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RESEARCH ARTICLE

On the set of elasticities in numerical monoids

Thomas Barron[1] · **Christopher O'Neill**[2] · **Roberto Pelayo**[3]

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Abstract In an atomic, cancellative, commutative monoid S, the elasticity of an element provides a coarse measure of its non-unique factorizations by comparing the largest and smallest values in its set of factorization lengths (called its length set). In this paper, we show that the set of length sets $\mathcal{L}(S)$ for any arithmetical numerical monoid S can be completely recovered from its set of elasticities $R(S)$; therefore, $R(S)$ is as strong a factorization invariant as $\mathcal{L}(S)$ in this setting. For general numerical monoids, we describe the set of elasticities as a specific collection of monotone increasing sequences with a common limit point of $\max R(S)$.

Keywords Factorization · Numerical monoid · Elasticity · Length set · Arithmetic sequence

1 Introduction

In studying the non-unique factorization theory of atomic monoids, the development of several invariants—such as delta sets [2] and ω-primality [7]—has provided significant insight. Of particular interest is the set of length sets $\mathcal{L}(S)$ for an atomic monoid S,

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✉ Roberto Pelayo
robertop@hawaii.edu

Thomas Barron
thomas.barron@uky.edu

Christopher O'Neill
coneill@math.tamu.edu

1 Mathematics Department, University of Kentucky, Lexington, KY 40506, USA

2 Mathematics Department, Texas A&M University, College Station, TX 77843, USA

3 Mathematics Department, University of Hawai'i at Hilo, Hilo, HI 96720, USA

which has as its elements the sets of factorization lengths of elements in S [1,5,8]. The following longstanding conjecture states that, with one exception, the set of length sets is a perfect invariant for the important class of *block monoids* $\mathcal{B}(G)$ of zero-sum sequences over a finite Abelian group G [6, Sect. 7.3].

Conjecture 1.1 *Given two finite Abelian groups G and G' with $|G|, |G'| > 3$, we have $\mathcal{L}(\mathcal{B}(G)) = \mathcal{L}(\mathcal{B}(G'))$ implies $\mathcal{B}(G) \cong \mathcal{B}(G')$.*

In contrast to the above conjecture, the authors of [1] show that two distinct numerical monoids (co-finite, additive submonoids of $\mathbb{N}$) can have the same length sets. In this paper, we investigate the elasticity $\rho(n)$ of elements n in a numerical monoid S. This invariant, computed as the quotient of the largest factorization length by the smallest, provides a coarse measure of an element's non-unique factorizations. We now state our main result concerning the set $R(S) = \{\rho(n) : n \in S\}$ of elasticities of S.

Theorem 1.2 *For distinct arithmetical numerical monoids $S = \langle a, a+d, \ldots, a+kd\rangle$ and $S' = \langle a', a'+d', \ldots, a'+k'd'\rangle$, the following are equivalent:*

1. $R(S) = R(S')$.
2. $\mathcal{L}(S) = \mathcal{L}(S')$.

Therefore, for the class of arithmetical numerical monoids (numerical monoids generated by an arithmetic sequence), the set of elasticities is as strong an invariant as the set of length sets. In contrast, we also provide Example 3.11, which gives two non-arithmetical numerical monoids with identical sets of elasticities, but distinct sets of length sets.

After developing our main result in Sect. 3, we provide a full characterization of the set of elasticities for any numerical monoid, thereby completing a coarser description provided by Chapman et al. [4]. This characterization (Corollary 4.5) demonstrates the stark contrast between the set of length sets, which is often very large and hard to compute, with the set of elasticities, which we describe as a union of monotonically increasing sequences with a common limit point of $\max R(S)$. For arithmetical numerical monoids, this characterization of $R(S)$ takes the form of a complete parametrization (Theorem 3.4).

2 Background

In this section, we provide definitions and previous results related to the elasticity of elements in a numerical monoid. In what follows, let $\mathbb{N}$ denote the set of non-negative integers. Unless otherwise stated, we will assume that S has minimal generating set $\{g_1, \ldots, g_k\}$ with $g_1 < \cdots < g_k$ and $\gcd(g_1, \ldots, g_k) = 1$.

Definition 2.1 Let $S = \langle g_1, \ldots, g_k\rangle$ be a numerical monoid with minimal generating set $\{g_1, \ldots, g_k\}$, and fix $n \in S$. An element $\vec{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$ is a *factorization* of n if $n = a_1g_1 + \cdots + a_kg_k$, and its *factorization set* is given by

$$\mathsf{Z}(n) = \left\{(a_1, \ldots, a_k) \in \mathbb{N}^k : a_1g_1 + \cdots + a_kg_k = n\right\}.$$

The *length* of the factorization $\vec{a}$, denoted $|\vec{a}|$, is given by $a_1 + \cdots + a_k$. For each n, the *length set* of n is the set $\mathsf{L}(n) = \{|a| \ : \vec{a} \in \mathsf{Z}(n)\}$, and the *set of length sets* of the monoid S is given by $\mathcal{L}(S) = \{\mathsf{L}(n) \ : \ n \in S\}$.

Remark 2.2 While the length set of an element in a numerical monoid is a helpful measure of its non-unique factorizations, some information is lost when passing from $\mathsf{Z}(n)$ to $\mathsf{L}(n)$. For example, in $S = \langle 3, 5, 7\rangle$, the element $10 \in S$ has as its two distinct factorizations $(1, 0, 1)$ and $(0, 2, 0)$, both of which have length 2. Thus, even though $\mathsf{L}(10) = \{2\}$ is singleton, the element 10 has multiple factorizations. This phenomenon is common in numerical monoids, especially those minimally generated by arithmetic sequences of length 3 or greater. See [3] for a more detailed analysis on this phenomenon.

