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RESEARCH ARTICLE

On some subtraction Menger algebras of multiplace functions

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Abstract We give an abstract characterization of algebras of some partial functions from A^n to A endowed with the operations of Menger superposition and set-theoretic difference of functions as subsets of A^{n+1} and prove that these classes constitute varieties.

Keywords Menger algebra · Algebra of multiplace functions · Subtraction algebra

1. Let A be a nonempty set. Any partial mapping from A^n into A is called a *partial n-place function* defined on A. The set of all such functions is denoted by $\mathcal{F}(A^n, A)$. On $\mathcal{F}(A^n, A)$ we define the operation of *Menger superposition* (*composition*) of n-place functions $\mathcal{O}\colon (f, g_1, \ldots, g_n) \mapsto f[g_1 \ldots g_n]$ by putting

$$(\bar{a}, c) \in f[g_1 \ldots g_n] \longleftrightarrow (\exists \bar{b})\Big((\bar{a}, b_1) \in g_1 \wedge \cdots \wedge (\bar{a}, b_n) \in g_n \wedge (\bar{b}, c) \in f\Big)$$

for all $\bar{a} \in A^n$, $\bar{b} = (b_1, \ldots, b_n) \in A^n$ and $c \in A$. If a set $\Phi \subseteq \mathcal{F}(A^n, A)$ is closed under this operation, the algebra $(\Phi, \mathcal{O})$ is called a *Menger algebra of n-place functions*. For $n = 1$ it is just an arbitrary semigroup of functions.

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If the set Φ is also closed under the set-theoretic difference $\setminus$ of functions (as subsets of A^{n+1}), the algebra $(\Phi, \mathcal{O}, \setminus)$ is called a *difference Menger algebra of n-place functions*, cf. [2]. For $n = 1$ it is a *difference semigroup of functions* in the sense of Schein [4].

The set of all *reversive* n-place functions, i.e., the set of all $f \in \mathcal{F}(A^n, A)$ such that

$$f(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n) = f(a_1, \dots, a_{i-1}, c, a_{i+1}, \dots, a_n) \longrightarrow b = c$$

for all $a_1, \dots, a_n, b, c \in A$, $i = 1, \dots, n$, is denoted by $\mathcal{R}(A^n, A)$.

2. A nonempty set G with one $(n+1)$-ary operation $o\colon (f, g_1, \dots, g_n) \mapsto f[g_1 \dots g_n]$ satisfying the so-called *superassociative law*:

$$x[y_1 \dots y_n][z_1 \dots z_n] = x[y_1[z_1 \dots z_n] \dots y_n[z_1 \dots z_n]],$$

is called a *Menger algebra of rank n* and is denoted by (G, o). A Menger algebra of rank 1 is just a semigroup.

A Menger algebra (G, o) of rank n is called *unitary* if there exist elements $e_1, \dots, e_n \in G$, called *selectors*, such that $x[e_1 \dots e_n] = x$ and $e_i[x_1 \dots x_n] = x_i$ for all $x, x_1, \dots, x_n \in G$ and $i = 1, \dots, n$. It is known (cf. [3, Thm. 2.1.12]) that each Menger algebra (G, o) of rank n can be isomorphically embedded into a unitary Menger algebra (G^*, o^*) of the same rank with the selectors $e_1, \dots, e_n$, which do not belong to G, such that $G \cup \{e_1, \dots, e_n\}$ is a generating set for (G^*, o^*).

Let (G, o) be a Menger algebra of rank n. Consider the set $T_n(G)$ of all expressions, called *polynomials*, in the alphabet $G \cup \{[\ ], x\}$, where the square brackets and x do not belong to G, defined inductively as follows:

(a) $x \in T_n(G)$;
(b) if $i \in \{1, \dots, n\}$, $a, b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n \in G$, $t \in T_n(G)$, then $a[b_1 \dots b_{i-1} t\, b_{i+1} \dots b_n] \in T_n(G)$.

Further, for simplicity, instead of $a[b_1 \dots b_{i-1} t\, b_{i+1} \dots b_n]$ we write $a[\bar{b}\,|_i t]$.

