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RESEARCH ARTICLE

Directed graphs of inner translations of semigroups

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Abstract A mapping $\alpha: S \to S$ is called a *Cayley function* if there exist an associative operation $\mu: S \times S \to S$ and an element $a \in S$ such that $\alpha(x) = \mu(a, x)$ for every $x \in S$. The aim of the paper is to give a characterization of Cayley functions in terms of their directed graphs. This characterization is used to determine which elements of the centralizer of a permutation on a finite set are Cayley functions. The paper ends with a number of problems.

Keywords Inner translations · Cayley functions · Functional digraphs

1 Introduction

Let S be a set equipped with a binary operation, say:

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$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	0
3	0	3	0	3

Both the rows and columns of this table can be viewed as full transformations on S. For example, the row of 1 induces the map λ_2 and the column of 2 induces the map ρ_1 as follows:

$$\rho_1 = \begin{pmatrix} 0\,1\,2\,3 \\ 0\,1\,0\,1 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0\,1\,2\,3 \\ 0\,0\,2\,0 \end{pmatrix}.$$

Observe that $\lambda_1\rho_2 = \rho_2\lambda_1$. In fact, one may check that in the Cayley table above, every row commutes with every column. Saying that the rows of $(S, \cdot)$ commute with its columns is just another way of saying that $(S, \cdot)$ is a semigroup (see [5]). This simple observation prompts an approach to a study of semigroups. However, before outlining the approach, we introduce some terminology and notation, and recall some facts.

Let S be a semigroup. For a fixed $a \in S$, the mapping $\lambda_a : S \to S[\rho_a : S \to S]$ defined by $\lambda_a(x) = ax[\rho_a(x) = xa]$ is called a left [right] *inner translation* of S. If S is a finite group, then λ_a is a regular permutation on S, that is, λ_a is a product of disjoint cycles of the same length. If S is an infinite group, then λ_a is a formal product of disjoint cycles of the same (possibly infinite) length (see [21, Definition 3.2] and [24, Definition 1.1]). The converse is also true, that is, if α is a regular permutation on a set S, then there is a group with universe S such that α is a left inner translation of the group S [28]. The same facts are true for right inner translations.

Let α be a transformation on a set S. Following [30], we say that α is a *Cayley function* on S if there is a semigroup with universe S such that α is an inner translation of the semigroup S. Note that α is a left inner translation of a semigroup $(S, \cdot)$ if and only if α is a right inner translation of the semigroup $(S, *)$, where for all $a, b \in S$, $a*b = b\cdot a$.

We now describe an approach to a study of semigroups prompted by the observation that in any semigroup, the mappings induced by the rows of the Cayley table (left inner translations) commute with the mappings induced by the columns (right inner translations). Let S be any set.

1. Find the Cayley functions on S.
2. Given a Cayley function α on S, find all transformations on S that commute with α, that is, describe the centralizer of α in the full transformation monoid $T(S)$ on S.
3. Given a Cayley function α on S, find all Cayley functions on S that commute with α.
4. Find pairs $\{\alpha, \beta\}$ of Cayley functions on S such that α and β occur as left inner translations of the same semigroup $(S, \cdot)$.
5. Let G_S be the simple graph whose vertices are the Cayley functions on S and the edges are pairs $\{\alpha, \beta\}$ such that α and β occur as left inner translations of the same semigroup $(S, \cdot)$. We will call the graph G_S the *common semigroup graph* of S.

Identify the maximal cliques of the common semigroup graph G_S.

6. Let H_S be the simple graph whose vertices are the Cayley functions on S and the edges are pairs $\{\alpha, \beta\}$ such that:
   - $\{\alpha, \beta\}$ is an edge in the common semigroup graph G_S, and
   - there exists a clique C of size $|S|$ in G_S such that α and β commute with every element of C.

   Identify the maximal cliques of the graph H_S.

The solution of problem 1 tells us which mappings can appear as rows or columns in the Cayley table of a semigroup. The solution of problem 2 gives us the candidates for the rows of the Cayley table of a semigroup provided a given mapping appears as a column. Problems 3 and 4 are steps toward solving problems 5 and 6.

Regarding problem 5, suppose that $(S, \cdot)$ is a semigroup. The left inner translations λ_a of S, where $a \in S$, form a clique in the common semigroup graph G_S. Hence, the solution of problem 5 would give us the candidates (the vertices of a maximal clique of G_S) from which the rows of the Cayley table of a semigroup can be selected. The same analysis applies to columns.

Regarding problem 6, suppose that we have the Cayley table of a semigroup $(S, \cdot)$. Then the rows of the table form a clique, say C_1, in the common semigroup graph G_S, and the columns also form a clique, say C_2, in G_S. Moreover, $|C_1| = |C_2| = |S|$ and each Cayley function in C_1 commutes with each Cayley function in C_2. Therefore, the rows [columns] of the Cayley table of $(S, \cdot)$ are contained in a maximal clique of the graph H_S. Therefore, the solution of problem 6 would give us the candidates (the vertices of a pair of maximal cliques in H_S) from which the Cayley table of a semigroup can be constructed. In other words, it would provide us with a tool to devise a method for *building* the Cayley tables of semigroups. In the process we might gain a deeper understanding of transformations and the Cayley tables of semigroups.

The current status of the solutions of these problems is as follows. Algebraic descriptions of transformations α that are Cayley functions (in terms of properties of powers of α) have been provided in [8,9,11,30]. The problem of describing the centralizer of a given transformation took longer, but it has been fully solved; the final stage was [6], but this is just the last step in a long process [1–4,14–23,25–27] (not claiming exhaustiveness). Given that problems 1 and 2 have been solved, one might think that problem 3 follows straightforwardly. This is not the case, though. One of the reasons is that while the description of the centralizers is geometric in nature, the available descriptions of the Cayley functions are not, thus making it difficult to combine the two results.

Hence, this paper has two goals. The first is to provide a geometric characterization of the Cayley functions, which can be connected to the existing geometric descriptions of centralizers. The second is to describe the Cayley functions that commute with a finite permutation. That is, we solve problem 3 in a special case. In other words, we describe the candidates for the rows of the Cayley table of a finite semigroup when one of its columns is a permutation. This is the part of the paper containing the most delicate considerations.

Any $\alpha : S \to S$ can be represented by a directed graph D_α with S as the set of vertices such that $x \to y$ is an arc in D_α if and only if $\alpha(x) = y$. A directed graph representing some transformation on its set of vertices is called a *functional digraph*. We will characterize the functional digraphs that represent Cayley functions. As a result, we will obtain a visual criterion for a transformation α on a set S to be a Cayley function: look at the digraph of α and see if it has desired properties.

To obtain a complete characterization of digraphs that represent Cayley functions, we first need to describe functional digraphs. We provide such a description in Sect. 2 (Propositions 2.5 and 2.6). Transformations on a set S can be divided into two types: those that have the so called stabilizer and those that do not. In Sect. 3, we characterize the functional digraphs that represent transformations with stabilizers (Theorem 3.10). In Sect. 4, we characterize the digraphs that represent Cayley functions (Theorems 4.6 and 4.10). Finally, in Sect. 5, we apply our characterization to centralizers of finite permutations (Theorem 5.6). We illustrate our results with examples and figures. We conclude the paper with a list of open problems (Sect. 6).

For the remainder of the paper, we fix a non-empty set S and denote by $T(S)$ the set of all transformations on S (mappings $\alpha\colon S \to S$).

2 Functional digraphs

A *directed graph* (or a *digraph*) is a pair $D = (S, \rho)$ where S is a non-empty set of vertices (not necessarily finite), which we denote by $V(D)$, and ρ is a binary relation on S. Any pair $(x, y) \in \rho$ is called an *arc* of D, which we will write as $x \to y$. A vertex x is called an *initial vertex* in D if there is no $y \in S$ such that $y \to x$; it is called a *terminal vertex* in D if there is no $y \in S$ such that $x \to y$.

A digraph D is called a *functional digraph* if there is $\alpha \in T(S)$ such that for all $x, y \in S$, $x \to y$ is an arc in D if and only if $\alpha(x) = y$. If such an α exists, then it is unique, and we will write $D = D_\alpha$ and refer to D as the digraph that represents α. In this section, we describe functional digraphs.

Let D be a digraph and let $\dots, x_{-1}, x_0, x_1, \dots$ be pairwise distinct vertices of D. Consider the following sub-digraphs of D:

$$x_0 \to x_1 \to \cdots \to x_{k-1} \to x_0 \tag{2.1}$$

$$x_0 \to x_1 \to \cdots \to x_m \tag{2.2}$$

$$x_0 \to x_1 \to x_2 \to \cdots \tag{2.3}$$

$$\cdots \to x_2 \to x_1 \to x_0 \tag{2.4}$$

$$\cdots \to x_{-1} \to x_0 \to x_1 \to \cdots \tag{2.5}$$

We call (2.1)–(2.5), respectively: a *cycle* of length $k (k \geq 1)$, written $(x_0\, x_1 \dots x_{k-1})$; a *chain* of length m, written $[x_0\, x_1 \dots x_m] (m \geq 0)$; a *right ray*, written $[x_0\, x_1\, x_2 \dots\rangle$; a *left ray*, written $\langle \dots x_2\, x_1\, x_0]$; and a *double ray*, written $\langle \dots x_{-1}\, x_0\, x_1 \dots\rangle$.