In a numerical monoid, length sets of elements are finite. Thus, analyzing the relationship between an element's maximal and minimal lengths provides a meaningful, albeit coarse, gauge of the non-uniqueness of its factorizations. This concept, known as the elasticity of an element, is defined below.

Definition 2.3 For an element $n \in S$ of a numerical monoid, we denote by

$$M_S(n) = \max \mathsf{L}(n) \quad \text{and} \quad m_S(n) = \min \mathsf{L}(n)$$

the *maximal* and *minimal length* of n, respectively. The ratio

$$\rho_S(n) = M_S(n)/m_S(n)$$

is called the *elasticity* of n. When there is no ambiguity, we omit the subscripts and simply write $M(n)$, $m(n)$, and $\rho(n)$. The *set of elasticities* of S is given by

$$R(S) = \{\rho(n) \ : \ n \in S\},$$

and the *elasticity of* S is given by the supremum of this set: $\rho(S) = \sup R(S)$.

Definition 2.4 A numerical monoid S is *arithmetical* if it is minimally generated by an arithmetic sequence of positive integers, that is,

$$S = \langle a, a + d, \ldots, a + kd\rangle$$

for positive integers a, d, and k. Unless otherwise stated, when the generating set of a numerical monoid is expressed in the form $a, a + d, \ldots, a + kd$, it is assumed that $\gcd(a, d) = 1$ and $1 \le k < a$.

We conclude this section by recalling some relevant results from the literature. Theorem 2.5 provides some coarse properties of the set of elasticites of a numerical monoid. Proposition 2.6 is a consequence of [1, Theorem 2.2], and characterizes the functions M_S and m_S for any arithmetical numerical monoid S.

Lastly, Theorem 2.7 appeared as [1, Theorem 3.2] and is vital to the proof of Theorem 1.2.

Theorem 2.5 [4, Theorem 2.1, Corollary 2.3] *If S is a numerical monoid minimally generated by $g_1 < \cdots < g_k$, then $\rho(S) = g_k/g_1$ is the unique accumulation point of $R(S)$, and there exists an $n \in S$ such that $\rho(n) = \rho(S)$.*

Proposition 2.6 *Fix an arithmetical numerical monoid $S = \langle a, a+d, \ldots, a+kd\rangle$ with $\gcd(a, d) = 1$ and $k < a$. For $n \in S$, we have the following.*

(a) *If $n = x(a+kd) - yd$ for $0 \le y < a+kd$, then $m(n) = x$.*
(b) *If $n = x'a + y'd$ for $0 \le y' < a$, then $M(n) = x'$.*

Theorem 2.7 [1, Theorem 3.2] *Fix two distinct numerical monoids $S = \langle a, a+d, \ldots, a+kd\rangle$ and $S' = \langle a', a'+d', \ldots, a'+k'd'\rangle$ for $\gcd(a, d) = \gcd(a', d') = 1$, $1 \le k < a$ and $1 \le k < a'$. The following statements are equivalent:*

(a) $\mathsf{L}(S) = \mathsf{L}(S')$, *and*
(b) $d = d'$, $\frac{a}{k} = \frac{a'}{k'}$, $\gcd(a, k) \ge 2$ *and* $\gcd(a', k') \ge 2$.

3 Elasticity sets for arithmetical numerical monoids

Remark 2.2 demonstrates that information is lost when passing from $\mathsf{Z}(n)$ to $\mathsf{L}(n)$. Since only the ratio of $\max \mathsf{L}(n)$ and $\min \mathsf{L}(n)$ is retained when passing from $\mathsf{L}(n)$ to $\rho(n)$, one might expect that further information is lost when passing from the set of length sets $\mathcal{L}(S)$ to the set of elasticities $R(S)$. While this is true in general (see Example 3.11), when S is an arithmetical numerical monoid, $\mathcal{L}(S)$ can be recovered from $R(S)$. This is the content of Theorem 1.2, the main result of this section.

For an arithmetical numerical monoid $S = \langle a, a+d, \ldots, a+kd\rangle$, Theorem 2.7 states that the values d and a/k can both be recovered from $\mathcal{L}(S)$, and that if $\gcd(a, k) = 1$, then $\mathcal{L}(S)$ cannot coincide with $\mathcal{L}(S')$ for any arithmetical numerical monoid $S' \ne S$. In order to prove Theorem 1.2 we show that each of these results also holds true for the set of elasticities $R(S)$.

The proof of Theorem 1.2 comes in two steps. First, Proposition 3.6 proves that d can be recovered from $R(S)$. This also implies the value of a/k can be recovered; see Remark 3.7. Second, Theorem 3.10 ensures that if $\gcd(a, k) = 1$, then $R(S)$ does not coincide with $R(S')$ for any arithmetical numerical monoid S'.

Example 3.1 Figure 1 plots the elasticities of elements of $S = \langle 7, 12, 17, 22\rangle$. Notice that the graph appears to be a collection of "wedges", each consisting of several "rows" of points with the same elasticity value. Theorem 3.4 uses Proposition 2.6 to eliminate much of the redundancy in $R(S)$ by reparametrizing in terms of these wedges and rows (Definition 3.2), thereby simplifying many computations in results throughout this section. See Remark 3.3 for a description of these values.

