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RESEARCH ARTICLE

Arens regularity of certain weighted semigroup algebras and countability

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Abstract In this paper, among other things, we show that for a large class of semigroups, the Arens regularity of the weighted semigroup algebra $\ell_1(S, \omega)$ implies the countability of S. This generalizes the main result of Craw and Young (Q J Math 25:351–358, 1974), stating that every countable semigroup admits a weight ω for which $\ell_1(S, \omega)$ is Arens regular and no uncountable group admits such a weight.

Keywords Arens regularity · Weighted semigroup algebra · Completely simple semigroup · Inverse semigroup

1 Introduction

Arens [2] introduced two (Arens) multiplications on the second dual of a Banach algebra turning it into a Banach algebra. A Banach algebra is said to be Arens regular if the Arens multiplications coincide on its second dual. The Arens regularity of the semigroup algebra $\ell_1(S)$ and the weighted semigroup algebra $\ell_1(S, \omega)$ have been

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investigated in [7] and [3,4], respectively. Recent developments on the Arens regularity of $\ell_1(S, \omega)$ are presented in [5].

In this paper we first show that the Arens regularity of a weighted semigroup algebra is stable under certain homomorphisms of semigroups (Proposition 3.1). Then we study those semigroups for which the Arens regularity of $\ell_1(S, \omega)$ necessities the countability of S (a known example for such a semigroup is actually a group; see [4]). As the main aim of the paper we shall show that for a wide variety of semigroups the Arens regularity of $\ell_1(S, \omega)$ implies that S is countable; (Theorem 4.5 infra).

2 Preliminaries

Let S be a semigroup and $\omega : S \to (0, \infty)$ be a weight on S, i.e. $\omega(st) \le \omega(s)\omega(t)$ for all $s, t \in S$. Then we define the mapping $\Omega : S \times S \to (0, 1]$ by $\Omega(s, t) = \frac{\omega(st)}{\omega(s)\omega(t)}, \quad (s, t \in S)$.

Definition 2.1 Let X, Y be sets and f be a complex-valued function on $X \times Y$.

(1) We say that f is cluster on $X \times Y$ if for each pair of sequences (x_n), (y_m) of distinct elements of X, Y, respectively

$$\lim_n \lim_m f(x_n, y_m) = \lim_m \lim_n f(x_n, y_m) \tag{2.1}$$

whenever both sides of (2.1) exist.

(2) If f is cluster and both sides of 2.1 are zero (respectively positive) in all cases, we say that f is 0-cluster (respectively positive cluster).

For a weight ω on a semigroup S,

$$\ell_1(S, \omega) := \left\{ f : S \to \mathbb{C} : \|f\|_{\omega,1} = \sum_{s \in S} |f(s)|\omega(s) < \infty \right\}$$

is a Banach algebra under the pointwise linear space operations, the convolution multiplication and the norm $\|\cdot\|_{\omega,1}$. Many aspects of the weighted semigroup algebra $\ell_1(S, \omega)$ are investigated in the extensive memoir [5]. For ease of reference we quote the following criterion from [3] which will be frequently used in the sequel.

Theorem 2.2 ([3, Lemma 3.1 and Theorem 3.3]) *For a weighted semigroup algebra $\ell_1(S, \omega)$, the following statements are equivalent.*

(i) *$\ell_1(S, \omega)$ is regular.*

(ii) *The map $(x, y) \mapsto h(xy)\Omega(x, y)$ is cluster on $S \times S$ for every bounded function h on S.*

(iii) *For each pair of sequences (x_n), (y_m) of distinct points of S there exist subsequences (a_n), (b_m) of (x_n), (y_m), respectively, such that*
- *either* $\lim_n \lim_m \Omega(a_n, b_m) = 0 = \lim_m \lim_n \Omega(a_n, b_m)$,
- *or the matrix $(a_n b_m)$ is of type C (i.e. either the rows or the columns of $(a_n b_m)$ are constant and distinct).*

In particular, if Ω is 0-cluster then $\ell_1(S, \omega)$ is regular.

3 Arens regularity of $\ell_1(S, \omega)$ and homomorphisms

Let $\psi : S \to T$ be a homomorphism of semigroups. If ω is a weight on T, then trivially $\overleftarrow{\omega}(s) := \omega(\psi(s))$ defines a weight on S.

If $\psi : S \to T$ is an epimorphism and ω is a bounded below (that is, $\inf \omega(S) > 0$) weight on S then a direct verification reveals that $\overrightarrow{\omega}(t) := \inf \omega(\psi^{-1}(t))$ defines a weight on T.

We present the next elementary result concerning to the stability of regularity under homomorphisms.