Definition 2.1 Let D_α be a functional digraph, where $\alpha \in T(S)$. A right ray $[x_0\, x_1\, x_2 \dots\rangle$ in D_α is called a *maximal right ray* if x_0 is an initial vertex of D_α.

Definition 2.2 Let D_α be a functional digraph, where $\alpha \in T(S)$.

- A left ray $L = \langle \dots x_2\, x_1\, x_0]$ in D_α is called an *infinite branch* of a cycle C [double ray W] in D_α if x_0 lies on $C[W]$ and x_1 does not lie on $C[W]$. We will refer to any such L as an infinite branch in D_α.
- A chain $P = [x_0\, x_1 \dots x_m]$ of length $m \geq 1$ in D_α is called a *finite branch* of a cycle C [double ray W, maximal right ray R, infinite branch L] in D_α if x_0 is an initial vertex of D_α, x_m lies on $C[W, R, L]$ and x_{m-1} does not lie on $C[W, R, L]$. If x_m lies on an infinite branch $L = \langle \dots y_2\, y_1\, y_0]$, we also require that $x_m \neq y_0$. We will refer to any such P as a finite branch in D_α.

By a *branch* in D_α we will mean a finite or infinite branch in D_α. Note that all branches of a maximal right ray R or an infinite branch L are finite. In other words, we only consider infinite branches of cycles and double rays.

Definition 2.3 Let $\alpha \in T(S)$, $x \in S$. The subgraph of D_α induced by the set

$$\{y \in S : \alpha^k(y) = \alpha^m(x) \text{ for some integers } k, m \geq 0\}$$

is called the *component* of D_α containing x. The components of D_α correspond to the connected components of the underlying undirected graph of D_α.

Definition 2.4 Let $\{D_i\}_{i \in I}$ be a collection of digraphs $D_i = (S_i, \rho_i)$. By the *join* of the digraphs D_i, denoted $\bigsqcup_{i \in I} D_i$, we mean the digraph $D = (S, \rho)$ such that $S = \bigcup_{i \in I} S_i$ and $\rho = \bigcup_{i \in I} \rho_i$. (That is, $x \to y$ is an arc in the join D if and only if $x \to y$ is an arc in some D_i.) If the index set I is finite, say $I = \{1, 2, \dots, m\}$, we will write $D_1 \sqcup D_2 \sqcup \dots \sqcup D_m$ for $\bigsqcup_{i \in I} D_i$.

The following two propositions, proved in [6, Propositions 2.10 and 2.13], describe the functional digraphs. (The first characterization of functional digraphs is due to F. Harary [10, Theorem 2].)

Proposition 2.5 *Let D_α be a functional digraph. Then for every component A of D_α, exactly one of the following three conditions holds:*

(a) *A has a unique cycle but not a double ray or right ray;*
(b) *A has a double ray but not a cycle; or*
(c) *A has a maximal right ray but not a cycle or double ray.*

Proposition 2.6 *Let D_α be a functional digraph. Then for every component A of D_α:*

(1) *if A has a (unique) cycle C, then A is the join of C and its branches;*
(2) *if A has a double ray W, then A is the join of W and its branches;*
(3) *if A has a maximal right ray R but not a double ray, then A is the join of R and its (finite) branches.*

Suppose that a component A of D_α has a right ray R but not a double ray. It is then clear by Proposition 2.6 that A is the join of its maximal right rays. We will say that such a component A is of type *rro* ("right rays only").

Figure 1 presents a component of a digraph that contains a cycle (necessarily unique). The cycle has two infinite and three finite branches. The infinite branch

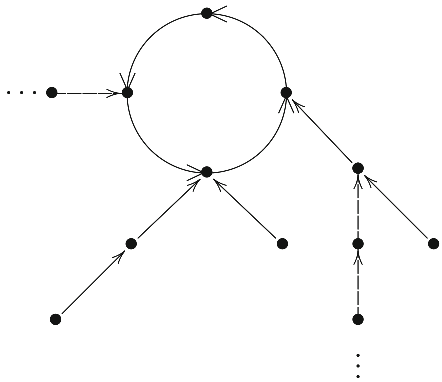

Fig. 1 A functional digraph component with a cycle

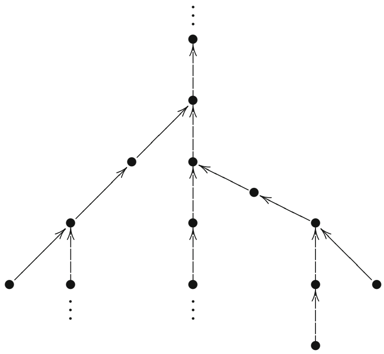

Fig. 2 A functional digraph component with a double ray

on the right has one finite branch. Figure 2 presents a component of a digraph that contains a double ray. The double ray in the middle has one infinite and three finite branches. The infinite branch has one finite branch and extends to the second double ray of the component. Figure 3 presents a component of a digraph that contains a maximal right ray but not a double ray (type *rro*). The maximal right ray in the middle has five (necessarily) finite branches.

3 Transformations with stabilizers

In this section, we describe the functional digraphs representing transformations that have the so called stabilizer.

Let $\alpha \in T(S)$. We denote by $\mathrm{im}(\alpha)$ the image of α. For an integer $n > 0$, we denote by α^n the nth power of α, that is, the composition of α with itself n times. As usual, α^0 denotes the identity transformation id_S on S.

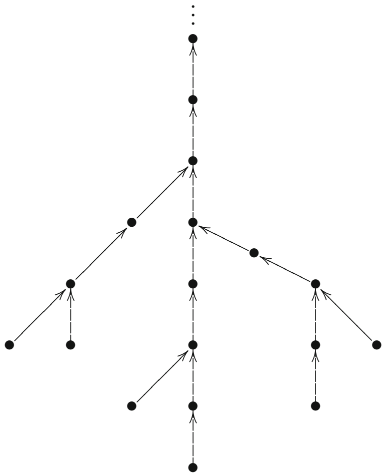

Fig. 3 A functional digraph component of type *rro*

Definition 3.1 Let $\alpha \in T(S)$. The *stable image* of α, denote $\mathrm{sim}(\alpha)$, is a subset of S defined by

$$\mathrm{sim}(\alpha) = \{x \in S : x \in \mathrm{im}(\alpha^n) \text{ for every } n \geq 0.\}.$$

(See [12, p. 42], where $\mathrm{sim}(\alpha)$ is called the stable range of α.)

Remark 3.2 If $\alpha \in T(S)$, then:

- $\mathrm{sim}(\alpha)$ consists of the vertices of D_α that lie on cycles, double rays, or infinite branches;
- $\mathrm{sim}(\alpha) = \emptyset$ if and only if each component of D_α is of type *rro*.

Definition 3.3 Following [30], we define the *stabilizer* of $\alpha \in T(S)$ as the smallest integer $s \geq 0$ such that $\mathrm{im}(\alpha^s) = \mathrm{im}(\alpha^{s+1})$. If such an s does not exist, we say that α has no stabilizer.

Remark 3.4 If $\alpha \in T(S)$, then:

- the stabilizer of α is the smallest integer $s \geq 0$ such that $\alpha^s(x) \in \mathrm{sim}(\alpha)$ for every $x \in S$;
- α has the stabilizer $s = 0$ if and only if $\mathrm{im}(\alpha) = \mathrm{sim}(\alpha) = S$, which happens if and only if each component A of D_α is either the join of a cycle C and the infinite branches of C or the join of a double ray W and the infinite branches of W;
- if α has the stabilizer s, then $\mathrm{sim}(\alpha) = \mathrm{im}(\alpha^s)$.

The transformations represented by the digraphs in Figs. 1 and 2 have stabilizers 2 and 4, respectively. If α is the transformation represented by the digraph in Fig. 3, then $\mathrm{sim}(\alpha) = \emptyset$ and the stabilizer of α does not exist.

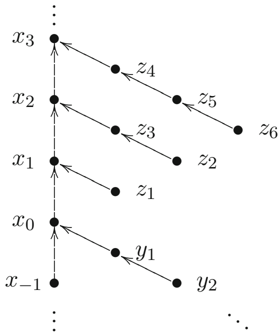

Fig. 4 The digraph of α from Example 3.5

Example 3.5 A transformation may have a non-empty stable image and no stabilizer. Let $S = \{\ldots, x_{-1}, x_0, x_1, \ldots\} \cup \{y_1, y_2, \ldots\} \cup \{z_1, z_2, \ldots\}$. Consider $\alpha \in T(S)$ represented by the digraph in Fig. 4. Then $\mathrm{sim}(\alpha) = \{\ldots, x_{-1}, x_0, x_1, \ldots\} \cup \{y_1, y_2, \ldots\}$ but α has no stabilizer because of the increasing lengths of the finite branches of the double ray $\langle \ldots\, x_{-1}\, x_0\, x_1 \ldots \rangle$.

For the rest of this section, our goal is to characterize the digraphs of transformations with stabilizers.