Definition 3.2 Fix an arithmetical numerical monoid $S = \langle a, a+d, \ldots, a+kd\rangle$ with $\gcd(a, d) = 1$ and $k < a$. An element $(c, s, x) \in \mathbb{Z}^3$ is an S-*elasticity tuple* if $c \ge 0$, $0 \le s < k$, and

$$\left\lceil \frac{sa}{k} \right\rceil \le x \le \left\lfloor \frac{sa + 2(a-1)}{k} \right\rfloor + d.$$

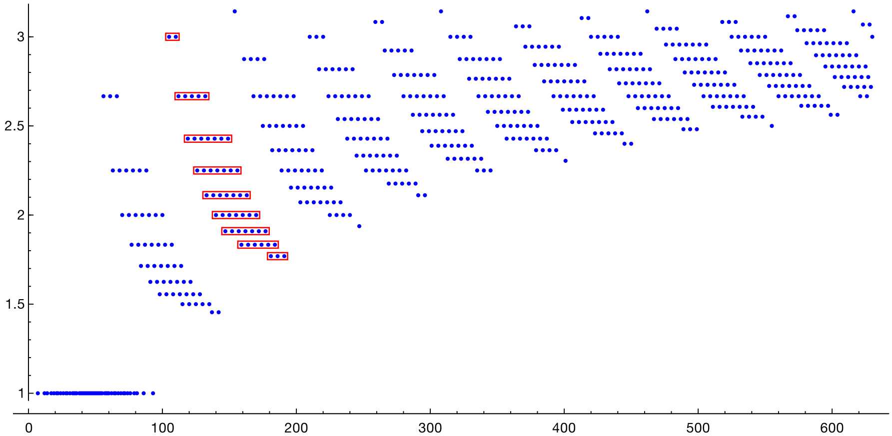

Fig. 1 Plot of elasticities of $S = \langle 7, 12, 17, 22\rangle$

The value $ck + s$ is the *wedge* of (c, s, x), and x is called the *row* of (c, s, x). An S-elasticity tuple (c, s, x) is *minimal* if $x = \lceil \frac{sa}{k} \rceil$ and *maximal* if $x = \left\lfloor \frac{sa+2(a-1)}{k} \right\rfloor + d$. Write $\mathcal{E}(S)$ for the set of S-elasticity tuples, and define $\rho_S : \mathcal{E}(S) \to \mathbb{Q}$ as

$$\rho_S(c, s, x) = \frac{c(a + kd) + x + sd}{ca + x}.$$

Remark 3.3 Let $S = \langle 7, 12, 17, 22\rangle$ as in Example 3.1. Each row in the 2nd wedge is boxed in the elasticity plot in Fig. 1, and each box corresponds to an S-elasticity tuples $(0, 2, x)$ for some $5 \leq x \leq 13$. Each such S-elasticity tuple corresponds to the elasticity $\rho_S(0, 2, x) = (x + 10)/x$ in the plot, yielding the rational values

$$\begin{array}{lll} \rho_S(0, 2, 5) = 3, & \rho_S(0, 2, 6) = 8/3, & \rho_S(0, 2, 7) = 17/7, \\ \rho_S(0, 2, 8) = 9/4, & \rho_S(0, 2, 9) = 19/9, & \rho_S(0, 2, 10) = 2, \\ \rho_S(0, 2, 11) = 21/11, & \rho_S(0, 2, 12) = 11/6, & \rho_S(0, 2, 13) = 23/13, \end{array}$$

appearing from top to bottom in the wedge.

For a general arithmetical $S = \langle a, a + d, \ldots, a + kd\rangle$, each S-elasticity tuple (c, s, x) corresponds to the elasticity $\rho_S(c, s, x)$ occuring in the $(ck + s)$th wedge (where every elasticity in the 0th wedge is 1). Within a given wedge, different values of x corresponds to a different elasticity occuring in that wedge. Minimal S-elasticity tuples (those with a minimal x value for their wedge) correspond to the largest elasticity in the wedge, and each successive value of x corresponds to the next row down in the wedge. Maximal S-elasticity tuples play a key role in Lemma 3.9 and Theorem 3.10; see Example 3.8.

Since the tuple (c, s, x) corresponds to the $(ck + s)$th wedge, it is tempting to use the ordered pair $(ck + s, x)$ in place of (c, s, x) in Definition 3.2. However, the individual values of c and s are used in nearly every proof in this section. In particular, the wedges whose S-elasticity tuples have $s = 0$ are precisely those whose highest elasticity value is sup $R(S)$. Indeed, the arithmetical numerical monoid S depicted in Fig. 1 has $k = 3$, and every third wedge has $\rho(S)$ as its highest value.

We now state Theorem 3.4, which ensures that the parametrization given in Definition 3.2 produces the correct elasticity set.

Theorem 3.4 *If $S = \langle a, a+d, \ldots, a+kd\rangle$ is arithmetical, then $R(S) = \rho_S(\mathcal{E}(S))$.*

Proof We begin by showing that for each $n \in S$, the elasticity $\rho(n) = \rho(c, s, x)$ for some $(c, s, x) \in \mathcal{E}(S)$. First, write $n = x'a + y'd = x''(a + kd) - y''d$ for $x', x'', y', y'' \geq 0$, $y' < a$, and $y'' < a + kd$. By Proposition 2.6, $M(n) = x'$ and $m(n) = x''$, and since $x', x'' \in \mathsf{L}(n)$, we have $d \mid x' - x''$ by [2, Theorem 3.9]. Fix $c \geq 0$ and $0 \leq s < k$ such that $x' - x'' = (ck+s)d$, and let $x = x'' - ca = x' - c(a+kd) - sd$. Notice that

$$\begin{aligned} x(a + kd) - y''d &= \left(x'' - ca\right)(a + kd) - y''d = n - ca(a + kd) \\ &= \left(x' - c(a + kd)\right) a + y'd = (x + sd)a + y'd \end{aligned}$$

which implies that $xk = sa + y' + y''$. Since $y' + y'' \leq 2a + kd - 2$, this means $sa \leq xk \leq sa + 2a + kd - 2$, which yields

$$\left\lceil \frac{sa}{k} \right\rceil \leq x \leq \left\lfloor \frac{sa + 2a + kd - 2}{k} \right\rfloor = \left\lfloor \frac{sa + 2(a-1)}{k} \right\rfloor + d.$$

This means $(c, s, x) \in \mathcal{E}(S)$ and

$$\rho_S(c, s, x) = \frac{c(a + kd) + sd + x}{ca + x} = \frac{x'}{x''} = \rho_S(n),$$

which proves $R(S) \subset \rho_S(\mathcal{E}(S))$.