A binary relation $\rho \subset G \times G$, where (G, o) is a Menger algebra of rank n, is called

- *stable* if for all $x, y, x_i, y_i \in G$, $i = 1, \dots, n$,

$$(x, y), (x_1, y_1), \dots, (x_n, y_n) \in \rho \longrightarrow (x[x_1 \dots x_n], y[y_1 \dots y_n]) \in \rho;$$

- *v-regular* if for all $x_i, y_i, z \in G$, $i = 1, \dots, n$,

$$(x_1, y_1), \dots, (x_n, y_n) \in \rho \longrightarrow (z[x_1 \dots x_n], z[y_1 \dots y_n]) \in \rho;$$

- *weakly steady* if for all $x, y, z \in G$, $t_1, t_2 \in T_n(G)$,

$$(x, y), (z, t_1(x)), (z, t_2(y)) \in \rho \longrightarrow (z, t_2(x)) \in \rho;$$

- *steady* if for all $x, y, z \in G$, $t_1, t_2 \in T_n(G)$,

$$(z, t_1(x)), (z, t_1(y)), (z, t_2(y)) \in \rho \longrightarrow (z, t_2(x)) \in \rho.$$

A subset H of a Menger algebra (G, o) is called an *l-ideal* if for all $x, h_1, \ldots, h_n \in G$,

$$(h_1, \ldots, h_n) \in G^n \setminus (G \setminus H)^n \longrightarrow x[h_1 \ldots h_n] \in H.$$

3. Any homomorphism P of a Menger algebra (G, o) of rank n onto a Menger algebra $(\Phi, \mathcal{O})$, where $\Phi \subset \mathcal{F}(A^n, A)$ (respectively, $\Phi \subset \mathcal{R}(A^n, A)$) and A is an arbitrary set, is called a *representation of* (G, o) *by* n-place functions (respectively, by *reversive n-place functions*). Thus, P is a representation if

$$P(x[y_1 \ldots y_n]) = P(x)[P(y_1) \ldots P(y_n)]$$

for all $x, y_1, \ldots, y_n \in G$. In the case when P is an isomorphism we say that this representation is *faithful*, cf. [3, Sect. 2.7].

Let $(P_i)_{i \in I}$ be a family of representations of a Menger algebra (G, o) of rank n by n-place (reversive) functions defined on pairwise disjoint sets $(A_i)_{i \in I}$. The *sum* of the family $(P_i)_{i \in I}$ is the map $P\colon g \mapsto P(g)$, denoted by $\sum_{i \in I} P_i$, from (G, o) into $\mathcal{F}(A^n, A)$ (respectively, into $\mathcal{R}(A^n, A)$), where $A = \bigcup_{i \in I} A_i$ and $P(g) = \bigcup_{i \in I} P_i(g)$ for every $g \in G$. It is not difficult to see that P is a representation of (G, o).

A *determining pair* of a Menger algebra (G, o) of rank n is any pair (ε^*, W), where ε is a v-regular equivalence on the algebra (G, o), W is an l-ideal of (G, o), which is an ε-class if $W \neq \varnothing$, and $\varepsilon^* = \varepsilon \cup \{(e_1, e_1), \ldots, (e_n, e_n)\}$, where $e_1, \ldots, e_n$ are selectors of (G^*, o^*). With each determining pair (ε^*, W) we associate a so-called *simplest representation* $P_{(\varepsilon^*, W)}$ of (G, o) defined in the following way. Let $\mathcal{H}_0$ be the collection of all ε-classes of G distinct from W and $\mathcal{H} = \mathcal{H}_0 \cup \{\{e_1\}, \ldots, \{e_n\}\}$. Each element $g \in G$ is associated with an n-place function $P_{(\varepsilon^*, W)}(g)$ on $\mathcal{H}$ defined by:

$$(H_1, \ldots, H_n, H) \in P_{(\varepsilon, W)}(g) \longleftrightarrow g[H_1 \ldots H_n] \subset H,$$

where $(H_1, \ldots, H_n) \in \mathcal{H}_0^n \cup \{(\{e_1\}, \ldots, \{e_n\})\}$ and $H \in \mathcal{H}$. One can prove (cf. [3, Sect. 2.7]) that $P_{(\varepsilon^*, W)}$ is a representation of (G, o) by n-place functions. Moreover, if the determining pair (ε^*, W) satisfies the condition

$$(u[\bar{w} \,|_i x], u[\bar{w} \,|_i y]) \in \varepsilon \wedge u[\bar{w} \,|_i x] \notin W \longrightarrow (x, y) \in \varepsilon, \tag{1}$$

then $P_{(\varepsilon^*, W)}$ is a representation by reversive n-place functions.

4. Now we recall some basic facts on so-called subtraction algebras studied in [1,4]. A nonempty set G with one binary operation "−" is called a *subtraction algebra* if it satisfies the following three axioms:

$$x - (y - x) = x, \tag{2}$$
$$x - (x - y) = y - (y - x), \tag{3}$$

$$(x - y) - z = (x - z) - y. \tag{4}$$

Notice that any subtraction algebra $(G, -)$ satisfies

$$x - x = y - y. \tag{5}$$

Indeed, according to (2) and (4), we have

$$\begin{aligned}(x - x) - ((x - x) - (y - y)) &= (x - x) - ((x - (y - y)) - x)\\ &= (x - ((x - (y - y)) - x)) - x = x - x.\end{aligned}$$

Swapping x and y, we obtain $(y - y) - ((y - y) - (x - x)) = y - y$. From the above, applying (3), we get

$$(x - x) - ((x - x) - (y - y)) = (y - y) - ((y - y) - (x - x)),$$

which implies (5).