Lemma 3.6 *Let $\alpha \in T(S)$ such that α has the stabilizer s. Suppose $[x_0\, x_1 \ldots x_s]$ is a chain in D_α such that x_s does not lie on a cycle. Then α has a left ray $\langle \ldots\, y_2\, y_1\, x_s]$.*

Proof We will construct a sequence $y_0, y_1, y_2, \ldots$ of elements of S such that $y_0 = x_s$ and for every $n \geq 0$,

(a) $[y_n \ldots y_0]$ is a chain in D_α, and
(b) $y_n \in \mathrm{sim}(\alpha)$.

Let $y_0 = x_s$. Then $[y_0]$ is a chain in D_α and $y_0 = x_s = \alpha^s(x_0) \in \mathrm{sim}(\alpha)$ (see Remark 3.4). Let $n \geq 0$ and suppose we have constructed vertices $y_0 = x_s, y_1, \ldots, y_n$ that satisfy (a) and (b). Since $\mathrm{sim}(\alpha) = \mathrm{im}(\alpha^s) = \mathrm{im}(\alpha^{s+1})$, we have $y_n \in \mathrm{im}(\alpha^{s+1})$. Thus, there are $z_0, \ldots, z_s, z_{s+1}$ in S such that

$$z_0 \to \cdots \to z_s \to z_{s+1} = y_n.$$

Let $y_{n+1} = z_s$. Then

$$y_{n+1} = z_s \to z_{s+1} = y_n \to y_{n-1} \to \cdots \to y_0.$$

Note that $y_{n+1} \notin \{y_n, \ldots, y_0\}$ since otherwise $x_s = y_0$ would lie on a cycle. Thus $[y_{n+1}\, y_n \ldots y_0]$ is a chain in D_α and $y_{n+1} = z_s = \alpha^s(z_0) \in \mathrm{sim}(\alpha)$.

The sequence $y_0 = x_s, y_1, y_2, \ldots$ that satisfies (a) and (b) for every $n \geq 0$ has been constructed. But then $\langle \ldots\, y_2\, y_1\, x_s]$ is a left ray in D_α. $\square$

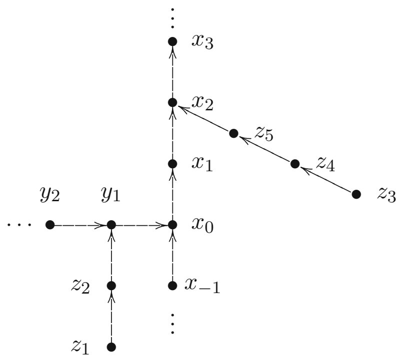

Fig. 5 The digraph of α from Example 3.9

The following lemma states that the digraph of a transformations with stabilizer cannot have a component of type *rro* (see the paragraph after Proposition 2.6 and Figure 3).

Lemma 3.7 *Let $\alpha \in T(S)$ with stabilizer s. Then for every component A of D_α, A has a (unique) cycle or a double ray.*

Proof Suppose that a component A of D_α does not have a cycle. Select any $x_0 \in A$. Since A has no cycle, $[x_0\, x_1\, x_2 \ldots\rangle$, where $x_1 = \alpha(x_0), x_2 = \alpha(x_1), \ldots$, is a right ray in A. By Lemma 3.6, A has a left ray $\langle \ldots\, y_2\, y_1\, x_s]$. For every $n \geq 0$, $y_n \notin \{x_{s+1}, x_{s+2}, \ldots\}$ since otherwise A would have a cycle. Hence $\langle \ldots\, y_2\, y_1\, x_s\, x_{s+1}\, x_{s+2} \ldots\rangle$ is a double ray in A. □

The converse of Lemma 3.7 is not true, that is, not every $\alpha \in T(S)$ such that every component of D_α has a cycle or a double ray has the stabilizer (see Example 3.5).

Definition 3.8 Let $\alpha \in T(S)$. A finite branch $[x_0\, x_1 \ldots x_m]$ in D_α is called a *twig* in D_α if $x_m \in \mathrm{sim}(\alpha)$ (that is, x_m lies on a cycle, double ray, or infinite branch) and $x_p \notin \mathrm{sim}(\alpha)$ for every $p \in \{0, \ldots, m-1\}$.

Example 3.9 Not every finite branch is a twig. Let $S = \{\ldots, x_{-1}, x_0, x_1, \ldots\} \cup \{y_1, y_2, \ldots\} \cup \{z_1, \ldots, z_5\}$, and consider $\alpha \in T(S)$ with the digraph in Figure 5. Then $[z_1\, z_2\, y_1\, x_0]$ is a branch in D_α but not a twig, while $[z_1\, z_2\, y_1]$ and $[z_3\, z_4\, z_5\, x_2]$ are twigs.

Theorem 3.10 *Let $\alpha \in T(S)$. Then α has the stabilizer if and only if:*

(1) *every component of D_α has a unique cycle or a double ray, and*
(2) *there is an integer $M \geq 0$ such that every twig in D_α has length $\leq M$.*

Proof Suppose α has the stabilizer, say s. Then (1) holds by Lemma 3.7. To prove (2), suppose to the contrary that such an integer M does not exist. Then there is a twig in D_α of length greater than s, say $[x_0 \ldots x_s \ldots x_m]$ with $m > s$. Since s is the stabilizer of α, we have $x_s = \alpha^s(x_0) \in \mathrm{sim}(\alpha)$ (see Remark 3.4), which contradicts the definition of a twig. We have proved (2).

Conversely, suppose that α satisfies (1) and (2). Let s be the smallest value of M from (2). We claim that $\mathrm{im}(\alpha^s) = \mathrm{im}(\alpha^{s+1})$. Let $z \in \mathrm{im}(\alpha^s)$. Suppose to the contrary that $z \notin \mathrm{sim}(\alpha^{s+1})$. Then, by (1), Proposition 2.6, and the fact that $z \in \mathrm{im}(\alpha^s)$, D_α has a finite branch $[x_0 \dots x_k \dots x_{k+s} = z \dots x_t]$, where $k \geq 0$, $t \geq k+s$, and x_t lies on a cycle or double ray. Consider the smallest $p \in \{0, \dots, t\}$ such that $x_p \in \mathrm{sim}(\alpha)$ and note that $p \leq k+s$. (Indeed, if $p > k+s$, then $[x_0 \dots x_{k+s} \dots x_p]$ would be a twig of length $p > s$, which is impossible.) But then $z = x_{k+s} = \alpha^{k+s-p}(x_p) \in \mathrm{sim}(\alpha)$, which is a contradiction. Thus $z \in \mathrm{sim}(\alpha)$, and so $z \in \mathrm{im}(\alpha^{s+1})$ by the definition of $\mathrm{sim}(\alpha)$. We proved that $\mathrm{im}(\alpha^s) \subseteq \mathrm{im}(\alpha^{s+1})$. Since the reverse inclusion is obvious, it follows that $\mathrm{im}(\alpha^s) = \mathrm{im}(\alpha^{s+1})$.

We claim that s is the stabilizer of α. The claim is clearly true if $s = 0$. Suppose $s > 0$. Then D_α has a twig $[x_0 \dots x_s]$. Suppose to the contrary that there exists $p, 0 \leq p < s$ such that $\mathrm{im}(\alpha^p) = \mathrm{im}(\alpha^{p+1})$. Then $x_p = \alpha^p(x_0) \in \mathrm{im}(\alpha^p)$. But $\mathrm{im}(\alpha^p) = \mathrm{im}(\alpha^{p+1}) = \mathrm{im}(\alpha^{p+2}) = \cdots$, which implies that $x_p \in \mathrm{im}(\alpha^n)$ for every $n \geq 0$. Thus $x_p \in \mathrm{sim}(\alpha)$, which is a contradiction since $p < s$ and $[x_0 \dots x_s]$ is a twig. The claim has been proved, which concludes the proof of the theorem. □

It follows from Theorem 3.10 and its proof that if $\alpha \in T(S)$ has the stabilizer s, then

$$s = \text{ the smallest } M \geq 0 \text{ such that every twig in } D_\alpha \text{ has length } \leq M. \tag{3.1}$$

4 Cayley functions

Let S be a set. Recall that a transformation $\alpha \in T(S)$ is called a *Cayley function* if there is a binary operation $*$ on S such that $(S, *)$ is a semigroup and α is an inner translation of the semigroup S; that is, there exists $a \in S$ such that for every $x \in S$, $\alpha(x) = a * x$.

The purpose of this section is to characterize the digraphs of the Cayley functions. To do this, we will use the algebraic description of the Cayley functions given in [30].

Definition 4.1 Suppose $\alpha \in T(S)$ has the stabilizer s. If $s > 0$, we define the subset Ω_α of S by:

$$\Omega_\alpha = \{a \in S : \alpha^s(a) \in \mathrm{sim}(\alpha) \text{ but } \alpha^{s-1}(a) \notin \mathrm{sim}(\alpha)\}.$$

If $s = 0$, we define Ω_α to be S.

Remark 4.2 If $\alpha \in T(S)$ has the stabilizer $s > 0$, then Ω_α consists of the initial vertices of the twigs of length s in D_α.

For example, for α defined in Example 3.9, $\Omega_\alpha = \{z_3\}$. The following result is due to Zupnik [30, Theorems 1–3].