Proposition 3.1 *Let $\psi : S \to T$ be a homomorphism of semigroups.*

(i) *If ψ is onto and ω is a bounded below weight on S then the regularity of $\ell_1(S, \omega)$ necessities the regularity of $\ell_1(T, \overrightarrow{\omega})$. Furthermore, if Ω is 0-cluster, then so is $\overrightarrow{\Omega}$.*

(ii) *For a weight ω on T if $\ell_1(S, \overleftarrow{\omega})$ is regular, then $\ell_1(T, \omega)$ is regular.*

Proof To prove (i), since ω is bounded below, we can assume that, $\inf \omega(S) \geq \varepsilon > 0$, for some $\varepsilon < 1$, so $\overrightarrow{\omega} \geq \varepsilon$. Let (x_n), (y_m) be sequences of distinct elements in T. Then there are sequences of distinct elements (s_n), (t_m) in S such that

$$\begin{cases} \overrightarrow{\omega}(x_n) > \omega(s_n)(1-\varepsilon) \quad \text{and} \quad \psi(s_n) = x_n, \\ \overrightarrow{\omega}(y_m) > \omega(t_m)(1-\varepsilon) \text{ and } \quad \psi(t_m) = y_m. \end{cases}$$

It follows that $\overrightarrow{\omega}(x_n)\overrightarrow{\omega}(y_m) > \omega(s_n)\omega(t_m)(1-\varepsilon)^2$ and so from $\overrightarrow{\omega}(x_n y_m) \leq \omega(s_n t_m)$ we get $\frac{\overrightarrow{\omega}(x_n y_m)}{\overrightarrow{\omega}(x_n)\overrightarrow{\omega}(y_m)} \leq \frac{1}{(1-\varepsilon)^2}\frac{\omega(s_n t_m)}{\omega(s_n)\omega(t_m)}$; or equivalently,

$$\overrightarrow{\Omega}(x_n, y_m) \leq \frac{1}{(1-\varepsilon)^2}\Omega(s_n, t_m), \quad (n, m \in \mathbb{N}). \tag{3.1}$$

Applying the inequality (3.1), a standard argument based on part (iii) of Theorem 2.2 shows that if $\ell_1(S, \omega)$ is regular then $\ell_1(T, \overrightarrow{\omega})$ is regular.

To prove (ii), let h be a bounded function on T. As $\ell_1(S, \overleftarrow{\omega})$ is regular, by Theorem 2.2, the mapping $(s, t) \to h \circ \psi(st)\overleftarrow{\Omega}(s, t)$ is cluster on $S \times S$. It follows that the mapping $(x, y) \mapsto h(xy)\Omega(x, y)$ is cluster on $T \times T$ which implies that $\ell_1(T, \omega)$ is regular. □

Corollary 3.2 *Let $\psi : S \to T$ be a homomorphism of semigroups. If $\ell_1(S)$ is Arens regular then $\ell_1(T, \omega)$ is Arens regular, for every weight function ω on T.*

Proof Let $\ell^1(S)$ be Arens regular and let ω be a weight on T. Then $\ell^1(S, \overleftarrow{\omega})$ is Arens regular by [3, Corollary3.4]. Proposition 3.1 now implies that $\ell_1(T, \omega)$ is Arens regular. □

4 Arens regularity of $\ell_1(S, \omega)$ and the countability of S

We commence with the following elementary lemma that will be used frequently in the sequel.

Lemma 4.1 *A nonempty set X is countable if and only if there exists a function $f : X \to (0, \infty)$ such that the sequence $(f(x_n))$ is unbounded for every sequence (x_n) of distinct elements in X.*

Proof If $X = \{x_n : n \in \mathbb{N}\}$ is countable the $f(x_n) = n$ is the desired function. For the converse, suppose that such a function $f : X \to (0, \infty)$ exists. Since $X = \cup_{n\in\mathbb{N}}\{x \in X : f(x) \leq n\}$ and each of the sets $\{x \in X : f(x) \leq n\}$ is countable, so X is countable. □

Applying the latter lemma we give the next result of [4] with a slightly simpler proof.

Corollary 4.2 (See [4, Corollary1]) *On every countable semigroup S there exists a bounded below weight ω such that Ω is 0-cluster. In particular, $\ell_1(S, \omega)$ is Arens regular.*

Proof Let F be the free semigroup generated by the countable semigroup $S = \{a_k : k \in \mathbb{N}\}$. For every element $x \in F$ (with the unique presentation $x = a_{k_1}a_{k_2}\cdots a_{k_r}$) set $\omega_1(x) = 1 + k_1 + k_2 + \cdots k_r$. A direct verification shows that ω_1 is a weight on F with $1 \leq \omega_1$, and that Ω_1 is 0-cluster. Let $\psi : F \to S$ be the canonical epimorphism. Set $\omega := \overrightarrow{\omega_1}$. By Proposition 3.1, ω is our desired weight on S. □

The next result is a converse to the Corollary 4.2.