This means that the value of $x - x$ is independent of x. Hence it can be denoted by 0. So, in any subtraction algebra we have

$$x - x = 0 \tag{6}$$

for all $x \in G$.

From (6), applying (2), we deduce the following two identities:

$$x - 0 = x, \quad 0 - x = 0. \tag{7}$$

Notice that the identities (6) and (7) were proved in [2], but the proof in [2] was based on the axiom $0 - 0 = 0$ which is omitted in our present definition.

On a subtraction algebra $(G, -)$ we define a relation $\leqslant$ by putting:

$$x \leqslant y \longleftrightarrow x - y = 0. \tag{8}$$

This relation is reflexive, transitive and antisymmetric. Hence this relation is an order and for all $x, y, z, v \in G$ satisfies the following conditions (cf. [1–3]):

$$0 \leqslant x, \tag{9}$$
$$x - y \leqslant x, \tag{10}$$
$$x \leqslant y \longleftrightarrow x - (x - y) = x, \tag{11}$$
$$x \leqslant y \longrightarrow x - z \leqslant y - z, \tag{12}$$
$$x \leqslant y \longrightarrow z - y \leqslant z - x, \tag{13}$$
$$x \leqslant y \wedge z \leqslant v \longrightarrow x - v \leqslant y - z, \tag{14}$$
$$(x - y) - y = x - y, \tag{15}$$
$$(x - y) - z = (x - z) - (y - z). \tag{16}$$

On $(G, -)$ we also define an additional binary operation $\curlywedge$ as follows:

$$x \curlywedge y = x - (x - y). \tag{17}$$

This operation is idempotent, commutative and associative, cf. [1]. Hence $(G, \curlywedge)$ is a semilattice. From (11) it follows that $x \leqslant y \longleftrightarrow x \curlywedge y = x$.

The following conditions are satisfied in $(G, -)$:

$$x \leqslant y \wedge x \leqslant z \longrightarrow x \leqslant y \curlywedge z, \tag{18}$$
$$x \leqslant y \longrightarrow x \curlywedge z \leqslant y \curlywedge z, \tag{19}$$
$$x \curlywedge y = 0 \longrightarrow x - y = x, \tag{20}$$
$$(x - y) \curlywedge y = 0, \tag{21}$$
$$x \curlywedge (y - z) = (x \curlywedge y) - (x \curlywedge z), \tag{22}$$
$$x - y = x - (x \curlywedge y), \tag{23}$$
$$(x \curlywedge y) - (y - z) = x \curlywedge y \curlywedge z, \tag{24}$$
$$(x \curlywedge y) - z = (x - z) \curlywedge (y - z), \tag{25}$$
$$(x \curlywedge y) - z = (x - z) \curlywedge y. \tag{26}$$

For details see [1].

5. Now we consider algebras with two operations: one $(n + 1)$-ary and one binary. An algebra $(G, o, -)$ of type $(n + 1, 2)$, where (G, o) is a Menger algebra of rank n and $(G, -)$ is a subtraction algebra is called

- a *weak subtraction Menger algebra of rank n* if the equalities

$$(x - y)[z_1 \ldots z_n] = x[z_1 \ldots z_n] - y[z_1 \ldots z_n], \tag{27}$$
$$u[\bar{w}|_i (x - (x - y))] = u[\bar{w}|_i\, x] - u[\bar{w}|_i (x - y)] \tag{28}$$

hold for all $x, y, u, z_1, \ldots, z_n \in G$, $\bar{w} \in G^n$ and $i = 1, \ldots, n$;

- a *subtraction Menger algebra of rank n* if it satisfies (27) and (28) for all $x, y, u, z_1, \ldots, z_n \in G$, $\bar{w} \in G^n$ and $i = 1, \ldots, n$, and the implication

$$x - y = 0 \wedge z - t_1(x) = 0 \wedge z - t_2(y) = 0 \longrightarrow z - t_2(x) = 0 \tag{29}$$

holds for all $x, y, z \in G$ and $t_1, t_2 \in T^n(G)$;

- a *strong subtraction Menger algebra of rank n* if it satisfies (27) for all $x, y, z_1, \ldots, z_n \in G$, and the equality

$$u[\bar{w}|_i (x - y)] = u[\bar{w}|_i\, x] - u[\bar{w}|_i\, y] \tag{30}$$

holds for all $x, y, u \in G$, $\bar{w} \in G^n$ and $i = 1, \ldots, n$.