Theorem 4.3 *Let $\alpha \in T(S)$. Then α is a Cayley function if and only if exactly one of the following conditions holds:*

(a) *α has no stabilizer and there exists $a \in S$ such that $\alpha^n(a) \notin \mathrm{im}(\alpha^{n+1})$ for every $n \geq 0$;*
(b) *α has the stabilizer s such that $\alpha|_{\mathrm{im}(\alpha^s)}$ is one-to-one and there exists $a \in \Omega_\alpha$ such that $\alpha^m(a) = \alpha^n(a)$ implies $\alpha^m = \alpha^n$ for all $m, n \geq 0$; or*
(c) *α has the stabilizer s such that $\alpha|_{\mathrm{im}(\alpha^s)}$ is not one-to-one and there exists $a \in \Omega_\alpha$ such that:*
 (i) *$\alpha^m(a) = \alpha^n(a)$ implies $m = n$ for all $m, n \geq 0$; and*
 (ii) *For every $n \geq s$, there are pairwise distinct elements $y_1, y_2, \ldots$ of S such that $\alpha(y_1) = \alpha^n(a), \alpha(y_k) = y_{k-1}$ for every $k \geq 2$, and if $n > 0$ then $y_1 \neq \alpha^{n-1}(a)$.*

Conditions (a)–(c) of Theorem 4.3 correspond to Theorems 1–3 of [30], respectively. The slight difference in phrasing is due to the fact that for transformations α with stabilizer s, Zupnik uses the set

$$B_\alpha = \{b \in S : \alpha^n(b) \in \mathrm{im}(\alpha^s) \text{ if and only if } n \geq s - 1\}$$

(although he does not specify exactly what that means when $s = 0$), while we use the set Ω_α from Definition 4.1. The set Ω_α is more natural for our purposes than B_α since if $s > 0$, then Ω_α consists of the initial vertices of the twigs in D_α of length s, whereas $B_\alpha = \{\alpha(a) : a \in \Omega_\alpha\}$; that is, B_α consists of the vertices that come after the initial vertices of such twigs.

Lemma 4.4 *Let $\alpha \in T(S)$. Then $\alpha|_{\mathrm{sim}(\alpha)}$ is one-to-one if and only if D_α does not have an infinite branch.*

Proof ($\Rightarrow$) We will prove the contrapositive. Suppose D_α has an infinite branch L. Then L is an infinite branch of a cycle or double ray. Suppose $L = \langle \ldots\, y_2\, y_1\, x_i]$ is an infinite branch of a cycle $C = (x_0 \ldots x_{k-1})$. Then $y_1, x_{i-1} \in \mathrm{sim}(\alpha)$ with $y_1 \neq x_{i-1}$ (since y_1 does not lie on C) and $\alpha(y_1) = \alpha(x_{i-1}) = x_i$, which implies that $\alpha|_{\mathrm{sim}(\alpha)}$ is not one-to-one. An argument in the case when L is an infinite branch of a double ray is similar.

($\Leftarrow$) Suppose D_α does not have an infinite branch. Let $x, y \in \mathrm{sim}(\alpha)$ be such that $\alpha(x) = \alpha(y)$. Then x and y are vertices of the same component of D_α, say A. By Proposition 2.5, A has a unique cycle or a double ray (since a component of type *rro* has no vertices that belong to $\mathrm{sim}(\alpha)$). Suppose A has a cycle C. Then, since $x, y \in \mathrm{sim}(\alpha)$ and D_α does not have an infinite branch, x and y must lie on C. Thus $\alpha(x) = \alpha(y)$ since, clearly, α restricted to the vertices of C is one-to-one. If A has a double ray, we prove that $\alpha(x) = \alpha(y)$ in a similar way. Hence $\alpha|_{\mathrm{sim}(\alpha)}$ is one-to-one. □

Lemma 4.5 *Let $\alpha \in T(S)$ be a Cayley function. Then every component of D_α has a unique cycle or a double ray if and only if α has the stabilizer.*

Proof Suppose every component of D_α has a unique cycle or a double ray. Suppose to the contrary that α has no stabilizer. Then, by Theorem 4.3, there is $a \in S$ such that $\alpha^n(a) \notin \mathrm{im}(\alpha^{n+1})$ for every $n \geq 0$. Let A be the component of D_α containing a.

Suppose A has a unique cycle, say C. Then, by Proposition 2.6, A is the join of C and its branches. It follows that there is an integer $k \geq 0$ such that $\alpha^k(a)$ lies on C. But then $\alpha^k(a) \in \operatorname{im}(\alpha^{k+1})$, which is a contradiction. If A has a double ray, we obtain a contradiction by a similar argument. Hence α has the stabilizer.

The converse follows from Theorem 3.10. □

If $\alpha \in T(S)$, then the digraph D_α has exactly one of the following features:

- D_α has a component of type *rro*;
- every component of D_α has a unique cycle or a double ray, and D_α does not have an infinite branch; or
- every component of D_α has a unique cycle or a double ray, and D_α has an infinite branch.

We will characterize the digraphs of Cayley functions considering each of the above three cases.

Theorem 4.6 *Let $\alpha \in T(S)$ be such that D_α has a component of type rro. Then α is a Cayley function if and only if D_α has a component A of type rro such that:*

(1) *A is the join of a maximal right ray $R = [x_0\, x_1\, x_2 \dots\rangle$ and its branches;*
(2) *for every $i \geq 1$, if $[y_0\, y_1 \dots\, y_m = x_i]$ is a branch of R, then $m \leq i$.*

Proof Suppose α is a Cayley function. Then α has no stabilizer by Lemma 4.5. Hence, by Theorem 4.3, there is $a \in S$ such that such that $\alpha^n(a) \notin \operatorname{im}(\alpha^{n+1})$ for every $n \geq 0$. Let A be the component of D_α containing a, and note that A is of type *rro* (see the proof of Lemma 4.5). Let $x_0 = a$ and $x_i = \alpha(x_{i-1})$ for every $i \geq 1$. Then x_0 is an initial vertex of D_α (since $x_0 = a \notin \operatorname{im}(\alpha)$) and $x_0, x_1, x_2, \dots$ are pairwise distinct (since A has no cycle). Thus $R = [x_0\, x_1\, x_2 \dots\rangle$ is a maximal right ray in A. By Proposition 2.6, A is the join of R and its branches, that is, (1) is satisfied. To prove (2), suppose to the contrary that for some $i \geq 1$, there is a branch $[y_0\, y_1 \dots\, y_m = x_i]$ of R with $m > i$. Then

$$\alpha^i(a) = \alpha^i(x_0) = x_i = y_m = \alpha^m(y_0) \in \operatorname{im}(\alpha^m) \subseteq \operatorname{im}(\alpha^{i+1}),$$

which is a contradiction. Hence (2) is satisfied.

Conversely, suppose that D_α has a component A of type *rro* such that (1) and (2) are satisfied. Then α has no stabilizer by Theorem 3.10. Let $a = x_0$. Suppose to the contrary that there is $n \geq 0$ such that $\alpha^n(a) \in \operatorname{im}(\alpha^{n+1})$. Then $x_n = \alpha^n(x_0) = \alpha^n(a) = \alpha^{n+1}(y)$ for some $y \in S$. Clearly y does not lie on R. Thus, by Proposition 2.6, there is a finite branch $[y_0 \dots\, y_k = y \dots\, y_m = x_i]$ of R with $0 \leq k < m$. By (2), we have $m \leq i$. Since $\alpha^{n+1}(y) = x_n$, we must have $n \geq i$. Thus

$$\alpha^{n-i}(x_i) = x_n = \alpha^{n+1}(y) = \alpha^{n+1}(y_k) = \alpha^{n-m+k+1}(\alpha^{m-k}(y_k)) = \alpha^{n-m+k+1}(x_i),$$

which implies $n - i = n - m + k + 1$. Thus $k + 1 = m - i \leq 0$ (since $m \leq i$), which is a contradiction. We have proved that $\alpha^n(a) \notin \operatorname{im}(\alpha^{n+1})$ for every $n \geq 0$. Hence α is a Cayley function by Theorem 4.3. □

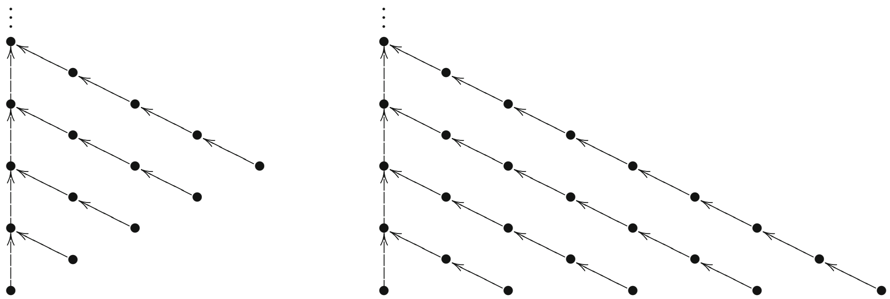

Fig. 6 A Cayley function (*left*) and not a Cayley function (*right*) (see Theorem 4.6)

Example 4.7 Consider components A_1 and A_2 in Fig. 6. By Theorem 4.6, any transformation α_1 whose digraph has component A_1 is a Cayley function, whereas any transformation α_2 whose only component of type *rro* is A_2 is not a Cayley function. Also, any α that has the component presented in Fig. 3 is a Cayley function.

