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SHORT NOTE

Infinite compact sets of idempotents in βS

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Abstract We show that if S is a countably infinite right cancellative semigroup and T is an infinite compact set of idempotents in the Stone–Čech compactification βS of S, then T contains an infinite compact left zero semigroup.

Keywords Idempotents · Left zero semigroup · Stone–Čech compactification

Given a discrete semigroup $(S, \cdot)$ we let βS be the Stone–Čech compactification of S with the operation $\cdot$ on βS which makes $(\beta S, \cdot)$ a compact right topological semigroup with S contained in its topological center. If the operation on S is denoted by another symbol such as $+$, we use the same symbol to denote the operation on βS. Because this is a short note, we refer the reader to Chapters 3 and 4 of [2] for basic information about the semigroup βS.

Properties of idempotents in βS play an important role in combinatorics and topological dynamics. (See, for example, [1].) It seems obvious to us that the semigroup $(\beta\mathbb{N}, +)$ cannot contain an infinite compact set of idempotents. Unfortunately, we cannot prove this assertion. Further, we should warn the reader that it was also

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obvious to us that there could not be an idempotent $p \in \beta\mathbb{N}$ which is both minimal and maximal with respect to the ordering of idempotents wherein $p \leq q$ if and only if $p = p+q = q+p$. But Zelenyuk [3] gave a ZFC proof that such idempotents exist.

It is known [2, Lemma 9.3] that if S is a countably infinite right cancellative semigroup, then βS does not contain an infinite compact right zero semigroup. It is also known [2, Theorems 6.15.2 and 6.44] that while any minimal left ideal in $(\beta\mathbb{N}, +)$ has $2^{\mathfrak{c}}$ idempotents which form a left zero semigroup, that semigroup is not compact.

We show in this note that $\beta\mathbb{N}$ contains an infinite compact set of idempotents if and only if it contains an infinite compact left zero subsemigroup. In fact, more generally, we have the following theorem.

Theorem 1 *Let S be a countably infinite right cancellative semigroup and assume that T is an infinite compact subset of βS consisting of idempotents. Then T contains an infinite compact left zero semigroup.*

Proof Let D be the set of right identities for S (which may, of course, be empty). If $D \neq \emptyset$, then D is a left zero semigroup so by [2, Exercise 4.2.1] $\overline{D}$ is a left zero semigroup. Therefore, if $T \cap \overline{D}$ is infinite, we are done. So we will assume that $T \cap \overline{D}$ is finite. Since $\overline{D}$ is open, we may presume that $T \cap \overline{D} = \emptyset$.

Pick a discrete sequence $\langle p_n \rangle_{n=1}^{\infty}$ in T and let $P = \{x_n : n \in \mathbb{N}\}$. (That is to say, the sequence $\langle p_n \rangle_{n=1}^{\infty}$ is injective and P is discrete.) We may assume that $T = c\ell_{\beta S} P$. Note that, since P is discrete and $T = c\ell_{\beta S} P$, we have that $T \setminus P$ is compact. Let $S' = S \setminus D$ and note that if $q \in T$, then $S' \in q$. We claim that if $a \in S'$ and $q \in T$, then $aq \neq q$. So suppose instead that $a \in S'$, $q \in T$, and $aq = q$. Then by [2, Theorem 3.35], $\{x \in S : ax = x\} \in q$ and in particular we may pick some $x \in S$ such that $ax = x$. Then since S is right cancellative, a is a right identity for S, a contradiction.