Notice that in the case $n = 1$ the implication (29) can be deduced from (2), (3), (4), (27), and (28). Thus, for semigroups, the concept of a weak subtraction semigroup coincides with the concept of a subtraction semigroup.

In our previous paper [2] a subtraction Menger algebra was defined as a weak subtraction Menger algebra (in the above sense) satisfying the conditions (29) and $0 - 0 = 0$. Now this last condition is omitted because, as mentioned earlier, it is a consequence of (2), (3), and (4). Notice, by the way, that in the proofs of some results in [2] the conditions (29) and $0 - 0 = 0$ were not used. So, these results are valid for our weak subtraction Menger algebras, too.

Since (28) follows from (30), any strong subtraction Menger algebra is a weak subtraction Menger algebra of the same rank. Obviously a subtraction Menger algebra is a weak subtraction Menger algebra. So, the class of subtraction Menger algebras is contained in the variety of weak subtraction Menger algebras. Below we prove that the class of subtraction Menger algebras is a variety containing as a subvariety the class of strong Menger algebras in which the relation $\leqslant$ defined by (8) is steady.

Notice that in a weak subtraction algebra $(G, o, -)$ the relation $\leqslant$ is stable with respect to the operation o. It also satisfies (12), (13), and (14), cf. [2].

Proposition 1 (cf. [2, Prop. 4]) *In the definition of a weak subtraction Menger algebra of rank n the axiom (28) can be replaced by each of the following equivalent conditions:*

$$x \leqslant y \longrightarrow u[\bar{w}|_i (y - x)] = u[\bar{w}|_i\, y] - u[\bar{w}|_i\, x], \tag{31}$$

$$x \leqslant y \longrightarrow t(y - x) = t(y) - t(x), \tag{32}$$

$$t(x - (x - y)) = t(x) - t(x - y) \tag{33}$$

for all $x, y, u \in G$, $\bar{w} \in G^n$, $i = 1, \ldots, n$, $t \in T_n(G)$.

In [2] this result was proved for subtraction Menger algebras, but the implication (29) was not used in the proof.

Proposition 2 *In a weak subtraction Menger algebra $(G, o, -)$ of rank n the formula*

$$t(x - y) = t(x) - t(x \curlywedge y) \tag{34}$$

is valid for each polynomial $t \in T_n(G)$ and all $x, y \in G$.

Proof According to (23), for each $t \in T_n(G)$ we have $t(x - y) = t(x - (x \curlywedge y))$. So, $x \curlywedge y \leqslant x$, by (32), implies $t(x - (x \curlywedge y)) = t(x) - t(x \curlywedge y)$. Hence, $t(x - y) = t(x) - t(x \curlywedge y)$ for all $x, y \in G$. □

Theorem 1 *A weak subtraction Menger algebra $(G, o, -)$ of rank n is a subtraction Menger algebra if and only if one of the following equivalent conditions is satisfied.*

(i) *The relation $\leqslant$ defined by (8) is weakly steady, i.e.,*

$$x \leqslant y \wedge z \leqslant t_1(x) \wedge z \leqslant t_2(y) \longrightarrow z \leqslant t_2(x) \tag{35}$$

for all $x, y, z \in G$ and $t_1, t_1 \in T_n(G)$.

(ii) *For all $x, y \in G$ and $t_1, t_2 \in T_n(G)$,*

$$t_1(x \curlywedge y) \curlywedge t_2(x - y) = 0. \tag{36}$$

(iii) *For all* $x, y \in G$ *and* $t_1, t_2 \in T_n(G)$,

$$t_1(x \curlywedge y) \curlywedge t_2(y) = t_1(x \curlywedge y) \curlywedge t_2(x \curlywedge y). \tag{37}$$

Proof In view of (8), conditions (35) and (29) are equivalent. So, a weak subtraction Menger algebra is subtraction Menger algebra if and only if it satisfies (35).

We now show that (i), (ii), and (iii) are equivalent.

$(i) \longrightarrow (ii)$ Let h denote the element $t_1(x \curlywedge y) \curlywedge t_2(x - y)$. Clearly, $h \leqslant t_1(x \curlywedge y)$ and $h \leqslant t_2(x - y)$. Since $t_2(x - y) \leqslant t_2(x)$, we have $h \leqslant t_2(x)$. So, $x \curlywedge y \leqslant x$, $h \leqslant t_1(x \curlywedge y)$ and $h \leqslant t_2(x)$, which, by (35), implies $h \leqslant t_2(x \curlywedge y)$. But $h \leqslant t_2(x - y)$, whence

$$h \leqslant t_2(x - y) \curlywedge t_2(x \curlywedge y). \tag{38}$$

Therefore:

$$\begin{aligned} t_2(x - y) - t_2(x \curlywedge y) &\overset{(34)}{=} (t_2(x) - t_2(x \curlywedge y)) - t_2(x \curlywedge y) \\ &\overset{(15)}{=} t_2(x) - t_2(x \curlywedge y) \overset{(34)}{=} t_2(x - y), \end{aligned}$$

and consequently,

$$\begin{aligned} t_2(x - y) \curlywedge t_2(x \curlywedge y) &\overset{(17)}{=} t_2(x - y) - (t_2(x - y) - t_2(x \curlywedge y)) \\ &= t_2(x - y) - t_2(x - y) = 0. \end{aligned}$$