Proposition 4.3 *If there exists a bounded below weight ω on S such that Ω is 0-cluster, then S is countable.*

Proof Let ω be a bounded below weight for which Ω is 0-cluster. Let $\epsilon > 0$ satisfy $\omega \geq \epsilon$. Let S be uncountable. By Lemma 4.1 there is a sequence (s_n) of distinct elements in S and $n_0 \in \mathbb{N}$ such that $\omega(s_n) \leq n_0$ for all $n \in \mathbb{N}$. For all subsequences $(s_{n_k}), (s_{m_l})$ of (s_n) we get

$$\Omega(s_{n_k}, s_{m_l}) = \frac{\omega(s_{n_k}s_{m_l})}{\omega(s_{n_k})\omega(s_{m_l})} \geq \frac{\epsilon}{n_0^2}, \quad (k, l \in \mathbb{N}),$$

contradicting the 0-clusterity of Ω. □

Combining Corollary 4.2 and Proposition 4.3 we arrive to the next result.

Corollary 4.4 *For every semigroup S the following assertions are equivalent.*

(1) *There is a bounded below weight ω on S such that Ω is* 0*-cluster.*
(2) *S is countable.*

As it has been noted in Theorem 2.2, if Ω is 0-cluster then $\ell_1(S, \omega)$ is Arens regular; and, the converse is not true, in general. For example, $\ell_1(S)$ is Arens regular for every zero semigroup S, but the weight $\omega = 1$ is not 0-cluster. However, the converse holds in the case where S is weakly cancellative; (see [3, Corollary3.8]).

In the sequel, we replace the condition "Ω is 0-cluster" by "$\ell_1(S, \omega)$ is Arens regular" and investigate how it influences the countability of S. Indeed, as we shall see in the next result, for a wide variety of semigroups (including Brandt semigroups, (0-)simple inverse semigroups and inverse semigroups with finite set of idempotents; see [6]) the Arens regularity of the weighted semigroup algebra $\ell^1(S,\omega)$ necessities the countability of S.

Theorem 4.5 *For every Brandt semigroup (resp. completely [0-]simple semigroup, [0-]simple inverse semigroup, inverse semigroup with finite set of idempotents) S, the following statements are equivalent.*

(1) *There is a bounded below weight ω on S such that $\ell_1(S,\omega)$ is Arens regular.*
(2) *S is countable.*

Proof If S is countable then Corollary 4.2 guarantees the existence of a weight ω satisfying (1). For the converse, let ω be a weight on S such that $\ell_1(S,\omega)$ is Arens regular. Then in either of the following cases ((i), (ii), (iii)) we shall show that S is countable.

(i) Let $S = M^0(G, I, I, \Delta)$ be an infinite Brandt semigroup (see [6]), and let $\omega \geq \epsilon > 0$. We show that $G \times I$ is countable. First, let (i_n) be an arbitrary sequence of distinct elements in I and let $x_n = (i_n, 1, i_1)$, $y_m = (i_1, 1, i_m)$. Then $x_n.y_m = (i_n, 1, i_m)$ and so for any subsequences $(a_n), (b_m)$ of $(x_n), (y_m)$, respectively, the matrix $(a_n b_m)$ is not of type C, so by Theorem 2.2, $\lim_n \lim_m \Omega(a_n, b_m) = 0 = \lim_m \lim_n \Omega(a_n, b_m)$. But $\Omega(a_n, b_m) \geq \frac{\epsilon}{\omega(a_n)\omega(b_m)}$, for each m, n, and this implies that either $(\omega(a_n))$ or $(\omega(b_m))$ is unbounded. Let $(\omega(a_n))$ be unbounded, and define $f(i) := \omega(i, 1, i_1) \quad (i \in I)$. By Lemma 4.1, I is countable.

Let (s_n) be an arbitrary sequence of distinct elements in G, and let $x_n = (i_0, s_n, i_0)$, $y_m = (i_0, s_m, i_0)$. Then $x_n.y_m = (i_0, s_n s_m, i_0)$ and for any subsequences $(a_n), (b_m)$ of $(x_n), (y_m)$, respectively, it is easy to verify that the matrix $(a_n b_m)$ is not of type C. Thus by Theorem 2.2, $\lim_n \lim_m \Omega(a_n, b_m) = 0 = \lim_m \lim_n \Omega(a_n, b_m)$. But $\Omega(a_n, b_m) \geq \frac{\epsilon}{\omega(a_n)\omega(b_m)}$, and this implies that either $(\omega(a_n))$ or $(\omega(b_m))$ is unbounded. Define $f(s) := \omega(i_0, s, i_0), \quad (s \in G)$. By Lemma 4.1, G is countable. By what we have shown the Brandt semigroup $S = M^0(G, I, I, \Delta)$ is countable.