Definition 4.8 Let $\alpha \in T(S)$ and let $\mathbb{N} = \{0, 1, 2, \ldots\}$. We define $\sup_b(\alpha), \sup_t(\alpha) \in \mathbb{N} \cup \{\infty\}$ by

$$\begin{aligned} \sup_b(\alpha) &= \sup\{m : m = 0 \text{ or } m \text{ is the length of a finite branch in } D_\alpha\}, \\ \sup_t(\alpha) &= \sup\{m : m = 0 \text{ or } m \text{ is the length of a twig in } D_\alpha\}. \end{aligned} \tag{4.1}$$

Lemma 4.9 *Let $\alpha \in T(S)$ be such that every component of D_α has a unique cycle or a double ray, and D_α does not have an infinite branch. Then* $\sup_b(\alpha) = \sup_t(\alpha)$.

Proof Since every twig is a finite branch (see Definition 3.8), $\sup_t(\alpha) \leq \sup_b(\alpha)$ in any D_α. To prove $\sup_b(\alpha) \leq \sup_t(\alpha)$, it suffices to show that if D_α has a finite branch of length m, then D_α has a twig of length $k \geq m$. Let $P = [y_0 \ldots y_m]$ be a finite branch of length m, and let A be the component of D_α containing P. If A has a unique cycle, then every finite branch in A is a twig (since D_α has no infinite branches and a component with a cycle does not have a double or right ray), so a desired twig of length at least m is P itself. Suppose A has a double ray, say W. Then $\mathrm{sim}(\alpha) \cap A = V(W)$ since D_α has no infinite branches. Thus, if y_m lies on W, then again P itself is a desired twig. Suppose y_m does not lie on W (which is possible since y_m may lie on a maximal right ray). Then, by Proposition 2.6, there is a finite branch $P_1 = [y_0 \ldots y_m \ldots y_k]$ such that $k > m$ and y_k lies on W. Thus P_1 is a twig of length bigger than m. Hence $\sup_b(\alpha) \leq \sup_t(\alpha)$. □

Lemma 4.9 is not true if D_α has an infinite branch. Let

$$S = \{\ldots\, x_{-1}\, x_0, x_1, \ldots\} \cup \{y_1, y_2, \ldots\} \cup \{z_1, z_2, \ldots\},$$

and consider $\alpha \in T(S)$ with digraph in Figure 7. Note that $\mathrm{sim}(\alpha) = \{\ldots, x_{-1}, x_0, x_1, \ldots\} \cup \{y_1, y_2, \ldots\}$. We have $\sup_b(\alpha) = \infty$ since for every $n \geq 1$, $[z_n\, y_n \ldots y_1\, x_0]$ is

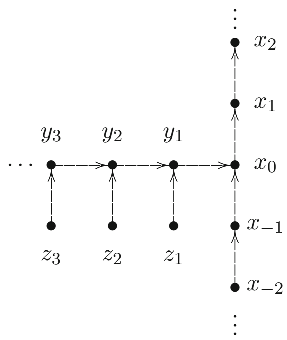

Fig. 7 A transformation α with $\sup_t(\alpha) < \sup_b(\alpha)$

a finite branch (of the double ray $\langle \dots x_{-1}\, x_0\, x_1 \dots \rangle$), but $\sup_t(\alpha) = 1$ since the only twigs in D_α are $[z_1\, y_1], [z_2\, y_2], [z_3\, y_3], \dots$.

Theorem 4.10 *Let $\alpha \in T(S)$ be such that every component of D_α has a unique cycle or a double ray, and D_α does not have an infinite branch. Then α is a Cayley function if and only if the following conditions are satisfied:*

(1) *$s = \sup_b(\alpha)$ is finite;*
(2) *if $s > 0$ and D_α has a double ray, then some double ray in D_α has a branch of length s;*
(3) *if D_α does not have a double ray, then there are integers $1 \le k_1 < k_2 < \dots < k_p$, $p \ge 1$, such that:*
   (a) *$\{k_1, \dots, k_p\}$ is the set of the lengths of the cycles in D_α;*
   (b) *k_i divides k_p for every $i \in \{1, \dots, p\}$; and*
   (c) *if $s > 0$, then some cycle of D_α of length k_p has a branch of length s.*

Proof Suppose α is a Cayley function. By Lemma 4.5, α has the stabilizer, say s_1. By (3.1), $s_1 = \sup_t(\alpha)$, and so $s_1 = \sup_b(\alpha) = s$ by Lemma 4.9. Thus (1) holds since the stabilizer of any transformation is finite by definition. By Lemma 4.4, $\alpha|_{\mathrm{sim}(\alpha)}$ is one-to-one, and so $\alpha|_{\mathrm{im}(\alpha^s)}$ is one-to-one since $\mathrm{sim}(\alpha) = \mathrm{im}(\alpha^s)$ (see Remark 3.4).

Suppose that $s > 0$ and that D_α has a double ray, say $W = \langle \dots x_{-1}\, x_0\, x_1 \dots \rangle$. By Theorem 4.3, there is $a \in \Omega_\alpha$ such that for all $m, n \ge 0$, if $\alpha^m(a) = \alpha^n(a)$ then $\alpha^m = \alpha^n$. Then a is the initial point of some twig P in D_α of length s (see Remark 4.2). Suppose to the contrary that no double ray in D_α has a branch of length s. Then P must be a branch of some cycle $(y_0 \dots y_{k-1})$, $k \ge 1$. Thus $\alpha^s(a) = \alpha^{s+k}(a)$, and so $\alpha^s = \alpha^{s+k}$. But then $x_s = \alpha^s(x_0) = \alpha^{s+k}(x_0) = x_{s+k}$, which is a contradiction, as W is a double ray. Hence some double ray of D_α must have a branch of length s, which proves (2).

Suppose D_α does not have a double ray. Then each component of D_α has a unique cycle. Let $a \in \Omega_\alpha$ be as in the proof of (2) (but here we do not assume that $s > 0$). Then there is a cycle $C = (y_0 \dots y_{k-1})$ in D_α such that, for some j, either $a = y_j$ (if $s = 0$) or C has a branch $[a \dots y_j]$ of length s (if $s > 0$). In either case, $\alpha^s(a) = \alpha^{s+k}(a)$, and so $\alpha^s = \alpha^{s+k}$. Let $(x_0 \dots x_{t-1})$ be any cycle in D_α, and let $i \in \{0, \dots, t-1\}$ be such that $\alpha^s(x_i) = x_0$. Then

$$x_k = \alpha^k(x_0) = \alpha^k(\alpha^s(x_i)) = \alpha^{s+k}(x_i) = \alpha^s(x_i) = x_0.$$

It follows that $k \equiv 0 \,(\text{mod } t)$, that is, t divides k. We have proved that the set of the lengths of the cycles in D_α is bounded above by k, and so it is finite, say $\{k_1, k_2, \ldots, k_p\}$ with $1 \le k_1 < k_2 < \ldots < k_p = k$. We have also proved that each $t = k_i$ divides $k = k_p$ and that C has a branch of length s (if $s > 0$). Thus (3) holds.

Conversely, suppose that (1)–(3) are satisfied. Then, by (1), (3.1), and Lemma 4.9, $s = \sup_b(\alpha)$ is the stabilizer of α. As in the proof of the first part, $\alpha|_{\mathrm{im}(\alpha^s)}$ is one-to-one.

Suppose D_α has a double ray. Then, by (2), there is $a \in \Omega_\alpha$ such that $\alpha^s(a)$ lies on some double ray. Then, for all $m, n \ge 0$, if $\alpha^m(a) = \alpha^n(a)$ then $m = n$, and so $\alpha^n = \alpha^m$.

Suppose D_α does not have a double ray. Then, by (3), there is $a \in \Omega_\alpha$ such that $\alpha^s(a)$ lies on a cycle C of length k_p. Suppose $\alpha^m(a) = \alpha^n(a)$, where $m, n \ge 0$. We may assume that $m \ge n$. Then $m \equiv n \,(\text{mod } k_p)$, that is, $m = n + qk_p$ for some $q \ge 0$. If $n < s$ (which is possible only if $s > 0$), then $\alpha^m(a) = \alpha^n(a)$ implies $m = n$, so $\alpha^m = \alpha^m$. Suppose $n \ge s$, and let $x \in S$. Since s is the stabilizer of α and $n \ge s$, $\alpha^n(x)$ lies on some cycle of D_α. Let $C_1 = (x_0 \ldots x_{k_i})$ be the cycle such that $\alpha^n(x) = x_j$ for some j. By (3), $k_p = rk_i$ for some $r \ge 1$. Thus

$$\alpha^m(x) = \alpha^{n+qk_p}(x) = \alpha^{n+qrk_i}(x) = \alpha^{qrk_i}(\alpha^n(x)) = \alpha^{qrk_i}(x_j) = x_j = \alpha^n(x),$$

which implies $\alpha^m = \alpha^n$ since x was an arbitrary element of S. Hence α is a Cayley function by Theorem 4.3. □

Example 4.11 Consider transformations α and β whose digraphs are presented in Figs. 8 and 9, respectively. Then α is a Cayley function, while β is not a Cayley function (since $s = 3$ and (2) of Theorem 4.10 is not satisfied). If we remove the component with the double ray from Fig. 8, the resulting transformation will not be a Cayley function since (3) of Theorem 4.10 will not be satisfied. If we remove the component with the double ray from Fig. 9, the resulting transformation will be a Cayley function.