Thus, $t_2(x - y) \curlywedge t_2(x \curlywedge y) = 0$. So, from (38) we obtain $h \leqslant 0$. However, $0 \leqslant h$, hence $h = 0$. This proves (36). So, (i) implies (ii).

$(ii) \longrightarrow (iii)$ Since $x \curlywedge y = y \curlywedge x$, it follows from (36) that

$$t_1(x \curlywedge y) \curlywedge t_2(y - x) = 0,$$

and thus, $t_1(x \curlywedge y) \curlywedge (t_2(y) - t_2(x \curlywedge y)) = 0$, by (34). From this, applying (22), we obtain $t_1(x \curlywedge y) \curlywedge t_2(y) - t_1(x \curlywedge y) \curlywedge t_2(x \curlywedge y) = 0$, which, according to (8), can be written in the form:

$$t_1(x \curlywedge y) \curlywedge t_2(y) \leqslant t_1(x \curlywedge y) \curlywedge t_2(x \curlywedge y). \tag{39}$$

Since the relation $\leqslant$ is stable and $x \curlywedge y \leqslant y$, we have $t_1(x \curlywedge y) \curlywedge t_2(x \curlywedge y) \leqslant t_2(x \curlywedge y) \leqslant t_2(y)$. Thus, $t_1(x \curlywedge y) \curlywedge t_2(x \curlywedge y) \leqslant t_1(x \curlywedge y) \curlywedge t_2(y)$, which together with (39) proves (37). So, (ii) implies (iii).

$(iii) \longrightarrow (i)$ To prove (35) suppose that $x \leqslant y$, $z \leqslant t_1(x)$, and $z \leqslant t_2(y)$ for some $x, y, z \in G$, $t_1, t_2 \in T_n(G)$. From $x \leqslant y$ we obtain $x \curlywedge y = x$. Therefore, $z \leqslant t_1(x \curlywedge y)$. From this, in view of (18) and $z \leqslant t_2(y)$, we get $z \leqslant t_1(x \curlywedge y) \curlywedge t_2(y)$, which, by (37), implies $z \leqslant t_1(x \curlywedge y) \curlywedge t_2(x \curlywedge y)$. Consequently, $z \leqslant t_1(x \curlywedge y) \curlywedge t_2(x) \leqslant t_2(x)$. This proves (35). Thus, (iii) implies (i). □

In terms of weak subtraction Menger algebras the conditions (36) and (37) can be rewritten in the form:

$$t_1(x-(x-y)) - \Big(t_1(x-(x-y)) - t_2(x-y)\Big) = z - z,$$
$$t_1(x-(x-y)) - \Big(t_1(x-(x-y)) - t_2(y)\Big)$$
$$= t_1(x-(x-y)) - \Big(t_1(x-(x-y)) - t_2(x-(x-y))\Big).$$

Since t_1, t_2 are arbitrary polynomials, each of the above identities defines an infinite set of axioms determining together with (27) and (28) the class of subtraction Menger algebras.

Corollary 1 *The class of subtraction Menger algebras of fixed rank is a variety.*

Proposition 3 *In a strong subtraction Menger algebra* $(G, o, -)$ *of rank n for each polynomial* $t \in T_n(G)$ *we have*

$$t(x \curlywedge y) = t(x) \curlywedge t(y). \tag{40}$$

Proof Indeed, each strong subtraction Menger algebra of rank n is a weak subtraction Menger algebra of the same rank, so it satisfies (28), which, by (17), can be rewritten as $u[\bar{w}|_i(x \curlywedge y)] = u[\bar{w}|_i\, x] - u[\bar{w}|_i(x-y)]$. From this, applying (30), we obtain $u[\bar{w}|_i(x \curlywedge y)] = u[\bar{w}|_i\, x] - (u[\bar{w}|_i\, x] - u[\bar{w}|_i\, y]) = u[\bar{w}|_i\, x] \curlywedge u[\bar{w}|_i\, y]$. Thus,

$$u[\bar{w}|_i(x \curlywedge y)] = u[\bar{w}|_i\, x] \curlywedge u[\bar{w}|_i\, y]$$

for all $x, y, u \in G$, $\bar{w} \in G^n$, $i = 1, \ldots, n$. This means that (40) is true for all polynomials of the form $t(x) = u[\bar{w}|_i\, x]$.