Conversely, fix $(c, s, x) \in \mathcal{E}(S)$. The assumptions on x ensure that

$$sa \leq xk \leq sa + 2(a - 1) + kd$$

meaning $0 \leq xk - sa \leq a + (a+kd) - 2$. Fix $y', y'' \geq 0$ such that $y' < a$, $y'' < a + kd$, and $y' + y'' = xk - sa$. Choosing

$$n = (c(a + kd) + x + sd)a + y'd = (ca + x)(a + kd) - y''d \in S$$

yields $\rho_S(n) = \rho_S(c, s, x)$, meaning $\rho_S(\mathcal{E}(S)) \subset R(S)$. □

In the terminology of Remark 3.3, Lemma 3.5(a) states that increasing the wedge of an S-elasticity tuple (c, s, x) (that is, increasing the value of $ck + s$) produces a larger elasticity. Additionally, Lemma 3.5(b) states that increasing the row of (c, s, x) (i.e. the value of x) yields a smaller elasticity, confirming that minimal and maximal S-elasticity tuples yield the smallest and largest elasticity within their wedge, respectively.

Lemma 3.5 *Fix an arithmetical numerical monoid $S = \langle a, a+d, \ldots, a+kd\rangle$.*

(a) *Any $(c, s, x), (c', s', x) \in \mathcal{E}(S)$ with $ck + s \le c'k + s'$ satisfy $\rho_S(c, s, x) \le \rho_S(c', s', x)$.*
(b) *Any $(c, s, x), (c, s, x') \in \mathcal{E}(S)$ with $x \le x'$ satisfy $\rho_S(c, s, x') \le \rho_S(c, s, x)$.*

Proof The claim follows directly upon comparing fractions and observing that

$$1 \le \rho_S(c, s, x) = \frac{c(a+kd) + x + sd}{ca + x} \le \frac{a+kd}{a}$$

for every $(c, s, x) \in \mathcal{E}(S)$. □

We now use the parametrization of $R(S)$ provided by Theorem 3.4 to prove Theorem 1.2. We begin with Proposition 3.6, which demonstrates how the value of d can be recovered from $R(\langle a, a+d, \ldots, a+kd\rangle)$.

Proposition 3.6 *Fix an arithmetical numerical monoid $S = \langle a, a+d, \ldots, a+kd\rangle$ with $\gcd(a, d) = 1$ and $1 \le k < a$. We have*

$$d = \frac{(g-1)(f-1)}{g-f}$$

where $1 < f < g$ are the three minimal values in $R(S)$.

Proof First, suppose $k = 1$. The maximal S-elasticity tuple $(1, 0, 2a+d-2)$ gives $f = \rho_S(1, 0, 2a+d-2)$ by Lemma 3.5. We claim $g = \rho_S(1, 0, 2a+d-1)$. Fix an S-elasticity tuple $(c, 0, x)$ with $\rho_S(c, 0, x) > f$. If $c = 1$, then by Lemma 3.5, $\rho_S(c, 0, x) \ge \rho_S(c, 0, 2a+d-1)$. If $c \ge 2$, then by Lemma 3.5, $\rho_S(c, 0, x)$ is minimal when $c = 2$ and when $(c, 0, x)$ is maximal, that is, when $x = 2a+d-2$. Notice that

$$\begin{aligned}(4a+3d-2)(3a+d-1) &= (4a+d-2)(3a+d-1) + d(6a+2d) - 2d\\ &\ge (4a+d-2)(3a+d-1) + d(4a+d) - 2d\\ &= (3a+2d-1)(4a+d-2),\end{aligned}$$

which means

$$\rho_S(2, 0, 2a+d-2) = \frac{4a+3d-2}{4a+d-2} \ge \frac{3a+2d-1}{3a+d-1} = \rho_S(1, 0, 2a+d-1).$$

Substituting these values for f and g gives

$$\frac{(g-1)(f-1)}{g-f} = \frac{d^2}{(3a+d-1)(3a+d-2)} \cdot \frac{(3a+d-1)(3a+d-2)}{d(3a+d-1) - d(3a+d-2)} = d,$$

as desired.

Now, suppose $k \geq 2$, and let $B = \lfloor (3a-2)/k \rfloor + d$. We will show that $f = (B+d)/B$ and $g = (B-1+d)/(B-1)$, from which the claim follows directly. Indeed, solving the first equality for B yields $B = d/(f-1)$, and substituting into the second yields

$$\begin{aligned} g &= (B-1+d)/(B-1) \\ &= (df - (f-1))/(d-(f-1)). \end{aligned}$$

Clearing the denominator on the right hand side yields

$$gd - fd = (g-1)(f-1),$$

and dividing by $g - f$ yields the desired equality.