(ii) Suppose that S is completely $0-$simple, then as it has been explained in [6], S has the presentation $S \cong M^0(G, I, \Lambda; P) = (I \times G \times \Lambda) \cup \{0\}$, equipped with the multiplication

$$(i, a, \lambda)(j, b, \mu) = \begin{cases} (i, a p_{\lambda j} b, \mu) & \text{if} \quad p_{\lambda j} \neq 0 \\ 0 & \text{if} \quad p_{\lambda j} = 0, \end{cases}$$
$$(i, a, \lambda)0 = 0(i, a, \lambda) = 0.$$

Fix $i_0 \in I$, $\lambda_0 \in \Lambda$ such that $p_{\lambda_0 i_0} \neq 0$ and define $f : I \times \Lambda \to (0, \infty)$ by $f(i, \lambda) = \omega(i, 1, \lambda_0)\omega(i_0, 1, \lambda)$. Let (i_n, λ_n) be a sequence of distinct elements in $I \times \Lambda$ and set $x_n = (i_n, 1, \lambda_0), \quad y_m = (i_0, 1, \lambda_m)$. It is readily verified that if $p_{\lambda_0 i_n} \neq 0$ then $x_n x_m = (i_n, p_{\lambda_0 i_0}, \lambda_m)$ for all $n, m \in \mathbb{N}$. So for any subsequences $(a_n), (b_m)$ of $(x_n), (y_m)$, respectively, the matrix $(a_n b_m)$ is not of type C. Thus by Theorem 2.2 $\lim_n \lim_m \Omega(a_n, b_m) = 0 = \lim_m \lim_n \Omega(a_n, b_m)$. But $\Omega(a_n, b_m) \geq \frac{\epsilon}{\omega(a_n)\omega(b_m)}$, and this implies $\omega(a_n)$ or $\omega(b_m)$ is unbounded. Thus $f(i_n, \lambda_n) = \omega(a_n)\omega(b_n)$ is unbounded sequence. Lemma 4.1 implies that $I \times \Lambda$ is countable.

We are going to show that G is also countable. Set $\omega_0(g) = \omega(i_0, gp^{-1}_{\lambda_0,i_0}, \lambda_0) \quad (g \in G)$. Then ω_0 is a weight on G such that Ω_0 is 0-cluster and by Corollary 4.4 G is countable. Therefore S is countable as claimed. The case where S is completely simple needs a similar argument.

(iii) Let S be an inverse semigroup such that the set of idempotents in S is finite. By [6] there is a principal series

$$S = S_0 \supsetneqq S_1 \supsetneqq \cdots \supsetneqq S_m = G \supsetneqq S_{m+1} = \emptyset,$$

for some group G, such that S_i/S_{i+1} is a Brandt semigroup, for each i, $\quad (0 \leq i \leq m)$. For each i set $\omega_i := \overrightarrow{\omega_{|S_i}}$ for the natural epimorphism $S_i \to S_i/S_{i+1}$. By Proposition 3.1 $\ell_1(S_i/S_{i+1}, \omega_i)$ is Arens regular and now part (i) of the proof implies that S_i/S_{i+1} is countable. We thus have that $S_i \setminus S_{i+1}$is countable, for $0 \leq i \leq m$. Therefore $S = \bigcup_{i=0}^{m}(S_i \setminus S_{i+1})$ is countable, as required. Proof for the 0-simple inverse semigroup case is similar. □

The next example illustrates that, a countable (Brandt) semigroup S may admits some weight ω for which $\ell_1(S, \omega)$ is not Arens regular.

Example 4.6 Let $S = M^0(\{e\}, \mathbb{N}, \mathbb{N}, \Delta)$. Define $\omega : S \to [1, 3]$ by

$$\omega(n, e, m) = 1 + \frac{1}{n} + \frac{1}{m}, \quad \omega(0) = 1.$$

Let $x_n = (n, e, 1)$, $y_m = (1, e, m)$. Then $x_n.y_m = (n, e, m)$ and so for any subsequences (a_n), (b_m) of (x_n), (y_m), respectively, the matrix $(a_n b_m)$ is not of type C and also

$$\lim_n \lim_m \Omega(a_n, b_m) = \lim_m \lim_n \Omega(a_n, b_m) \neq 0.$$

Thus by Theorem 2.2, $\ell_1(S, \omega)$ is not Arens regular.

As an immediate consequence of Theorem 4.5 we present the next result which has already proved in [1].

Corollary 4.7 *For every Brandt semigroup (resp. completely [0-]simple semigroup, [0-]simple inverse semigroup, inverse semigroup with finite set of idempotents) S, the following statements are equivalent.*

(1) *$\ell_1(S)$ is Arens regular.*
(2) *S is finite.*

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