Suppose now that (40) holds for some $t' \in T_n(G)$, i.e., $t'(x \curlywedge y) = t'(x) \curlywedge t'(y)$ for all $x, y \in G$. Then for any $t(x) = u[\bar{w}|_i\, t'(x)]$, where $\bar{w} \in G^n$, $i \in 1, \ldots, n$, we have

$$t(x \curlywedge y) = u[\bar{w}|_i\, t'(x \curlywedge y)] = u[\bar{w}|_i(t'(x) \curlywedge t'(y))]$$
$$= u[\bar{w}|_i\, t'(x)] \curlywedge u[\bar{w}|_i\, t'(y)] = t(x) \curlywedge t(y).$$

Thus, (40) is true for all polynomials $t \in T_n(G)$. □

Theorem 2 *In a strong subtraction Menger algebra* $(G, o, -)$ *of rank n the following conditions are equivalent.*

(a) *The relation* $\leqslant$ *defined by (8) is steady on* $(G, o, -)$*, i.e.,*

$$z \leqslant t_1(x) \wedge z \leqslant t_1(y) \wedge z \leqslant t_2(y) \longrightarrow z \leqslant t_2(x) \tag{41}$$

for all $x, y, z \in G$ *and* $t_1, t_1 \in T_n(G)$.

(b) *For all* $x, y \in G$ *and* $t_1, t_2 \in T_n(G)$ *holds:*

$$t_1(x \curlywedge y) \curlywedge t_2(y) = t_1(x \curlywedge y) \curlywedge t_2(x). \tag{42}$$

Proof $(a) \longrightarrow (b)$ First we show that the relation $\leqslant$ defined by (8) is weakly steady. For this let $x \leqslant y$, $z \leqslant t_1(x)$, and $z \leqslant t_2(y)$. As $x \curlywedge y = x$, then $z \leqslant t_1(x \curlywedge y)$, consequently, by (40), $z \leqslant t_1(x) \curlywedge t_1(y)$. This implies $z \leqslant t_1(x)$ and $z \leqslant t_1(y)$. Taking into account that $z \leqslant t_2(y)$ and applying (41), we get $z \leqslant t_2(x)$. This means that the relation $\leqslant$ is weakly steady and, by Theorem 1, it satisfies (37) for all $x, y \in G$ and $t_1, t_2 \in T_n(G)$. Thus, we have $t_1(x \curlywedge y) \curlywedge t_2(y) = t_1(x \curlywedge y) \curlywedge t_2(x \curlywedge y) \leqslant t_1(x \curlywedge y) \curlywedge t_2(x)$. On the other hand, we similarly obtain $t_1(x \curlywedge y) \curlywedge t_2(x) \leqslant t_1(x \curlywedge y) \curlywedge t_2(y)$. Therefore, $t_1(x \curlywedge y) \curlywedge t_2(y) = t_1(x \curlywedge y) \curlywedge t_2(x)$. This means that (a) implies (b).

$(b) \longrightarrow (a)$ Suppose that the premise of the implication (41) is satisfied. Then from $z \leqslant t_1(x)$ and $z \leqslant t_1(y)$, according to (18) and (40), we obtain $z \leqslant t_1(x) \curlywedge t_1(y) = t_1(x \curlywedge y)$. Since $z \leqslant t_2(y)$, the above implies

$$z \leqslant t_1(x \curlywedge y) \curlywedge t_2(y) \overset{(42)}{=} t_1(x \curlywedge y) \curlywedge t_2(x) \leqslant t_2(x),$$

which proves (41). Thus, (b) implies (a). □

Since in terms of the algebra $(G, o, -)$ the condition (42) has the form:

$$\begin{aligned} &t_1(x - (x - y)) - (t_1(x - (x - y)) - t_2(y)) \\ &\quad = t_1(x - (x - y)) - (t_1(x - (x - y)) - t_2(x)), \end{aligned}$$

it may be replaced by an infinite sets of identities. So, the class of strong subtraction Menger algebras of rank n in which the relation $\leqslant$ is steady is a variety contained in the variety of subtraction Menger algebras, and consequently, in the variety of weak Menger algebras of the same rank.

6. In our previous paper [2] we proved that each difference Menger algebra of n-place functions is a weak subtraction Menger algebra of rank n. Comparing this result with [2, Thm. 2], we conclude that a weak subtraction Menger algebra of rank n is isomorphic to a difference Menger algebra of n-place functions if and only if it satisfies the condition (29), i.e., if and only if it is a subtraction Menger algebra. Thus, a weak subtraction Menger algebra of rank n is isomorphic to a difference Menger algebra of n-place functions if and only if it satisfies one of the conditions of Theorem 1 of the present paper.