By Theorem 3.4, $f = \rho_S(c, s, x)$ for some S-elasticity tuple (c, s, x). By Lemma 3.5, (c, s, x) is maximal, and since $f > 1$, we have $c = 0$ and $s = 1$. This gives the desired form for f. It remains to prove that $g = (B-1+d)/(B-1)$.

Fix a S-elasticity tuple (c, s, x) and let $g' = \rho_S(c, s, x)$. By Lemma 3.5, it suffices to assume (c, s, x) is maximal. If $c = s = 0$, then $g' = 1$, and if $c = 0$ and $s = 1$, then $g' = g$. First, suppose $k = 2$. By Lemma 3.5, we can assume $c = 1$ and $s = 0$, meaning $x = \lfloor (2a-2)/2 \rfloor + d = a + d - 1$. Notice that

$$2B = 2\lfloor 3a/2 \rfloor + 2d - 2 \geq 3a + 2d - 2 \geq 2a + 2d + 1 \geq 2a + d + 1.$$

Manipulating the above inequality yields

$$(2a + 3d - 1)(B - 1) \geq (2a + d - 1)(B + d - 1),$$

which gives

$$g' = \frac{c(a+kd) + x + sd}{ca + x} = \frac{2a + 3d - 1}{2a + d - 1} \geq \frac{B + d - 1}{B - 1} = g.$$

Now, suppose $k > 2$. By Lemma 3.5, it suffices to assume $c = 0$ and $s = 2$, and maximality of x gives $x = \lfloor (4a-2)/k \rfloor + d$. Notice that

$$\begin{aligned} 2B - d - 2 &= 2\lfloor (3a-2)/k \rfloor + d - 2 \geq 2\lfloor (3a-2)/k \rfloor - 1 \geq \lfloor (6a-4)/k \rfloor - 2 \\ &\geq \lfloor (4a-2)/k \rfloor + 2\lfloor (a-1)/k \rfloor - 2 \geq \lfloor (4a-2)/k \rfloor = x - d. \end{aligned}$$

Manipulating the inequality above yields

$$(x + 2d)(B - 1) \geq x(B - 1 + d)$$

which gives

$$\frac{x + 2d}{x} \geq \frac{B - 1 + d}{B - 1} = g.$$

This completes the proof. □

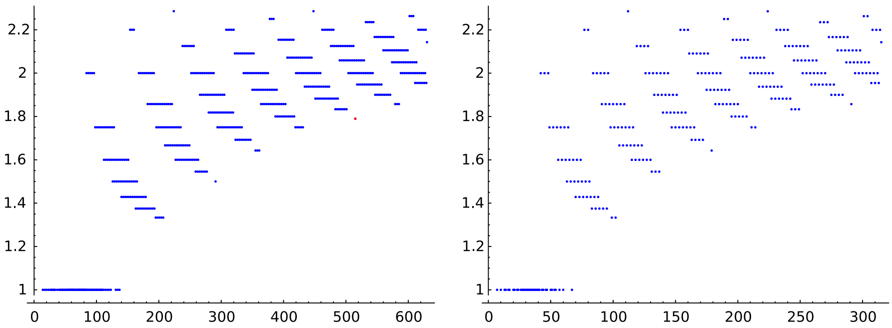

Fig. 2 Plot of elasticities of $S = \langle 14, 17, 20, 23, 26, 29, 32\rangle$ (*left*) and $S' = \langle 7, 10, 13, 16\rangle$ (*right*)

Remark 3.7 Fix an arithmetical numerical monoid $S = \langle a, a+d, \ldots, a+kd\rangle$. Since $\sup R(S) = (a+kd)/a = 1 + d(k/a)$ by Theorem 2.5, Proposition 3.6 also implies that we can recover a/k from $R(S)$.

Example 3.8 The majority of the proof of Theorem 3.10 considers two arithmetical numerical monoids $S = \langle a, a+d, \ldots, a+kd\rangle$ and $S' = \langle a', a'+d', \ldots, a'+k'd'\rangle$ satisfying $d = d'$, $\frac{a}{k} = \frac{a'}{k'}$, $\gcd(a', k') = 1$, and $\gcd(a, k) \geq 2$. In this case, the sets $R(S')$ and $R(S)$ are nearly identical, as are the elasticities achieved within their respective wedges. Figure 2 plots the elasticities of $S = \langle 14, 17, \ldots, 32\rangle$ and $S' = \langle 7, 10, 13, 16\rangle$, and the red points mark the (sparse) elasticities in $R(S) \setminus R(S')$. In general, every elasticity that lies in $R(S) \setminus R(S')$ is achieved by a maximal S-elasticity tuple. Lemma 3.9 produces an $(c, s, x) \in \mathcal{E}(S)$ such that $\rho_S(c, s, x) \notin R(S')$, and the proof of Theorem 3.10 verifies that this is the case.

Lemma 3.9 *Fix an arithmetical numerical monoid $S = \langle a, a+d, \ldots, a+kd\rangle$ with $\gcd(a, d) = 1$ and $1 \leq k < a$, and suppose $\gcd(a, k) \geq 2$. Write $g = \gcd(a, k)$, $k' = k/g$, and $a' = a/g$. There exists a maximal S-elasticity tuple (c, s, x) such that*

(a) $a'(s+2) \equiv 1 \bmod k'$, *and*
(b) $\gcd(ca + x, ck + s) = 1$.