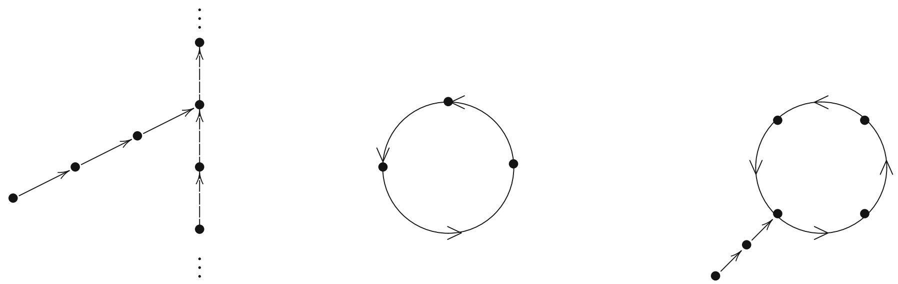

Fig. 8 A Cayley function described by Theorem 4.10

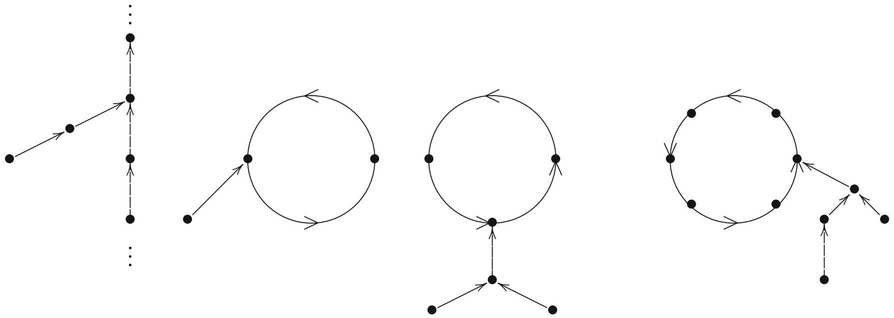

Fig. 9 Not a Cayley function (see Theorem 4.10)

It is of interest to apply Theorem 4.10 to permutations. Let $\mathrm{Sym}(S)$ be the set of permutations on S, that is, bijections from S to S. Let $\alpha \in \mathrm{Sym}(S)$. Then α is an inner translation of a group with universe S if and only if D_α is either a join of disjoint double rays or a join of cycles of the same length [28, Theorem 2]. The following is an immediate corollary of Theorem 4.10.

Corollary 4.12 *Let $\alpha \in \mathrm{Sym}(S)$. Then α is a Cayley function if and only if exactly one of the following conditions holds:*

(a) *D_α has a double ray; or*
(b) *D_α is a join of cycles, and there are integers $1 \leq k_1 < k_2 < \ldots < k_p$, $p \geq 1$, such that*
  (i) *$\{k_1, \ldots, k_p\}$ is the set of the lengths of the cycles in D_α;*
  (ii) *k_i divides k_p for every $i \in \{1, \ldots, p\}$.*

Example 4.13 Let $S = \{1, 2, \ldots, 11\}$. Consider

$$\alpha = (1\,2\,3\,4\,5\,6)(7\,8\,9)(10\,11) \text{ and } \beta = (1\,2\,3\,4)(5\,6\,7\,8)(9\,10\,11)$$

in $T(S)$. Then α is a Cayley function, while β is not a Cayley function.

The last case to consider is when D_α has no component of type *rro* but it does have an infinite branch.

Theorem 4.14 *Let $\alpha \in T(S)$ be such that every component of D_α has a unique cycle or a double ray, and D_α has an infinite branch. Then α is a Cayley function if and only if the following conditions are satisfied:*

(1) *$s = \sup_t(\alpha)$ is finite;*
(2) *D_α has a double ray $W = \langle \ldots x_{-1}\, x_0\, x_1 \ldots \rangle$ such that for some x_i:*
  (a) *if $s > 0$ then W has a finite branch at x_i of length s; and*
  (b) *W has an infinite branch at each x_j with $j > i$.*

Proof Suppose α is a Cayley function. By Lemma 4.5, α has the stabilizer, say s_1. By (3.1), $s_1 = \sup_t(\alpha) = s$, and so (1) holds since the stabilizer of any transformation is finite by definition. By Lemma 4.4, $\alpha|_{\mathrm{sim}(\alpha)}$ is not one-to-one, and so $\alpha|_{\mathrm{im}(\alpha^s)}$ is not one-to-one since since $\mathrm{sim}(\alpha) = \mathrm{im}(\alpha^s)$. Thus, there exists $a \in \Omega_\alpha$ such that (i) and (ii) of Theorem 4.3 hold. Let A be the component of D_α containing a. Suppose to the contrary that A has a cycle, say $C = (x_0 \dots x_{k-1})$. Then, by Proposition 2.6, $\alpha^p(a) = x_j$ for some j and $p \geq 0$, and so $\alpha^{k+p}(a) = \alpha^k(\alpha^p(a)) = \alpha^k(x_j) = x_j = \alpha^p(a)$. Thus $k + p = p$ by (i) of Theorem 4.3, which is a contradiction.

Hence A has a double ray, and so $\mathrm{sim}(\alpha) \cap V(A)$ consists of the vertices x of A such that x lies on some double ray in A. Thus, since $\alpha^s(a) \in \mathrm{sim}(\alpha)$, there is a double ray $W = \langle \dots x_{-1}\, x_0\, x_1 \dots \rangle$ in A such that $\alpha^s(a) = x_i$ for some i. If $s > 0$, then $[a\, \alpha(a)\, \alpha^2(a) \dots \alpha^s(a) = x_i]$ is a finite branch of W of length s (by the definition of Ω_α). Let $j > i$ and note that $x_j = \alpha^n(a)$ for $n = s + j - i$. By (ii) of Theorem 4.3, there is a left ray $L = \langle \dots y_2\, y_1\, \alpha^n(a) = x_j]$ in A such that $y_1 \neq \alpha^{n-1}(a) = x_{j-1}$. Hence y_1 does not lie on W, and so L is an infinite branch of W at x_j. We have proved (2).

Conversely, suppose (1) and (2) are satisfied. Then, by (1) and (3.1), $s = \sup_t(\alpha)$ is the stabilizer of α. By Lemma 4.4, $\alpha|_{\mathrm{im}(\alpha^s)}$ is not one-to-one. Let $W = \langle \dots x_{-1}\, x_0\, x_1 \dots \rangle$ be a double ray from (2). If $s > 0$, then W has a finite branch $[y_0 \dots y_s = x_i]$. Set $a = y_0$ if $s > 0$, and $a = x_i$ if $s = 0$. In either case, $a \in \Omega_\alpha$. For all $m, n \geq 0$, if $\alpha^m(a) = \alpha^n(a)$ then $m = n$ since a is in a component of D_α that does not have a cycle. Let $n \geq s$. We want to prove that there are pairwise distinct elements $y_1, y_2, \dots$ of S such that $\alpha(y_1) = \alpha^n(a)$, $\alpha(y_k) = y_{k-1}$ for every $k \geq 2$, and if $n > 0$ then $y_1 \neq \alpha^{n-1}(a)$. If $n = s$, then we can take $y_1 = x_{i-1}, y_2 = x_{i-2}, \dots$. Suppose $n > s$. Then $\alpha^n(a) = x_j$ for $j = i + n - s > i$. By (2b), W has an infinite branch $\langle \dots z_2\, z_1\, x_j]$, and we can take $y_1 = z_1, y_2 = z_2, \dots$. Hence α is a Cayley function by Theorem 4.3. □

For example, the transformation with digraph in Fig. 10 is a Cayley function (with $s = 4$).

Let $\alpha \in T(S)$ be such that every component of D_α has a unique cycle or a double ray. By Theorems 4.10 and 4.14, if α is a Cayley function, then either every component of D_α has *at most* one double ray or some component of D_α has infinitely many double rays. So, transformations with digraphs in Figs. 2, 4, 5, and 7 are not Cayley functions.

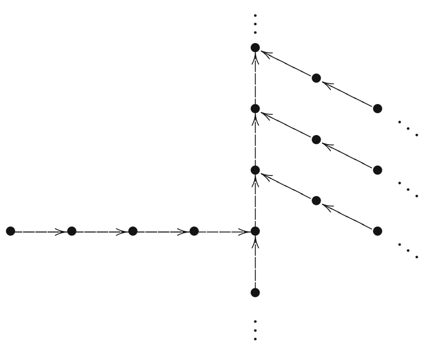

Fig. 10 A Cayley function described by Theorem 4.14

5 The solution of problem 3 for finite permutations

Let S be a non-empty set. For a transformation $\alpha \in T(S)$, the *centralizer* $C(\alpha)$ of α in $T(S)$ is defined by $C(\alpha) = \{\beta \in T(S) : \alpha\beta = \beta\alpha\}$. Elements of $C(\alpha)$, for an arbitrary $\alpha \in T(S)$, were characterized in [6]. In this section, we initiate the study of the following problem: given $\alpha \in T(S)$, which transformations $\beta \in C(\alpha)$ are Cayley functions? We show how our description of Cayley functions can be used to solve this problem in the special case when S is finite and α is a permutation. As will be clear by the end of this section, problem 3 is very difficult, even with graph descriptions of the centralizers and Cayley functions.