Proposition 4 *In each difference Menger algebra* $(\Phi, \mathcal{O}, \setminus)$ *of reversive n-place functions we have:*

$$u[\bar{w}\,|_i(f \setminus g)] = u[\bar{w}|_i\, f] \setminus u[\bar{w}|_i\, g] \tag{43}$$

for all $f, g \in \Phi$, $\bar{w} \in \Phi^n$, $i = 1, \ldots, n$.

Proof Let $(\bar{a}, c) \in u[\bar{w}|_i(f \setminus g)]$ for some $\bar{a} \in A^n$, $c \in A$. This means that there is $\bar{b} = (b_1, \ldots, b_n) \in A^n$ such that $(\bar{a}, b_i) \in f \setminus g$, $(\bar{a}, b_j) \in w_j$ for $j \in \{1, \ldots, n\} \setminus \{i\}$ and $(\bar{b}, c) \in u$. Hence, we have $(\bar{a}, b_i) \in f$, $(\bar{a}, b_i) \notin g$ and $(\bar{a}, c) \in u[\bar{w}|_i\, f]$.

Assume that $(\bar{a}, c) \in u[\bar{w}|_i\, g]$. Then there is $\bar{d} = (d_1, \ldots, d_n) \in A^n$ such that $(\bar{a}, d_i) \in g$, $(\bar{a}, d_j) \in w_j$ with $j \in \{1, \ldots, n\} \setminus \{i\}$, and $(\bar{d}, c) \in u$. Since all w_j are functions, from $(\bar{a}, b_j) \in w_j$, $(\bar{a}, d_j) \in w_j$, $j \in \{1, \ldots, n\} \setminus \{i\}$, it follows $b_j = d_j$, where $j \in \{1, \ldots, n\} \setminus \{i\}$. Thus, $(\bar{b}, c) \in u$ and $(\bar{b}|_i d_i, c) \in u$. Consequently, $u(\bar{b}) = c = u(\bar{b}|_i\, d_i)$. Since u is a reversive function, the last implies $b_i = d_i$. So, $(\bar{a}, b_i) \in g$, which contradicts the fact that $(\bar{a}, b_i) \notin g$. Therefore, $(\bar{a}, c) \notin u[\bar{w}|_i\, g]$. Thereby $(\bar{a}, c) \in u[\bar{w}|_i f] \setminus u[\bar{w}|_i g]$. Thus,

$$u[\bar{w}|_i (f \setminus g)] \subset u[\bar{w}|_i f] \setminus u[\bar{w}|_i g].$$

Conversely, let $(\bar{a}, c) \in u[\bar{\omega}|_i f] \setminus u[\bar{\omega}|_i g]$. Then $(\bar{a}, c) \in u[\bar{\omega}|_i f]$ and $(\bar{a}, c) \notin u[\bar{\omega}|_i g]$. The first of these two conditions means that there is a vector $\bar{b} = (b_1, \ldots, b_n) \in A^n$ such that $(\bar{b}, c) \in u$, $(\bar{a}, b_i) \in f$ and $(\bar{a}, b_j) \in \omega_j$ for each $j \in \{1, \ldots, n\} \setminus \{i\}$. The second condition $(\bar{a}, c) \notin u[\bar{\omega}|_i g]$ means that for every $\bar{d} = (d_1, \ldots, d_n) \in A^n$ the following implication

$$\bigwedge_{j=1, j \neq i}^{n} (\bar{a}, d_j) \in \omega_j \wedge (\bar{d}, c) \in u \longrightarrow (\bar{a}, d_i) \notin g$$

is valid. From this implication for $\bar{d} = \bar{b}$ we obtain $(\bar{a}, b_i) \notin g$. So $(\bar{a}, b_i) \in f \setminus g$. Thus, $(\bar{a}, c) \in u[\bar{\omega}|_i (f \setminus g)]$. Consequently,

$$u[\bar{w}\,|_i f] \setminus u[\bar{w}\,|_i\, g] \subset u[\bar{w}\,|_i (f \setminus g)],$$

which together with the previous inclusion proves (43). □

As a consequence of the above proposition and [2, Thm. 1] we obtain

Corollary 2 *A difference Menger algebra $(\Phi, \mathcal{O}, \setminus)$ of reversive n-place functions is a strong subtraction Menger algebra of rank n.*

7. Let $(G, o, -)$ be a weak (strong) subtraction Menger algebra of rank n. A nonempty subset $F \subset G$ is called a *filter* of $(G, o, -)$ if:

(1) $0 \notin F$;
(2) $x \in F \wedge x \leqslant y \longrightarrow y \in F$;
(3) $x \in F \wedge y \in F \longrightarrow x \curlywedge y \in F$

for all $x, y \in G$.