Proof Let s denote the integer satisfying $0 \leq s < k'$ and $a'(s+2) \equiv 1 \bmod k'$, and let

$$x = \left\lfloor \frac{(s+2)a - 2}{k} \right\rfloor + d = \left\lfloor \frac{(s+2)a' - (2/g)}{k'} \right\rfloor + d = \frac{(s+2)a' - 1}{k'} + d$$

denote the value such that $(0, s, x)$ is a maximal S-elasticity tuple. Since $\gcd(a', k') = 1$, there exist integers p and q such that $pa' + qk' = 1$. Notice that

$$xk' = (s+2)a' - 1 + k'd > sa',$$

so fix $b \in \mathbb{Z}$ such that $b(sa' - xk') > px + qs$. For $m = 1 - (p+bk')x - (q-ba')s > 0$, we have

$$\left(p + bk'\right)\left(ma' + x\right) + \left(q - ba'\right)\left(bk' + s\right) = m + \left(p + bk'\right)x + \left(q - ba'\right)s = 1,$$

meaning $\gcd(ma' + x, mk' + s) = 1$. Write $m = cg + r$ for $0 \le r < g$. We see that the S-elasticity tuple $(c, s + rk', x + ra')$ is maximal since

$$x + ra' = \frac{(s+2)a' - 1}{k'} + d + ra' = \frac{\left(s + rk' + 2\right) a' - 1}{k'} + d,$$

and $\gcd(ca + (x + ra'), ck + (s + rk')) = \gcd(ma' + x, mk' + s) = 1$, as desired. □

Theorem 3.10 *If $S = \langle a, a+d, \ldots, a+kd \rangle$ and $S' = \langle a', a'+d', \ldots, a'+k'd' \rangle$ are distinct arithmetical numerical monoids, then $R(S) = R(S')$ if and only if $d = d'$, $\frac{a}{k} = \frac{a'}{k'}$, $\gcd(a, k) \ge 2$ and $\gcd(a', k') \ge 2$.*

Proof If the given conditions are satisfied, then Theorem 2.7 implies $\mathcal{L}(S) = \mathcal{L}(S')$ and thus $R(S) = R(S')$. Conversely, if $R(S) = R(S')$, then $d = d'$ by Proposition 3.6 and $\frac{a}{k} = \frac{a'}{k'}$ by Remark 3.7. To complete the proof, it suffices to show that if $d = d'$, $\frac{a}{k} = \frac{a'}{k'}$, $\gcd(a', k') = 1$ and $\gcd(a, k) \ge 2$, then $R(S) \supsetneq R(S')$.

Since $\frac{a}{k} = \frac{a'}{k'}$, we have $a = ga'$ and $k = gk'$ for $g = \gcd(a, k)$. Define a map $\phi : \mathcal{E}(S') \to \mathcal{E}(S)$ given by $(c', s', x') \mapsto (q, s' + rk', x' + ra')$, where $c' = qg + r$ for $0 \le r < g$. Notice that $(q, s' + rk', x' + ra') \in \mathcal{E}(S)$ since $0 \le s' + rk' < k' + (g-1)k' = k$,

$$\left\lceil \frac{\left(s' + rk'\right) a}{k} \right\rceil = \left\lceil \frac{s'a'}{k'} \right\rceil + ra' \le x' + ra', \tag{3.1}$$

and

$$\begin{aligned} x' + ra' &\le \left\lfloor \frac{\left(s' + 2\right) a' - 2)}{k'} \right\rfloor + d + ra' = \left\lfloor \frac{(s' + rk' + 2)a - 2g)}{k} \right\rfloor + d \\ &\le \left\lfloor \frac{\left(s' + rk' + 2\right) a - 2}{k} \right\rfloor + d. \end{aligned} \tag{3.2}$$

We also have $\rho_{S'}(c', s', x') = \rho_S(q, s' + rk', x' + ra')$, so ϕ preserves elasticity values.

Now, by Lemma 3.9, there exists a maximal S-elasticity tuple (c, s, x) satisfying $a'(s+2) \equiv 1 \bmod k'$ and $\gcd(ca + x, ck + s) = 1$. If $(c, s, x) \in \mathrm{Im}(\phi)$, then it is the image of $(cg + r, s', x - ra') \in \mathcal{E}(S')$. In particular, since

$$\begin{aligned} x - ra' &= \left\lfloor \tfrac{(s+2)a-2}{k} \right\rfloor + d - ra' = \left\lfloor \tfrac{(s+2)a' - (2/g)}{k'} \right\rfloor + d - ra' \\ &= 1 + \left\lfloor \tfrac{(s+2)a'-2}{k'} \right\rfloor + d - ra' = 1 + \left\lfloor \tfrac{(s'+2)a'-2}{k'} \right\rfloor + d, \end{aligned}$$

we must have $(c, s, x) \notin \mathrm{Im}(\phi)$. Moreover, for $(c_0, s_0, x_0) \in \mathcal{E}(S)$, if $c_0k + s_0 > ck + s$, then $\rho_S(c, s, x) < \rho_S(c_0, s_0, x_0)$ by Lemma 3.5, and if $c_0k + s_0 < ck + s$, then

$$c_0a + x_0 \ne \frac{(ca + x)(c_0k + s_0)}{ck + s}$$

as the right hand side is not an integer. This means

$$\rho_S(c, s, x) = \frac{ca + x + d(ck + s)}{ca + x} \neq \frac{c_0 a + x_0 + d(c_0 k + s_0)}{c_0 a + x_0} = \rho_S(c_0, s_0, x_0).$$

We conclude that the elasticity $\rho_S(c, s, x) \in R(S)$ is only achieved by (c, s, x), which implies $\rho_S(c, s, x) \notin R(S')$ and completes the proof. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2 Apply Theorems 2.7 and 3.10. □

We conclude this section with Example 3.11, which shows the "arithmetical" hypothesis in Theorem 1.2 cannot be omitted.