For the remainder of this section, S will denote a finite non-empty set. The following theorem is [6, Corollary 6.4]. We agree that if $(y_0 \dots y_{m-1})$ is a cycle and i is an integer, then y_i means y_r where $r \equiv i \pmod m$ and $0 \le r < m$.

Theorem 5.1 *Let $\alpha \in \mathrm{Sym}(S)$ and $\beta \in T(S)$. Then $\beta \in C(\alpha)$ if and only if for every cycle $\sigma = (x_0 \dots x_{k-1})$ in α, there exists a cycle $\theta = (y_0 \dots y_{m-1})$ in α such that m divides k and β* wraps *σ around θ at some y_t, that is, $\beta(x_i) = y_{t+i}$ for every $i \in \{0, 1, \dots, k-1\}$.*

Let $\alpha \in \mathrm{Sym}(S)$ and let $\mathcal{C}_\alpha$ be the set of cycles in α (including the 1-cycles). For $\beta \in C(\alpha)$, we define a transformation ψ_β on $\mathcal{C}_\alpha$ by

$$\psi_\beta(\sigma) = \text{ the unique } \theta \in \mathcal{C}_\alpha \text{ such that } \beta \text{ wraps } \sigma \text{ around } \theta.$$

Note that ψ_β is well defined by Theorem 5.1, that is, $\psi_\beta \in T(\mathcal{C}_\alpha)$, and that the vertices of the digraph D_{ψ_β} are the cycles in α.

The following lemma, which we will use implicitly in the subsequent arguments, follows immediately from the definition of ψ_β and Theorem 5.1.

Lemma 5.2 *Let $\alpha \in \mathrm{Sym}(S)$ and $\beta \in C(\alpha)$. Suppose that A is a component of D_{ψ_β} with cycle $(\sigma_0\,\sigma_1 \dots \sigma_{k-1})$, and that Z is the set of all elements $x \in S$ such that x is in some $\sigma \in A$. Then:*

(1) *the cycles $\sigma_0, \sigma_1, \dots, \sigma_{k-1}$ have the same length;*
(2) *for every $x \in Z$, $\beta(x) \in Z$, that is, $\beta|_Z \in T(Z)$;*
(3) *if $\sigma, \theta \in A$ with $\psi_\beta(\sigma) = \theta$, then for every x in σ, $\beta(x)$ is in θ;*
(4) *if $x \in Z$ is not in any σ_i, then x does not lie on any cycle of $D_{\beta|_Z}$.*

Lemma 5.3 *Let $\alpha \in \mathrm{Sym}(S)$ and $\beta \in C(\alpha)$. Suppose that A is a component of D_{ψ_β} with cycle $(\sigma_0\,\sigma_1 \dots \sigma_{k-1})$, each σ_i having length p. Let Z be as in* Lemma 5.2 *and let $\sigma_0 = (x_0\,x_1 \dots x_{p-1})$. Then:*

(1) *$\beta^k(x_0) = x_l$ for some $l \in \{0, 1, \dots, p-1\}$;*
(2) *every cycle in $D_{\beta|_Z}$ has length $\frac{kp}{\gcd(p,l)}$.*

Proof Since $(\sigma_0\,\sigma_1 \dots \sigma_{k-1})$ is a cycle in D_{ψ_β}, $\psi_\beta^k(\sigma_0) = \sigma_0$. Hence $\beta^k(x_0)$ is in σ_0, and so $\beta^k(x_0) = x_l$ for some l. This proves (1).

To prove (2), recall that $\beta|_Z \in T(Z)$ by Lemma 5.2. Let t be the smallest positive integer such that $\beta^t(x_0) = x_0$. Then $\psi^t_\beta(\sigma_0) = \sigma_0$, and hence $t = ku$ for some integer u. As α and β commute, we have

$$\begin{aligned} x_0 &= \beta^t(x_0) = \beta^{ku}(x_0) = \beta^{k(u-1)}(x_l) = \beta^{k(u-1)}(\alpha^l(x_0)) \\ &= \alpha^l\beta^{k(u-1)}(x_0) = \cdots = \alpha^{lu}(x_0). \end{aligned}$$

The smallest positive integer u for which $\alpha^{lu}(x_0) = x_0$ is $\frac{p}{\gcd(p,l)}$. Hence $D_{\beta|_Z}$ contains a cycle of length $t = ku = \frac{kp}{\gcd(p,l)}$.

We will show that every cycle in $D_{\beta|_Z}$ has length $\frac{kp}{\gcd(p,l)}$. Let $x \in Z$. If x is not a vertex of any σ_i, then x is not a vertex of any cycle of $D_{\beta|_Z}$ by Lemma 5.2.

Suppose that x is a vertex of σ_0, say $x = x_i$. Since $\beta^k \in C(\alpha)$ and $\beta^k(x_0) = x_l$, we have $\beta^k(x_i) = x_{i+l}$ by Theorem 5.1. We can renumber the vertices of σ_0 in such a way that x_i becomes x_0. Then, by the foregoing argument, we obtain a cycle $(x\ldots)$ in $D_{\beta|_Z}$ of length $\frac{kp}{\gcd(p,l)}$.

Suppose that x is a vertex of σ_m with $m \neq 0$. Renumber the vertices of $\sigma_0, \sigma_1, \ldots, \sigma_{k-1}$ in such a way that for each i, $\sigma_i = (x^i_0\ldots)$, $x = x^m_0$, and $\beta^i(x^0_0) = x^i_0$. We have already shown that $\beta^k(x^0_0) = x^0_l$. Thus, since $\beta^m \in C(\alpha)$,

$$\beta^k(x^m_0) = \beta^k(\beta^m(x^0_0)) = \beta^m(\beta^k(x^0_0)) = \beta^m(x^0_l) = x^m_l.$$

Hence, by the foregoing argument, we obtain a cycle $(x\ldots)$ in $D_{\beta|_Z}$ of length $\frac{kp}{\gcd(p,l)}$. Since we have considered all possible cycles in $D_{\beta|_Z}$, (2) follows.

Remark 5.4 Let Z and $\sigma_0, \sigma_1, \ldots, \sigma_{k-1}$ be as in Lemma 5.3, and let $x \in Z$. By the proof of Lemma 5.3, we have:

(1) if x is not in any σ_i, then x does not lie on any cycle of $D_{\beta|_Z}$;
(2) if x is in some σ_i, then $D_{\beta|_Z}$ has a cycle $(x\ldots)$ of length $\frac{kp}{\gcd(p,l)}$, where $l \in \{0, 1, \ldots, p-1\}$ is such that $\beta^k(x) = \alpha^l(x)$.

Lemma 5.5 *Let* $\alpha \in \mathrm{Sym}(S)$ *and* $\beta \in C(\alpha)$. *Suppose that* A *is a component of* D_{ψ_β} *with cycle* $(\sigma_0\,\sigma_1\ldots\sigma_{k-1})$. *Let* Z *be as in* Lemma 5.2. *Assume that the maximum length of a branch in* A *is* s *(with* $s = 0$ *if* A *has no branches). Then:*

(1) *if* $s = 0$, *then* $D_{\beta|_Z}$ *has no branches;*
(2) *if* $s > 0$, *then every branch in* $D_{\beta|_Z}$ *has length at most* s, *and there exists a branch in* $D_{\beta|_Z}$ *of length* s.

Proof If A has no branches, then every element of Z lies on some cycle in $D_{\beta|_Z}$ by Remark 5.4. This shows the first assertion.

To prove (2), suppose that $s > 0$ and let $[\theta_0\,\theta_1\ldots\theta_m = \sigma_i]$ be a branch in A, so $m \leq s$. Let x be in θ_j, where $j \in \{0, 1\ldots, m-1\}$. Since $m \leq s$, we have $\psi^s_\beta(\theta_j) = \sigma_{i+s-m+j}$, and so $\beta^s(x)$ is in $\sigma_{i+s-m+j}$. Thus, by Remark 5.4, $\beta^s(x)$ lies on a cycle in $D_{\beta|_Z}$. Hence every branch in $D_{\beta|_Z}$ has length at most s.

Now let $[\theta_0\,\theta_1\ldots\theta_s = \sigma_i]$ be a branch of A whose length realizes the maximum value s. Let x be in θ_0. Then $\beta^s(x)$ is in σ_i (since $\psi_\beta(\theta_0) = \sigma_i$) and for every

$u \in \{0, 1, \ldots, s-1\}$, $\beta^u(x)$ is in θ_u (since $\psi_\beta(\theta_0) = \theta_u$), and so $\beta^u(x)$ does not lie on any cycle of $D_{\beta|_Z}$ by Remark 5.4. Thus $[x\, \beta(x)\, \beta^2(x) \ldots \beta^s(x)]$ is a branch in $D_{\beta|_Z}$ of length s. □

With Theorem 4.10 and the lemmas of this section, we obtain the following characterization.