Next we claim that for all $q \in T \setminus P$, $q \in c\ell_{\beta S}\big((T \cap Tq) \setminus \{q\}\big)$. Suppose instead we have $q \in T \setminus P$ and $A \in q$ such that $\overline{A} \cap \big((T \cap Tq) \setminus \{q\}\big) = \emptyset$. We claim that $q \in c\ell_{\beta S}(P \cap \overline{A}) \cap c\ell_{\beta S}(S'q \cap \overline{A})$. To see this, let $B \in q$. Then $A \cap B \in q$ so $\overline{A \cap B} \cap P \neq \emptyset$, that is $\overline{B} \cap (P \cap \overline{A}) \neq \emptyset$. Also $q = qq$ so $q \in \overline{S'}q = c\ell_{\beta S} S'q$. Therefore $\overline{A \cap B} \cap S'q \neq \emptyset$, that is $\overline{B} \cap (S'q \cap \overline{A}) \neq \emptyset$. Thus $q \in c\ell_{\beta S}(P \cap \overline{A}) \cap c\ell_{\beta S}(S'q \cap \overline{A})$ as claimed.

Thus by [2, Theorem 3.40] we have either

(1) There exist some $q' \in c\ell_{\beta S}(P \cap \overline{A})$ and some $a \in S'$ such that $q' = aq$ or
(2) There exists $n \in \mathbb{N}$ such that $p_n \in \overline{A}$ and $p_n \in c\ell_{\beta S}(S'q \cap \overline{A})$.

In case (1), $q'q = aqq = aq = q'$ and $aq \neq q$ so $q' \in \overline{A} \cap \big((T \cap Tq) \setminus \{q\}\big)$, a contradiction. In case (2) $p_n = rq$ for some $r \in \overline{S'}$ so $p_n q = rqq = rq = p_n$ and $p_n \neq q$ so $p_n \in \overline{A} \cap \big((T \cap Tq) \setminus \{q\}\big)$, a contradiction.

Thus we have established that for all $q \in T \setminus P$, $q \in c\ell_{\beta S}\big((T \cap Tq) \setminus \{q\}\big)$. Next we claim that for all $q \in T \setminus P$, $(T \cap Tq) \setminus P$ is infinite. So let $q \in T \setminus P$. We have that $q \in c\ell_{\beta S}\big((T \cap Tq) \setminus \{q\}\big)$ so $T \cap Tq$ is infinite and closed so $|T \cap Tq| = 2^{\mathfrak{c}}$ and in particular, $T \cap Tq$ is uncountable so $(T \cap Tq) \setminus P$ is infinite as claimed.

Next we claim that there is some $q \in T \setminus P$ which is $\leq_L$-minimal in $T \setminus P$; that is, for all $r \in T \setminus P$, if $r \leq_L q$, then $q \leq_L r$, where $r \leq_L q$ means that $r = r + q$. To see this, let $\mathcal{A} = \{B \subseteq T \setminus P : B$ is a chain with respect to $<_L\}$, where $p <_L r$ means that $p \leq_L r$ and it is not the case that $r \leq_L p$. Note that for $p, r \in T$, $p \leq_L r$ if and only if $\beta Sp \subseteq \beta Sr$. If $p \in T \setminus P$, then $\{p\} \in \mathcal{A}$. By Zorn's Lemma, pick a maximal member B of $\mathcal{A}$. Now $\{(\beta Sp) \cap (T \setminus P) : p \in B\}$ is a collection of closed subsets of βS with the finite intersection property, so pick $q \in \bigcap_{p \in B}(\beta Sp) \cap (T \setminus P)$. Then $q \leq_L p$ for each $p \in B$. If we had some $r \in T \setminus P$ with $r <_L q$, then $B \cup \{r\}$ would be a larger member of $\mathcal{A}$.

We have that $(T \cap Tq) \setminus P$ is infinite and compact. We claim it is a left zero semigroup. So let $r, s \in (T \cap Tq) \setminus P$. Then $r \in \beta Sq$ so $r \leq_L q$ and thus $q \leq_L r$. Consequently $\beta Sq \subseteq \beta Sr$, and therefore $\beta Sr = \beta Sq$. Similarly $\beta Ss = \beta Sq$. Since $\beta Sr = \beta Ss$ we have $sr = s$ and $rs = r$ as required. □

We close by stating the obvious question.

Question 2 *Does* $(\beta\mathbb{N}, +)$ *contain an infinite compact set of idempotents?*

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