Example 3.11 Theorem 1.2 shows that for an arithmetical numerical monoid S, computation of the elasticity set $R(S)$ (which is given in Theorem 3.4) is just as useful as computing the entire set of length sets $\mathcal{L}(S)$. This need not be true in general. Let $S = \langle 6, 10, 13, 14\rangle$ and $S' = \langle 6, 11, 13, 14\rangle$. A simple computation shows that

$$\{4, 6\} \in \mathcal{L}(S) \setminus \mathcal{L}(S') \text{ and } \rho_S(S \cap [1, 266]) = \rho_{S'}(S' \cap [1, 266]),$$

after which Theorems 4.2 and 4.3 guarantee that $R(S) = R(S')$.

Remark 3.12 It remains an interesting question to characterize which numerical monoids S and S' satisfy $R(S) = R(S)$ and $\mathcal{L}(S) \neq \mathcal{L}(S')$. Investigating this phenomenon for general numerical monoids—or even for specific classes such as those with three minimal generators—would be of much interest.

4 The set of elasticities for general numerical monoids

While Theorem 2.5 provides a concise description of the maximal elasticity attained in a numerical monoid S and a coarse topological property of the set of elasticities of S, it does not give a full description of $R(S)$. In this section, we provide such a description by showing that the functions $M(n)$ and $m(n)$ enjoy a powerful linearity property.

We begin with a combinatorial lemma.

Lemma 4.1 *Let $k \geq 0$, and fix $c_1, c_2, \ldots, c_r \in \mathbb{Z}$ with $r \geq k$. There exists $T \subsetneq \{1, \ldots, r\}$ satisfying $\sum_{i \in T} c_i \equiv \sum_{i=1}^{r} c_i \bmod k$.*

Proof Let $s_j = \sum_{n=1}^{j} c_n$ for $j \in \{0, \ldots, r\}$. The sequence $s_0, s_1, \cdots, s_r$ has length $r + 1 > k$, so by the pigeonhole principle, $s_i \equiv s_j \bmod k$ for some $i < j$. This means $s_j - s_i \equiv 0 \bmod k$, so choosing $T = \{1, \ldots, r\} \setminus \{i + 1, \ldots, j\}$ completes the proof. □

Theorem 4.2 *Given a numerical monoid $S = \langle g_1, \ldots, g_k\rangle$ minimally generated by $g_1 < \cdots < g_k$, the maximal factorization length function $M : S \to \mathbb{N}$ satisfies*

$$M(n) = M(n - g_1) + 1$$

for all $n > (g_1 - 1)g_k$.

Proof Fix a factorization $\vec{a}$ for n, and suppose that $a_2 + \cdots + a_k \geq g_1$. Since $a_1 g_1 + a_2 g_2 + \cdots + a_k g_k = n$, we have $a_2 g_2 + \cdots + a_k g_k \equiv n \bmod g_1$. Viewing this sum as $a_2 + \cdots + a_k$ integers selected from $\{g_2, \ldots, g_k\}$, Lemma 4.1 guarantees the existence of $b_2, \ldots, b_k \geq 0$ such that (i) $b_i \leq a_i$ for each $i > 1$, (ii) $\sum_{i=2}^k a_i > \sum_{i=2}^k b_i$, and (iii) $b_2 g_2 + \cdots + b_k g_k \equiv n \bmod g_1$. This implies $b_2 g_2 + \cdots + b_k g_k < a_2 g_2 + \cdots + a_k g_k$, so there exists $b_1 \geq 0$ so that $\vec{b} = (b_1, b_2, \ldots, b_k) \in \mathsf{Z}(n)$. This gives

$$(b_1 - a_1) g_1 = \sum_{i=2}^{k} (a_i - b_i) g_i > \sum_{i=2}^{k} (a_i - b_i) g_1,$$

from which canceling g_1 yields $|\vec{b}| > |\vec{a}|$.

Now, suppose that $\vec{a} \in \mathsf{Z}(n)$ is maximal. The above argument implies that $a_2 + \cdots + a_k < g_1$. In particular, if $n > (g_1 - 1) g_k$, we must have $a_1 > 0$. This means $\vec{a} - \vec{e}_1 \in \mathsf{Z}(n - g_1)$, so we have $M(n - g_1) \geq |\vec{a}| - 1$, and since $\vec{a}$ has maximal length, we have $M(n - g_1) = |\vec{a}| - 1$. This completes the proof. □

The proof of the following analogous result is almost identical to the proof of Theorem 4.2 and hence omitted.

Theorem 4.3 *Given a numerical monoid $S = \langle g_1, \ldots, g_k \rangle$ minimally generated by $g_1 < \cdots < g_k$, the minimal factorization length function $m : S \to \mathbb{N}$ satisfies*

$$m(n) = m(n - g_k) + 1$$

for all $n > (g_k - 1) g_{k-1}$.

Example 4.4 If a numerical monoid S has g_1 as its smallest minimal generator then Theorems 4.2 and 4.3 state that $M : S \to \mathbb{N}$ as a function will eventually manifest graphically as a collection of g_1 discrete lines with a common slope of $1/g_1$. Similarly, if g_k is the largest minimal generator, then the graph of $m : S \to \mathbb{N}$ will eventually appear as a collection of g_k discrete lines with common slope $1/g_k$. Figure 3, which shows the functions $M(n)$ and $m(n)$ for $S = \langle 5, 16, 17, 18, 19 \rangle$, demonstrates this concept.