If $a, b \in G$ and $a \nleqslant b$, then $[a) = \{x \in G \mid a \leqslant x\}$ is a filter such that $a \in [a)$ and $b \notin a$. By Zorn's Lemma, the family of all filters that contain a but not b has a maximal element, which we denote by $F_{a,b}$. Using this maximal element we define:

$$W_{a,b} = \{x \in G \mid (\forall t \in T_n(G))\, t(x) \notin F_{a,b}\},$$
$$\varepsilon_{a,b} = \{(x, y) \in G \times G \mid x \curlywedge y \notin W_{a,b} \vee x, y \in W_{a,b}\},$$
$$\varepsilon^*_{a,b} = \varepsilon_{a,b} \cup \{(e_1, e_1), \ldots, (e_n, e_n)\}.$$

The pair $(\varepsilon^*_{a,b}, W_{a,b})$ is a determining pair for the Menger algebra (G, o) provided that the relation $\leqslant$ defined by (8) is weakly steady [2, Prop. 10]. In the proof of Theorem 2 we have shown that steadiness of the relation $\leqslant$ implies its weak steadiness. Therefore all properties of the determining pair $(\varepsilon^*_{a,b}, W_{a,b})$ connected with steadiness of $\leqslant$ remain true. The sum

$$P = \sum_{a,b\in G, a\nleqslant b} P_{(\varepsilon^*_{a,b}, W_{a,b})} \tag{44}$$

of the simplest representations $\left(P_{(\varepsilon^*_{a,b}, W_{a,b})}\right)_{a,b\in G, a\nleqslant b}$ of the algebra (G, o) is a representation of (G, o) by n-place functions (cf. [3, Sect. 2.7]). So, P is a faithful representation of the weak subtraction Menger algebra $(G, o, -)$ by n-place functions (cf. [2, Thm. 2]).

Proposition 5 *If in a strong subtraction Menger algebra* $(G, o, -)$ *of rank* n *the relation* $\leqslant$ *defined by* (8) *is steady, then*

$$(u[\bar{w}|_i\, x], u[\bar{w}|_i\, y]) \in \varepsilon_{a,b} \ \wedge \ u[\bar{w}|_i\, x] \notin W_{a,b} \longrightarrow (x, y) \in \varepsilon_{a,b} \tag{45}$$

for all $a, b \in G$, *i.e., the determining pair* $(\varepsilon^*_{a,b}, W_{a,b})$ *satisfies* (1).

Proof Let the premise of (45) be satisfied. Then, $u[\bar{w}|_i\, x] \curlywedge u[\bar{w}|_i\, y] \notin W_{a,b}$. Thus, $t(u[\bar{w}|_i\, x] \curlywedge u[\bar{w}|_i\, y]) \in F_{a,b}$ for some polynomial $t \in T_n(G)$. But

$$t(u[\bar{w}|_i\, x] \curlywedge u[\bar{w}|_i\, y]) = t(u[\bar{w}|_i\, x]) \curlywedge t(u[\bar{w}|_i\, y]) = t(u[\bar{w}|_i(x \curlywedge y)],$$

by Proposition 3. Hence, $t(u[\bar{w}|_i(x \curlywedge y)] \in F_{a,b}$, which means that $x \curlywedge y \notin W_{a,b}$. So, $(x, y) \in \varepsilon_{a,b}$. □

From Proposition 5 it follows that the simplest representation $P_{(\varepsilon^*_{a,b}, W_{a,b})}$ of a strong subtraction Menger algebra of rank n in which the relation $\leqslant$ is steady is in fact a representation by reversive n-place functions. So, the representation P given by (44) is an isomorphism between the strong subtraction Menger algebra $(G, o, -)$ of rank n and a difference Menger algebra $(\Phi, \mathcal{O}, \setminus)$ of reversive n-place functions, where $\Phi = \{P(g) \mid g \in G\}$. Thus, according to Theorem 2, we have proved the following theorem:

Theorem 3 *A strong subtraction Menger algebra of rank* n *is isomorphic to a difference Menger algebra of reversive* n*-place functions if and only if it satisfies one of the conditions of Theorem* 2.

Corollary 3 *The class of strong subtraction Menger algebras* $(G, o, -)$ *of rank* n *isomorphic to difference Menger algebras of reversive* n*-place functions is a subvariety of the variety of subtraction Menger algebras.*

The variety of weak subtraction Menger algebras of rank n is defined by the superassociativity and the identities (2), (3), (4), (27) and (28). The variety of strong subtraction Menger algebras of rank n is defined by the superassociativity and the identities (2), (3), (4), (27) and (30).

Problem 1 Is the variety of subtraction Menger algebras of rank n finitely based?

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