assumption on the velocity space.

$$\begin{cases} \text{The velocity space } V \text{ can be written} \\ \text{as a partition of } (\Omega - \Omega)\text{-finite subsets of order } n_0. \end{cases} \tag{4.7}$$

Remark 4.2 If $d\mu$ denotes the Lebesgue measure on $\mathbb{R}^N$, then both of the subsets $B(a, r) := \{v - a \in \mathbb{R}^N \text{ such that } \|v\| \leq r\}$, $r > 0$, $a \in \mathbb{R}^N$, and $C_{r_1,r_2} := \{v \in \mathbb{R}^N \text{ such that } r_1 \leq \|v\| \leq r_2\}$, $0 \leq r_1 \leq \|v\| \leq r_2$, are $(\Omega - \Omega)$-finite subsets of order one [and hence each of them satisfies the geometrical assumption (4.7)]. Therefore, since $\cup_{n\in\mathbb{N}} C_{n,n+1} = \mathbb{R}^N$, this velocity space satisfies also the geometrical assumption (4.7).

The following theorem is the main result of this section.

Theorem 4.1 *Assume that Ω is bounded, $0 \leq \sigma(\cdot)$ and the Radon measure $d\mu$ satisfies conditions* (4.5) *and* (4.6). *If the collision operator is regular and the velocity space satisfies the assumption* (4.7)*, then for every $m \geq 4$, the operator $U_m(t)$ is compact for each $t \geq 0$ and therefore, the semigroups $(U(t))_{t\geq 0}$ and $(V(t))_{t\geq 0}$ have the same critical type ω_{crit} and the same essential type ω_{ess}.*

Remark 4.3 The result of Theorem 4.1 completes the results of [14, Theorem 10 and Corollary 11] and that of [12, Theorem 2.1]. Furthermore, the result of Theorem 4.1 can be generalized directly to L^p-spaces, $1 \leq p < +\infty$, and thus our result improves Sbihi and Mokhtar-Kharroubi's results.

For every $\varepsilon > 0$, small, we consider the operator [see (3.6)]

$$\vartheta_0^\varepsilon(T, K)(\lambda), \ \text{Re}\lambda > \omega(U).$$

To prove Theorem 4.1, we need the following

Proposition 4.1 [15, Theorem 4.4] *Assume that Ω is bounded and the collision operator K is regular. If the Radon measure $d\mu$ satisfies (4.5), then*

$$K\vartheta_0^\varepsilon(T, K)(\lambda)K \textit{ is weakly compact on } X.$$

Remark 4.4 Let K be a collision operator given by (4.3). It can be written in the form

$$K = [ReK^{+} - ReK^{-}] + i[ImK^{+} - ImK^{-}],$$

where $ReK^{\pm}$ and $ImK^{\pm}$ denote, respectively, the collision operators with kernels

$$Re\,\kappa^{\pm}(x, v, v') = \max\{\pm Re\,\kappa(x, v, v'), 0\}$$

and

$$Im\,\kappa^{\pm}(x, v, v') = \max\{\pm Im\,\kappa(x, v, v'), 0\}.$$

Therefore, since the collision operator K is assumed to be regular and since $V(t)-U(t)$ depends linearly and continuously on K, we can assume that K is positive and regular. Now, since the semigroup $(U^{H}(t))_{t\geq 0}$ is positive, similar reasoning as in [14, p. 1240] using domination arguments, by linearity and according to Condition (4.7), we can assume that K is a one rank collision operator with kernel in the form

$$\kappa(x, v, v') = \chi_{V_0}(v),$$

where V_0 is a Borelian $(\Omega - \Omega)$-finite subset of V having the order n_0, i.e., $K = K_0$, where

$$K_0 : X_p \ni \psi \longrightarrow K\psi(x, v) = \chi_{V_0}(v) \int_V \psi(x, v')\, d\mu(v') \in X_p. \tag{4.8}$$

For every $\omega > \omega(U)$, we consider the half-plane of $\mathbb{C}$:

$$\mathcal{R}_{\omega} := \{\lambda \in \mathbb{C} :\ \mathrm{Re}\lambda \geq \omega\}.$$

We are going to prove the following auxiliary result.

Proposition 4.2 *Assume that* $\sigma(\cdot) = \underline{\sigma} \in \mathbb{R}$ *and let* $\varepsilon > 0$ *and* V_0 *be a Borelian* $(\Omega - \Omega)$*-finite subset of* V *having the order* n_0*. If the Radon measure* $d\mu$ *satisfies Condition* (4.6)*, then for every* $\omega > \omega(U)$*, there exists* $C_\varepsilon > 0$ *such that*

$$|\lambda + \underline{\sigma}|\, \|K_0 \vartheta_0^{\varepsilon}(T, K)(\lambda) K_0\| \leq C_\varepsilon \textit{ for every } \lambda \in \mathcal{R}_{\omega},$$

and consequently,

$$|Im\,\lambda|\, \|K_0 \vartheta_0^{\varepsilon}(T, K)(\lambda) K_0\| \leq C_\varepsilon \textit{ for every } \lambda \in \mathcal{R}_{\omega}.$$