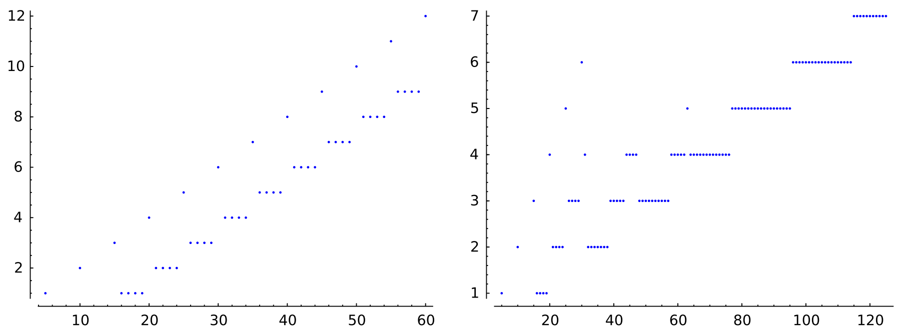

Fig. 3 Plots of $M(n)$ (*left*) and $m(n)$ (*right*) for the numerical monoid $\langle 5, 16, 17, 18, 19 \rangle$

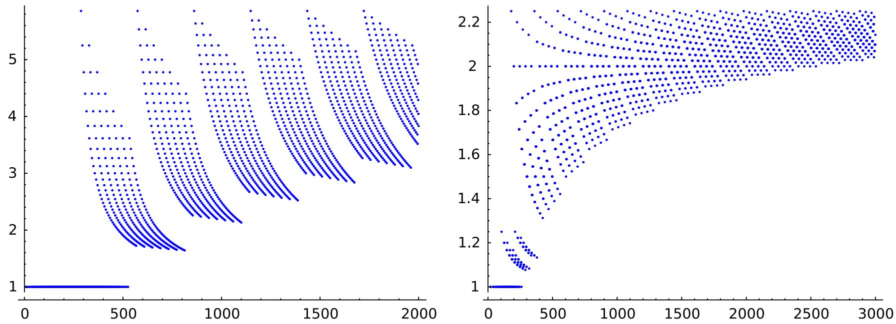

Fig. 4 Plots of the elasticities of elements from the monoids $\langle 7, 41\rangle$ (*left*) and $\langle 20, 21, 45\rangle$ (*right*)

Since the elasticity of an element n in a numerical monoid is given by the quotient of $M(n)$ and $m(n)$, we use Theorems 4.2 and 4.3 to provide a characterization of $R(S)$.

Corollary 4.5 *Fix a numerical monoid S minimally generated by $g_1 < \cdots < g_k$.*

(a) *For $n \geq g_{k-1}g_k$, we have*

$$\rho(n + g_1 g_k) = \frac{M(n) + g_k}{m(n) + g_1}.$$

(b) *The set $R(S)$ is the union of a finite set and a collection of $g_1 g_k$ monotone increasing sequences, each converging to g_k/g_1.*

Proof Part (a) follows directly from Theorems 4.2 and 4.3. From this, it follows that for $g_{k-1}g_k \leq n < g_{k-1}g_k + g_1 g_k$, the sequence

$$\rho(n), \rho(n + g_1 g_k), \rho(n + 2g_1 g_k), \ldots$$

is monotone increasing and converges to g_k/g_1. This completes the proof. □

Remark 4.6 Theorem 2.5 states that the only accumulation point of the elasticity set $R(S)$ is its maximum. Corollary 4.5, on the other hand, gives a characterization of the entire set $R(S)$, from which several other results from [4] can be recovered. In particular, the characterization of the set of elasticities provided in Corollary 4.5 describes $R(S)$ as a union of a finite set and $g_1 g_k$ monotone increasing sequences, each converging to g_k/g_1, which clearly implies that the only accumulation point is g_k/g_1.

Example 4.7 The elasticity graphs for numerical monoids $\langle 7, 41\rangle$ and $\langle 20, 21, 45\rangle$ are given in Fig. 4. The latter of numerical monoids is not arithmetical and demonstrates that the uniformity of the "wedges" enjoyed by arithmetical numerical monoids is not present, especially for smaller values. Regardless, for any numerical monoid $S = \langle g_1, \ldots, g_k\rangle$, the characterization of $R(S)$ provided in Corollary 4.5 shows that ρ can be eventually described as $g_1 g_k$ monotone increasing sequences that limit to g_k/g_1, where each sequence contains precisely one point in each "wedge."

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References

1. Amos, J., Chapman, S., Hine, N., Paixão, J.: *Sets of lengths do not characterize numerical monoids*, Integers **7**, #A50 (2007)
2. Bowles, C., Chapman, S., Kaplan, N., Reiser, D.: On delta sets of numerical monoids. J. Algebr. Appl. **5**, 1–24 (2006)
3. Chapman, S., García-Sánchez, P., Llena, D., Marshall, J.: Elements in a numerical semigroup with factorizations of the same length. Can. Math. Bull. **54**(1), 39–43 (2011)
4. Chapman, S.T., Holden, M.T., Moore, T.A.: Full elasticity in atomic monoids and integral domains. Rocky Mountain J. Math. **36**, 1437–1455 (2006)
5. Geroldinger, A.: A structure theorem for sets of lengths. Colloq. Math. **78**, 225–259 (1998)
6. Geroldinger, A., Halter-Koch, F.: Nonunique Factorization, Algebraic, Combinatorial and Analytic Theory, Pure and Applied Mathematics, vol. 278. Chapman & Hall/CRC, Boca Raton (2006)
7. O'Neill, C., Pelayo, R.: On the linearity of ω-primality in numerical monoids. J. Pure Appl. Algebr. **218**, 1620–1627 (2014)
8. Schmid, W.: A realization theorem for sets of lengths. J. Number Theory **129**(5), 990–999 (2009)