Theorem 5.6 *Let $\alpha, \beta \in T(S)$, where S is finite, α is a permutation, and $\beta \in C(\alpha)$. Suppose that $\{A_1, A_2, \ldots, A_t\}$ is the set of components of D_{ψ_β} and $s = \sup_b(\psi_\beta)$. Let M be the set of numbers of the form $\frac{k_i p_i}{\gcd(p_i, l_i)}$, $1 \le i \le t$, where k_i is the length of the cycle C_i in A_i, p_i is the length of each cycle of α that occurs in C_i, and l_i is the unique number in $\{0, 1, \ldots, p_i - 1\}$ such that $\beta^{k_i}(x) = \alpha^{l_i}(x)$, where x is any element of any cycle of α that occurs in C_i. Then β is a Cayley function if and only if the following conditions are satisfied:*

(1) *the largest element m of M is a multiple of every element of M;*
(2) *if $s > 0$, then some component A_r of D_{ψ_β} such that $\frac{k_r p_r}{\gcd(p_r, l_r)} = m$ has a branch of length s.*

Proof Suppose that β is a Cayley function. By Lemma 5.3, M is the set of the lengths of cycles in D_β. Thus (1) follows from Theorem 4.10. Suppose that $s > 0$. By Lemma 5.5, $s = \sup_b(\beta)$, and so, by Theorem 4.10 and the foregoing observation about M, some cycle C of D_β of length m has a branch of length s. By Lemma 5.3 and its proof, there exists a component A_r of D_{ψ_β} such that $m = \frac{k_r p_r}{\gcd(p_r, l_r)}$ and all vertices of C are contained in Z, where Z is as in Lemma 5.2 (with $A = A_r$). Finally, by Lemma 5.5, A_r has a branch of length s.

Conversely, suppose that conditions (1) and (2) are satisfied. We have already observed that $s = \sup_b(\beta)$ and that M is the set of the lengths of cycles in D_β. Thus, 3(a) and 3(b) of Theorem 4.10 hold by (1). By Lemmas 5.3 and 5.5, D_β has a cycle of length m with a branch of length s. Hence 3(c) of Theorem 4.10 holds, and so β is a Cayley function. □

Theorem 5.6 enables us to decide if a given $\beta \in C(\alpha)$ is a Cayley function by analyzing the components of the digraph of ψ_β and the corresponding numbers from the set M.

Example 5.7 Let $S = \{x_0, x_1, x_2, x_3, y_0, y_1, z_0, z_1, w_0, w_1\}$. Consider

$$\alpha = (x_0\, x_1\, x_2\, x_3)(y_0\, y_1)(z_0\, z_1)(w_0\, w_1) \in \mathrm{Sym}(S).$$

Let $\theta = (x_0\, x_1\, x_2\, x_3)$, $\sigma_1 = (y_0\, y_1)$, $\sigma_2 = (z_0\, z_1)$, and $\sigma_3 = (w_0\, w_1)$.

Let $\beta \in C(\alpha)$ such that D_{ψ_β} is given in Figure 11, $\beta(x_0) = y_0, \beta(y_0) = y_0, \beta(z_0) = w_1$, and $\beta(w_0) = z_0$. (Note that the remaining values of β are determined uniquely by Theorem 5.1.) The components of D_{ψ_β} and the corresponding numbers from M are:

- $A_1 = \{\theta, \sigma_1\}$ with $\frac{k_1 p_1}{\gcd(p_1, l_1)} = \frac{1 \cdot 2}{\gcd(2,0)} = 1$,
- $A_2 = \{\sigma_2, \sigma_3\}$ with $\frac{k_2 p_2}{\gcd(p_2, l_2)} = \frac{2 \cdot 2}{\gcd(2,1)} = 4$.

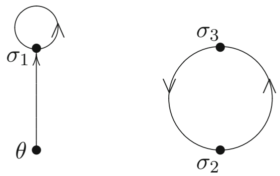

Fig. 11 Digraph D_{ψ_β} from Example 5.7

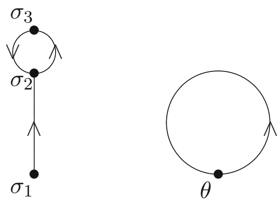

Fig. 12 Digraph D_{ψ_γ} from Example 5.7

Thus $M = \{1, 4\}$ with $m = 4$, and so (1) of Theorem 5.6 holds. However, we have $s = 1$ and A_2, which is the only component with the corresponding number equal to $m = 4$, does not have a branch of length s. Hence (2) of Theorem 5.6 does not hold, and so β is not a Cayley function.

Now consider $\gamma \in C(\alpha)$ such that D_{ψ_γ} is given in Fig. 12, $\gamma(x_0) = x_2$, $\gamma(y_0) = z_0$, $\gamma(z_0) = w_1$, and $\gamma(w_0) = z_0$. The components of D_{ψ_γ} and the corresponding numbers from M are:

- $A_1 = \{\theta\}$ with $\frac{k_1 p_1}{\gcd(p_1,l_1)} = \frac{1\cdot 4}{\gcd(4,2)} = 2$,
- $A_2 = \{\sigma_1, \sigma_2, \sigma_3\}$ with $\frac{k_2 p_2}{\gcd(p_2,l_2)} = \frac{2\cdot 2}{\gcd(2,1)} = 4$.

Thus $M = \{2, 4\}$ with $m = 4$, and so (1) of Theorem 5.6 holds. Further, we have $s = 1$ and A_2, whose corresponding number is $m = 4$, has a branch of length s. Hence (2) of Theorem 5.6 also holds, and so γ is a Cayley function.

6 Problems

In this paper, we have solved a special case of problem 3 of the approach outlined on page 2. The solution for the case of finite permutations opens some natural questions.

Problem 6.1 Let σ be a permutation on a finite set S. Describe the Cayley idempotents that commute with σ. Describe the Cayley permutations δ of S such that $\sigma\delta$ and $\delta\sigma$ are Cayley permutations.

Moving from permutations to idempotents, the following problem is also natural.

Problem 6.2 Find the Cayley functions on a finite set that commute with a given Cayley idempotent.

The main open question regarding the content of this paper is problem 3 in its full generality.

Problem 6.3 Describe the Cayley functions on a set S that commute with a given Cayley function on S.

Problem 4 of the approach does not need the solution of problem 3, and hence can be immediately attempted. Recall that the size of the image of a transformation on S is called its *rank*.

Problem 6.4 Characterize the pairs $\{\alpha, \beta\}$ of Cayley functions on S such that α and β occur as left inner translations of the same semigroup (possibly with some constrains added, such as pairs of permutations, idempotents, or maps of a given rank).

The ultimate goal is to carry out all steps of the approach outlined on page 2.

Problem 6.5 Carry out the steps 3–6 of the approach on page 2 for some special types of maps (permutations, idempotents, maps of a fixed maximum rank) on a finite set.

Problem 6.6 Carry out the steps 3–6 of the approach for all Cayley functions on an arbitrary set.

Other problems about Cayley functions present themselves.

Problem 6.7 Characterize the ordered pairs (α, β) of Cayley functions on S such that in the same semigroup $(S, \cdot)$, $\alpha = \lambda_a$ and $\beta = \rho_a$ for some $a \in S$.

Problem 6.8 Given a Cayley function α on S, find all Cayley functions β on S such that $\alpha\beta$ and $\beta\alpha$ are Cayley functions.

If α and β occur as left inner translations of the same semigroup, then the semigroup generated by α and β consists of Cayley functions. The converse is not necessarily true.

Problem 6.9 Find the pairs of Cayley functions [permutations, idempotents, functions of a given rank] that generate a semigroup of Cayley functions.

Let S be a semigroup. A *left translation* of S is a transformation λ on S such that $\lambda(xy) = (\lambda(x))y$ for all $x, y \in S$; similarly, a *right translation* of S is a transformation ρ on S such that $\rho(xy) = x(\rho(y))$ for all $x, y \in S$. A left translation λ and a right translation ρ of S form a *linked pair* (λ, ρ) if $x(\lambda(y)) = (\rho(x))y$ for all $x, y \in S$. For every $a \in S$, the pair of inner translations (λ_a, ρ_a) is a linked pair. (See [7, page 10].) The following problem is a generalization of Problem 6.7.

Problem 6.10 Characterize the pairs (α, β) of transformations on a set S such (α, β) is a linked pair for some semigroup with universe S.

We can also replace inner translations of a semigroup with endomorphisms of a semigroup or some other algebraic structure.

Problem 6.11 Describe the transformations α on a set S such that α is an endomorphism of some semigroup [group, inverse semigroup, completely regular semigroup, band, partially ordered set, etc.] with universe S.

Some research along these lines has already been done [29].

In this paper, we deal with full transformations on a set S, but analogous problems can also be considered for partial injective transformations on S. A partial injective transformation α on a set S is said to be a *Vagner-Preston function* if there exists an inverse semigroup with universe S such that α appears in the Vagner-Preston representation of S (see [13, Theorem 5.1.7]).

Problem 6.12 Describe Vagner-Preston functions and their digraphs. For Vagner-Preston functions, solve problems analogous to those problems posed above for which an analogous version makes sense and is not trivial.

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