Before proving this result, we need the following

Lemma 4.1 *Assume that* $\sigma(\cdot) = \underline{\sigma} \in \mathbb{R}$ *and let* $\varepsilon > 0$ *and* V_0 *be a Borelian* $(\Omega - \Omega)$*-finite subset of* V *having the order* n_0*. Then, for every* $\omega > \omega(U)$*, there exists* $C_{\varepsilon, n_0, 1} > 0$ *such that*

$$|\lambda+\underline{\sigma}|\left|\int_{\varepsilon}^{+\infty} e^{-(\lambda+\underline{\sigma})s}\,\chi_{V_0}\left(\frac{x-x'}{s}\right)\frac{ds}{s^N}\right| \leq C_{\varepsilon,n_0,1}$$

uniformly on $x, x' \in \Omega$ and $\lambda \in \mathcal{R}_\omega$.

Proof Since V_0 is a Borelian $(\Omega-\Omega)$-finite subset of V having the order n_0, for every $x, x' \in \Omega$, there exists $m_0(x-x') < n_0$ satisfying

$$\int_{\varepsilon}^{+\infty} e^{-(\lambda+\underline{\sigma})s}\,\chi_{V_0}\left(\frac{x-x'}{s}\right)\frac{ds}{s^N} = \sum_{j=1}^{m_0(x-x')}\int_{a_j(x-x')}^{b_j(x-x')} e^{-(\lambda+\underline{\sigma})s}\,\chi_{V_0}\left(\frac{x-x'}{s}\right)\frac{ds}{s^N},$$

where the (a_j, b_j) are bounded intervals included in $(\varepsilon, +\infty)$. Simple integrations by part to the integrals

$$\int_{a_j(x-x')}^{b_j(x-x')} e^{-(\lambda+\underline{\sigma})s}\,\chi_{V_0}\left(\frac{x-x'}{s}\right)\frac{ds}{s^N},\ 1 \leq j \leq m_0,$$

yields

$$\begin{aligned}\int_{\varepsilon}^{+\infty} e^{-(\lambda+\underline{\sigma})s}\,\chi_{V_0}\left(\frac{x-x'}{s}\right)\frac{ds}{s^N} &= \sum_{j=1}^{m_0(x-x')}\left[-\frac{1}{\lambda+\underline{\sigma}}\,e^{-(\lambda+\underline{\sigma})s}\,\frac{1}{s^N}\right]_{a_j(x-x')}^{b_j(x-x')}\\ &\quad-\sum_{j=1}^{m_0(x-x')}\int_{a_j(x-x')}^{b_j(x-x')}\frac{1}{\lambda+\underline{\sigma}}\,e^{-(\lambda+\underline{\sigma})s}\,\frac{N ds}{s^{N+1}}.\end{aligned}$$

Consequently, by passing to the module, we get

$$\begin{aligned}|\lambda+\underline{\sigma}|\left|\int_{\varepsilon}^{+\infty} e^{-(\lambda+\underline{\sigma})s}\,\chi_{V_0}\left(\frac{x-x'}{s}\right)\frac{ds}{s^N}\right| &\leq \frac{2m_0(x-x')}{\varepsilon}+\int_{\varepsilon}^{+\infty}\frac{N ds}{s^{N+1}}\\ &\leq \frac{2n_0+1}{\varepsilon^N} := C_{\varepsilon,n_0,1},\end{aligned}$$

uniformly on $x, x' \in \Omega$ and $\lambda \in \mathcal{R}_\omega$. This ends the proof. □

Proof of Proposition 4.2 Note that according to [16, p. 12], the operator $K_0\vartheta_0^\varepsilon(T, K_0)(\lambda)K_0$ is an integral operator from $L^1(\Omega\times V, d\nu(x)\otimes d\mu(v))$ onto X, whose kernel is

$$\chi_{V_0}(v)\,\chi_{\left\{|x-x'|\leq\tilde{s}\left(x,\frac{x-x'}{|x-x'|}\right)\right\}}\int_{\varepsilon}^{+\infty} e^{-(\lambda+\underline{\sigma})s}\,\chi_{V_0}\left(\frac{x-x'}{s}\right)\frac{ds}{s^N},$$

where $\tilde{s}(x, v) = \inf\{s > 0 : x - sv \notin \Omega\}$. Therefore, the result follows from Lemma 4.1 on the basis of Assumption (4.6). □

Finally, we are ready to give the

Proof of Theorem4.1 Since $0 \leq \sigma(\cdot)$, the semigroup $(U(t))_{t\geq 0}$ is positive and the collision operator K can be chosen to be the positive collision operator K_0 given by (4.8) (see, Remark 4.4). By using comparison arguments, we need only to prove the result for $\sigma(\cdot) = 0$. Now, the result follows from Propositions 4.1 and 4.2 by applying Corollary 3.6 of Theorem 3.2. □

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